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Minimal enclosing discs, circumcircles, and circumcenters in normed planes (Part I) ☆

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ABSTRACT

It is surprising that there are almost no results on the precise location of (all) minimal enclosing balls, circumballs, and circumcenters of simplices in finite-dimensional real Banach spaces. In this paper and a subsequent second part of it we give the starting point in this direction, also for computational investigations. More precisely, we present the first thorough study of these topics for triangles in arbitrary normed planes. In the present Part I we lay special emphasize on a complete description of possible locations of the circumcenters, and as a needed tool we give also a modernized classification of all possible shapes of the intersection that two homothetic norm circles can create. Based on this, we give in Part II the complete solution of the strongly related subject to find all minimal enclosing discs of triangles in arbitrary normed planes.

1. Introduction

It is well known that already elementary results and constructions from Euclidean geometry have non-trivial analogues and extensions in Minkowski geometry, i.e., in the geometry of finite-dimensional real Banach spaces. Simple notions like bisectors, circumcenters, circumballs, minimal enclosing balls and angular bisectors are still interesting subjects of research in real Banach spaces. The main reason is that their "convenient" geometric properties get lost when switching to sufficiently general norms. E.g., bisectors are, in general, no longer (topological) hyperplanes, and circumballs of simplices need not be unique, like also their minimal enclosing balls. Even in the planar situation there exist only a few observations and results in this direction.

A well known problem from Location Science and Computational Geometry is the so-called minimax or 1-center problem: for m given points in $\mathbb{R}^n$, find the (unique) point that minimizes its maximal distance to the given points. Basic references to this (also algorithmically studied) problem are [8,24,29,9,25,31]; but except for the very special class of norms considered in [26], until now this problem was not investigated for normed planes and spaces! It is easy to see that the solution of the 1-center problem yields the center of the minimal enclosing circle or ball of the given point set. We note that, historically, the minimal enclosing circle problem goes back to Sylvester [28]. In the present paper and a subsequent Part II we present, for triangles given in arbitrary normed planes, the first approach to the location of their *minimal enclosing circles* that always exist. Related to this we study also *circumcircles* and *circumcenters* of triangles; note that these do not

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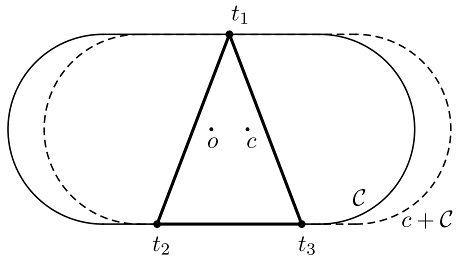


Fig. 1. More than one circle through t_1, t_2, t_3.

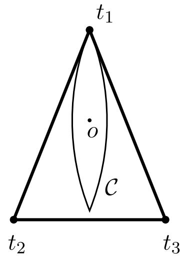


Fig. 2. No circle through t_1, t_2, t_3.

always exist. Since such circles (related to fixed triangles) are also not necessarily unique, *minimal circumcircles* are also studied.

In particular, the present Part I contains the first complete description of possible locations of circumcenters of triangles (Theorems 4.1 and 4.2). As it turns out, the complete classification of the possible intersection shapes that two homothetic norm circles can create plays an essential role for this purpose. Therefore we also reprove and refine results on the shapes of these intersections which are due to Grünbaum [13] and Banasiak [3]. Namely, it was shown by Grünbaum [13] that the intersection of two circles in a normed plane is always the union of two segments, either disjoint or having a point in common, where such a segment may degenerate to a point or even to the empty set; see also Banasiak [3]. This is closely related to the fact that, in contrast to the Euclidean situation, a circle of minimal radius containing a segment need not be unique; see Fig. 1. On the other hand, there are normed planes (even strictly convex) with triangles that have no circumcircle; see Fig. 2. In fact, a normed plane is smooth if and only if any triangle in it has at least one circumcircle; see [15] and, for a wider discussion of that result, § 7.1 in [22]. One implication was extended to higher dimensions and even to *gauges* (i.e., to spaces whose unit balls are still convex, but not necessarily centered at the origin, thus creating a *general convex distance function*). Namely, already Gromov [11] proved that if the n-dimensional convex unit ball of a gauge is smooth, then at least one respective $(n-1)$-sphere passes through any $n+1$ non-collinear points. Makeev [20] reproved this; see also [21] for a local version. Strongly related results can be found in [10,14,17–19]; see also the discussion on page 125 of the survey [23].

Our results should be taken as starting points for extensions to more complicated given point sets (instead of triangles), to higher dimensions, and for algorithmical investigations. Also the reader should note that results of this type are basic for further research on the following notions, problems and fields with regard to finite-dimensional real Banach spaces: unit distance graphs [6], bisectors and Voronoi diagrams [23, Section 4], coresets, also in view of Approximation Theory and Computational Geometry (see [2,1]), and Location Science.

Minimal enclosing balls also have a long history in Classical Convexity (see [5, § 35 and § 44] and [27, § 14]), particularly in view of Jung's theorem (cf. [16, § 78], [12, p. 49], and [4]) and related Geometric Inequalities (see [7, § 11]).

Since this paper refers only to triangles in normed planes, we can easily define the following basic notions: Any circle containing three non-collinear points t_1, t_2, t_3 in a normed plane is called *circumcircle*, and its center *circumcenter*, of the triangle $t_1t_2t_3$; and any circle of smallest possible radius containing in its closure these three points is a *minimal enclosing circle* of the set t_1, t_2, t_3 or of the triangle $t_1t_2t_3$.

2. Preliminaries

Since our paper refers to the geometry of finite-dimensional real Banach spaces, also called Minkowski geometry, we cite, for general background, the monograph [30] and the survey [22]. Let $(\mathbb{R}^2, \|\cdot\|)$ be a two-dimensional space of such type, called a *normed plane*, with *unit disc* $\mathcal{D}$ and *unit circle* $\mathcal{C}$. Recall that $\mathcal{D}$ is a compact, convex set with interior points and centered at the origin o. Homothetic copies of $\mathcal{D}$ and $\mathcal{C}$ are said to be *discs* and *circles*, respectively, and we write $\mathcal{C}(c,\rho)$ for the circle with center c and radius ρ.

A normed plane is called *strictly convex* if $\mathcal{C}$ contains no segments, and *smooth* if there is a unique supporting line of $\mathcal{D}$ at each point of $\mathcal{C}$.

For different points $p, q \in \mathbb{R}^2$, we denote the line through p and q, the segment with endpoints p and q, and the ray with origin p passing through q by $\langle p,q\rangle$, $[p,q]$ and $[p,r\rangle$, respectively. For brevity, we will refer to the relative interior of a segment in $\mathbb{R}^2$ simply as the interior of such segment. If p and q are two different points and x is a point not in $\langle p,q\rangle$, then the closed half-plane bounded by $\langle p,q\rangle$ and containing x will be denoted by $HP_x^+(p,q)$, the opposite one by $HP_x^-(p,q)$. For $x, y \in (\mathbb{R}^2, \|\cdot\|)$, we say that x is *Birkhoff orthogonal* to y, denoted by $x \dashv y$, if $\|x+\lambda y\| \geqslant \|x\|$ for every $\lambda \in \mathbb{R}$, i.e., if the line through x with direction of the vector y supports the circle with center o and radius $\|x\|$ at x.

Lemma 2.1. *(Monotonicity lemma; cf. [22, Proposition 31].) Let $p, q, r \in (\mathbb{R}^2, \|\cdot\|)$ be three points different from the origin o, $p \neq r$, with $[o,q\rangle$ between $[o,p\rangle$ and $[o,r\rangle$, and suppose that $\|q\| = \|r\|$. Then $\|p-q\| \leqslant \|p-r\|$, with equality if and only if either*

(i) *$q = r$,*
(ii) *or o and q are on opposite sides of $\langle p, r\rangle$, and $[\frac{r-p}{\|r-p\|}, \frac{q}{\|q\|}]$ is a segment on $\mathcal{C}$,*
(iii) *or o and q are on the same side of $\langle p, r\rangle$, and $[\frac{r-p}{\|r-p\|}, \frac{-r}{\|r\|}]$ is a segment on $\mathcal{C}$.*

Let t_1, t_2, t_3 be three non-collinear points of $\mathbb{R}^2$. Referring to the incidence of the points t_1, t_2, t_3 with a circle of $(\mathbb{R}^2, \|\cdot\|)$, the following situations are possible:

(A) A unique circle passes through t_1, t_2, t_3. This happens always when the plane $(\mathbb{R}^2, \|\cdot\|)$ is strictly convex and smooth; see, e.g., [22, Proposition 14 and Proposition 41].
(B) At least two circles pass through t_1, t_2, t_3 (see Fig. 1). This is only possible when the plane $(\mathbb{R}^2, \|\cdot\|)$ is not strictly convex; see [22, Proposition 14].
(C) There exists no circle passing through t_1, t_2, t_3 (see Fig. 2). This is only possible when the plane $(\mathbb{R}^2, \|\cdot\|)$ is not smooth; see [22, Proposition 41].

3. The intersection of two circles

The intersection of two circles in $(\mathbb{R}^2, \|\cdot\|)$ was studied by Grünbaum [13] and Banasiak [3], where Theorem 3.1 was obtained. Here we will give a new proof of this theorem which yields more information about the different ways that two circles can intersect each other.

Let $\mathcal{C}$ be the unit circle of $(\mathbb{R}^2, \|\cdot\|)$, and let $\mathcal{C}_1 = \mathcal{C}(c_1, \rho_1)$ and $\mathcal{C}_2 = \mathcal{C}(c_2, \rho_2)$ be two different homothetic copies of $\mathcal{C}$. For $i = 1, 2$, let u_i and v_i be the intersection points of $\langle c_1, c_2\rangle$ and $\mathcal{C}_i$.

Theorem 3.1. *(See [3].) The intersection of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ can have only one of the following forms.*

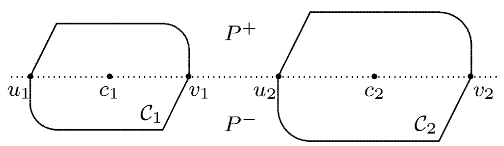
(a) *$\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.*
(b) *$\mathcal{C}_1 \cap \mathcal{C}_2$ consists of two closed, disjoint segments (one of them or both may be reduced to a point) lying on opposite sides of $\langle c_1, c_2\rangle$.*
(c) *$\mathcal{C}_1 \cap \mathcal{C}_2$ consists of two segments (one of them or both may be reduced to a point) with common point u_1 or v_1.*

Proof. To show in detail how $\mathcal{C}_1 \cap \mathcal{C}_2$ can look like, we assume, without loss of generality, that $\mathcal{C}_1 = \mathcal{C}$, i.e., $c_1 = 0$ and $\rho_1 = 1$, and that $\rho_2 \geqslant 1$. Let $\prec$ be an orientation of $\langle c_1, c_2\rangle$. If $p \prec q$ or $p = q$, we use the notation $p \preccurlyeq q$. We assume that $c_1 \prec c_2$ and $u_i \prec c_i \prec v_i$, $i = 1, 2$. We will describe the set $\mathcal{C}_1 \cap \mathcal{C}_2$ according to the position of all these points over $\langle c_1, c_2\rangle$. Table 1 "Circle intersections" summarizes all the possible situations. Observe that all cases can be achieved with the same unit circle $\mathcal{C}$.

