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Multistep Methods for Nonlinear Boundary-Value Problems with Parameters

TADEUSZ JANKOWSKI

4 Rylkego Strasse, Gdansk 80-307, Poland

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Multistep methods combined with iterative ones are applied to find a numerical solution of ordinary differential equations with parameters. This paper deals with the convergence of such methods. There are some estimations of errors too. © 1990 Academic Press, Inc.

1. INTRODUCTION

We assume throught that

$$y'(t) = f(t, y(t), \lambda), \qquad t \in I = [\alpha, \beta], \alpha < \beta, \tag{1}$$

is a system of q ordinary differential equations. The right-hand side of this system depends on p parameters λ, so $\lambda = [\lambda_1, ..., \lambda_p]^T \in R^p$. Together with (1) the following boundary conditions are given and they are of the form

$$y(\alpha) = y_0, \tag{2}$$

$$g(\lambda, y(\beta)) = \theta, \tag{3}$$

where y_0 is given in R^q and θ is zero vector in R^p. The function g is nonlinear. Now the exact solution (φ, λ) of BVP (1)–(3) consists of such $\varphi \in C(I, R^q)$ and $\lambda \in R^p$ that both the Eq. (1) and the conditions (2)–(3) are satisfied ($C(I, R^q)$ denotes the collection of all continuous functions from I into R^q).

A question of the existence and uniqueness of solutions of the problems (1)–(3) was considered by many authors (for example, see [10, 12, 14–17]). Our task is a problem of a numerical solution for (1)–(3). So due to this fact we assume that BVP (1)–(3) has the unique solution $(\varphi, \lambda) \in C(I, R^q) \times R^p$.

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To describe a numerical method, we first subdivide the interval I into N subintervals all of the same length $h=(\beta-\alpha)/N$. The points of subdivision will be denoted by $t_{h0}, t_{h1}, ..., t_{hN}$, where the ith such point is defined by $t_{hi}=\alpha+ih$, $i \in R_N=\{0, 1, ..., N\}$.

Now we suggest a multistep method for y_h combined with an iterative method for λ_{hj} to determine the numerical solution (y_h, λ_{hj}) of (1)–(3). This method can be written as

$$\begin{aligned} &\lambda_{h0}=\lambda_0, \qquad \lambda_0 \text{ is given}, \\ &\lambda_{h,j+1}=\lambda_{hj}-B^{-1}g(\lambda_{hj}, y_h(\beta; \lambda_{hj})), \qquad j=0, 1, ..., \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\sum_{i=0}^{k} a_i(t, h)\, y_h(t+ih; \lambda_{hj}) \\ &\quad = hF(t, ..., t+kh-h, h, y_h(t; \lambda_{hj}), ..., y_h(t+kh-h; \lambda_{hj}), \lambda_{hj}) \\ &\quad \equiv h\mathscr{F}(t, h, y_h, \lambda_{hj}), \qquad a_k \equiv 1, \text{ for } t=t_{hn}, n \in R_{N-k}, j=0, 1, ... \end{aligned} \tag{5}$$

Conditions for a_i, F, and B will be defined later.

The use of (5) requires that the approximations of y_h for $t_{h0}, t_{h1}, ..., t_{h,k-1}$ be computed first. We can apply any one-step method including the Runge–Kutta method as a natural for this purpose. Once the values of y_h for $t_{h0}, ..., t_{h,k-1}$ are available, the formula (5) can be employed to compute the rest. Now knowing the approximate solution y_h for $t=t_{hN}=\beta$ we are able to use (4) for determining the new value $\lambda_{h,j+1}$ and then the corresponding numerical solution y_h on the mesh points.

The purpose of this paper is to give sufficient conditions for the convergence of (4)–(5). To get it, Lipschitz or Peron conditions are needed on F. Indeed it is necessary to assume that the method (4)–(5) is consistent. Some estimations of errors are given.

The linear case,

$$g(\lambda, y)=\tilde{M}\lambda+\tilde{N}y-\tilde{K}, \qquad \tilde{M}_{p\times p}, \tilde{N}_{p\times q}, \tilde{K}_{p\times 1}, \tag{6}$$

was discussed in [8] (one-step methods) and in [7, 9] (multistep methods). You can find there some numerical examples too.

