On a Ring of invariant Polynomials on a Hermitian Symmetric Space

KENNETH D. JOHNSON

Department of Mathematics, University of Georgia, Athens, Georgia 30602

Communicated by J. Tits
Received December 8, 1978

1. INTRODUCTION AND NOTATION

Suppose $\Omega$ is a noncompact irreducible Hermitian symmetric space. Let $G$ be the group of holomorphic transformations of $\Omega$. Fix $e \in \Omega$ and let $K$ be the isotropy of $e$ in $G$. Then $K$ is a maximal compact subgroup of $G$ and $G/K = \Omega$. Let $\mathfrak{g}$ ($\mathfrak{k}$ respectively) denote the Lie algebra of $G$ ($K$ respectively). There is a Cartan decomposition $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$, and complexifying $\mathfrak{g}$ we have $\mathfrak{g}_{c} = \mathfrak{f}_{c} + \mathfrak{p}_{+} + \mathfrak{p}_{-}$, where $\mathfrak{p}_{c} = \mathfrak{p}_{+} + \mathfrak{p}_{-}$, $[\mathfrak{p}_{\pm}, \mathfrak{p}_{\pm}] = 0$, $[\mathfrak{p}_{\pm}, \mathfrak{f}_{c}] = 0$ and $[\mathfrak{f}_{c}, \mathfrak{p}_{c}] = \mathfrak{p}_{c}$ (see [6]).

Select $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{f}_{c}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}_{c}$. If $a \in \mathfrak{h}^* \sim \{0\}$ we say that $a$ is a root if the space

$$\mathfrak{g}_a = \{U \in \mathfrak{g}_c : [U, H] = a(H)U \text{ for all } H \in \mathfrak{h}\}$$

is $\neq \{0\}$. If $\Phi$ is the set of all roots and $a \in \Phi$ we say that $a$ is compact (resp. noncompact) if $\mathfrak{g}_a \subset \mathfrak{f}_c$ (resp. $\mathfrak{g}_a \subset \mathfrak{p}_c$). According to Harish-Chandra [3] there is an ordering on $\Phi$ so that if $a$ is a positive (resp. negative) noncompact root $\mathfrak{g}_a \subset \mathfrak{p}$ (resp. $\mathfrak{g}_a \subset \mathfrak{p}_c$).

The roots $a$ and $\beta$ are called strongly orthogonal if neither $a + \beta$ nor $a - \beta$ are roots. There is a set $A$ of strongly orthogonal noncompact positive roots such that $|A| = r$, where $r$ is the rank of the symmetric space $\Omega$. It is well known (see [6]) that the elements of $A$ are linearly independent elements of $\mathfrak{h}^*$, and no other set of strongly orthogonal noncompact positive roots has more than $r$ elements.

Fix $\mathfrak{n}$ a nilpotent subalgebra of $\mathfrak{f}_c$ so that $\mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{f}_c$. We may assume that $\mathfrak{n}$ is spanned by the positive compact root spaces. We now fix $A = \{\gamma_1, \ldots, \gamma_r\}$ so that $\gamma_1$ is the lowest positive noncompact root and assuming $\gamma_1, \ldots, \gamma_k$ have been chosen select $\gamma_{k+1}$ to be the lowest positive noncompact root strongly orthogonal to $\gamma_1, \ldots, \gamma_k$.

* Research partially supported by a grant from the National Science Foundation.

0021-8693/80/110072-10$02.00/0
Copyright © 1980 by Academic Press, Inc.
All rights of reproduction in any form reserved.
In general, if $V$ is a complex vector space let $P(V)$ be the ring of polynomials on $V$. Dualizing the action of $\mathfrak{t}_c$ on $p_+$ we obtain an action of $\mathfrak{t}_c$ on $P(p_+)$. Let $I$ be the ring of $\mathfrak{n}$-invariant polynomials in $P(p_+)$. By the theory of the highest weight, we know that $I$ is spanned by the highest weight vectors of the irreducible representations of $\mathfrak{t}_c$, which occur in $P(p_+)$. Fixing $\Omega$ an irreducible Hermitian symmetric space of rank $r$ and $\Delta = \{\gamma_1, \ldots, \gamma_r\}$ ordered as above we now state the main theorem of this paper.