Let $v_1^* \in \mathcal{C}$ be such that $v_1 \dashv v_1^*$, and let P^+ and P^- be the half-planes defined by $\langle c_1, c_2\rangle$ that contain v_1^* and $-v_1^*$, respectively, i.e., $P^+ = HP^+_{v_1^*}(c_1, c_2)$ and $P^- = HP^-_{v_1^*}(c_1, c_2)$. Let $\theta \in [0, 2\pi] \to x(\theta) \in \mathcal{C}$ be an angle parametrization on $\mathcal{C}$ where θ is the angle between $v_1 = x(0) = x(2\pi)$ and $x(\theta)$, and $v_1^* = x(\theta^*)$ with $0 < \theta^* < \pi$.

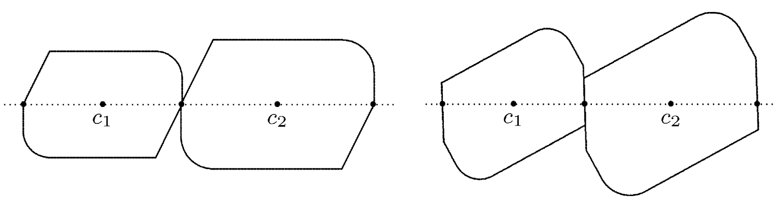
Table 1
Circle intersections.

CASE 1: $u_1 \prec c_1 \prec v_1 \prec u_2 \prec c_2 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: empty set.

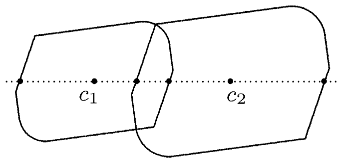
CASE 2: $u_1 \prec c_1 \prec v_1 = u_2 \prec c_2 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: v_1; a segment with v_1 in its interior.

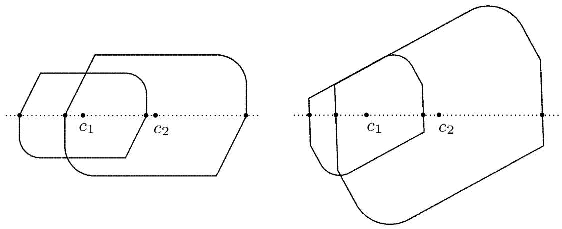
Table 1 (*continued*)

CASE 3: $u_1 \prec c_1 \preccurlyeq u_2 \prec v_1 \preccurlyeq c_2 \prec v_2$



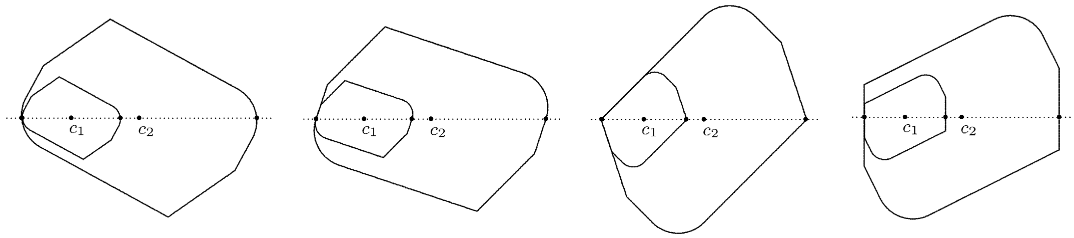
$\mathcal{C}_1 \cap \mathcal{C}_2$: two points; a point and a segment of a line that cuts $\langle c_1, c_2 \rangle$ in $p \prec u_1$ (if $\rho_1 < \rho_2$); two segments parallel to $\langle c_1, c_2 \rangle$ (if $\rho_1 = \rho_2$).

CASE 4: $u_1 \prec u_2 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$



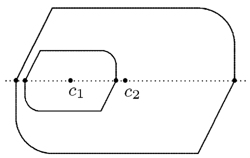
$\mathcal{C}_1 \cap \mathcal{C}_2$: two points; a point and a segment of a line that cuts $\langle c_1, c_2 \rangle$ in $p \prec u_1$.

CASE 5: $u_1 = u_2 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$



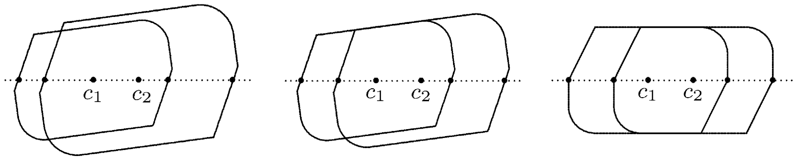
$\mathcal{C}_1 \cap \mathcal{C}_2$: u_1; one or two segments with extreme u_1; a segment with u_1 in its interior.

CASE 6: $u_2 \prec u_1 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: empty set.

CASE 7: $u_1 \prec u_2 \prec c_1 \prec c_2 \prec v_1 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: two points; a point and a segment of a line that cuts $\langle c_1, c_2 \rangle$ in $p \prec u_1$ (if $\rho_1 < \rho_2$); two segments parallel to $\langle c_1, c_2 \rangle$ (if $\rho_1 = \rho_2$).

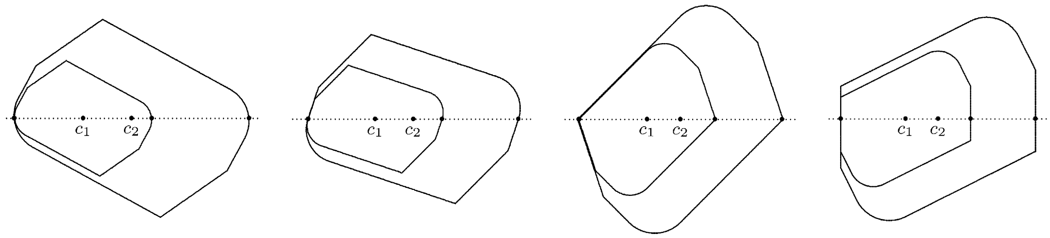
(*continued on next page*)

Assume first that $v_1 \preccurlyeq c_2$. According to the position of u_2, the following cases are possible.

Case 1. $u_1 \prec c_1 \prec v_1 \prec u_2 \prec c_2 \prec v_2$. Then it is obvious that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, because the lines $v_1 + \lambda v_1^*$ and $u_2 + \lambda v_1^*$ support $\mathcal{C}_1$ and $\mathcal{C}_2$ at the points v_1 and u_2, respectively, and the distance between these two lines is $\|u_2 - v_1\| > 0$.

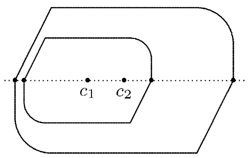
Table 1 (*continued*)

CASE 8: $u_1 = u_2 \prec c_1 \prec c_2 \prec v_1 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: u_1; one or two segments with extreme u_1; a segment with u_1 in its interior.

CASE 9: $u_2 \prec u_1 \prec c_1 \prec c_2 \prec v_1 \prec v_2$



$\mathcal{C}_1 \cap \mathcal{C}_2$: empty set.

Case 2. $u_1 \prec c_1 \prec v_1 = u_2 \prec c_2 \prec v_2$. Then $c_2 = (1+\rho_2)v_1$ and $v_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$. Assume that there exists another point $x \in \mathcal{C}_1 \cap \mathcal{C}_2$, $x \neq v_1$. Then

$$\frac{c_2 - x}{\|c_2 - x\|} = \frac{(1+\rho_2)v_1 - x}{\rho_2} = v_1 + \frac{1}{\rho_2}(v_1 - x).$$

Thus x, v_1 and $\frac{c_2-x}{\|c_2-x\|}$ are three aligned points in $\mathcal{C}_1$, which implies that $I_1 := [x, v_1 + \frac{1}{\rho_2}(v_1 - x)] \subset \mathcal{C}_1$. Moreover, x, v_1 and $c_2 - \rho_2 x = v_1 + \rho_2(v_1 - x)$ are aligned points in $\mathcal{C}_2$, which implies that $I_2 := [x, v_1 + \rho_2(v_1 - x)] \subset \mathcal{C}_2$. Since $I_1 \subset I_2$, we obtain that $I_1 \subset \mathcal{C}_1 \cap \mathcal{C}_2$. Since v_1 is an interior point of I_1, we get also that the vector v_1^* is parallel to that segment. Any other point in $\mathcal{C}_1 \cap \mathcal{C}_2$ has to lie in a segment of the line $v_1 + \lambda v_1^*$ with v_1 as interior point. Therefore $\mathcal{C}_1 \cap \mathcal{C}_2$ is a closed segment parallel to v_1^* with v_1 in its interior.

Case 3. $u_1 \prec c_1 \preccurlyeq u_2 \prec v_1 \preccurlyeq c_2 \prec v_2$. First observe that the common points of $\mathcal{C}_1$ and C_2 have to lie between the parallel lines $u_2 + \lambda v_1^*$ and $v_1 + \lambda v_1^*$. Moreover,

$$\begin{aligned}\big\|x(0) - c_2\big\| &= \big\|v_1 - \|c_2\|v_1\big\| = \|c_2\| - 1 < \|c_2\| - \|u_2\| = \rho_2 \\ &< \|c_2\| + 1 = \big\|\big(1 + \|c_2\|\big)v_1\big\| = \|-v_1 - c_2\| = \big\|x(\pi) - c_2\big\|.\end{aligned}$$

Therefore, there must exist θ_1 and $\bar{\theta}_1$ such that $0 < \theta_1 \leqslant \theta^* < \pi < \theta^* + \pi \leqslant \bar{\theta}_1 < 2\pi$ and $\|x(\theta_1) - c_2\| = \|x(\bar{\theta}_1) - c_2\| = \rho_2$. That is, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect each other in at least two points, one in P^+ and the other in P^-.