2. Definitions and Assumptions

We introduce the following basic definitions.

Definition 1. The method (4)–(5) is said to be convergent to the solution (φ, λ) of BVP (1)–(3) if

$$\lim_{\substack{N \to \infty \\ j \to \infty}} \max_{i \in R_N} \|\varphi(t_{hi}; \lambda) - y_h(t_{hi}; \lambda_{hj})\| = 0,$$

$$\lim_{\substack{N \to \infty \\ j \to \infty}} \|\lambda_{hj} - \lambda\| = 0.$$

DEFINITION 2. The method (4)–(5) is said to be consistent with the problem (1)–(3) on the solution (φ, λ) if there exists a function $\varepsilon: J_h \times H \to R_+ = [0, \infty)$, $J_h = [\alpha, \beta - kh]$ such that

$$\left\| \sum_{i=0}^{k} a_i(t, h)\, \varphi(t + ih; \lambda) - h\mathscr{F}(t, h, \varphi, \lambda) \right\| \leqslant \varepsilon(t, h), \qquad t \in J_h,$$

$$\lim_{N \to \infty} \sum_{i=0}^{N-k} \varepsilon(t_{hi}, h) = 0.$$

Let

$$A_n^h = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{0n}^h & a_{1n}^h & a_{2n}^h & \cdots & a_{k-1,n}^h \end{bmatrix},$$

where $a_{in}^h = -a_i(t_{hn}, h)$, $i \in R_{k-1}$.

Now we have the following assumptions.

ASSUMPTION H_1. *Suppose that*

1° $F: I^k \times H \times R^{qk} \times R^p \to R^q$, $f: I \times R^q \times R^p \to R^q$, $H = [0, h_0]$, $h_0 > 0$, $g: R^p \times R^q \to R^p$;

2° *there exist constants* $L_i \geqslant 0$, $i \in R_k$, *and a function* $\varepsilon_F: I \times H \to R_+$ *such that for* $(s_0, ..., s_{k-1}, h) \in I^k \times H$ *and* $z_i, \bar{z}_i \in R^q$, $i \in R_{k-1}$, $\mu, \bar{\mu} \in R^p$ *we have*

$$\|F(s_0, ..., s_{k-1}, h, z_0, ..., z_{k-1}, \mu) - F(s_0, ..., s_{k-1}, h, \bar{z}_0, ..., \bar{z}_{k-1}, \bar{\mu})\|$$
$$\leqslant \sum_{i=0}^{k-1} L_i \|z_i - \bar{z}_i\| + L_k \|\mu - \bar{\mu}\| + \varepsilon_F(s_0, h),$$

and

$$\lim_{N \to \infty} h \sum_{i=0}^{N-k} \varepsilon_F(t_{hi}, h) = 0;$$

3° *there exist a nonsingular square matrix B of order p and a constant* $m_1 < 1$ *such that*

$$\|\mu_1 - \mu_2 - B^{-1}[g(\mu_1, y) - g(\mu_2, y)]\| \leqslant m_1 \|\mu_1 - \mu_2\|$$

for $\mu_1, \mu_2 \in R^p$, $y \in R^q$;

4° $\|B^{-1}[g(\mu, y_1) - g(\mu, y_2)]\| \leqslant m_2 \|y_1 - y_2\|$, *for* $\mu \in R^p$, y_1, $y_2 \in R^q$.

ASSUMPTION H_2. *Suppose that*

1° *the conditions* 1°, 3°, *and* 4° *of Assumption* H_1 *are satisfied*;

2° $\|A_n^h\| \leqslant 1 + h\tilde{R}$ (*maximum norm*) *for* $n \in R_{N-k}$, $h \in H$, *where* $\tilde{R}$ *is a nonnegative constant*;

3° *there exist functions* $\Omega: I^k \times H \times R_+^{k+1} \to R_+$, $\varepsilon_F: H \to R_+$, $\lim_{h \to 0} \varepsilon_F(h) = 0$, *such that*

$$\|F(s_0, ..., s_{k-1}, h, z_0, ..., z_{k-1}, \mu) - F(s_0, ..., s_{k-1}, h, \bar{z}_0, ..., \bar{z}_{k-1}, \bar{\mu})\|$$
$$\leqslant \Omega(s_0, ..., s_{k-1}, h, \|z_0 - \bar{z}_0\|, ..., \|z_{k-1} - \bar{z}_{k-1}\|, \|\mu - \bar{\mu}\|) + \varepsilon_F(h),$$

for $(s_0, ..., s_{k-1}, h) \in I^k \times H$, z_i, $\bar{z}_i \in R^q$, $i \in R_{k-1}$, μ, $\bar{\mu} \in R^p$;