**Theorem A.** There exist $r$ homogeneous polynomials $p_1, \ldots, p_r$ in $Z$ with $\deg p_i = i$ for each $i$ such that:

(i) $p_i$ is of weight $\lambda_i$, where $\lambda_i = -\gamma_1 - \gamma_2 - \cdots - \gamma_i$ with respect to $\mathfrak{h}$; and,

(ii) $Z$ is the free polynomial algebra on $p_1, \ldots, p_r$ (i.e., $I = C[p_1, \ldots, p_r]$).

As a corollary to this theorem we obtain the following result which is due to Schmid [8] in general and Hua [4] for the classical domains.

**Corollary.** Any irreducible representation of $\mathfrak{t}_c$ which occurs in $P(p_+)$ occurs with multiplicity one and it must have a highest weight with respect to $\mathfrak{h}$ of the form

$$k_1\lambda_1 + \cdots + k_r\lambda_r,$$

where $k_1, \ldots, k_r$ are integers $\geq 0$. Moreover, this representation occurs in polynomials of degree

$$k_1 + 2k_2 + \cdots + rk_r.$$

In Section 2, we prove a result which will greatly simplify the proof of Theorem A. In Section 3, we prove Theorem A for the irreducible classical domains and in Section 4 we prove Theorem A for the two irreducible exceptional domains. (Recall that noncompact irreducible Hermitian symmetric spaces are irreducible bounded symmetric domains and conversely by a result of Harish-Chandra [3]). For the classical domains we will prove our result by explicitly constructing the polynomials $p_1, \ldots, p_r$.

Although our main theorem is Theorem A, the primary result of this paper is the explicit construction of the polynomials $p_1, \ldots, p_r$ when $\Omega$ is a classical domain. This, we feel, is important to invariant theory. Finally, it should be mentioned that these polynomials are actually implicit in the book of Hua [4].
2. A Simplification of the Proof

In this section \( \mathcal{Q} \) will be an irreducible Hermitian symmetric space of noncompact type of rank \( r \). Then if \( \mathfrak{a} \) is a maximal abelian subalgebra of \( \mathfrak{p} \) it has dimension \( r \). If \( \mathbb{P}(\mathfrak{p}) \) is the ring of polynomials with complex coefficients on the real vector space \( \mathfrak{p} \) the restriction map \( R: \mathbb{P}(\mathfrak{p}) \to \mathbb{P}(\mathfrak{a}) \) is an isomorphism which carries \( \mathfrak{t}_c \)-irreducible modules to \( \mathfrak{t} \)-irreducible modules and hence \( \mathfrak{t}_c \)-invariant polynomials to \( \mathfrak{t} \)-invariant polynomials.

If \( \mathcal{M} \) is the centralizer of \( a \) in \( K \) and \( \mathcal{M}' \) is the normalizer of \( a \) in \( K \), \( \mathcal{W} = \mathcal{M}'/\mathcal{M} \) is a finite group called the restricted Weyl group and \( \mathcal{W} \) acts on \( a \) and hence on \( \mathbb{P}(\mathfrak{a}) \). By a theorem of Chevalley the restriction map

\[
R: \mathbb{P}(\mathfrak{p}) \to \mathbb{P}(\mathfrak{a})
\]

gives an isomorphism of \( \mathbb{P}(\mathfrak{p})^K \) with \( \mathbb{P}(\mathfrak{a})^W \).