Suppose now that $\mathcal{C}_1 \cap \mathcal{C}_2$ contains at least three points. Thus we can assume that, without loss of generality, there exists θ_2 such that $0 < \theta_2 < \theta_1 \leqslant \theta^* < \pi$ and $\|x(\theta_2) - c_2\| = \rho_2$. Let θ_1' and θ_2' be such that $x(\theta_1') = \frac{x(\theta_1)-c_2}{\|x(\theta_1)-c_2\|}$ and $x(\theta_2') = \frac{x(\theta_2)-c_2}{\|x(\theta_2)-c_2\|}$. Then $\theta_1 \leqslant \theta^* < \theta_1' < \pi$ and $\theta_2 < \theta^* < \theta_2' < \pi$. Moreover, $\theta_1' \neq \theta_2'$ because $x(\theta_1') \neq x(\theta_2')$. Assume that $\theta_2' > \theta_1'$. From Lemma 2.1 it follows that $[x(\theta_1), x(\theta_1' + \pi)]$ is a segment of $\mathcal{C}_1$ that contains $x(\theta_2)$ and cuts $\langle c_1, c_2\rangle$ in v_1. Similarly, we obtain that $[x(\theta_1), c_2 - \rho_2 x(\theta_1)]$ is a segment of $\mathcal{C}_2$ that contains $x(\theta_2)$ and u_2, which implies $v_1 = u_2$, against the hypothesis. Therefore $\theta_2' < \theta_1'$. Again it follows from Lemma 2.1 that $[x(\theta_2), x(\theta_1')] \subset \mathcal{C}_1$, and since $\theta_2 < \theta_1 \leqslant \theta^* < \theta_2' < \theta_1'$, we obtain that $[x(\theta_1), x(\theta_2)] \subset \mathcal{C}_1 \cap \mathcal{C}_2$. Moreover, since $x(\theta_1)$ and $x(\theta_1')$ are in $\langle x(\theta_1), x(\theta_2)\rangle$, $\rho_1 = 1 = \|x(\theta_1')\| \leqslant \rho_2 = \|x(\theta_1) - c_2\|$ and the lines $\langle x(\theta_1'), c_1\rangle$ and $\langle x(\theta_1), c_2\rangle$ are parallel, it follows that either $\langle x(\theta_1), x(\theta_2)\rangle$ is parallel to $\langle c_1, c_2\rangle$ (if $\rho_1 = \rho_2$) or intersects the ray $u_1 + \lambda(u_1 - c_1)$, $\lambda > 0$ (if $\rho_1 < \rho_2$).

Thus, we have shown that if there are two points in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$, then the segment having these points as extremes belongs to $\mathcal{C}_1 \cap \mathcal{C}_2$. This implies that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ is a segment. Moreover, the line that contains this segment either is parallel to $\langle c_1, c_2\rangle$ (if $\rho_1 = \rho_2$) or intersects $\langle c_1, c_2\rangle$ (if $\rho_1 < \rho_2$) in a point $p \prec u_1$.

Assume now that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$ also contains two points. From the above we know that if $x(\bar{\theta}_1)$ and $x(\bar{\theta}_2)$, $\pi < \bar{\theta}_1 < \bar{\theta}_2 < 2\pi$, belong to $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$, then these two points are in a segment of $\mathcal{C}_1$ that contains $x(\theta^* + \pi)$, and therefore is parallel to $\langle x(\theta_1), x(\theta_2)\rangle$. Also we know that $\langle x(\bar{\theta}_1), x(\bar{\theta}_2)\rangle$ is either parallel to $\langle c_1, c_2\rangle$ or intersects $\langle c_1, c_2\rangle$ at $\bar{p} \prec u_1$. But the latter contradicts the parallelity of $\langle x(\theta_1), x(\theta_2)\rangle$ and $\langle x(\bar{\theta}_1), x(\bar{\theta}_2)\rangle$. Thus $\rho_1 = \rho_2$, which implies that the segments $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$ are parallel to $\langle c_1, c_2\rangle$ and symmetric with respect $\frac{1}{2}(c_1 + c_2)$.

Case 4. $u_1 \prec u_2 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$. In this case, necessarily $\rho_1 < \rho_2$ holds. Moreover,

$$\begin{aligned}\|x(0) - c_2\| &= \|v_1 - \|c_2\| v_1\| = \|c_2\| - 1 < \|c_2\| + \|u_2\| \\ &= \|c_2 - u_2\| = \rho_2 < \rho_2 + \|u_1 - u_2\| = \|x(\pi) - c_2\|,\end{aligned}$$

which implies that there exist θ_1 and $\bar\theta_1$ such that $0 < \theta_1 < \pi < \bar\theta_1 < 2\pi$ and $\|x(\theta_1) - c_2\| = \|x(\bar\theta_1) - c_2\| = \rho_2$. Therefore $\mathcal{C}_1 \cap \mathcal{C}_2$ has at least one point in P^+ and another point in P^-.

Assume that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ contains at least two points. Then it follows, as in Case 3, that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ is a segment and, since $\rho_1 < \rho_2$, the line that contains this segment intersects $\langle c_1, c_2\rangle$ in a point $p \prec u_1$. Moreover, $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$ has only one point.

Case 5. $u_1 = u_2 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$. Then $u_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$. Consider the parametrization of $\mathcal{C}_2$, $y(\theta) = c_2 + \rho_2 x(\theta)$, $0 \leqslant \theta \leqslant 2\pi$. From Lemma 2.1 it follows that the function $\|y(\theta)\|$ is decreasing in $[0, \pi]$ and increasing in $[\pi, 2\pi]$. Then, $\|y(\theta)\| \geqslant \|y(\pi)\| = \|c_2 + \rho_2 u_1\| = \|u_2\| = \|u_1\| = \rho_1$ for $0 \leqslant \theta \leqslant 2\pi$. Therefore, for any $x \in \mathcal{C}_2$ we have $\|x\| \geqslant \rho_1$.

Assume that in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ there exists another point $x \neq u_1$. Again from Lemma 2.1 it follows that $[u_1, x] \subset \mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$, and therefore $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ is a segment with extreme u_1. The same situation holds with $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$.

Thus, the set $\mathcal{C}_1 \cap \mathcal{C}_2$ can have the following forms: (a) the point u_1; (b) a segment with extreme u_1; (c) two non-aligned segments with u_1 as common extreme; (d) a segment with u_1 as interior point.

Case 6. $u_2 \prec u_1 \prec c_1 \prec v_1 \preccurlyeq c_2 \prec v_2$. Considering the parametrization of $\mathcal{C}_2$ given in Case 5 we obtain that $\|y(\theta)\| \geqslant \|y(\pi)\| = \|u_2\| > \|u_1\| = \rho_1$ for $0 \leqslant \theta \leqslant 2\pi$, which implies that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

Assume now that $c_2 \prec v_1$. Since we are assuming that $\rho_1 \leqslant \rho_2$, we have that $v_1 \prec v_2$ and $u_2 \prec c_1$. Thus only the following cases are possible.

Case 7. $u_1 \prec u_2 \prec c_1 \prec c_2 \prec v_1 \prec v_2$. This case is very similar to Case 3. Now we have $\|x(0) - c_2\| = \rho_1 - \|c_1 - c_2\| < \rho_1 + \|c_1 - c_2\| = \|x(\pi) - c_2\|$. Therefore, at least one point exists in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$, and another point in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$. Assume that $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ contains two points. Then there exist $0 < \theta_2 < \theta_1 < \pi$ such that $\|x(\theta_1) - c_2\| = \|x(\theta_2) - c_2\| = \rho_2$. Let θ_1' and θ_2' be such that $x(\theta_1') = \frac{x(\theta_1) - c_2}{\|x(\theta_1) - c_2\|}$ and $x(\theta_2') = \frac{x(\theta_2) - c_2}{\|x(\theta_2) - c_2\|}$. Then $\theta_1' \neq \theta_2'$, $\theta_1 < \theta_1' < \pi$ and $\theta_2 < \theta_2' < \pi$. Assume that $\theta_2' > \theta_1'$. Then $[x(\theta_1), x(\theta_1' + \pi)] \subset \mathcal{C}_1$ and $[x(\theta_1), c_2 - \rho_2 x(\theta_1)] \subset \mathcal{C}_2$. But since both segments contains $x(\theta_2)$ and intersect $\langle c_1, c_2\rangle$, it follows that $\mathcal{C}_1$ and $\mathcal{C}_2$ have a common point in $\langle c_1, c_2\rangle$, which contradicts the hypothesis. Therefore $\theta_2' < \theta_1'$, which implies that $[x(\theta_2), x(\theta_1')] \subset \mathcal{C}_1$, and since $\theta_1, \theta_2' \in (\theta_2, \theta_1')$, we get that $[x(\theta_1), x(\theta_2)] \subset \mathcal{C}_1 \cap \mathcal{C}_2$. Therefore $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ is a segment. As in Case 3, this segment is parallel to $\langle c_1, c_2\rangle$ or is in a line that intersects $\langle c_1, c_2\rangle$ in a point $p \prec u_1$, depending on whether $\rho_1 = \rho_2$ or $\rho_1 < \rho_2$. As in Case 3, we have again that if $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$ contains also two points, then $\rho_1 = \rho_2$ and the segments $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^+$ and $\mathcal{C}_1 \cap \mathcal{C}_2 \cap P^-$ are parallel to $\langle c_1, c_2\rangle$ and symmetric with respect $\frac{1}{2}(c_1 + c_2)$.

Case 8. $u_1 = u_2 \prec c_1 \prec c_2 \prec v_1 \prec v_2$. This case is like Case 5, since there we have not used that $v_1 \preccurlyeq c_2$.

Case 9. $u_2 \prec u_1 \prec c_1 \prec c_2 \prec v_1 \prec v_2$. As in Case 6, it follows that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. □

The following statements directly follow from Theorem 3.1.

Corollary 3.1. *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed plane with unit circle $\mathcal{C}$. Assume that $\mathcal{C}_1 = \mathcal{C}(c_1, \rho_1)$ and $\mathcal{C}_2 = \mathcal{C}(c_2, \rho_2)$ are two different circles whose intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ contains three non-collinear points t_1, t_2 and t_3. Let u_1 and v_1 be the points in which the line $\langle c_1, c_2\rangle$ intersects $\mathcal{C}_1$.*

(a) *If $\rho_1 = \rho_2$, then $\mathcal{C}_1 \cap \mathcal{C}_2$ consists of two closed, disjoint non-degenerate segments, parallel to $\langle c_1, c_2\rangle$ and symmetric with respect to $\frac{1}{2}(c_1 + c_2)$.*
(b) *If $\rho_1 \neq \rho_2$, then $\mathcal{C}_1 \cap \mathcal{C}_2$ can have only one of the following forms:*
  (b.1) *$\mathcal{C}_1 \cap \mathcal{C}_2$ consists of a non-degenerate segment and a point lying on the opposite sides of $\langle c_1, c_2\rangle$. Moreover, the segment is not parallel to $\langle c_1, c_2\rangle$.*
  (b.2) *$\mathcal{C}_1 \cap \mathcal{C}_2$ consists of two non-degenerate segments, lying on the opposite sides of $\langle c_1, c_2\rangle$, with common endpoint u_1 or v_1.*

Therefore, in all three situations one of the segments contains a side of the triangle $t_1t_2t_3$.

