Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 886 (2014) 1029–1045
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Explicit boundary form factors: The scaling Lee–Yang model

L. Hollo [a,*], Z.B. Laczko [b], Z. Bajnok [a]

[a] *MTA Lendület Holographic QFT Group, Wigner Research Centre for Physics, P.O.B. 49, H-1525 Budapest 114, Hungary*
[b] *Roland Eötvös University, Pázmány Péter sétány 1/A, 1117 Budapest, Hungary*

Received 24 June 2014; accepted 17 July 2014
Available online 23 July 2014
Editor: Hubert Saleur

Abstract

We provide explicit expressions for boundary form factors in the boundary scaling Lee–Yang model for operators with the mildest ultraviolet behavior for all integrable boundary conditions. The form factors of the boundary stress tensor take a determinant form, while the form factors of the boundary primary field contain additional explicit polynomials.

1. Introduction

A complete solution of a $1+1$ dimensional integrable QFT means the construction of all of its Wightman functions and the procedure to reach this goal is called the bootstrap programme.

The first step is the so-called S-matrix bootstrap which determines the multiparticle scattering matrix. This S-matrix connects the asymptotic in and out states, and factorizes into pairwise elastic scatterings satisfying unitarity, crossing symmetry and the Yang–Baxter equations. Maximal analyticity is also required: poles of the S-matrix have to be located on the imaginary axis on the rapidity plane and have to be explained by bound states or some Coleman–Thun diagrams (for detailed review see [1–3]). Assuming that the only particle of the model appears also as a bound state the S-matrix bootstrap programme results in the S-matrix of the Lee–Yang model.

* Corresponding author.

http://dx.doi.org/10.1016/j.nuclphysb.2014.07.021

Integrable impurities with nontrivial bulk S-matrices can be either purely transmittive defects or purely reflective boundaries [4,5]. In these cases the S-matrix bootstrap has to be complemented by the determination of the defect transmission or boundary reflection matrices via the T-matrix or R-matrix bootstrap. Requiring unitarity, crossing unitarity and the defect/boundary Yang–Baxter equations together with maximal analiticity determines these matrices up to CDD factors which can be fixed by some physical input [6,7].

The final step in the bootstrap programme is the form factor bootstrap. Form factors are the matrix elements of local operators between asymptotic states. An axiomatic formulation of form factors was initiated in [8] in the bulk and was then extended to the boundary [9] and defect [10] cases. In the bootstrap framework the determination of the form factors consists of finding all solutions of the form factor axioms. These axioms are functional relations which also connect form factors with different particle numbers.

In this axiomatic approach a family of form factor solutions, called tower, corresponds to a local field. As the axioms do not contain any information about these fields the identification is not obvious. However, first for the Ising model [11] and later for the scaling Lee–Yang model [12–14] the space of the form factor solutions was shown to be isomorphic to the space of local operators (see also [15]) and later this isomorphism has been shown level by level [16]. This counting argument was then extended to the boundary Lee–Yang model in [17] and also to the boundary Sinh-Gordon model with Dirichlet boundary condition at the self dual-point [18].

The importance of explicit form factor solutions lies in the fact that they can be used to build up correlation functions by their spectral representations (for review see [8]). However, the explicit solution for towers of form factors is not easy and so far this goal was achieved only for a few models such as for the scaling Lee–Yang model [19] and its defect version [20], for the Sinh-Gordon model [21–23] and for some homogeneous Sine-Gordon models [24,25]. In [18] a closed formula was conjectured for all n-particle form factors for some operators in the boundary Sinh-Gordon model with Dirichlet boundary condition at special coupling. It is worth mentioning that in [26] the boundary one-particle minimal form factors were calculated for the A_n affine Toda field theories and in case of the A_2 theory the solutions of the axioms were given up to four particles for some operators. In this paper, for the first time, we give explicit *boundary* form factor solutions for the lowest lying fields for all possible integrable boundary conditions in the scaling Lee–Yang model.

The paper is organized as follows: In Section 2 we briefly review the scaling Lee–Yang model, the boundary form factor axioms and introduce a useful Ansatz for the form factors. In Section 3 we give the explicit solutions in three steps: first we give the form factor tower for the energy–momentum tensor in case of the identity boundary in a determinant-form. Then by the fusion method we extend this result for the other integrable boundary conditions. Finally, we derive the form factor tower for the remaining boundary primary field. In Section 4 we briefly summarize our results and conclude. The details of the proof for the identity boundary condition are relegated to Appendix A.

2. Boundary form factors: axioms and parametrization

In this section, following [9], we recall the boundary form factor axioms and a parametrization which fulfills them. We analyze the scaling Lee–Yang theory, which is the simplest integrable quantum field theory containing one particle type with mass m. The multiparticle scattering matrix factorizes into pairwise scatterings, which depends on the difference of the rapidities of the particles $S(\theta_1 - \theta_2)$ and takes the following simple form:

$$S(\theta)=\frac{\sinh\theta+i\sin\frac{\pi}{3}}{\sinh\theta-i\sin\frac{\pi}{3}}\equiv-(2)_\theta(4)_\theta;\qquad (x)_\theta=\frac{\sinh(\frac{\theta}{2}+\frac{i\pi x}{12})}{\sinh(\frac{\theta}{2}-\frac{i\pi x}{12})} \tag{2.1}$$

where $p=m\sinh\theta$. The pole of the scattering matrix at $\theta=\frac{2i\pi}{3}$:

$$S(\theta)=i\frac{\Gamma^2}{\theta-\frac{2i\pi}{3}}+\text{reg.};\qquad \Gamma=i\sqrt{2\sqrt{3}} \tag{2.2}$$

signals a bound-state, while the fusion relation

$$S(\theta)=S\left(\theta-i\frac{\pi}{3}\right)S\left(\theta+i\frac{\pi}{3}\right) \tag{2.3}$$

ensures that the bound-state is the particle itself.

In the presence of integrable boundaries the scattering matrix has to be supplemented by the one particle reflection factor which satisfies

$$R(\theta)=R(-\theta)^{-1}=S(2\theta)R(i\pi-\theta) \tag{2.4}$$

In the Lee–Yang theory there are two types of boundary conditions:

- The identity boundary condition, denoted by $\mathbb{I}$, does not have a parameter and its reflection factor is

$$R(\theta)_{\mathbb{I}}=(1)_\theta(3)_\theta(-4)_\theta \tag{2.5}$$

  The pole at $i\frac{\pi}{2}$

$$R(\theta)_{\mathbb{I}}=\frac{g_{\mathbb{I}}^2}{2\theta-i\pi}+\text{reg.};\qquad g_{\mathbb{I}}=-2i\sqrt{(2\sqrt{3}-3)} \tag{2.6}$$

  shows that it can emit a virtual particle with zero energy but there are no bound-states on this boundary.
- The other type of boundary condition can accommodate a parameter, b, as

$$R(\theta)_\Phi=R(\theta)_{\mathbb{I}}(b+1)_\theta(b-1)_\theta(5-b)_\theta(7-b)_\theta \tag{2.7}$$

  which can also emit a virtual zero energy particle with rate

$$g_\Phi(b)=\frac{\tan((b+2)\frac{\pi}{12})}{\tan((b-2)\frac{\pi}{12})}g_{\mathbb{I}} \tag{2.8}$$

  This boundary has bound states for $b>-1$, see [27] for the details.