4° *the function* Ω *has the properties*

(i) Ω *is continuous and bounded and it is nondecreasing with respect to the last* $k+1$ *variables and* $\Omega(s_0, ..., s_{k-1}, 0, ..., 0) = 0$;

(ii) *there exists a function* $\xi: H \to R_+$, $\lim_{h \to 0} \xi(h) = 0$ *such that the inequality*

$$\int_t^{t+h} \Omega(s, ..., s, h, v, ..., v, \bar{v})\, ds + h\xi(h)$$
$$\geqslant h\Omega(t, ..., t + kh - h, h, v, ..., v, \bar{v})$$

holds for $(t, h, v, \bar{v}) \in J_h \times H \times R_+ \times R_+$;

(iii) *the function* $v(t) \equiv 0$ *is the only continuous solution of the problem*

$$v'(t) = \Omega(t, ..., t, 0, v(t), ..., v(t), u) + \tilde{R}v(t), \qquad t \in I,$$
$$v(\alpha) = 0, \qquad (1 - m_1)u = m_2 v(\beta),$$

where m_1 *and* m_2 *are defined in Assumption* H_1.

3. Convergence of (4)–(5)

In this section we wish to examine the convergence behaviour as $N \to \infty$ (or $h \to 0$) and $j \to \infty$ of the approximate solution (y_h, λ_{hj}). First it will be assumed that the function F satisfies a Lipschitz condition with suitable constants. We can prove the following main theorem:

THEOREM 1. *If Assumption* H_1 *is satisfied and if*

1° *there exists the unique solution* (φ, λ) *of BVP* (1)–(3);

2° $\|A_n^h\| \leqslant 1 + h\tilde{R}$ *(maximum norm), for* $n \in R_{N-k}$, $h \in H$, *where* $\tilde{R}$ *is a nonnegative constant*;

3° $d = m_1 + m_2 A < 1$, *where*

$$A = \frac{L_k}{L}(D-1), \qquad D = \exp(L(\beta - \alpha)), \qquad L = \tilde{R} + \sum_{i=0}^{k-1} L_i;$$

4° *there exists a function* $\eta: H \to R_+$, $\lim_{h \to 0} \eta(h) = 0$, *such that*

$$\max_j \max_{s \in R_{k-1}} \|y_h(t_{hs}; \lambda_{hj}) - \varphi(t_{hs}; \lambda)\| \leqslant \eta(h);$$

5° *the method* (4)–(5) *is consistent with BVP* (1)–(3) *on the solution* (φ, λ);

then the method (4)–(5) *is convergent to the solution* (φ, λ) *of BVP* (1)–(3) *and the estimations*

$$\|\lambda_{hj} - \lambda\| \leqslant u_j(h), \qquad j = 0, 1, \ldots, \tag{7}$$

$$\max_{n \in R_N} \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\| \leqslant A u_j(h) + D w(h), \qquad j = 0, 1, \ldots, \tag{8}$$

hold true with

$$u_j(h) = d^j \|\lambda_0 - \lambda\| + m_2 D w(h) \frac{1 - d^j}{1 - d}$$

$$w(h) = \eta(t) + \sum_{i=0}^{N-k} [\varepsilon(t_{hi}, h) + h\varepsilon_F(t_{hi}, h)].$$

Proof. Put

$$z_{hn}^j = \|y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)\|, \qquad n \in R_{N-k},\ j = 0, 1, \ldots,$$

$$e_{hn}^j = \max_{s \in R_{k-1}} z_{h,n+s}^j,$$

$$\tilde{\varepsilon}(t, h) = \varepsilon(t, h) + h\varepsilon_F(t, h).$$

Repeating the proof of the first part of Theorem 2 in [9], we have

$$e_{hn}^{j} \leqslant A\,\|\lambda_{hj}-\lambda\| + D\left[e_{h0}^{j} + \sum_{i=0}^{n-1} \tilde{\varepsilon}(t_{hi}, h)\right], \qquad n \in R_{N-k+1},\ j=0, 1, \ldots.$$