4. Where is the circumcenter?

Let t_1, t_2, t_3 be three non-collinear points in a normed plane $(\mathbb{R}^2, \|\cdot\|)$. Assume that there exists a circle which passes through the three points. Theorem 4.1 describes the region of the plane where the center of that circle has to be located. (See also Fig. 3.) Let

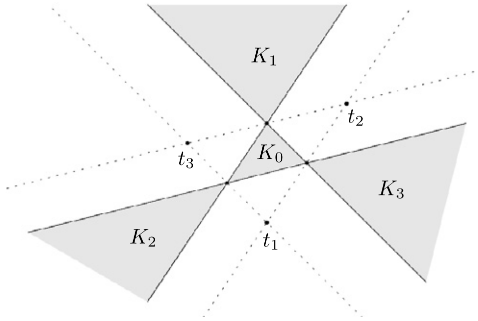


Fig. 3. The region where the center of a circle passing through t_1, t_2, t_3 has to be located.

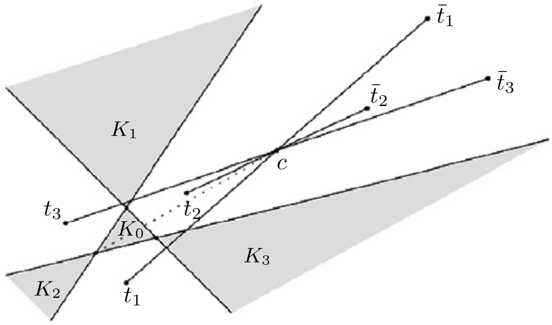


Fig. 4. Proof of Theorem 4.1, $c \notin \bigcup_{i=0}^{3} K_i(t_1, t_2, t_3)$.

$$K_0(t_1, t_2, t_3) = \operatorname{conv}\left\{\frac{t_1 + t_2}{2}, \frac{t_2 + t_3}{2}, \frac{t_3 + t_1}{2}\right\},$$

and define, for $\{i, j, k\} = \{1, 2, 3\}$, the cones

$$K_i(t_1, t_2, t_3) = \left\{t_i + x_1(t_j - t_i) + x_2(t_k - t_i)\colon x_1 \geqslant \frac{1}{2},\ x_2 \geqslant \frac{1}{2}\right\}.$$

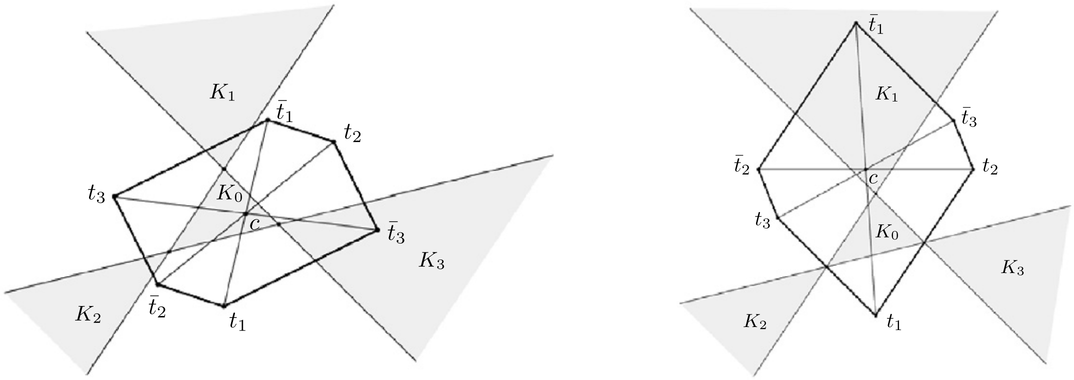
Theorem 4.1. *Let t_1, t_2, t_3 be three non-collinear points in $\mathbb{R}^2$. There exists a norm $\|\cdot\|$ and a circle $\mathcal{C}(c, \rho)$ in $(\mathbb{R}^2, \|\cdot\|)$ passing through the three points if and only if $c \in \bigcup_{i=0}^{3} K_i(t_1, t_2, t_3)$.*

Proof. Assume that $c \notin \bigcup_{i=0}^{3} K_i(t_1, t_2, t_3)$ and that there exists $\rho > 0$ such that $\mathcal{C}(c, \rho)$ passes through the three points. We will get a contradiction. Without loss of generality, we can assume that

$$c = \frac{t_1 + t_3}{2} + \mu\left[\beta\left(\frac{t_1 + t_2}{2}\right) + (1 - \beta)\left(\frac{t_2 + t_3}{2}\right) - \frac{t_1 + t_3}{2}\right]$$

with $\mu > 1$ and $0 < \beta < 1$ (see Fig. 4). According to the values of μ and β, we consider the following six cases:

1. Assume that $1 < \mu < 2$. Then $0 < \frac{\mu - 1}{\mu} < \frac{1}{\mu} < 1$.
   1.1. Assume that $0 < \beta < \frac{\mu - 1}{\mu}$. Taking $\gamma = 1 - 2\beta$ and $\delta = \frac{\mu - 1 - \mu\beta}{\mu(1 - 2\beta)}$, we have that
   $$c - t_2 = \gamma\big(\delta(c - t_1) + (1 - \delta)(t_3 - c)\big).$$
   1.2. Assume that $\frac{\mu - 1}{\mu} \leqslant \beta \leqslant \frac{1}{\mu}$. Taking $\gamma = \frac{2 - \mu}{\mu}$ and $\delta = \frac{1 - \mu + \mu\beta}{2 - \mu}$, we have that
   $$c - t_2 = \gamma\big(\delta(t_1 - c) + (1 - \delta)(t_3 - c)\big).$$
   1.3. Assume that $\frac{1}{\mu} < \beta < 1$. Taking $\gamma = 2\beta - 1$ and $\delta = \frac{1 - \mu + \mu\beta}{\mu(2\beta - 1)}$, we have that
   $$c - t_2 = \gamma\big(\delta(t_1 - c) + (1 - \delta)(c - t_3)\big).$$



(A) $c \in K_0(t_1, t_2, t_3)$ (B) $c \in K_1(t_1, t_2, t_3)$

Fig. 5. Proof of Theorem 4.1.

2. Assume that $\mu \geqslant 2$. Then $0 < \frac{1}{\mu} \leqslant \frac{\mu-1}{\mu} < 1$.
   2.1. Assume that $0 < \beta < \frac{1}{\mu}$. Taking $\gamma = 1 - 2\beta$ and $\delta = \frac{\mu-1-\mu\beta}{\mu(1-2\beta)}$, we have that
   $$c - t_2 = \gamma\big(\delta(c - t_1) + (1 - \delta)(t_3 - c)\big).$$
   2.2. Assume that $\frac{1}{\mu} \leqslant \beta \leqslant \frac{\mu-1}{\mu}$. Taking $\gamma = \frac{\mu-2}{\mu}$ and $\delta = \frac{\mu-1-\mu\beta}{\mu-2}$, we have that
   $$c - t_2 = \gamma\big(\delta(c - t_1) + (1 - \delta)(c - t_3)\big).$$
   2.3. Assume that $\frac{\mu-1}{\mu} < \beta < 1$. Taking $\gamma = 2\beta - 1$ and $\delta = \frac{1-\mu+\mu\beta}{\mu(2\beta-1)}$, we have that
   $$c - t_2 = \gamma\big(\delta(t_1 - c) + (1 - \delta)(c - t_3)\big).$$

   In the six cases we have $0 \leqslant \gamma < 1$, $0 \leqslant \delta \leqslant 1$, and
$$\rho = \|c - t_2\| \leqslant \gamma\big(\delta\|c - t_1\| + (1 - \delta)\|c - t_3\|\big) = \gamma\big(\delta\rho + (1 - \delta)\rho\big) = \gamma\rho < \rho,$$
which is absurd.

Conversely, assume that $c \in \bigcup_{i=0}^{3} K_i(t_1, t_2, t_3)$. For $i = 1, 2, 3$, let $\bar{t}_i$ be the point symmetric to t_i with respect to c, i.e., $\bar{t}_i = 2c - t_i$. We assume first that $c \in K_0(t_1, t_2, t_3)$. Then for $\{i, j, k\} = \{1, 2, 3\}$ we have that $\bar{t}_i \in \operatorname{conv}\{-t_i + t_j + t_k, t_j, t_k\}$ (see Fig. 5(A)), which implies that the hexagon with consecutive vertices t_1, $\bar{t}_3$, t_2, $\bar{t}_1$, t_3, $\bar{t}_2$ is convex and symmetric with respect to c, i.e., it is a sphere of a certain norm in $\mathbb{R}^2$ centered at c. Observe that this hexagon is reduced to a rectangle if c is the midpoint of a side of the triangle $t_1t_2t_3$. Assume now that $c \in K_i(t_1, t_2, t_3)$ with $i \in \{1, 2, 3\}$. Without loss of generality, we can assume that $c \in K_1(t_1, t_2, t_3)$ (see Fig. 5(B)). We will show that the hexagon with consecutive vertices t_1, t_2, $\bar{t}_3$, $\bar{t}_1$, $\bar{t}_2$, t_3 is convex. For that purpose it is enough to see that $t_1 \in HP_c^-(t_2, t_3) \cap HP_c^+(\bar{t}_2, t_3) \cap HP_c^+(t_2, \bar{t}_3)$, which is equivalent to the situation that $t_1 = \delta t_2 + (1 - \delta)t_3 + \gamma(t_2 - \bar{t}_3)$, with $0 \leqslant \delta \leqslant 1$ and $\gamma \geqslant 0$. Since $c \in K_1(t_1, t_2, t_3)$, we know that $c = t_1 + x_1(t_2 - t_1) + x_2(t_3 - t_1)$, with $x_1 \geqslant \frac{1}{2}$ and $x_2 \geqslant \frac{1}{2}$. If $x_1 + x_2 = 1$, then $x_1 = x_2 = \frac{1}{2}$, i.e., $c = \frac{t_2+t_3}{2} \in K_0(t_1, t_2, t_3)$. Therefore we can assume that $x_1 + x_2 > 1$. Taking
$$\gamma = \frac{1}{2(x_1 + x_2 - 1)}, \qquad \delta = \frac{2x_1 - 1}{2(x_1 + x_2 - 1)},$$
we have $\gamma > 0$, $0 \leqslant \delta \leqslant 1$, and $\delta t_2 + (1 - \delta)t_3 + \gamma(t_2 - \bar{t}_3) = t_1$, and the proof is complete. □

Now we consider the case in which two circles $\mathcal{C}(c, \rho)$ and $\mathcal{C}(c', \rho')$ pass through three non-collinear points t_1, t_2, t_3. Recall that then (Corollary 3.1) at least one side of the triangle $t_1t_2t_3$ belongs to both circles. Theorem 4.2 will show that in this situation the region where the centers c and c' can be located is more restricted. This long theorem gives a complete (analytical and geometrical) description of this region. It is divided in four sections according to the location of c and c' in the sets $K_i(t_1, t_2, t_3)$, $i = 0, 1, 2, 3$. The case in which $c, c' \in K_2(t_1, t_2, t_3) \setminus K_0(t_1, t_2, t_3)$ is not considered because it

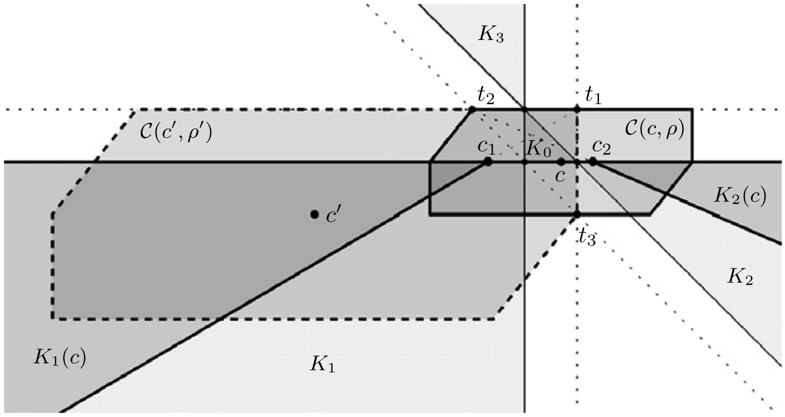


Fig. 6. Theorem 4.2(b).

is completely similar to the case (c), in which $c, c' \in K_1(t_1,t_2,t_3) \setminus K_0(t_1,t_2,t_3)$. Figs. 6, 7 and 8 illustrate the different situations.