The scaling Lee–Yang theory is the only relevant perturbation of the conformally invariant Lee–Yang model. This model has two conformal invariant boundary conditions and the reflection factors correspond to their perturbations. The identity conformal boundary has no relevant operator as only the conformal descendents of the identity operator can live on it. In contrast, the Φ conformal boundary additionally contains the descendents of the boundary operator ϕ with weight $h=-\frac{1}{5}$. In the perturbed theory the integrable boundary perturbation introduces the parameter b, see [27] for the relation between the perturbed CFT and the scattering theory.

2.1. Axioms

Elementary boundary form factors are the matrix elements of local boundary operators between the vacuum and asymptotic states:

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, \dots, \theta_n) = \langle 0|\mathcal{O}(0)|\theta_1, \theta_2, \dots, \theta_n\rangle \tag{2.9}$$

These form factors satisfy the following functional relations [9], which are postulated as axioms[1]:

$$F_n(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S(\theta_i - \theta_{i+1}) F_n(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) \tag{2.10}$$

$$F_n(\theta_1, \dots, \theta_{n-1}, \theta_n) = R(\theta_n) F_n(\theta_1, \dots, \theta_{n-1}, -\theta_n) \tag{2.11}$$

$$F_n(\theta_1, \theta_2, \dots, \theta_n) = R(i\pi - \theta_1) F_n(2i\pi - \theta_1, \theta_2, \dots, \theta_n) \tag{2.12}$$

Additionally, they have singularities with residues, which are related to form factors with less particles:

$$-i \operatorname*{Res}_{\theta=\theta'} F_{n+2}\big(\theta + i\pi, \theta', \theta_1, \dots, \theta_n\big) = \left(1 - \prod_{a=1}^{n} S(\theta - \theta_a) S(\theta + \theta_a)\right) F_n(\theta_1, \dots, \theta_n) \tag{2.13}$$

$$-i \operatorname*{Res}_{\theta=0} F_{n+1}\left(\theta + i\frac{\pi}{2}, \theta_1, \dots, \theta_n\right) = \frac{g}{2}\left(1 - \prod_{a=1}^{n} S\left(i\frac{\pi}{2} - \theta_a\right)\right) F_n(\theta_1, \dots, \theta_n) \tag{2.14}$$

$$-i \operatorname*{Res}_{\theta=\theta'} F_{n+2}\left(\theta + i\frac{\pi}{3}, \theta' - i\frac{\pi}{3}, \theta_1, \dots, \theta_n\right) = \Gamma F_{n+1}(\theta, \theta_1, \dots, \theta_n) \tag{2.15}$$

Each solution of these functional relations with the appropriate asymptotic properties corresponds to a local boundary operator of the theory [17]. In the following we would like to present the simplest explicit solutions. In doing so we introduce first a useful parametrization.

2.2. Parametrization of the form factors

A parametrization, which fulfills the boundary form factor axioms is given by [9]:

$$F_n(\theta_1, \theta_2, \dots, \theta_n) = H_n Q_n(y_1, y_2, \dots, y_n) \prod_{i=1}^{n} \frac{r(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_i - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j}, \tag{2.16}$$

where $f(\theta)$ is the minimal solution of the bulk two particle form factor equations

$$f(\theta) = S(\theta) f(-\theta); \qquad f(i\pi - \theta) = f(i\pi + \theta) \tag{2.17}$$

The function $r(\theta)$ is the minimal one particle boundary form factor which satisfies

$$r(\theta) = R(\theta) r(-\theta); \qquad r(i\pi - \theta) = R(\theta) r(i\pi + \theta) \tag{2.18}$$

and Q_n is a symmetric polynomial of its arguments $y_i = e^{\theta_i} + e^{-\theta_i}$. The polynomiality of Q_n ensures for the correlation functions in the conformal (short distance) limit to exhibit a polynomial separation dependence. Finally, H_n is some appropriately chosen normalization constant proportional to the vacuum expectation value of the operator.

[1] To streamline the notation we suppress the operator in the form factor, if it does not lead to any confusion.

In particular, for the scaling Lee–Yang model the minimal solution of the bulk two particle form factor equation is

$$f(\theta) = \frac{y-2}{y+1} v(i\pi - \theta) v(-i\pi + \theta); \quad y = e^{\theta} + e^{-\theta} \tag{2.19}$$

where

$$\log v(\theta) = 2 \int_0^{\infty} \frac{dt}{t} \frac{\sinh \frac{t}{2} \sinh \frac{t}{3} \sinh \frac{t}{6}}{\sinh^2 t} e^{i \frac{\theta t}{\pi}} \tag{2.20}$$

The minimal one particle form factor depends on the boundary condition. For the identity boundary

$$r_{\mathbb{I}}(\theta) = 4 i u(\theta) \sinh \theta \tag{2.21}$$

with

$$\log u(\theta) = \int_0^{\infty} \frac{dt}{t} \frac{\sinh t - \cos((\frac{i}{2} - \frac{\theta}{\pi})t)(\sinh \frac{5t}{6} + \sinh \frac{t}{2} - \sinh \frac{t}{3})}{\sinh \frac{t}{2} \sinh t} \tag{2.22}$$

and a useful normalization is $H_n = \langle \mathcal{O} \rangle (\frac{i \sqrt[4]{3}}{v(0)\sqrt{2}})^n$.

With these choices the form factor axioms Eqs. (2.10)–(2.12) are automatically satisfied, while the singularity axioms provide recursion relations for the polynomials Q_n:

$$Q_{n+2}(y_+, y_-, y_1, \ldots, y_n) = D_n(y \mid y_1, \ldots, y_n) Q_{n+1}(y, y_1, \ldots, y_n) \tag{2.23}$$

$$Q_{n+2}(y, -y, y_1, \ldots, y_n) = P_n(y \mid y_1, \ldots, y_n) Q_n(y_1, \ldots, y_n) \tag{2.24}$$

$$Q_{n+1}(0, y_1, \ldots, y_n) = B_n(y_1, \ldots, y_n) Q_n(y_1, \ldots, y_n) \tag{2.25}$$

with

$$D_n(y \mid y_1, \ldots, y_n) = \prod_{i=1}^{n} (y + y_i) \tag{2.26}$$

$$P_n(y \mid y_1, \ldots, y_n) = \frac{\prod_{i=1}^{n} (y_i - y_-)(y_i + y_+) - \prod_{i=1}^{n} (y_i + y_-)(y_i - y_+)}{2(y_+ - y_-)} \tag{2.27}$$

$$B_n(y_1, \ldots, y_n) = \frac{\prod_{i=1}^{n} (y_i + \sqrt{3}) - \prod_{i=1}^{n} (y_i - \sqrt{3})}{2\sqrt{3}} \tag{2.28}$$

Here we introduced $y_{\pm} = 2\cosh(\theta \pm i \frac{\pi}{3})$. In the next section we explicitly construct a polynomial solution of these recursion relations.