Now using the definition of λ_{hj} and Assumptions 3° and 4° of H_1 we note

$$\begin{aligned}\|\lambda_{h,j+1}-\lambda\| &= \|\lambda_{hj}-\lambda - B^{-1}[g(\lambda_{hj}, \varphi(\beta;\lambda)) - g(\lambda, \varphi(\beta;\lambda))] \\ &\quad + B^{-1}[g(\lambda_{hj}, \varphi(\beta;\lambda)) - g(\lambda_{hj}, y_h(\beta;\lambda_{hj}))]\| \\ &\leqslant m_1\,\|\lambda_{hj}-\lambda\| + m_2 z_{hN}^{j}\end{aligned}$$

or

$$\|\lambda_{h,j+1}-\lambda\| \leqslant d\,\|\lambda_{hj}-\lambda\| + m_2\,Dw(h).$$

Hence, by Lemma 1.2 in [5] we have the estimation (7) and then (8). The convergence follows directly from (7)–(8).

Remark 1. Instead of the modified Newton method (4) we may take

$$\lambda_{h,j+1} = \lambda_{hj} - B^{-1}(\lambda_{hj}, y_h(\beta;\lambda_{hj}))\, g(\lambda_{hj}, y_h(\beta;\lambda_{hj})), \qquad j=0, 1, \ldots.$$

Using a slight modification we may get its convergence provided that the matrix $B_{p\times p}$ is nonsingular for each pair (λ_{hj}, y_h).

Remark 2. It follows from the proof that Theorem 1 remains true if condition 3° of Assumption H_1 is satisfied only on the solution φ, i.e., if $y=\varphi(\beta;\lambda)$.

Remark 3. Put $p=q$. Assume that for all $u, v \in R^q$ the matrix

$$P(u, v) = D_u\,g(u, v) + D_v\,g(u, v),$$

$$D_u\,g(u, v) = \left[\frac{\partial g_i(u, v)}{\partial u_j}\right], \qquad D_v\,g(u, v) = \left[\frac{\partial g_i(u, v)}{\partial v_j}\right],$$

has a representation of the form

$$P(u, v) = P_0(I + Z(u, v))$$

with a constant nonsingular matrix P_0 and there are constants v_1, v_2, $v_1 + v_2 < 1$ such that

$$\|Z(u, v)\| \leqslant v_1, \qquad \|P_0^{-1}\,D_v\,g(u, v)\| \leqslant v_2 \qquad \text{for all} \quad u, v \in R^q.$$

Now with a suitable choice of B, namely $B = P_0$, condition 3° of

Assumption H_1 is satisfied with $m_1 = v_1 + v_2$. Such case was considered in [18, see p. 476].

Indeed, we have

$$\begin{aligned}\mu_1 - \mu_2 - P_0^{-1}[g(\mu_2 + \mu_1 - \mu_2, y) - g(\mu_2, y)]\\ = \mu_1 - \mu_2 - P_0^{-1} D_u g(\tau(\mu_1 - \mu_2), y)(\mu_1 - \mu_2)\\ = \mu_1 - \mu_2 - P_0^{-1}[P(\tau(\mu_1 - \mu_2), y) - D_v g(\tau(\mu_1 - \mu_2), y)](\mu_1 - \mu_2)\\ = [-Z(\tau(\mu_1 - \mu_2), y) + P_0^{-1} D_v g(\tau(\mu_1 - \mu_2), y)](\mu_1 - \mu_2),\end{aligned}$$

and hence we have our assertion.

We note that for

$$g(u, v) = \tilde{M}u + \tilde{N}v - \tilde{K},$$

if $\tilde{M} + \tilde{N}$ is a nonsingular square matrix of order q, $\tilde{K} \in R^q$, we have

$$P_0 = \tilde{M} + \tilde{N}, \qquad Z = \theta, \ v_1 = 0, \qquad \|(\tilde{M} + \tilde{N})^{-1} \tilde{N}\| \leqslant v_2 = m_1.$$

This linear case was discussed in [8] for one-step methods for y_h combined with an iterative method for λ_{hj}.