Theorem 4.2. *Let t_1, t_2, t_3 be three non-collinear points in $(\mathbb{R}^2, \|\cdot\|)$. Assume that there are two different circles, $\mathcal{C}(c,\rho)$ and $\mathcal{C}(c',\rho')$ that pass through the three points, and that $[t_1,t_2] \subset \mathcal{C}(c,\rho) \cap \mathcal{C}(c',\rho')$. Then the following statements hold true:*

(a) *No center c or c' is in $K_3(t_1,t_2,t_3)$.*
(b) *If $c \in K_0(t_1,t_2,t_3)$, then $c \in [\frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}]$ and $\rho = \frac{\|t_1-t_3\|}{2} = \frac{\|t_2-t_3\|}{2}$. Moreover, if $c = \alpha(\frac{t_1+t_3}{2}) + (1-\alpha)(\frac{t_2+t_3}{2})$, with $0 \leqslant \alpha \leqslant 1$, then $c' \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle \cup K_1(c) \cup K_2(c)$, where*

$$K_1(c) = \left\{ t_1 + y_1(t_2-t_1) + y_2(t_3-t_1)\colon \frac{1}{2} \leqslant y_2 \leqslant \frac{y_1}{1+\alpha} \right\} \subset K_1(t_1,t_2,t_3)$$

and

$$K_2(c) = \left\{ t_2 + y_1(t_1-t_2) + y_2(t_3-t_2)\colon \frac{1}{2} \leqslant y_2 \leqslant \frac{y_1}{2-\alpha} \right\} \subset K_2(t_1,t_2,t_3).$$

Moreover, if $c' = t_i + x'_1(t_{3-i}-t_i) + x'_2(t_3-t_i) \in K_i(c)$, with $i = 1$ or 2, then $\rho' = 2x'_2\rho$. Besides, the sets $K_1(c)$ and $K_2(c)$ can be equivalently defined by

$$K_1(c) = \left\{ c_1 + \bar{\alpha}(t_2-t_1) + \bar{\beta}(c_1-t_1)\colon \bar{\alpha} \geqslant 0,\ \bar{\beta} \geqslant 0 \right\}$$

and

$$K_2(c) = \left\{ c_2 + \bar{\alpha}(t_1-t_2) + \bar{\beta}(c_2-t_2)\colon \bar{\alpha} \geqslant 0,\ \bar{\beta} \geqslant 0 \right\},$$

where $c_1 = t_2 + t_3 - c$ and $c_2 = t_1 + t_3 - c$ (see Fig. 6).
(c) *Assume that $c, c' \in K_1(t_1,t_2,t_3) \setminus K_0(t_1,t_2,t_3)$, i.e.,*

$$c = t_1 + x_1(t_2-t_1) + x_2(t_3-t_1), \quad x_1 \geqslant \frac{1}{2},\ x_2 \geqslant \frac{1}{2},\ (x_1,x_2) \neq \left(\frac{1}{2},\frac{1}{2}\right),$$

$$c' = t_1 + x'_1(t_2-t_1) + x'_2(t_3-t_1), \quad x'_1 \geqslant \frac{1}{2},\ x'_2 \geqslant \frac{1}{2},\ \left(x'_1,x'_2\right) \neq \left(\frac{1}{2},\frac{1}{2}\right).$$

Then $x_2\rho' = x'_2\rho$. Moreover, if $x_1 + x_2 \leqslant x'_1 + x'_2$, then

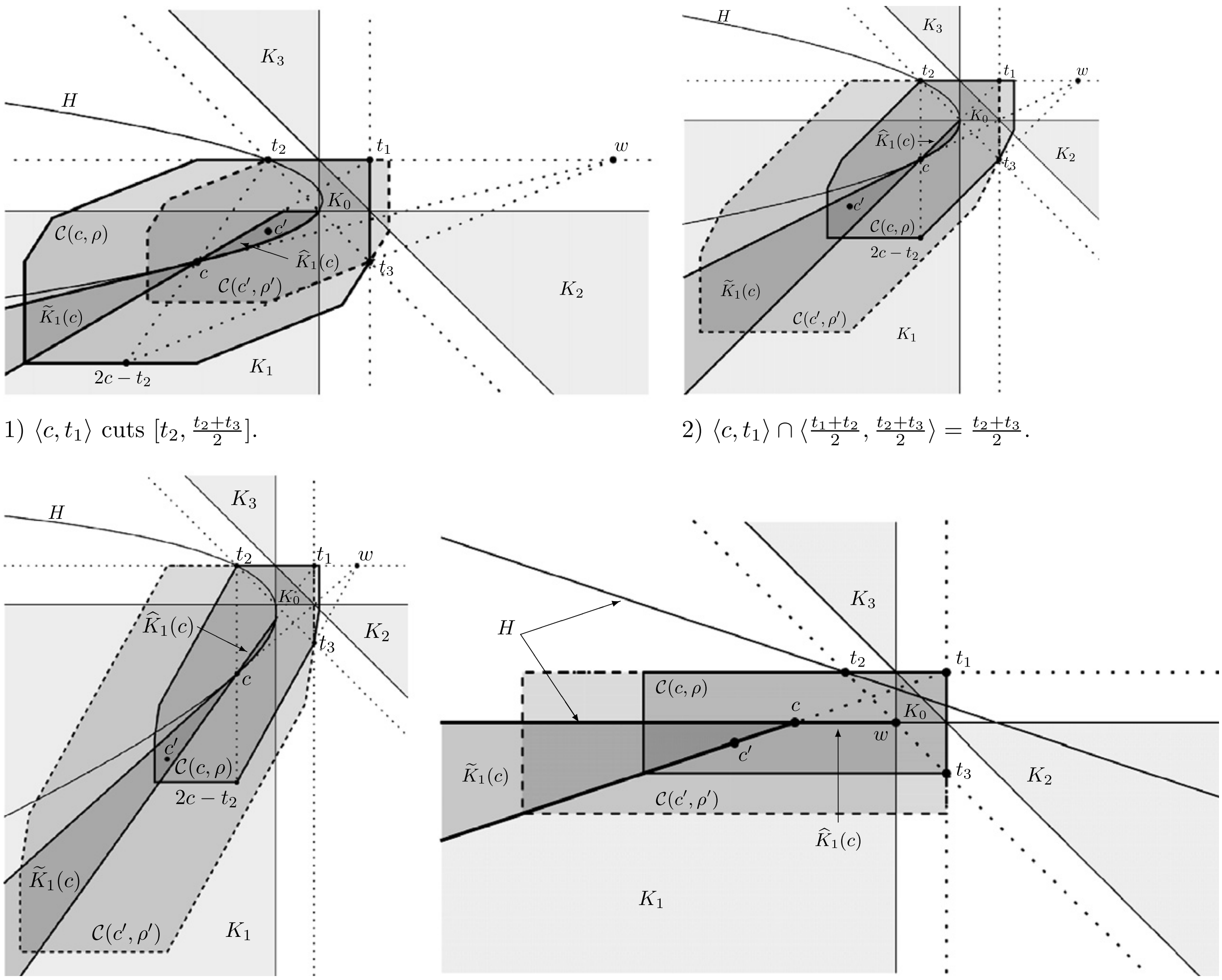
$$\frac{(2x_2^2 - x_2)x'_1 + x_2(2x_1-1)}{x_1(2x_2+1)-1} \leqslant x'_2 \leqslant \frac{x_2}{x_1}x'_1,$$

whereas if $x_1 + x_2 \geqslant x'_1 + x'_2$, then

$$\frac{x_2}{x_1}x'_1 \leqslant x'_2$$

and

$$2x_1{x'_2}^2 + 2(1-x_2)x'_1x'_2 - x_2x'_1 - (x_1+1)x'_2 + x_2 \leqslant 0.$$



1) $\langle c, t_1\rangle$ cuts $[t_2, \frac{t_2+t_3}{2}]$.

2) $\langle c, t_1\rangle \cap \langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle = \frac{t_2+t_3}{2}$.

3) $\langle c, t_1\rangle$ cuts $[\frac{t_2+t_3}{2}, t_3]$.

4) $c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}\rangle$ and H degenerates to two lines.

Fig. 7. Theorem 4.2(c).

Geometrically this means that, with respect to c, the point c' must be in the region $(\widetilde{K}_1(c) \cup \widehat{K}_1(c)) \cap K_1(t_1, t_2, t_3)$, where $\widetilde{K}_1(c)$ is the cone

$$\widetilde{K}_1(c) = \{c + \bar{\alpha}(c - t_1) + \bar{\beta}(c - w)\colon\ \bar{\alpha} \geqslant 0,\ \bar{\beta} \geqslant 0\},$$

being

$$w = \begin{cases} \langle t_1, t_2\rangle \cap \langle 2c - t_2, t_3\rangle & \text{if } c \notin \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}\rangle, \\ \frac{t_2+t_3}{2} & \text{if } c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}\rangle, \end{cases}$$

and $\widehat{K}_1(c)$ is the region limited by the line $\langle c, t_1\rangle$ and the conic that goes through the points t_2, $\frac{t_2+t_3}{2}$, $\langle c, t_1\rangle \cap \langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle$ and is tangent to $\langle c, w\rangle$ at c (see Fig. 7). If $\langle c, t_1\rangle \cap \langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle = \frac{t_2+t_3}{2}$, then the conic is also tangent to $\langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle$ at $\frac{t_2+t_3}{2}$ (see Fig. 7(2)). If $x_2 = 1$, the conic is a parabola. In other case, the conic is a hyperbola that degenerates to two lines when $x_2 = \frac{1}{2}$, i.e., when $c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}\rangle$ (see Fig. 7(4)). If $c, c' \in K_2(t_1, t_2, t_3) \setminus K_0(t_1, t_2, t_3)$, then the situation is similar.