Let us summarize the similar parametrization for the Φ boundary. We distinguish the form factors and polynomials from those of the identity boundary by a tilde:

$$\tilde{F}_n(\theta_1, \ldots, \theta_n) = H_n \prod_{i=1}^{n} \frac{r_{\Phi}(\theta_i)}{y_i} \prod_{i<j} \frac{f(\theta_1 - \theta_j) f(\theta_i + \theta_j)}{y_i + y_j} \tilde{Q}_n(y_1, \ldots, y_n) \tag{2.29}$$

Here the one particle form factor has poles corresponding to the possible bound-states and takes the form

$$r_\Phi(\theta) = \frac{i\sinh\theta}{(\sinh\theta - i\sin((b-1)\frac{\pi}{6}))(\sinh\theta - i\sin((b+1)\frac{\pi}{6}))}u(\theta) \tag{2.30}$$

Due to this factor the recursion relations are slightly modified:

$$\tilde{Q}_{n+2}(y_+, y_-, y_1, \ldots, y_n) = \big(y^2 - 3 + \alpha\big) D_n(y \mid y_1, \ldots, y_n) \tilde{Q}_{n+1}(y, y_1, \ldots, y_n) \tag{2.31}$$

$$\tilde{Q}_{n+2}(y, -y, y_1, \ldots, y_n) = \big(y^4 - (3+\alpha)y^2 + \alpha^2\big) P_n(y \mid y_1, \ldots, y_n) \tilde{Q}_n(y_1, \ldots, y_n) \tag{2.32}$$

$$\tilde{Q}_{n+1}(0, y_1, \ldots, y_n) = \alpha B_n(y_1, \ldots, y_n) \tilde{Q}_n(y_1, \ldots, y_n) \tag{2.33}$$

where $\alpha = 1 + 2\cos\frac{b\pi}{3}$.

Finally we note that the solutions of the form factor axioms and the space of operators are related. We call the tower of form factor solutions [22,17] the set $\mathcal{F}_\mathcal{O} = \{F_n^\mathcal{O}(\theta_1, \ldots, \theta_n)\}_{n\in\mathbb{N}}$, which satisfies the form factor axioms (2.10)–(2.15). It was pointed out for the bulk Lee–Yang model [14,13] and also for the boundary case [17] that there is a one-to-one correspondence between these towers and the boundary operator content of the ultraviolet conformal field theory.

Every such tower starts with a so-called kernel solution. An nth level kernel solution is defined as a polynomial of n variables whose value is zero at each pole and in case of the boundary Lee–Yang model is given as [17]

$$\sigma_{k_1}^{(n)} \ldots \sigma_{k_l}^{(n)} \prod_{1\le i<j\le n} (y_i + y_j) \prod_{1\le i<j\le n} \big(y_i^2 + y_i y_j + y_j^2 - 3\big) \prod_{i=1}^{n} y_i \tag{2.34}$$

where $0 < k_1 \le k_2 \le \ldots \le k_l \le n$.

3. Explicit boundary form factor solutions

In this section we explicitly solve the recurrence relations for operators with the mildest ultraviolet behavior. As Q_n and $\tilde{Q}_n$ are symmetric polynomials we introduce the following basis of homogeneous symmetric polynomials:

$$\prod_{i=1}^{n}(y + y_i) = \sum_k y^{n-k} \sigma_k^{(n)}(y_1, \ldots, y_n) \tag{3.1}$$

With this definition $\sigma_k^{(n)} = 0$ if $k < 0$ or $k > n$.

3.1. The identity boundary

In this subsection we solve explicitly the recurrence relations (2.23)–(2.25) for the operator which has the mildest ultraviolet behavior, i.e. the off-critical version of the boundary stress-tensor. This is the tower of form factors built over the first level kernel solution σ_1 [9].

3.1.1. Formulating the conjecture

The lowest lying solutions are given as [17]

$$Q_1^T = \sigma_1, \qquad Q_2^T = \sigma_1, \qquad Q_3^T = \sigma_1^2, \qquad Q_4^T = \sigma_1^2(\sigma_2 + 3). \tag{3.2}$$

As the σ_1 symmetric polynomial has the properties

$$\begin{aligned}
&\sigma_1^{(n+2)}(y,-y,y_1,\ldots,y_n)=\sigma_1^{(n)}(y_1,\ldots,y_n)\\
&\sigma_1^{(n+2)}(y_+,y_-,y_1,\ldots,y_n)=\sigma_1^{(n+1)}(y,y_1,\ldots,y_n)\\
&\sigma_1^{(n+1)}(0,y_1,\ldots,y_n)=\sigma_1^{(n)}(y_1,\ldots,y_n)
\end{aligned}\tag{3.3}$$

the tower $Q_n^T\sigma_1^k$ for $k>0$ will also satisfy the recursion. This tower is claimed to correspond to the operator $\partial^k T$ [17].

In what follows we give explicit formulae for these Q_n^T polynomials. Even if our method is special for the Lee–Yang model and demand some guesswork one can expect that similar method adapted to some other integrable models could also work.

We start with the leading order analysis. Let us take all the rapidities and shift them uniformly $\theta_i\to\theta_i+\Lambda$. The Q_n polynomials are functions of the variables $y_i=x_i+x_i^{-1}$ with $x_i=e^{\theta_i}$, therefore they are large if $\Lambda\to\pm\infty$. The recursion polynomials can then be expanded in powers of $\lambda=e^{\Lambda}$ and one finds

$$\begin{aligned}
&D_n(y\mid y_1,\ldots,y_n)\sim\lambda^{\pm n}V_n\big(x^{\pm1}\bigm| x_1^{\pm1},\ldots,x_n^{\pm1}\big)\\
&P_n(y\mid y_1,\ldots,y_n)\sim\lambda^{\pm(2n-1)}(-1)^{n+1}U_n\big(x^{\pm1}\bigm| x_1^{\pm1},\ldots,x_n^{\pm1}\big);\qquad \Lambda\to\pm\infty\\
&B_n(y_1,\ldots,y_n)\sim\lambda^{\pm n}0
\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}
&U_n(x\mid x_1,\ldots,x_n)=\frac{\prod_{i=1}^n(x+\omega x_i)(x-\omega^{-1}x_i)-\prod_{i=1}^n(x-\omega x_i)(x+\omega^{-1}x_i)}{2x(\omega-\omega^{-1})}\\
&V_n(x\mid x_1,\ldots,x_n)=\prod_{i=1}^n(x+x_i)
\end{aligned}\tag{3.5}$$

are the polynomials appearing in the bulk recursion [19]. Here we introduced $\omega=e^{\frac{i\pi}{3}}$. Its solution is given in a determinant form

$$\begin{aligned}
&\det\Sigma(x,-x,x_1,\ldots,x_n)=(-1)^{n+1}U_n(x\mid x_1,\ldots,x_n)\det\Sigma(x_1,\ldots,x_n)\\
&\det\Sigma\big(\omega x,\omega^{-1}x,x_1,\ldots x_n\big)=V_n(x\mid x_1,\ldots,x_n)\det\Sigma(x,x_1,\ldots,x_n)
\end{aligned}\tag{3.6}$$

for $n\geq4$ where the Σ matrix is given as $\Sigma_{i,j}=\sigma_{3j-2i+1}$. Here we need to comment on the notations. The Σ matrix is expressed in terms of the σ_k symmetric polynomials and its form is universal for all n. In what follows we use the following abbreviation: if the arguments are not written explicitly we denote by $\Sigma(n)$ the matrix with n general entries. The n dependence appears through the size of the matrix and also comes from the range of the elementary symmetric polynomials as $\sigma_k^{(n)}=0$ if $k>n$, thus we have

$$\Sigma(n)_{i,j}=\sigma_{3j-2i+1}^{(n)},\qquad 1\leq i,j\leq n-3\tag{3.7}$$

In the bulk case the determinant form is motivated by the clustering property of the form factors [22]. For the boundary problem the simplest idea is to try to find some lower order corrections to the Σ matrix and interestingly, even though the clustering argument fails we managed to obtain the matrix explicitly.