Recall that \( \lambda \in \mathfrak{a}^* \) is a restricted root if the space \( \mathfrak{g}_\lambda = \{ X \in \mathfrak{g} : [H,X] = \lambda(H) \text{ for all } H \in \mathfrak{a} \} \neq \{0\} \). By a theorem of Moore [7], \( \Omega \) has the restricted root systems of type \( B_r \) or of type \( C_r \). Hence \( \mathbb{P}(\mathfrak{a})^W \) is isomorphic to the \( \text{ad } B_r \)-invariant polynomials on \( B_r \), or the \( \text{ad } C_r \)-invariant polynomials in \( C_r \). In either case we have

**Lemma 2.1.** \( \mathbb{P}(\mathfrak{p})^K \cong \mathbb{P}(\mathfrak{a})^W \cong \mathbb{C}[\Delta_1, \ldots, \Delta_r] \), where \( \Delta_i \) is a \( K \)-invariant homogeneous polynomial on \( \mathfrak{p} \) of degree \( 2i \).

**Theorem 2.2.** Suppose \( Z = \mathbb{C}[p_1, \ldots, p_r] \), where each \( p_i \) is homogeneous of weight \( \lambda_i \) and degree \( i \) with \( \lambda_1 \), \ldots, \( \lambda_r \) linearly independent. Then \( Z = \mathbb{C}[p_1, \ldots, p_r] \).

**Proof:** In a polynomial ring \( R \) let \( R_n \) denote the homogeneous elements of degree \( n \). Let \( A = \mathbb{C}[\Delta_1, \ldots, \Delta_r] \) in Lemma 2.1. Clearly, \( \dim \mathbb{C}[p_1, \ldots, p_r]_n = \dim A_n \leq \dim I_n \) but we claim that in general \( \dim I_n \leq \dim A_n \).

Let \( P(\mathfrak{p}_+) \cong V_1 \oplus \ldots \oplus V_m \) where each \( V_j \) \( j < r \) is a \( \mathfrak{t}_c \)-irreducible module. The natural map from \( P(\mathfrak{p}_+ \otimes P(\mathfrak{p}_-)) \to P(\mathfrak{p}_+) \) is an isomorphism and a \( \mathfrak{t}_c \)-invariant pairing. Hence the number of times the trivial representation of \( \mathfrak{t}_c \) occurs in \( P(\mathfrak{p}_+) \) is at least \( m \). Thus \( \dim I_n \leq \dim A_n \) in general. Hence \( I_n = \mathbb{C}[p_1, \ldots, p_r]_n \) for all \( n \) and since \( Z \) is generated by homogeneous elements we have our result.

From the theory of the highest weight, we have the following result.

**Corollary.** There exist homogeneous polynomials \( p_1 \), \ldots, \( p_r \) in \( Z \) with each \( p_i \) of degree \( i \) and of weight \( \lambda_i \) with \( \lambda_1 \), \ldots, \( \lambda_r \) linearly independent if and only if there exists an irreducible \( \mathfrak{t}_c \)-module \( V_i \) in \( P(\mathfrak{p}_+) \) of highest weight \( \lambda_i \) for \( i \leq r \) with \( \lambda_1 \), \ldots, \( \lambda_r \) independent.

**Remark.** The reader should compare Theorem 2.2 to Herman Weyl's use of the Capelli identities in [9].
3. The Proof of Theorem A (The Classical Domains)

There are four classes of irreducible Hermitian symmetric spaces of noncompact type $SU(m, n)/SU(m) \times SU(n)$ and $SO^*(2n)/U(n)$. We will prove Theorem A for each of these classes but first we introduce some notations.

Suppose $Z$ is an $m \times n$-matrix $m \geq n$ with complex entries so $Z = (Z_{ij})$. If $k \leq n$ let $Z_k$ be the $k \times k$-matrix whose $(i, j)$th entry is $z_{m-k+i+1, n-k+j+1}$. That is, $Z_k$ is the $k \times k$-matrix in the lower right-hand corner of $Z$.

Case 1. $G = SU(m, n)$, $K = SU(m) \times SU(n)$ ($m \geq n$), rank $\Omega = n$.