(d) *Assume that $c \in K_1(t_1, t_2, t_3)$ and $c' \in K_2(t_1, t_2, t_3)$, i.e.,*

$$c = t_1 + x_1(t_2 - t_1) + x_2(t_3 - t_1), \quad x_1 \geqslant \frac{1}{2},\ x_2 \geqslant \frac{1}{2},$$

$$c' = t_2 + x_1'(t_1 - t_2) + x_2'(t_3 - t_2), \quad x_1' \geqslant \frac{1}{2},\ x_2' \geqslant \frac{1}{2}.$$

Then $x_2\rho' = x_2'\rho$. Moreover, the following situations are possible (see Fig. 8):

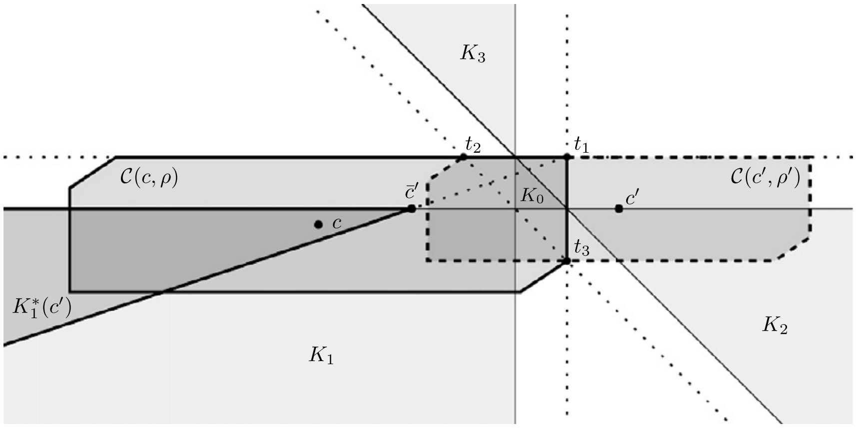


Fig. 8. Theorem 4.2(d).

(d.1) *Both centers c and c' can be simultaneously in $\langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$. In such a case $\rho = \rho'$.*

(d.2) *If $c \notin \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$, then $c' \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$, i.e., $x_2' = \frac{1}{2}$. Moreover, $x_1 \geqslant x_2(1+2x_1')$, which geometrically means that*

$$c \in K_1^*(c') = \{\bar{c}' + y_1(t_2 - t_1) + y_2(\bar{c}' - t_1):\ y_1 \geqslant 0,\ y_2 \geqslant 0\},$$

where $\bar{c}' = t_2 + t_3 - c'$ is the point symmetric to c' with respect to $\frac{t_2+t_3}{2}$.

(d.3) *If $c' \notin \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$, then $c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$, i.e., $x_2 = \frac{1}{2}$. Moreover, $x_1' \geqslant x_2'(1+2x_1)$, which geometrically means that*

$$c' \in K_2^*(c) = \{\bar{c} + y_1(t_1 - t_2) + y_2(\bar{c} - t_1):\ y_1 \geqslant 0,\ y_2 \geqslant 0\},$$

where $\bar{c} = t_1 + t_3 - c$ is the point symmetric to c with respect to $\frac{t_1+t_3}{2}$.

Proof. (a) Assume that $c \in K_3(t_1,t_2,t_3)$, i.e., $c = t_3 + x_1(t_1 - t_3) + x_2(t_2 - t_3)$, with $x_1 \geqslant \frac{1}{2}$ and $x_2 \geqslant \frac{1}{2}$. Then $(x_1 + x_2 - 1) \times (t_3 - c) = (x_1 + x_2)(\frac{x_1}{x_1+x_2} t_1 + \frac{x_2}{x_1+x_2} t_2 - c)$, and since $[t_1,t_2] \subset \mathcal{C}(c,\rho)$, we get that $(x_1 + x_2 - 1)\rho = (x_1 + x_2)\rho$, which is absurd. Obviously, a similar absurdity holds when $c' \in K_3(t_1,t_2,t_3)$.

(b) Assume that $c \in K_0(t_1,t_2,t_3)$, i.e.,

$$c = \frac{t_2 + t_3}{2} + x_1(t_1 - t_2) + x_2(t_1 - t_3),$$

with $0 \leqslant x_1 \leqslant \frac{1}{2}$ and $0 \leqslant x_2 \leqslant \frac{1}{2}$. Then

$$(1 - 2x_2)(c - t_3) = (1 + 2x_2)\left(\frac{2x_1 + 2x_2}{1 + 2x_2} t_1 + \frac{1 - 2x_1}{1 + 2x_2} t_2 - c\right),$$

and recalling that $[t_1,t_2] \subset \mathcal{C}(c,\rho)$ we get $(1 - 2x_2)\rho = (1 + 2x_2)\rho$. Therefore $x_2 = 0$, and then

$$c = \alpha\left(\frac{t_1 + t_3}{2}\right) + (1 - \alpha)\left(\frac{t_2 + t_3}{2}\right),$$

where $\alpha = 2x_1$, i.e., $c \in [\frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}]$. Moreover, from the identities

$$\frac{t_1 - t_3}{2} = \left(\frac{1 + 2x_1}{2}\right) t_1 + \left(\frac{1 - 2x_1}{2}\right) t_2 - c$$

and

$$\frac{t_2 - t_3}{2} = x_1 t_1 + (1 - x_1) t_2 - c$$

it follows that $\rho = \frac{\|t_1 - t_3\|}{2} = \frac{\|t_2 - t_3\|}{2}$.

Now, from Theorem 4.1 we know that $c' \in \bigcup_{i=0}^{3} K_i(t_1,t_2,t_3)$, but from (a) we know that $c' \notin K_3(t_1,t_2,t_3)$. Assume that $c' \in K_0(t_1,t_2,t_3)$. Then, as it happened with c, we have that $c' \in [\frac{t_1+t_3}{2}, \frac{t_2+t_3}{2}] \subset \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$. Assume that $c' \in K_1(t_1,t_2,t_3)$, i.e.,

$$c' = t_1 + x_1'(t_2 - t_1) + x_2'(t_3 - t_1),$$

with $x_1' \geqslant \frac{1}{2}$ and $x_2' \geqslant \frac{1}{2}$. First we will obtain the relation between ρ and ρ'. For that purpose, consider the identities

$$t_2 - c' = 2x_2'\big(\gamma(t_1 - c) + (1-\gamma)(t_2 - c)\big) \tag{1}$$

and

$$t_2 - c = \left(\gamma + \frac{1}{2x_2'}\right)(t_2 - c') + \gamma\big(c' - t_1\big), \tag{2}$$

where

$$\gamma = \frac{x_1' + x_2' - 1 + \alpha x_2'}{2x_2'} \geqslant 0.$$

Now assume that $\gamma \leqslant 1$. Since $[t_1, t_2] \subset \mathcal{C}(c, \rho)$, it follows from (1) that $\rho' = 2x_2'\rho$. On the other hand, assume that $\gamma > 1$. Then, again from (1), it follows that

$$\rho' = 2x_2'\big\|\gamma(t_1 - c) + (1-\gamma)(t_2 - c)\big\| \geqslant 2x_2'\big(\gamma\rho - (\gamma - 1)\rho\big) = 2x_2'\rho,$$

and from (2) we get that

$$\rho \geqslant \left(\gamma + \frac{1}{2x_2'}\right)\rho' - \gamma\rho' = \frac{\rho'}{2x_2'}.$$

Therefore, in any case,

$$\rho' = 2x_2'\rho. \tag{3}$$

Now, having (3) in mind, we obtain the identity

$$x_2'\big(x_1' + x_2' + \alpha x_2'\big)\big(c' - t_3\big) = \frac{x_1'\rho'}{\rho}(t_2 - c) + x_2'\big(x_1' + x_2' - 1 + \alpha\big(x_2' - 1\big)\big)\big(c' - t_1\big),$$

and then

$$x_2'\big(x_1' + x_2' + \alpha x_2'\big)\rho' \leqslant x_1'\rho' + x_2'\big|x_1' + x_2' - 1 + \alpha\big(x_2' - 1\big)\big|\rho'. \tag{4}$$

Assume that $x_1' + x_2' - 1 + \alpha(x_2' - 1) \geqslant 0$. Then it follows from (4) that

$$x_2' \leqslant \frac{x_1'}{1+\alpha},$$

and then $c' \in K_1(c)$. Assume now that $x_1' + x_2' - 1 + \alpha(x_2' - 1) < 0$. We will see that in this case $c' \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$. From the identity

$$\big(x_1' + x_2' - 1 + \alpha x_2'\big)\big(c' - t_3\big) = 2\big(x_1' + x_2' - 1\big)(t_2 - c) + \big(1 - x_1' - x_2' - \alpha\big(x_2' - 1\big)\big)\big(t_2 - c'\big)$$

it follows that

$$\big(x_1' + x_2' - 1 + \alpha x_2'\big)\rho' \leqslant 2\big(x_1' + x_2' - 1\big)\rho + \big(1 - x_1' - x_2' - \alpha\big(x_2' - 1\big)\big)\rho',$$

and having (3) in mind, we get

$$x_2'\big(2\big(x_1' + x_2' - 1\big) + \alpha\big(2x_2' - 1\big)\big) \leqslant x_1' + x_2' - 1,$$

which yields $x_2' \leqslant \frac{1}{2}$. Therefore $x_2' = \frac{1}{2}$, and then $c' = (1 - 2x_1')(\frac{t_1+t_3}{2}) + 2x_1'(\frac{t_2+t_3}{2})$.

Finally, taking $\bar{\alpha} = y_1 - (1+\alpha)y_2$ and $\bar{\beta} = 2y_2 - 1$, it follows that

$$t_1 + y_1(t_2 - t_1) + y_2(t_3 - t_1) = c_1 + \bar{\alpha}(t_2 - t_1) + \bar{\beta}(c_1 - t_1),$$

which gives the equivalent definition of $K_1(c)$.

Similarly we can see that if $c' \in K_2(t_1, t_2, t_3)$, then $c' \in K_2(c) \cup \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$, and also that the two definitions of $K_2(c)$ are equivalent.