One can determine the Q_n^T polynomials for small n and then organize them into a determinant

$$Q_n^T(y_1,\dots,y_n)=\sigma_1^2(y_1,\dots,y_n)\det\Xi(y_1,\dots,y_n),\quad n\geq 4 \tag{3.8}$$

where the first few are

$$\Xi(4)=\left(\sigma_2^{(4)}+3\right);\qquad \Xi(5)=\begin{pmatrix}\sigma_2^{(5)}+3 & \sigma_5^{(5)}\\ 1 & 3\sigma_1^{(5)}+\sigma_3^{(5)}\end{pmatrix}$$

$$\Xi(6)=\begin{pmatrix}\sigma_2^{(6)}+3 & \sigma_5^{(6)} & -3\sigma_6^{(6)}\\ 1 & 3\sigma_1^{(6)}+\sigma_3^{(6)} & \sigma_6^{(6)}\\ 0 & \sigma_1^{(6)} & 3\sigma_2^{(6)}+\sigma_4^{(6)}+9\end{pmatrix}$$

$$\Xi(7)=\begin{pmatrix}\sigma_2^{(7)}+3 & \sigma_5^{(7)} & -3\sigma_6^{(7)} & 9\sigma_7^{(7)}\\ 1 & 3\sigma_1^{(7)}+\sigma_3^{(7)} & \sigma_6^{(7)} & -3\sigma_7^{(7)}\\ 0 & \sigma_1^{(7)} & 3\sigma_2^{(7)}+\sigma_4^{(7)}+9 & \sigma_7^{(7)}\\ 0 & 0 & \sigma_2^{(7)}+6 & 9\sigma_1^{(7)}+3\sigma_3^{(7)}+\sigma_5^{(7)}\end{pmatrix}$$

and we adapted the same notations as we explained for Σ. We determined the form factors and packed them into similar matrix forms up to $n=16$.

These matrices already suggest some properties for the Ξ matrix. We can see that at least for small n every term in the last column of $\Xi(n)$ are proportional to $\sigma_n^{(n)}$ except the right bottom corner, which does not contain $\sigma_n^{(n)}$ at all. Suppose that this is the case and let us analyze the boundary kinematical relation (2.25). As the symmetric polynomials have the properties $\sigma_k^{(n+1)}(0,y_1,\dots,y_n)=\sigma_k^{(n)}(y_1,\dots,y_n)$ if $k<n+1$ and $\sigma_{n+1}^{(n+1)}(0,y_1,\dots,y_n)=0$ we have

$$\Xi(0,y_1,\dots,y_n)=\left(\begin{array}{ccc|c} & & & 0\\ & \Xi(y_1,\dots,y_n) & & \vdots\\ & & & 0\\ \hline * & \dots & * & B_n\end{array}\right) \tag{3.9}$$

Expanding the determinant along the last column we can conclude that the element on the right bottom corner is the boundary polynomial B_n thus these are the diagonal elements of Ξ.

A uniform shift $\theta_k\to\theta_k+i\pi$ leads to the transformation $y_k\to -y_k$. Apparently as in the bulk case the Q_n^T polynomials have a definite parity under this transformation which is consistent with the fact that the recursion polynomials do also have a definite and compatible parity. This can be reached if in Ξ every matrix element has a definite parity thus it contains only odd or only even σ polynomials.

Finally, we can observe that if the correction to a specific matrix element has the form of

$$\sigma_k+a_1\sigma_{k-2}+a_2\sigma_{k-4}\dots=\sigma_k+\sum_{j=1}a_j\sigma_{k-2j} \tag{3.10}$$

where σ_k is the leading order term, then the coefficient can be written as $a_j=3^jb_j$ with some integer b_j. Now if we arrange the b_j numbers to a table (up to $n=16$) one can recognize the elements of Pascal's triangle. Then if we define

$$\binom{m}{k}=\begin{cases}\frac{m(m-1)(m-2)\dots(m-k+1)}{k!} & \text{if } k\in\mathbb{N}\\ 0 & \text{otherwise}\end{cases} \tag{3.11}$$

we can formulate our conjecture, namely

$$\Xi(n)_{i,j} = \sum_{k\in\mathbb{Z}} 3^k \binom{i-j+k}{k} \sigma^{(n)}_{3j-2i+1-2k} \qquad 1 \le i, j \le n-3 \tag{3.12}$$

We prove in Appendix A, that (3.8) indeed satisfies all the recurrence relations (2.23)–(2.25).

3.2. The Φ boundary

In this subsection we use the fusion method to extend the previous results for the Φ-boundary. We then determine the form factors of the boundary primary operator ϕ, which lives only on this boundary.

3.2.1. Fusion method

From the perturbed CFT point of view it follows that the space of operators living on the identity boundary is the Verma module built over the conformal vacuum, while the space of operators living on the Φ boundary is the direct sum of the Verma module of the conformal vacuum and the Verma module built over the only other highest weight conformal vector ϕ of weight $-\frac{1}{5}$.

In [28] it was proven that the Φ-boundary can be thought of if the only non-trivial defect (the Φ-defect) were fused to the identity boundary. In the language of the form factor solutions [10] this means that the form factors of the operators which are present on both boundaries (i.e. operators form the vacuum module) are related as[2]

$$\tilde{F}_n(\theta_1, \dots, \theta_n) = \prod_{i=1}^{n} T_-(\theta_i) F_n(\theta_1, \dots, \theta_n) \tag{3.13}$$

where T_- is the defect transmission factor which is

$$T_-(\theta) = S\left(\theta - i(3-b)\frac{\pi}{6}\right) = -\frac{\sinh(\frac{\theta}{2} + (b+1)\frac{i\pi}{12})\sinh(\frac{\theta}{2} + (b-1)\frac{i\pi}{12})}{\sinh(\frac{\theta}{2} + (b-5)\frac{i\pi}{12})\sinh(\frac{\theta}{2} + (b-7)\frac{i\pi}{12})}. \tag{3.14}$$

Here b is the defect parameter which, after fusion, becomes the parameter of the Φ-boundary. Plugging back the Ansatz (2.16) and (2.29) to (3.13) we get relations between the polynomials Q and $\tilde{Q}$, namely

$$\begin{aligned}\tilde{Q}_n(y_1, \dots, y_n) &= \left(\prod_{i=1}^{n} T_-(\theta_i)\frac{r_{\mathbb{I}}(\theta_i)}{r_{\Phi}(\theta_i)}\right) Q_n(y_1, \dots, y_n) \\ &= \prod_{i=1}^{n} \left(\alpha - \sqrt{3}\sqrt{\alpha+1}\, y_i + y_i^2\right) Q_n(y_1, \dots, y_n)\end{aligned} \tag{3.15}$$

It is straightforward to check that if Q_n satisfies Eqs. (2.23)–(2.25) then $\tilde{Q}_n$ indeed satisfies (2.31)–(2.33).