Then

$$ t_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathfrak{gl}(m, C), B \in \mathfrak{gl}(n, C), \text{tr } A + \text{tr } B = 0 \right\}, $$

and

$$ p_+ = \left\{ \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} : Z \text{ is an } m \times n \text{-matrix with complex entries} \right\}. $$

Let $\mathfrak{g}$ be the diagonal matrices in $t_c$ and let $n$ be the matrices in $t_c$ of the form

$$ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, $$

where $A$ is upper triangular, $B$ lower triangular and both $A$ and $B$ have $O$'s down the diagonal. It is easy to see that the homogeneous polynomials $p_r$, where

$$ p_r \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} = \det Z_r, $$

are $n$-invariant of degree $r$. Let $HE \mathfrak{h}$ with

$$ H = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y_n \end{pmatrix}. $$

Then $A = \{y_1, \ldots, y_n\}$, where $y_k(H) = x_{m-k+1} - y_{n-k+1}$. 
Case 2. \( G = \text{Sp}(n, \mathbb{R}), K = U(n), \text{rank } \Omega = n \)

The group \( G \) is the group of all real \( 2n \times 2n \)-matrices \( g \) such that \( gJg' = J \), where \( J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \) and \( I_n \) is the \( n \times n \)-identity. Then

\[
\mathfrak{t}_c = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in \mathfrak{gl}(n, \mathbb{C}) \text{ and } A' = -A, B = B' \right\},
\]
\[
\mathfrak{p}_+ = \left\{ \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} : Z = Z \in \mathfrak{gl}(n, \mathbb{C}) \right\}.
\]

Observe that \( \mathfrak{t}_c \) is isomorphic with \( \mathfrak{gl}(n, \mathbb{C}) \) via the map

\[
\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + iB,
\]

and identify \( \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} \) with \( Z \). Now if \( U = A + iB \) with \( \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{t}_c \)

\[
\begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} = \begin{pmatrix} UZ + ZU' & i(UZ + ZU') \\ i(UZ + ZU') & -(UZ + ZU') \end{pmatrix}.
\]

Let \( \mathfrak{h} \) be the diagonal elements of \( \mathfrak{gl}(n, \mathbb{C}) \) and \( n \) the upper triangular elements of \( \mathfrak{gl}(n, \mathbb{C}) \) with \( 0 \)'s down the diagonal. For \( k \leq n \) let

\[
P_k \left( \begin{pmatrix} Z & iZ \\ iZ & -Z \end{pmatrix} \right) = \det Z_k.
\]

Each \( P_k \) is a homogeneous polynomial of degree \( k \) and is \( n \)-invariant.

If

\[
H = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}
\]

\( \Delta = \{\gamma_1, \ldots, \gamma_n\}, \text{ where } \gamma_k(H) = 2x_{n-k+1} \).

Case 3. \( G = SO^*(2n), K = U(n), \text{rank } \Omega = [n/2] \)

The group \( G \) is the group of all complex \( 2n \times 2n \)-matrices \( g \in SO(2n, \mathbb{C}) \) such that \( gJ^*g = J \). Then

\[
\mathfrak{t}_c = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A, B \in \mathfrak{gl}(n, \mathbb{C}) \text{ and } A' = -A, B = B' \right\},
\]
\[
\mathfrak{p}_c = \left\{ i \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A, B \in \mathfrak{gl}(n, \mathbb{C}) \text{ and } A' = -A, B = -B' \right\}.
\]
and

\[ p_+ = \left\{ \begin{pmatrix} Z & iZ \\
Z & -Z \end{pmatrix} : Z = -Z' \in \mathfrak{gl}(n, \mathbb{C}) \right\}. \]

Take \( \mathfrak{k}_c \cong \mathfrak{gl}(n, \mathbb{C}) \), \( \mathfrak{h} \) and \( n \) as in Case 2. Then again the homogeneous polynomials

\[ q_k \left( \begin{pmatrix} Z & iZ \\
Z & -Z \end{pmatrix} \right) = \det Z_k \]

are \( n \)-invariant. However, as \( Z_k = -Z_k', q_k = 0 \) if \( k \) is odd. Writing \( Z_{2k} \) as an element of \( \bigwedge^2 \mathbb{C}^{2k} \) we have that the Pfaffian of \( Z_{2k}, Pf(Z_{2k}) = p_k(Z) \), is a well-defined (see [9, p. 82]) square root of \( q_{2k} \). Hence the \( p_1, \ldots, p_{[n/2]} \) are \( n \)-invariant homogeneous polynomials with \( p_k \) of degree \( k \).