Now assuming a Peron condition for F, the corresponding result for convergence of (4)–(5) is given in the following theorem:

THEOREM 2. *If both Assumption* H_2 *and conditions* 1°, 4°, *and* 5° *of Theorem* 1 *are satisfied with*

$$\varepsilon(t, h) = h\varepsilon(h), \qquad \varepsilon(h) \to 0,$$

then the method (4)–(5) *is convergent to the solution* (φ, λ) *of BVP* (1)–(3) *and*

$$\lim_{\substack{j \to \infty \\ N \to \infty}} \sum_{i=0}^{j} z_{hN}^i m_1^{j-i} = 0, \tag{9}$$

where z_{hN}^i *is defined in the proof of Theorem* 1.

Proof. We note that

$$\begin{aligned}\sum_{i=0}^{k} a_i(t, h)[y_h(t + ih; \lambda_{hj}) - \varphi(t + ih; \lambda)]\\ = h\mathscr{F}(t, h, y_h, \lambda_{hj}) - h\mathscr{F}(t, h, \varphi, \lambda)\\ + h\mathscr{F}(t, h, \varphi, \lambda) - \sum_{i=0}^{k} a_i(t, h)\, \varphi(t + ih; \lambda).\end{aligned} \tag{10}$$

So we have a family of recurrent equations of order k,

$$\sum_{i=0}^{k} a_i(t_{hn}, h)\, v^j_{h,n+i} = c^j_{hn}, \qquad n \in R_{N-k}, \tag{11}$$

where

$$v^j_{hn} = y_h(t_{hn}; \lambda_{hj}) - \varphi(t_{hn}; \lambda)$$

and c^j_{hn} is defined by the right-hand side of (10) for $t = t_{hn}$. Indeed (11) may be written by

$$V^j_{h,n+1} = A^h_n V^j_{hn} + W^j_{hn}, \qquad n \in R_{N-k}, \tag{12}$$

where

$$V^j_{hn} = [v^j_{hn}, ..., v^j_{h,n+k-1}]^{\mathrm{T}}, \qquad W^j_{hn} = [\theta, ..., \theta, c^j_{hn}]^{\mathrm{T}}, \qquad \theta \in R^q,$$

with the matrix A^h_n defined before the assumptions. So we have

$$\| V^j_{h,n+1} \| \leqslant \| A^h_n \| \; \| V^j_{hn} \| + \| W^j_{hn} \|$$

or

$$\begin{aligned} e^j_{h,n+1} = \max_{s \in R_{k-1}} z^j_{h,n+s+1} &\leqslant (1 + h\tilde{R}) e^j_{hn} + h[\varepsilon(h) + \varepsilon_F(h)] \\ &+ h\Omega(t_{hn}, ..., t_{h,n+k-1}, h, e^j_{hn}, ..., e^j_{hn}, \delta^h_j) \overset{\mathrm{df}}{=} w^j_{h,n+1}, \end{aligned}$$

where

$$\delta^h_j = \| \lambda_{hj} - \lambda \| \qquad \text{and} \qquad w^j_{h0} = \eta(h).$$

Now we consider the problem

$$\begin{aligned} &\omega'(t) = \tilde{R}\omega(t) + \Omega(t, ..., t, h, \omega(t), ..., \omega(t), \delta^h_j) + \xi(h) + \varepsilon(h) + \varepsilon_F(h) \\ &\omega(\alpha) = \eta(h). \end{aligned} \tag{13}$$

This problem has a solution, $\omega_h(t; \delta^h_j)$, which is a nondecreasing and continuous function. We are able to prove

$$\omega_h(t_{hn}; \delta^h_j) \geqslant w^j_{hn}, \qquad n \in R_{N-k}, \; j = 0, 1, ...$$