3.2.2. Form factors of the boundary primary field

At the conformal point the Φ-boundary contains also operators from the module of the conformal primary field ϕ. In [17] it was argued by a counting argument that there is a one-to-one

[2] In the CFT limit the defect is not seen by the energy momentum tensor, i.e. it is continuous there.

correspondence between the conformal boundary fields and the solutions of the form factor equations. It was also argued that the form factor solution with the mildest ultraviolet behavior corresponds to the primary field ϕ. The first few solutions of the Φ-boundary recurrence relations for ϕ are

$$\tilde{Q}_1^\phi = \sigma_1, \qquad \tilde{Q}_2^\phi = \sigma_1(\sigma_2+\alpha), \qquad \tilde{Q}_3^\phi = \sigma_1\big(\alpha\sigma_1(\alpha+\sigma_2)+(\sigma_2+3)\sigma_3\big)$$
$$\tilde{Q}_4^\phi = \alpha\sigma_1(\sigma_2+3)\big(\sigma_1\big(\alpha^2+\alpha\sigma_2\big)+(\sigma_2+3)\sigma_3\big)+\sigma_1(\sigma_2+3)(3\sigma_1+\sigma_3)\sigma_4 \tag{3.16}$$

We attempt to determine the whole tower of solutions based on these first members (3.16). To this end we take a similar Ansatz as was proposed for the defect case [20], namely

$$\tilde{Q}_n^\phi = \sigma_1^{(n)} S_n \det \Xi(n) \quad n \geq 4 \tag{3.17}$$

The S_n polynomials are defined only for $n \geq 4$ and have to satisfy the recurrence relations

$$S_{n+2}(y_+, y_-, y_1, \dots, y_n) = \big(\alpha - 3 + y^2\big) S_{n+1}(y, y_1, \dots, y_n)$$
$$S_{n+2}(y, -y, y_1, \dots, y_n) = \big(y^4 - (3+\alpha)y^2 + \alpha^2\big) S_n(y_1, \dots, y_n)$$
$$S_{n+1}(0, y_1, \dots, y_n) = \alpha S_n(y_1, \dots, y_n) \tag{3.18}$$

When it does not lead to any confusion we do not write out explicitly the arguments of S_n keeping in mind that S_n always has n arguments. We can compute explicitly the first few solutions which can be cast to the form

$$S_n = \sum_{k\in\mathbb{Z}} p_k(\alpha)\kappa_{n-2k-1}^{(n)} \tag{3.19}$$

where we introduced the

$$\kappa_k^{(n)} = \sum_{l=0}^{k} \alpha^l \sigma_{n-l}^{(n)} \sigma_{k-l}^{(n)} = \sum_{l\in\mathbb{Z}} \alpha^l \sigma_{n-l}^{(n)} \sigma_{k-l}^{(n)} \tag{3.20}$$

polynomials. Here in the last equation we used the fact that the symmetric polynomials are defined to be zero whenever their index is negative or have less arguments then their index. These polynomials have the properties

$$\begin{aligned}
\kappa_k^{(n+2)}(y_+, y_-, y_1, \dots, y_n) &= \alpha^2\kappa_{k-2}^{(n)} + \alpha^2 y\kappa_{k-3}^{(n)} + \alpha^2\big(y^2-3\big)\kappa_{k-4}^{(n)} + \alpha y\kappa_{k-1}^{(n)} + \alpha y^2\kappa_{k-2}^{(n)} \\
&\quad + \alpha y\big(y^2-3\big)\kappa_{k-3}^{(n)} + \big(y^2-3\big)\kappa_k^{(n)} \\
&\quad + y\big(y^2-3\big)\kappa_{k-1}^{(n)} + \big(y^2-3\big)^2\kappa_{k-2}^{(n)} \\
\kappa_k^{(n+1)}(y, y_1, \dots, y_n) &= \alpha\kappa_{k-1}^{(n)} + \alpha y\kappa_{k-2}^{(n)} + y\kappa_k^{(n)} + y^2\kappa_{k-1}^{(n)} \\
\kappa_k^{(n+2)}(y, -y, y_1, \dots, y_n) &= \alpha^2\kappa_{k-2}^{(n)} - y^2\alpha^2\kappa_{k-4}^{(n)} - y^2\kappa_k^{(n)} + y^4\kappa_{k-2}^{(n)} \\
\kappa_k^{(n+1)}(0, y_1, \dots, y_n) &= \alpha\kappa_{k-1}^{(n)}
\end{aligned} \tag{3.21}$$

where on the right hand side of (3.21) the arguments of every $\kappa^{(n)}$ are taken to be $(y_1, \dots, y_n)$. The functions $p_k(\alpha)$ appearing on the right hand side of Eq. (3.19) are polynomials of α and the first few are

$$p_0(\alpha) = 1; \qquad p_1(\alpha) = 3$$
$$p_2(\alpha) = 9 + 3\alpha - \alpha^2; \qquad p_3(\alpha) = 27 + 18\alpha - 3\alpha^2 - \alpha^3 \tag{3.22}$$

For negative indices they are defined to be zero, $p_k = 0$ if $k < 0$, which means that the sum on the right hand side of (3.19) is finite.

Plugging back the Ansatz (3.19) to the recursion (3.18) and taking advantage of the identities (3.21) one can easily derive that the recursion (3.18) is satisfied provided

$$p_{k+1}(\alpha) - (\alpha + 3)p_k(\alpha) + \alpha^2 p_{k-1}(\alpha) = 0 \quad k \geq 1. \tag{3.23}$$

This is an ordinary second order recursion which can be solved by usual techniques.

The final solution for the p_k polynomials has a simpler form in terms of the boundary parameter b:

$$p_k(b) = C(b)r(b)^k + D(b)q(b)^k \tag{3.24}$$

with

$$r(b) = 4\cos^2\left((b+1)\frac{\pi}{6}\right); \qquad q(b) = 4\cos^2\left((b-1)\frac{\pi}{6}\right)$$
$$C(b) = \frac{1}{\sqrt{3}}\frac{\cos\frac{(b+1)\pi}{6}}{\cos\frac{b\pi}{6}}; \qquad D(b) = \frac{1}{\sqrt{3}}\frac{\cos\frac{(b-1)\pi}{6}}{\cos\frac{b\pi}{6}} \tag{3.25}$$

Note that if $b \in 6\mathbb{Z} - 3$ the polynomial p_k is simply the limit of the above formulae which is $p_k = 1 + 2k$.

4. Conclusion

In this paper we gave explicit closed formulae for the form factors of the boundary fields with the lowest scaling dimensions in the scaling Lee–Yang model for all integrable boundary conditions. We first determined the generic n-particle form factor of the energy–momentum tensor for the identity boundary in a determinant form. We then applied the fusion idea to derive the corresponding n-particle form factors in case of the Φ-boundary and proved the consistency of the solutions. Finally, based on our experience with defect form factors, we presented the only remaining form factors for the boundary primary field ϕ. We emphasize these results are the first explicit *boundary* form factor solutions.

It is very remarkable that the form factor solution we found takes a determinant form. Actually in finding the solution we exploited the fact that for large rapidities the boundary form factor reduces to the bulk one, which took already a determinant form. We then systematically determined the lower order entries of this determinant. We expect that similar method can work for other models including the boundary form factors of the exponential operators in the sinh-Gordon and sine-Gordon models [29,30].

The explicit form of the form factors are very useful. They can be used to calculate correlation functions in infinite and also in finite volumes [31] or describe the exact finite volume/temperature boundary vacuum expectation values following [32].