Let

\[ H = \begin{pmatrix} x_1 & \cdots & 0 \\
& \ddots & \vdots \\
0 & \cdots & x_n \end{pmatrix} \in \mathfrak{h} \]

Then \( A = \{ \gamma_1, \ldots, \gamma_{[n/2]} \} \), where \( \gamma_k(H) = x_{n-2k+1} + x_{n-2k+2} \).

**Case 4.** \( G = SO^0(n, 2), K = SO(n) \times O(2)^0, \) rank \( \Omega = 2 \)

Then

\[ \mathfrak{k}_c = \left\{ \begin{pmatrix} A & 0 \\
0 & B \end{pmatrix} : A \in \mathfrak{o}(n) \text{ and } B \in \mathfrak{o}(2) \right\}, \]

and

\[ p_+ = \left\{ \begin{pmatrix} 0 & \overline{Z} - i\overline{Z} \\
\overline{Z}' & 0 \\
Z & i \overline{Z}' \end{pmatrix} : Z \in \mathbb{C}^n \right\}. \]

Choose \( \mathfrak{h} \) to be the \( \text{artan} \) subalgebra consisting of all matrices in \( \mathfrak{k}_c \) of the form

\[
\begin{pmatrix}
0 & ix_1 & 0 & 0 \\
-ix_1 & 0 & i x_2 & 0 \\
0 & -ix_2 & 0 & \ddots \\
0 & 0 & \cdots & 0 & iy \\
0 & iy & 0 & 0
\end{pmatrix}
\]

with 0 in the \((n, n)\)th position if \( n \) is odd.
If \( n \) is even \( n = 2k \), we write elements of \( \mathfrak{f}_c \) as \((A_0 \quad 0)\), where \( A_0 = (a,...) \) \( u, v \leq k \). where \( a_{uv} \) is a 2 \( \times \) 2-matrix and \( a_{uv} = a'_{uv} \). In this case, we take \( \mathfrak{n} \) as all elements of \( \mathfrak{f}_c \), where \( A_0 = (a,...) \) and if \( u \leq v \ a_{uv} = r_{uv}(\frac{1}{-i} \quad \frac{i}{1}) + s_{uv}(\frac{1}{-i} \quad \frac{i}{1}) \) with \( r_{uv}, s_{uv} \in \mathbb{R} \). If \( n = 2k + 1 \) every element of \( \mathfrak{f}_c \) may be written as

\[
\begin{pmatrix}
A_0 & a \\
-a' & 0 \\
0 & B
\end{pmatrix}
\]

where \( A_0 \) is as above and \( a \in \mathbb{C}^{2k} \). In this case \( \mathfrak{n} \) consists of elements of the form where \( A_0 = (a_{uv}) \) is as above and \( a' = (a,...,a_{2k}) \) with \( a_{2j+1} = -ia_{2j} \).

Then in either case, we may take

\[
p_1 \begin{pmatrix} 0 & Z & -iZ \\ Z' & 0 & 0 \\ -iZ' & 0 & 0 \end{pmatrix} = z_1 - iz_2
\]

and

\[
p_2 \begin{pmatrix} 0 & Z & -iZ \\ Z' & 0 & 0 \\ -iZ' & 0 & 0 \end{pmatrix} = z_1^2 + z_2^2 + \cdots + z_n^2.
\]

Both are \( n \)-invariant and if \( H \in \mathfrak{h} \) is chosen as above \( A = \{y_1, y_2\} \), where \( y_1(H) = -8x \) and \( y_2(H) = y - x_1 \).

The proof of Theorem A for \( \mathfrak{Q} \) a classical irreducible bounded symmetric domain now follows from Theorem 2.2 and our above constructions.