It is obviously true for $n = 0$. Assuming that it is true for fixed s and integrating (13) from t_{hs} to $t_{h,s+1}$, we have

$$\omega_h(t_{h,s+1};\delta_j^h)=\omega_h(t_{hs};\delta_j^h)+\tilde{R}\int_{t_{hs}}^{t_{h,s+1}}\omega_h(\tau;\delta_j^h)\,d\tau$$
$$+\int_{t_{hs}}^{t_{h,s+1}}\Omega(\tau,...,\tau,h,\omega_h(\tau;\delta_j^h),...,\omega_h(\tau;\delta_j^h),\delta_j^h)\,d\tau$$
$$+h[\xi(h)+\varepsilon(h)+\varepsilon_F(h)]$$
$$\geqslant w_{hs}^j+\tilde{R}hw_{hs}^j+\int_{t_{hs}}^{t_{h,s+1}}\Omega(\tau,...,\tau,h,w_{hs}^j,...,w_{hs}^j,\delta_j^h)\,d\tau$$
$$+h[\xi(h)+\varepsilon(h)+\varepsilon_F(h)]$$
$$\geqslant(1+\tilde{R}h)e_{hs}^j+h\Omega(t_{hs},...,t_{h,s+k-1},h,e_{hs}^j,...,e_{hs}^j,\delta_j^h)$$
$$+h[\varepsilon(h)+\varepsilon_F(h)]=w_{h,s+1}^j.$$

Now as in the proof of Theorem 1 we have

$$\delta_{j+1}^h\leqslant m_1\delta_j^h+m_2z_{hN}^j,\qquad j=0,1,.... \tag{14}$$

Let

$$\bar{\delta}_0^h=\frac{\max(m_2S,\|\lambda_0-\lambda\|)}{1-m_1},\qquad \omega_h(\beta;\delta)\text{ is bounded by }S,$$
$$\bar{\delta}_{j+1}^h=m_1\bar{\delta}_j^h+m_2\omega_h(\beta;\bar{\delta}_j^h),\qquad j=0,1,....$$

Indeed,

$$\delta_j^h\leqslant\bar{\delta}_j^h,\qquad j=0,1,....$$

It is easy to see

$$\bar{\delta}_1^h=m_1\bar{\delta}_0^h+m_2\omega_h(\beta;\bar{\delta}_0^h)\leqslant m_1\bar{\delta}_0^h+m_2S$$
$$\leqslant m_1\bar{\delta}_0^h+\max(m_2S,\|\lambda_0-\lambda\|)=\bar{\delta}_0^h.$$

It means the sequence $\{\bar{\delta}_j^h\}$ is nonincreasing with respect to j, i.e.,

$$\bar{\delta}_{j+1}^h\leqslant\bar{\delta}_j^h\leqslant\cdots\leqslant\bar{\delta}_0^h.$$

It has a limit u^h, where

$$u^h=\frac{m_2}{1-m_1}\,\omega_h(\beta;u^h).$$

But according to assumption 4° (iii) of H_2 we note that $\lim_{h\to0}\omega_h(t;u^h)\equiv0$ so $u=0$ and we have the convergence of our method. Now the estimation (9) follows directly from (14). The proof is completed.

Remark 4 (see [6]). The condition 4° (ii) of Assumption H_2 remains true if we add:

(i) Ω is the continuous function with respect to the variables 1st, r_1th, ..., r_sth uniformly with respect to the left variables $(1 < r_1 < \cdots < r_s \leqslant k)$;

(ii) Ω is the non-increasing function with respect to the variables n_1th, ..., n_qth, where $\{n_1, ..., n_q\} = \{2, ..., k\} \setminus \{r_1, ..., r_s\}$; or

(iii) Ω is the continuous function with respect to the variables r_1th, ..., r_sth uniformly with respect to the left variables;

(iv) Ω is the non-decreasing function with respect to the first variable and it is the non-increasing function with respect to the variables n_1th, ..., n_qth.

Remark 5. Let there exist constants $L_i \geqslant 0$ such that

$$\Omega(s_0, ..., s_{k-1}, h, u_0, ..., u_k) = \sum_{i=0}^{k} L_i u_i.$$

Theorem 2 remains true though the function Ω is not bounded. In this case the function

$$v(t) = \frac{L_k}{L} u[\exp(L(t-\alpha)) - 1]$$

is the solution of the initial-value problem given in 4° (iii) of Assumption H_2. Now adding the boundary condition

$$(1 - m_1)u = m_2 v(\beta)$$

we have

$$(1 - m_1 - m_2 A)u = 0$$

and if condition 3° of Theorem 1 is satisfied then $u = 0$ and hence really $v(t) \equiv 0$ is the solution of BVP given in 4° (iii) of Assumption H_2.

Remark 6. Some numerical examples for one-step methods you can find in [8].

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