Acknowledgements

We thank Gábor Takacs and Máté Lencsés for the useful discussions. We were supported by a Lendület Grant number LP2012-18/2014 and by OTKA 81461.

Appendix A. Proof of the conjecture

Throughout the appendix we do not display twice the number of arguments of any object. This means for an elementary symmetric polynomial that we use the new notation $\sigma_k^{(n)} \equiv \sigma_k(y_1, \dots, y_n)$. Let us recall our notations: The Ξ matrix is considered to be an infinite matrix and expressed in terms of the σ_k symmetric polynomials. Its form is universal for all n. If we consider the matrix Ξ with n general arguments, then we cut the infinite matrix into an $(n-3) \times (n-3)$ submatrix and denote it by $\Xi(n)$. The n-dependence appears through the size of the matrix and also comes from the range of the elementary symmetric polynomials as $\sigma_k^{(n)} = 0$ if $k > n$, see (3.12). During the proof we use the notation

$$\Xi_{i,j}^{(s)} = \sum_{k \in \mathbb{Z}} 3^k \binom{i-j+k}{k} \sigma_{3j-2i+1-2k-s} \tag{A.1}$$

and also $\Xi^{(s)}(n)$ with similar conventions.

In this appendix we also consider the recursion polynomials as functions of the elementary symmetric polynomials:

$$D_n = \sum_i y^{n-i} \sigma_i \tag{A.2}$$

$$P_n = \sum_{q<p} \mathcal{P}_{p,q} \sigma_{n-p} \sigma_{n-q} \tag{A.3}$$

$$B_n = \sum_k 3^k \sigma_{n-1-2k} \tag{A.4}$$

where

$$\mathcal{P}_{p,q} = \frac{y_+^p y_-^q ((-1)^q - (-1)^p) + y_+^q y_-^p ((-1)^p - (-1)^q)}{2(y_+ - y_-)}$$

When these quantities are taken at specified arguments we use similar abbreviation as for Ξ. Note that these functions reduce to the recursion polynomials (2.26)–(2.28) only if the appropriate number of arguments is chosen, for example $D_n(y \mid y_1, \dots, y_n) = D_n(n)$.

A.1. General ideas

Since the solution of the recursion equations (2.23), (2.24), (2.25) is given in a determinant form (without the factor σ_1^2) the proof of the conjecture should reflect this property. Namely, we first expand the elements of the determinants with special arguments on the left hand side of these equations by exploiting the properties of elementary symmetrical polynomials:

$$\sigma_k(y, -y, y_1, \dots, y_n) = \sigma_k^{(n)} - y^2 \sigma_{k-2}^{(n)} \tag{A.5}$$

$$\sigma_k(y_+, y_-, y_1, \dots, y_n) = \sigma_k^{(n)} + y \sigma_{k-1}^{(n)} + \left(y^2 - 3\right) \sigma_{k-2}^{(n)} \tag{A.6}$$

$$\sigma_k(0, y_1, \dots, y_n) = \sigma_k^{(n)} \tag{A.7}$$

We then manipulate the rows and columns systematically, until they get into a form in which the equations hold true explicitly. In the following paragraphs we present these desired forms and the manipulation algorithms.

The exact proofs would get rather technical later on, therefore in these cases only brief outlines are given. During the proof we use the fact that the determinant of a matrix does not change if we add to a row (column) any scalar time an other row (column) and by such steps we reduce the matrices to a special form.

A.2. Boundary recursion

By construction of the matrix (3.12) the boundary recursion equation holds trivially, see Subsection 3.1.1.

A.3. Dynamical recursion

The desired reduced form of $\Xi(y_+, y_-, y_1, \ldots, y_n)$ after expanding its elementary symmetric polynomials using (A.6) is

$$\left(\begin{array}{ccc|c} & & & 0 \\ & \Xi(y, y_1, \ldots, y_n) & & \vdots \\ & & & 0 \\ \hline * & \cdots & * & D_n \end{array}\right) \tag{A.8}$$

where the argument of D_n is $D_n(y \mid y_1, \ldots, y_n)$. This is so because after calculating the determinant by expanding it along the last column only $D_n(y \mid y_1, \ldots, y_n)\det(\Xi(y, y_1, \ldots, y_n))$ remains, which is exactly what the dynamical recursion requires.

The algorithm consists of two steps, whose order is arbitrary. During the column operation we add recursively from left to right each column times $y(y^2-3)$ to the one to its right. The row operations consist of adding from top to bottom to each row $(3-y^2)$ times the one below it, (i.e. not recursively). Note that while the column operation results in a cumulative sum the row operation does not.

The expansion of the elementary symmetric polynomials in $\Xi(y_+, y_-, y_1, \ldots, y_n)$ can be written as

$$\Xi(y_+, y_-, y_1, \ldots, y_n) = \Xi^{(0)}(n) + y\Xi^{(1)}(n) + \bigl(y^2-3\bigr)\Xi^{(2)}(n) \tag{A.9}$$

After the row operation the (i, j) element of our matrix becomes

$$\Xi^{(0)}_{i,j} + y\Xi^{(1)}_{i,j} + \bigl(y^2-3\bigr)\bigl(\Xi^{(2)}_{i,j} - \Xi^{(0)}_{i+1,j}\bigr) - y\bigl(y^2-3\bigr)\Xi^{(1)}_{i+1,j} - \bigl(y^2-3\bigr)^2\Xi^{(2)}_{i+1,j} \tag{A.10}$$

Now we focus on the first $n-2$ rows. In particular, the row manipulation already transforms the first column into the required form which can be seen by explicit calculation (only the upper two elements are non-zero). Then we can complete the proof by induction after carrying out the column operation, which has no effect on the first column. The induction step is that the column operation up to the jth column transforms the $(j-1)$th column into the required form, thus after the next addition, what we really add to the jth column is $y(y^2-3)(\Xi^{(0)}_{i,j-1} + y\Xi^{(1)}_{i,j-1})$. By this we get

$$\begin{aligned} &\Xi^{(0)}_{i,j} + y\Xi^{(1)}_{i,j} + \bigl(y^2-3\bigr)\bigl[\Xi^{(2)}_{i,j} - \Xi^{(0)}_{i+1,j} + 3\Xi^{(1)}_{i,j-1}\bigr] \\ &\quad - y\bigl(y^2-3\bigr)\bigl[\Xi^{(1)}_{i+1,j} - \Xi^{(0)}_{i,j-1}\bigr] - \bigl(y^2-3\bigr)^2\bigl[\Xi^{(2)}_{i+1,j} - \Xi^{(1)}_{i,j-1}\bigr] \end{aligned} \tag{A.11}$$

where it can be shown that all the [ ] brackets vanish separately by using their definitions and relabeling the summation indices. We can conclude that the upper-left $(n-2) \times (n-2)$

submatrix what we got by applying the operations to $\Xi(y_+, y_-, y_1, \ldots, y_n)$ is nothing but $\Xi(y, y_1, \ldots, y_n) = \Xi^{(0)}(n) + y\Xi^{(1)}(n)$, and because the last column (except for its bottom element) only contains terms proportional to $\sigma_k(y_1, \ldots, y_n)$ with $k > n$, they all vanish.