(c) To simplify the notation, we consider $e_1 = t_2 - t_1$ and $e_2 = t_3 - t_1$. Assume first that $x_1 + x_2 \leqslant x_1' + x_2'$. Consider the convex function $f(\mu) = \|t_2 - c - \mu e_1\|$. Since $[t_1, t_2] \subset \mathcal{C}(c, \rho)$, we have that $f(\mu) = \rho$ for all $\mu \in [0, 1]$. Taking

$$\mu_1 = 1 - x_1 + \frac{x_2 x_1'}{x_2'}, \qquad \mu_2 = \mu_1 - \frac{x_2}{x_2'},$$

we have that

$$f(\mu_1)=\|t_2-c-\mu_1e_1\|=\left\|\frac{x_2}{x_2'}\big(t_1-c'\big)\right\|=\frac{x_2}{x_2'}\rho'=\left\|\frac{x_2}{x_2'}\big(t_2-c'\big)\right\|=\|t_2-c-\mu_2e_1\|=f(\mu_2).$$

Now we will see that $x_1x_2'\leqslant x_2x_1'$. Assume on the contrary that $x_1x_2'>x_2x_1'$. Then $\mu_2<\mu_1<1$, and since $f(\mu_2)=f(\mu_1)$, we get from the convexity of $f(\mu)$ that $f(\mu_2)=f(\mu_1)=f(0)=\rho$, which implies that

$$\rho'=\frac{x_2'}{x_2}\rho. \tag{5}$$

Consider now the identity

$$\big(x_1'+x_1x_2'-x_2x_1'\big)\big(t_3-c'\big)=x_1'(t_3-c)+\big(x_1x_2'-x_1'x_2+x_1'-x_1\big)\big(t_1-c'\big). \tag{6}$$

Since

$$\begin{aligned}x_1x_2'-x_1'x_2+x_1'-x_1&=\left(\frac{x_1}{x_1+x_2}\right)(x_1+x_2)\big(x_1'+x_2'-1\big)+x_1'(1-x_1-x_2)\\&\geqslant\left(\frac{x_1'}{x_1'+x_2'}\right)(x_1+x_2)\big(x_1'+x_2'-1\big)+x_1'(1-x_1-x_2)\\&=\frac{x_1'(x_1'+x_2'-x_1-x_2)}{x_1'+x_2'}\geqslant 0\end{aligned}$$

(recall that we are assuming $x_1'+x_2'\geqslant x_1+x_2$), we get from (6) that

$$\big(x_1'+x_1x_2'-x_2x_1'\big)\rho'\leqslant x_1'\rho+\big(x_1x_2'-x_1'x_2+x_1'-x_1\big)\rho',$$

and from (5) it follows that

$$\frac{(x_2x_1'-x_1x_2')\rho'}{x_2'}\geqslant 0,$$

which is absurd. Therefore,

$$x_2'\leqslant\frac{x_2}{x_1}x_1'. \tag{7}$$

Then $\mu_1\geqslant 1$, and since

$$\mu_2=\mu_1-\frac{x_2}{x_2'}\geqslant\mu_1-\frac{x_2(x_1'+x_2')}{x_2'(x_1+x_2)}=\frac{(x_1+x_2-1)(x_1'x_2-x_1x_2')}{x_2'(x_1+x_2)}\geqslant 0,$$

again the convexity of $f(\mu)$ implies that (5) holds.

To get the left bound of x_2', assume first that $x_2x_1'-x_1x_2'+x_1-x_1'>0$. From the identity

$$\big(x_1+x_2x_1'-x_1x_2'\big)(t_3-c)=x_1\big(t_3-c'\big)+\big(x_2x_1'-x_1x_2'+x_1-x_1'\big)(t_1-c),$$

(5), and (7) we get that

$$\big(x_1+x_2x_1'-x_1x_2'\big)\rho\leqslant x_1\rho'+\big(x_2x_1'-x_1x_2'+x_1-x_1'\big)\rho=\frac{x_1x_2'}{x_2}\rho+\big(x_2x_1'-x_1x_2'+x_1-x_1'\big)\rho,$$

from which it follows that $0\leqslant(x_2x_1'-x_1x_2')\rho\leqslant 0$, and therefore $x_2x_1'=x_1x_2'$. But then $0<x_2x_1'-x_1x_2'+x_1-x_1'=x_1-x_1'$, which implies $x_1'<x_1$, and then $x_1'+x_2'=\dfrac{x_1'(x_1+x_2)}{x_1}<x_1+x_2$, against the hypothesis. Therefore $x_2x_1'-x_1x_2'+x_1-x_1'\leqslant 0$. Now, from the identity

$$\big(x_2x_1'-x_1x_2'+x_1+x_2'-1\big)(t_3-c)=\big(x_1x_2'-x_2x_1'+x_1'-x_1\big)(c-t_2)+(x_1+x_2-1)\big(t_3-c'\big) \tag{8}$$

it follows that

$$\big(x_2x_1'-x_1x_2'+x_1+x_2'-1\big)\rho\leqslant\big(x_1x_2'-x_2x_1'+x_1'-x_1\big)\rho+(x_1+x_2-1)\rho'.$$

Having in mind (5), we obtain

$$\frac{(2x_2^2-x_2)x_1'+x_2(2x_1-1)}{x_1(2x_2+1)-1}\leqslant x_2'. \tag{9}$$

(Observe that $x_1\geqslant\frac{1}{2}$, $x_2\geqslant\frac{1}{2}$, $(x_1,x_2)\neq(\frac{1}{2},\frac{1}{2})$ implies that $x_1(2x_2+1)-1>0$.)

Assume now that $x_1 + x_2 \geqslant x'_1 + x'_2$. Interchanging the roles of x_i and x'_i $(i = 1, 2)$, we get (5) again, and from (7) and (9) it follows that

$$\frac{x_2}{x_1} x'_1 \leqslant x'_2 \tag{10}$$

and

$$2x_1 x_2'^2 + 2(1 - x_2)x'_1 x'_2 - x_2 x'_1 - (x_1 + 1)x'_2 + x_2 \leqslant 0. \tag{11}$$

We now come to the geometric interpretation of this case. First, let us consider the case $x_1 + x_2 \leqslant x'_1 + x'_2$. We will show that then $c' \in \widetilde{K}_1(c)$. For that purpose, we will see that $\widetilde{K}_1(c) = K'_1(c)$, where

$$K'_1(c) = \left\{ t_1 + x'_1 e_1 + x'_2 e_2\colon \frac{(2x_2^2 - x_2)x'_1 + x_2(2x_1 - 1)}{x_1(2x_2 + 1) - 1} \leqslant x'_2 \leqslant \frac{x_2}{x_1} x'_1 \right\}.$$

Recall that we have defined

$$\widetilde{K}_1(c) = \left\{ c + \bar{\alpha}(c - t_1) + \bar{\beta}(c - w)\colon \bar{\alpha} \geqslant 0,\ \bar{\beta} \geqslant 0 \right\},$$

where

$$w = \begin{cases} \langle t_1, t_2 \rangle \cap \langle 2c - t_2, t_3 \rangle & \text{if } c \notin \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle, \\ \frac{t_2+t_3}{2} & \text{if } c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle. \end{cases}$$

Assume that $c \notin \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$. Then $2x_2 - 1 > 0$, and the intersection point of $\langle t_1, t_2 \rangle$ and $\langle 2c - t_2, t_3 \rangle$ is

$$w = t_3 + \gamma(t_3 - 2c + t_2) = t_1 + \gamma'(t_1 - t_2),$$

where $\gamma = 1/(2x_2 - 1)$ and $\gamma' = (2x_1 - 1)/(2x_2 - 1)$. Let $c' \in \widetilde{K}_1(c)$, i.e., $c' = c + \bar{\alpha}(c - t_1) + \bar{\beta}(c - w)$ with $\bar{\alpha} \geqslant 0$ and $\bar{\beta} \geqslant 0$. Straightforward computations show that $c' = t_1 + x'_1 e_1 + x'_2 e_2$, where

$$x'_1 = \frac{(2x_2 - 1)(2x_1 - 1 + 2\bar{\alpha}x_1) + 2\bar{\beta}(2x_1 x_2 + x_1 - 1)}{2(2x_2 - 1)} + \frac{1}{2},$$

$$x'_2 = (1 + \bar{\alpha} + \bar{\beta})x_2.$$

Since $x_1 \geqslant \frac{1}{2}$ and $x_2 \geqslant \frac{1}{2}$, we have that $x'_1 \geqslant \frac{1}{2}$ and $x'_2 \geqslant \frac{1}{2}$. Moreover,

$$\frac{x_2}{x_1} x'_1 - x'_2 = \frac{\bar{\beta} x_2(2x_1 - 1)}{x_1(2x_2 - 1)} \geqslant 0$$

and

$$x'_2 - \frac{(2x_2^2 - x_2)x'_1 + x_2(2x_1 - 1)}{x_1(2x_2 + 1) - 1} = \frac{\bar{\alpha} x_2(2x_1 - 1)}{x_1(2x_2 + 1) - 1} \geqslant 0,$$

which implies that $c' \in K'_1(c)$. Conversely, assume that $c' \in K'_1(c)$, i.e., $c' = t_1 + x'_1 e_1 + x'_2 e_2$ with

$$\delta := \frac{(2x_2^2 - x_2)x'_1 + x_2(2x_1 - 1)}{x_1(2x_2 + 1) - 1} \leqslant x'_2 \leqslant \frac{x_2}{x_1} x'_1.$$

Taking

$$\bar{\alpha} = \frac{(x'_2 - \delta)(2x_2 x_1 + x_1 - 1)}{x_2(2x_1 - 1)}, \qquad \bar{\beta} = \frac{(2x_2 - 1)(x_2 x'_1 - x_1 x'_2)}{x_2(2x_1 - 1)}$$

if $x_1 > \frac{1}{2}$, and $\bar{\alpha} = \bar{\beta} = \frac{2x'_1 - 1}{2}$ if $x_1 = \frac{1}{2}$ (observe that in this case $w = t_1$ and $\delta = x'_2 = 2x_2 x'_1$), we have that $c' = c + \bar{\alpha}(c - t_1) + \bar{\beta}(c - w)$, with $\bar{\alpha} \geqslant 0$ and $\bar{\beta} \geqslant 0$. Thus $c' \in \widetilde{K}_1(c)$. So we have $\widetilde{K}_1(c) = K'_1(c)$.

Assume now that $c \in \langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$. Then $x_2 = \frac{1}{2}$ (and therefore $x_1 > \frac{1}{2}$) and $w = \frac{t_2+t_3}{2}$. Taking $\bar{\alpha} = 2x'_2 - 1$ and $\bar{\beta} = \frac{4(x_2 x'_1 - x_1 x'_2)}{2x_1 - 1}$, we have that $c + \bar{\alpha}(c - t_1) + \bar{\beta}(c - w) = t_1 + x'_1 e_1 + x'_2 e_2$, which shows that $K'_1(c) \subset \widetilde{K}_1(c)$. On the other hand, taking $x'_1 = x_1(1 + \bar{\alpha}) + \bar{\beta}(x_1 - \frac{1}{2})$ and $x'_2 = \frac{1+\bar{\alpha}}{2}$, we get the same identity, which shows that $\widetilde{K}_1(c) \subset K'_1(c)$.