Now we have to prove that the bottom right element of the modified matrix is D_n which is done by induction. Have we applied only the column operations the $(n-2, n-2)$ element would be by induction D_{n-1}. Denote the $(n-1, n-2)$ element which is below this one by A_n what is a cumulative sum (of the last row's elements) made by the column operations. This element is not effected by the row operations. Now we apply the row operations: we add $(3 - y^2)A_n$ to D_{n-1}, and – as explained in the previous paragraph – this must produce

$$D_{n-1}(n) + \left(3 - y^2\right)A_n = \Xi^{(0)}_{n-2,n-2}(n) + y\Xi^{(1)}_{n-2,n-2}(n) \tag{A.12}$$

On the other hand the only effect of the algorithm on the original bottom right element $\Xi^{(0)}_{n-1,n-1}(n) + y\Xi^{(1)}_{n-1,n-1}(n) + (y^2 - 3)\Xi^{(2)}_{n-1,n-1}(n)$ is the addition of $y(y^2 - 3)A_n$ which should turn it into something that we conjecture to be D_n. Now substituting the form of A_n from (A.12), after some algebra, we get

$$\begin{aligned} &\Xi^{(0)}_{n-1,n-1} + y\Xi^{(1)}_{n-1,n-1} + \left(y^2 - 3\right)\Xi^{(2)}_{n-1,n-1} - y\Xi^{(0)}_{n-2,n-2} - y^2\Xi^{(1)}_{n-2,n-2} \\ &\quad = D_n - yD_{n-1} \end{aligned} \tag{A.13}$$

The left hand side of (A.13) is manifestly zero which can be checked explicitly after shifting summation indices. As the D_n polynomials are also thought of as functions of the σ_k symmetric polynomials (A.2) it is easy to see that the right hand side also vanishes.

A.4. Kinematical recursion

The desired form into which $\Xi_{n+2}(y, -y, y_1, \ldots, y_n)$ should be reduced is

$$\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & \Xi(y_1, \ldots, y_n) & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline * & \ldots & * & K_n & L_n \\ * & \ldots & * & M_n & N_n \end{array}\right) \tag{A.14}$$

where the 0s denote an $(n-3) \times 2$ zero matrix and the right bottom 2×2 submatrix satisfies $K_nN_n - L_nM_n = P_n(y \mid y_1, \ldots, y_n)$ as after taking the determinant it would reduce to the kinematical recursion equation.

In this case we also apply row and column operations. The column operation is as follows: takes the first and second column, the third and fourth, etc. and form pairs out of them, then add the first pair times $y^2(y^2 - 3)^2$ to the second pair (the first to the third, the second to the fourth), then the new second pair to the third pair, and so on (each time take the latest modified pair, and add it to the next pair after multiplying it by $y^2(y^2 - 3)^2$). The row operation consists of adding to each row y^2 times the row below and $y^2(y^2 - 3)$ times the second row below of the original matrix. Unlike to the column operation it is not a cumulative sum. The order of the operations is again arbitrary.

First we expand the elementary symmetrical polynomials in $\Xi(y, -y, y_1, \ldots, y_n)$ according to (A.5)

$$\Xi(y, -y, y_1, \ldots, y_n) = \Xi(n) - y^2\Xi^{(2)}(n) \tag{A.15}$$

and we again proceed the proof by induction similarly to the dynamical case. Now we focus on the first $n-3$ rows of this matrix. We execute the row operation which brings the first two columns into the required form and since they are not affected by the column operation we can start the induction on the columns. We cumulatively add $y^2(y^2-3)^2$ times the $(j-2)$th column to the jth one and as the induction assumption we suppose that the $(j-2)$th column have had already the good form (i.e. the same as the $(j-2)$th column of $\Xi(n)$). After the row operation (i,j) element becomes

$$\Xi_{i,j} - y^2\Xi^{(2)}_{i,j} + y^2\Xi_{i+1,j} - y^4\Xi^{(2)}_{i+1,j} + y^2\big(y^2-3\big)\Xi_{i+2,j} - y^4\big(y^2-3\big)\Xi^{(2)}_{i+2,j} \tag{A.16}$$

Now applying the column operation and using the induction assumption we get finally for $1 \le i, j \le n-3$

$$\begin{aligned} &\Xi_{i,j} + y^2\big[\Xi_{i+1,j} - \Xi^{(2)}_{i,j}(n) - 3\Xi_{i+2,j} + 9\Xi_{i,j-2}\big] \\ &\quad + y^4\big[\Xi_{i+2,j} - \Xi^{(2)}_{i+1,j} + 3\Xi^{(2)}_{i+2,j} - 6\Xi_{i,j-2}\big] + y^6\big[\Xi_{i,j-2} - \Xi^{(2)}_{i+2,j}\big] \end{aligned} \tag{A.17}$$

again each [] bracket vanishes separately which can be seen by shifting the summation indices in the elements of Ξ. This proves that after the reduction of the determinant of $\Xi(y,-y,y_1,\dots,y_n)$ its upper-left $(n-3)\times(n-3)$ block is $\Xi(n)$. This argument is valid for the whole $\Xi(y,-y,y_1,\dots,y_n)$ matrix except for the two bottom rows as in this cases the row operation is different. In the last two columns of this $(n-3)\times 2$ submatrix we have only terms which are proportional to $\sigma_{n+1}(y_1,\dots,y_n)$ or $\sigma_{n+2}(y_1,\dots,y_n)$ which are zeros, therefore we have an $(n-3)\times 2$ zero-block at the upper-right corner.

Now all that is left to prove that the determinant formed by the bottom right 2×2 elements is equal to $P_n(y \mid y_1,\dots,y_n)$. For that reason consider the $(n-2)\times(n-2)$ matrix $\Xi(y,-y,y_1,\dots,y_{n-1})$ and apply the row and column operations. As already proved we end up with a matrix of the form

$$\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\Xi(n-1)} & 0 & 0 \\ \hline * & * & K_{n-1} & L_{n-1} \\ * & R_{n-1} & M_{n-1} & N_{n-1} \end{array}\right) \tag{A.18}$$

Now take the $(n-1)\times(n-1)$ matrix $\Xi(y,-y,y_1,\dots,y_n)$. Have we applied all the operation except from the addition of the last row to any other row would result in (A.18) as the upper left block. Now apply this last row operation which results in

$$\left(\begin{array}{cc|cc} \Xi(n-1) & 0 & 0 & * \\ * & K_{n-1} & L_{n-1} & * \\ \hline * & M_{n-1} & N_{n-1} & * \\ * & R_n & M_n & N_n \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\Xi(n)} & 0 & 0 \\ \hline * & * & K_n & L_n \\ * & R_n & M_n & N_n \end{array}\right) \tag{A.19}$$

The only difference between the matrices is that we added y^2 times the last row to the one above and also added $y^2(y^2-3)$ times the last row to the second to the last row. Then we got the relations

$$\begin{aligned} &K_{n-1} + y^2\big(y^2-3\big)R_n = \Xi_{n-3,n-3} \\ &L_{n-1} + y^2\big(y^2-3\big)M_n = 0 \\ &N_{n-1} + y^2 M_n = K_n \end{aligned} \tag{A.20}$$

Two other relations can be derived if we consider how we got the L_n and N_n elements: we took the original elements at the kinematical pole, added the $y^2(y^2-3)^2$ times the $(n-3)$th column and y^2 times the row below which leads to

$$N_n = \Xi_{n-1,n-1} - y^2\Xi^{(2)}_{n-1,n-1} + y^2\left(y^2-3\right)^2 R_n$$
$$L_n = -y^2\Xi^{(2)}_{n-2,n-1} + y^2 N_n + y^2\left(y^2-3\right)^2 M_{n-1} \tag{A.21}$$