Now we consider the case $x_1 + x_2 \geqslant x'_1 + x'_2$. We will show that then $c' \in \widehat{K}_1(c)$. But first let us see that $x_1 \geqslant x'_1$ and $x_2 \geqslant x'_2$. From (10) it follows that $x'_1 \leqslant x_1 + x_2 - x'_2 \leqslant x_1 + x_2 - \frac{x_2 x'_1}{x_1}$, and then

$$x'_1 \leqslant \frac{x_1 + x_2}{1 + \frac{x_2}{x_1}} = x_1.$$

Moreover, from (11) we get that

$$x_2 \geqslant x_2'\left(\frac{2x_1x_2' + 2x_1' - x_1 - 1}{2x_1'x_2' + x_1' - 1}\right) = x_2'\left(1 + \frac{(2x_2' - 1)(x_1 - x_1')}{2x_1'x_2' + x_1' - 1}\right) \geqslant x_2'. \tag{12}$$

Let us consider the function

$$h(y_1, y_2) = 2x_1y_2^2 + 2(1 - x_2)y_1y_2 - x_2y_1 - (x_1 + 1)y_2 + x_2 = (1 \quad y_1 \quad y_2)\, A \begin{pmatrix} 1 \\ y_1 \\ y_2 \end{pmatrix},$$

where

$$A = \begin{pmatrix} x_2 & \frac{-x_2}{2} & \frac{-(x_1+1)}{2} \\ \frac{-x_2}{2} & 0 & 1 - x_2 \\ \frac{-(x_1+1)}{2} & 1 - x_2 & 2x_1 \end{pmatrix},$$

and the conic

$$H = \left\{t_1 + y_1e_1 + y_2e_2\colon h(y_1, y_2) = 0\right\}.$$

Since $\det A = -\frac{1}{2}x_2(x_1 + x_2 - 1)(2x_2 - 1)$, $\det\begin{pmatrix} 0 & 1-x_2 \\ 1-x_2 & 2x_1 \end{pmatrix} = -(1 - x_2)^2$, and $x_2(x_1 + x_2 - 1) > 0$, it follows that if $x_2 \neq 1$, then H is a hyperbola (that degenerate into two lines if $x_2 = \frac{1}{2}$). If $x_2 = 1$, H is a parabola. It is immediate to see that H passes through the points

$$c = t_1 + x_1e_1 + x_2e_2,$$
$$t_2 = t_1 + e_1,$$
$$\frac{t_2 + t_3}{2} = t_1 + \frac{1}{2}e_1 + \frac{1}{2}e_2,$$
$$s = t_1 + \frac{1}{2}e_1 + \frac{x_2}{2x_1}e_2,$$

where $\{s\} = \langle c, t_1\rangle \cap \langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle$, and straightforward computations show that its tangent at c coincides with the line $\langle c, w\rangle$. Moreover, if $s = \frac{t_2+t_3}{2}$, i.e., $x_1 = x_2$, then H is also tangent to the line $\langle \frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}\rangle$ at s.

Now we have to consider several cases. Assume that $x_2 = x_2'$. Since we can consider that $c' \neq c$, it follows from (12) that $x_2 = x_2' = \frac{1}{2}$, and then $h(x_1', x_2') = h(x_1', \frac{1}{2}) = 0$, i.e., $c' \in H$. Assume that $2x_1x_2' - x_2 = 0$. Then it follows from (10) that $0 = 2x_1x_2' - x_2 \geqslant 2x_2x_1' - x_2 = x_2(2x_1' - 1) \geqslant 0$, which implies that $x_1' = \frac{1}{2}$ and $h(x_1', x_2') = \frac{1}{2}(2x_2' - 1)(2x_1x_2' - x_2) = 0$, and we get also that $c' \in H$. Finally, assume that $x_2 > x_2'$ and $2x_1x_2' - x_2 > 0$, and let

$$p = t_1 + \left(\frac{x_1x_2'}{x_2}\right)e_1 + x_2'e_2, \qquad q = t_1 + \left(\frac{x_2 - x_2' - x_1x_2' + 2x_1x_2'^2}{x_2'(2x_2 - 1) + x_2 - x_2'}\right)e_1 + x_2'e_2.$$

Then we have $p \in \langle c, t_1\rangle$ and $q \in H$. Moreover, $c' = \mu p + (1 - \mu)q$ with

$$\mu = \frac{-h(x_1', x_2')x_2}{(x_2 - x_2')(2x_1x_2' - x_2)} = \frac{-h(x_1', x_2')x_2}{-h(x_1', x_2')x_2 + (x_1x_2' - x_2x_1')(x_2 - 2x_2' + 2x_2x_2')},$$

and since $x_2 - 2x_2' + 2x_2x_2' \geqslant x_2' - 2x_2' + 2x_2x_2' = x_2'(2x_2 - 1) \geqslant 0$, it follows from (10) and (11) that $0 \leqslant \mu \leqslant 1$. This which implies that $c' \in \widehat{K}_1(c)$.

(d) Since

$$\begin{aligned}\left(x_1x_2' + x_2\left(x_1' + x_2'\right)\right)\rho &= \left\|\left(x_1x_2' + x_2\left(x_1' + x_2'\right)\right)(c - t_2)\right\| = \left\|\left(x_2x_1' + x_2'(x_1 + x_2 - 1)\right)(c - t_1) + x_2\left(c' - t_2\right)\right\| \\ &\leqslant \left(x_2x_1' + x_2'(x_1 + x_2 - 1)\right)\rho + x_2\rho',\end{aligned}$$

we get that $x_2'\rho \leqslant x_2\rho'$. On the other hand, since

$$\begin{aligned}\left(x_1x_2' + x_2\left(x_1' + x_2'\right)\right)\rho' &= \left\|\left(x_1x_2' + x_2\left(x_1' + x_2'\right)\right)\left(c' - t_1\right)\right\| = \left\|\left(x_1x_2' + x_2\left(x_1' + x_2' - 1\right)\right)\left(c' - t_2\right) + x_2'(c - t_1)\right\| \\ &\leqslant \left(x_1x_2' + x_2\left(x_1' + x_2' - 1\right)\right)\rho' + x_2'\rho,\end{aligned}$$

we get that $x_2\rho' \leqslant x_2'\rho$. Therefore,

$$x_2\rho' = x_2'\rho. \tag{13}$$

The cases (d.1)–(d.3) will follow from the four cases that result from considering the sign of the following quantities:

$$\lambda = x_1(x'_2 - 1) + x_2(x'_1 + x'_2 - 1),$$
$$\lambda' = x'_1(x_2 - 1) + x'_2(x_1 + x_2 - 1).$$

Case 1. Assume that $\lambda \geqslant 0$ and $\lambda' \geqslant 0$. We will see that this case is not possible. From the identities

$$(\lambda + x_1 + x_2)(c' - t_3) = x'_1(t_1 - c) + \lambda(c' - t_2)$$

and

$$(\lambda' + x'_1 + x'_2)(t_3 - c) = x_1(c' - t_2) + \lambda'(t_1 - c)$$

it follows that

$$(\lambda + x_1 + x_2)\rho' \leqslant x'_1\rho + \lambda\rho'$$

and

$$(\lambda' + x'_1 + x'_2)\rho \leqslant x_1\rho' + \lambda'\rho.$$

By summing up the above inequalities, we obtain the absurdity $x_2\rho' + x'_2\rho \leqslant 0$.

Case 2. Assume that $\lambda \leqslant 0$ and $\lambda' \leqslant 0$. In this case, $x_2 - 1 \leqslant 0$ and $x'_2 - 1 \leqslant 0$. Therefore,

$$x_1x'_2 + (x_2 - 1)(x'_1 + x'_2 - 1) \geqslant x_1x'_2 + \frac{(x_2 - 1)x_1(1 - x'_2)}{x_2} = \frac{x_1(x_2 + x'_2 - 1)}{x_2} \geqslant 0$$

and

$$x_2x'_1 + (x'_2 - 1)(x_1 + x_2 - 1) \geqslant x_2x'_1 + \frac{(x'_2 - 1)x'_1(1 - x_2)}{x'_2} = \frac{x'_1(x_2 + x'_2 - 1)}{x'_2} \geqslant 0,$$

and from the identities

$$x'_2(t_3 - c) = (x_1x'_2 + (x_2 - 1)(x'_1 + x'_2 - 1))(c' - t_2) - \lambda'(c' - t_1)$$

and

$$x_2(c' - t_3) = (x_2x'_1 + (x'_2 - 1)(x_1 + x_2 - 1))(t_1 - c) - \lambda(t_2 - c),$$

as well as (13), it follows that

$$x_2\rho' = x'_2\rho \leqslant (x_1x'_2 + (x_2 - 1)(x'_1 + x'_2 - 1) - \lambda')\rho' = (1 - x_2)\rho'$$

and

$$x'_2\rho = x_2\rho' \leqslant (x_2x'_1 + (x'_2 - 1)(x_1 + x_2 - 1) - \lambda)\rho = (1 - x'_2)\rho.$$

Therefore, $x_2 \leqslant \frac{1}{2}$ and $x'_2 \leqslant \frac{1}{2}$, and then $x_2 = x'_2 = \frac{1}{2}$. This corresponds to case (d.1).

Case 3. Assume that $\lambda \leqslant 0$ and $\lambda' \geqslant 0$. From the identity

$$(\lambda' + x'_1 + x'_2)(t_3 - c) = x_1(c' - t_2) + \lambda'(t_1 - c)$$

it follows that

$$(x'_1 + x'_2)\rho \leqslant x_1\rho', \tag{14}$$

and from the identity

$$(\lambda' + x'_1 + x'_2)(c' - t_3) = x'_1(t_1 - c) + \lambda(c' - t_2)$$

and (13) it follows that

$$(\lambda + \lambda' + x'_1 + x'_2)\rho' \leqslant x'_1\rho = \frac{x'_1x_2\rho'}{x'_2}.$$

Therefore,

$$0 \geqslant x_2'\big(\lambda + \lambda' + x_1' + x_2'\big) - x_1' x_2 = \big(2x_2' - 1\big)\big(x_1 x_2' + x_2\big(x_1' + x_2'\big)\big),$$

and we obtain $x_2' = \frac{1}{2}$. Moreover, from (13) and (14) we get that $x_1 \geqslant x_2(1 + 2x_1')$, corresponding to case (d.2). The geometric interpretation follows from the identity

$$c = \bar{c}' + \big(x_1 - x_2\big(1 + 2x_1'\big)\big)(t_2 - t_1) + (2x_2 - 1)\big(\bar{c}' - t_1\big).$$

Case 4. Assume that $\lambda \geqslant 0$ and $\lambda' \leqslant 0$. This case is completely analogous to the above one by considering the identities

$$(\lambda + x_1 + x_2)\big(c' - t_3\big) = x_1'(t_1 - c) + \lambda\big(c' - t_2\big)$$

and

$$(\lambda + x_1 + x_2)(t_3 - c) = x_1\big(c' - t_2\big) - \lambda'(c - t_1),$$

yielding case (d.3). □

The next corollary follows easily from Theorem 4.2.

Corollary 4.1. *Under the hypothesis of Theorem 4.2, the two centers c and c' are in $\langle \frac{t_1+t_3}{2}, \frac{t_2+t_3}{2} \rangle$ if and only if $\rho = \rho'$.*

Remark 4.1. From the proof of Theorem 4.2 one can also deduce that if the centers c and c' are situated in the described regions, then it is possible to define a polygonal norm such that c and c' are the centers of two circles that meet at t_1, t_2, and t_3. In Figs. 6, 7, and 8 it is shown how such circles look like.

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