After eliminating R_n and using (A.1) we get

$$N_n = \sigma_n - \left(y^2-3\right)K_{n-1}; \qquad M_n = -\frac{1}{y^2(y^2-3)}L_{n-1}$$
$$K_n = N_{n-1} - \frac{1}{y^2-3}L_{n-1}; \qquad L_n = -y^2\left(y^2-3\right)K_{n-1} + y^2\left(y^2-3\right)^2 M_{n-1} \tag{A.22}$$

It can be also rearranged to a fourth order recursion

$$L_n = -y^2\left(y^2-3\right)\sigma_{n-2} + \left(6-y^2\right)L_{n-2} - \left(y^2-3\right)^2 L_{n-4} \tag{A.23}$$

Let our induction hypothesis be

$$P_{n-1} = \det\mathcal{M}_{n-1} = K_{n-1}N_{n-1} - L_{n-1}M_{n-1}$$

which can be easily checked for small values of n. With the relations (A.22) we can reformulate our conjecture as

$$P_n + \left(y^2-3\right)P_{n-1} = \sigma_n N_{n-1} - \frac{\sigma_n}{y^2-3}L_{n-1}$$

If we take into account (A.3) it reduces to

$$\sum_{p=1}^{n}\mathcal{P}_{p,0}\sigma_{n-p} = N_{n-1} - \frac{1}{y^2-3}L_{n-1} = \sigma_{n-1} + 3\left(N_{n-3} - \frac{1}{y^2-3}L_{n-3}\right) + L_{n-3}$$

which is indeed true for small values of n. At the second equation we used (A.22). By induction suppose that we already proved that

$$\sum_{p=1}^{n-2}\mathcal{P}_{p,0}\sigma_{n-2-p} = N_{n-3} - \frac{1}{y^2-3}L_{n-3}$$

and what remains is to prove that

$$L_n = \sum_{p=1}^{n+3}\mathcal{P}_{p,0}\sigma_{n+3-p} - 3\sum_{p=1}^{n+1}\mathcal{P}_{p,0}\sigma_{n+1-p} - \sigma_{n+2} \tag{A.24}$$

A lengthy but straightforward calculation shows that the right hand side of (A.24) satisfies the recursion (A.23) and one can check explicitly that for small n the two sides of (A.24) are indeed equal that proves the validity of (A.24), and so the conjecture.

References

[1] P. Dorey, Exact S Matrices, 1996, pp. 85–125.
[2] Z. Bajnok, L. Samaj, Introduction to integrable many-body systems III, Acta Phys. Slovaca 61 (2011) 129–271.
[3] L. Samaj, Z. Bajnok, Introduction to the Statistical Physics of Integrable Many-Body Systems, Cambridge University Press, 2013.
[4] G. Delfino, G. Mussardo, P. Simonetti, Scattering theory and correlation functions in statistical models with a line of defect, Nucl. Phys. B 432 (1994) 518–550.
[5] O.A. Castro-Alvaredo, A. Fring, F. Gohmann, On the absence of simultaneous reflection and transmission in integrable impurity systems, arXiv:hep-th/0201142, 2002.
[6] S. Ghoshal, A.B. Zamolodchikov, Boundary S matrix and boundary state in two-dimensional integrable quantum field theory, Int. J. Mod. Phys. A 9 (1994) 3841–3886.
[7] Z. Bajnok, A. George, From defects to boundaries, Int. J. Mod. Phys. A 21 (2006) 1063–1078.
[8] F.A. Smirnov, Form-factors in completely integrable models of quantum field theory, Adv. Ser. Math. Phys. 14 (1992) 1–208.
[9] Z. Bajnok, L. Palla, G. Takacs, On the boundary form-factor program, Nucl. Phys. B 750 (2006) 179–212.
[10] Z. Bajnok, O. el Deeb, Form factors in the presence of integrable defects, Nucl. Phys. B 832 (2010) 500–519.
[11] J.L. Cardy, G. Mussardo, Form-factors of descendent operators in perturbed conformal field theories, Nucl. Phys. B 340 (1990) 387–402.
[12] P. Christe, Factorized characters and form-factors of descendant operators in perturbed conformal systems, Int. J. Mod. Phys. A 6 (1991) 5271–5286.
[13] A. Koubek, Form-factor bootstrap and the operator content of perturbed minimal models, Nucl. Phys. B 428 (1994) 655–680.
[14] A. Koubek, The space of local operators in perturbed conformal field theories, Nucl. Phys. B 435 (1995) 703–734.
[15] F.A. Smirnov, Counting the local fields in SG theory, Nucl. Phys. B 453 (1995) 807–824.
[16] G. Delfino, G. Niccoli, Isomorphism of critical and off-critical operator spaces in two-dimensional quantum field theory, Nucl. Phys. B 799 (2008) 364–378.
[17] M. Szots, G. Takacs, Spectrum of local boundary operators from boundary form-factor bootstrap, Nucl. Phys. B 785 (2007) 211–233.
[18] O.A. Castro-Alvaredo, Boundary form factors of the sinh-Gordon model with Dirichlet boundary conditions at the self-dual point, J. Phys. A 39 (2006) 11901–11914.
[19] A.B. Zamolodchikov, Two point correlation function in scaling Lee–Yang model, Nucl. Phys. B 348 (1991) 619–641.
[20] Z. Bajnok, F. Buccheri, L. Hollo, J. Konczer, G. Takacs, Finite volume form factors in the presence of integrable defects, 2013.
[21] A. Fring, G. Mussardo, P. Simonetti, Form-factors for integrable Lagrangian field theories, the sinh-Gordon theory, Nucl. Phys. B 393 (1993) 413–441.
[22] A. Koubek, G. Mussardo, On the operator content of the sinh-Gordon model, Phys. Lett. B 311 (1993) 193–201.
[23] M. Pillin, The form-factors in the sinh-Gordon model, Int. J. Mod. Phys. A 13 (1998) 4469–4486.
[24] O.A. Castro-Alvaredo, A. Fring, Identifying the operator content, the homogeneous sine-Gordon models, Nucl. Phys. B 604 (2001) 367–390.
[25] O.A. Castro-Alvaredo, A. Fring, Decoupling the SU(n)(2) homogeneous Sine-Gordon model, Phys. Rev. D 64 (2001) 085007.
[26] O.A. Castro-Alvaredo, Form factors of boundary fields for A(2)-affine Toda field theory, J. Phys. A 41 (2008) 194005.
[27] P. Dorey, R. Tateo, G. Watts, Generalizations of the Coleman–Thun mechanism and boundary reflection factors, Phys. Lett. B 448 (1999) 249–256.
[28] Z. Bajnok, Zs. Simon, Solving topological defects via fusion, Nucl. Phys. B 802 (2008) 307–329.
[29] G. Takacs, Form-factors of boundary exponential operators in the sinh-Gordon model, 2008.
[30] M. Lencses, G. Takacs, Breather boundary form factors in sine-Gordon theory, Nucl. Phys. B 852 (2011) 615–633.
[31] M. Kormos, G. Takacs, Boundary form-factors in finite volume, Nucl. Phys. B 803 (2008) 277–298.
[32] G. Takacs, Finite temperature expectation values of boundary operators, Nucl. Phys. B 805 (2008) 391–417.