4. The Proof of Theorem A (the Exceptional Domains)

There are exactly two irreducible bounded symmetric domains of noncompact type. One is given as \( G/K \) where \( G \) is a noncompact real group of type \( E_6 \) and \( K \) is \( \text{Spin} \) 10 \( T \) with \( T \) a one dimensional circle group. The second case occurs where \( G \) is a noncompact real group of type \( E_6 \) and \( K \) is of type \( E_6 \) \( T \). Throughout this section, we shall have occasion to use Dynkin diagrams and the classification results of Wolf [10].

Case 1. \( G \) is of type \( E_6 \), \( K = \text{Spin}(10) \) \( T \), rank \( \mathfrak{Q} = 2 \).

Fixing a Cartan subalgebra of \( \mathfrak{f}_c \) we decompose \( \mathfrak{g}_c \) under the action of \( \mathfrak{h} \) and obtain the following Dynkin diagram
where the $\alpha_i$'s are the compact simple roots and $\gamma_i$ is the noncompact simple root. A simple calculation yields $A = \{\gamma_1, \gamma_2\}$, where

\[ \gamma_2 = \gamma_1 + a_2 + 2a_3 + 2a_4 + a_5. \]

Now $\dim(G/K) = 32$ and $\dim_{\mathbb{C}} p_+ = 16$. Also, $t_c$ acts irreducibly on $p_+$ with lowest weight $\gamma_1$ and so dualizing we have that the highest weight of the representation of $t'_c$ on $P(p_+)_1$ is given by $\lambda_1 = -\gamma_1$.

From the Weyl dimension formula the degree of the representation of highest weight $2\lambda_1$ is found to be 136. Since

\[ \dim P(p_+)_2 = 146 \]

the corollary of Section 2 yields that an irreducible representation of $t'_c$ of dimension 10 must occur. An elementary calculation yields that this representation has highest weight $\lambda_2 = -\gamma_1 - \gamma_2$ and this proves Theorem A for this domain.

**Case 2.** $G$ is of type $E_6$, $K$ is of type $E_6 T$, rank $\Omega = 3$, and $\dim p_+ = 27$.

Fixing $h$ a Cartan subalgebra of $t_c$ we decompose $g_c$ under the action of $h$ and obtain the following Dynkin diagram

\[ a_6 \]

\[ a_1 a_2 a_3 a_4 a_5 \gamma_1 \]

where the $\alpha_i$'s are the compact simple roots and $\gamma_i$ is the noncompact simple root. A simple calculation now yields $A = \{\gamma_1, \gamma_2, \gamma_3\}$, where

\[ \gamma_2 = \gamma_1 + a_2 + 2a_3 + 2a_4 + 2a_5 + a_6. \]
and

\[ \gamma_3 = \gamma_1 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6. \]

Now \( e_6 \), the semisimple part of \( \mathfrak{t}_c \) acts irreducibly on \( \mathfrak{p}_+ \). Since \( \dim_\mathbb{C} \mathfrak{p}_+ = 27 \) the representation of \( e_6 \) on \( \mathfrak{p}_+ \) is the same as the representation of \( e_6 \) on the complexification of \( J_c \), the exceptional simple Jordan algebra. However, in [2], Freudenthal showed that there is polynomial \( N \) of degree 3 on \( J_c \) such that

\[ e_6 = \{ X \in \mathfrak{gl}(J_c) : XN = 0 \}. \]

Since the representations of \( e_6 \) on \( \mathfrak{p}_+ \) and \( J_c \) are equivalent the trivial representation of \( e_6 \) must occur in \( P(\mathfrak{p}_+, J_c) \). As the natural map

\[ P(\mathfrak{p}_+)_1 \otimes P(\mathfrak{p}_+)_2 \rightarrow P(\mathfrak{p}_+)_3 \]

is \( \mathfrak{t}_c \)-invariant and a fortiori \( e_6 \)-invariant the dual representation to that of \( e_6 \) on \( P(\mathfrak{p}_+)_1 \) must occur in \( P(\mathfrak{p}_+)_2 \).

Now the representation of \( \mathfrak{t}_c \) with highest weight \( \lambda_1 = -\gamma_1 \) occurs in \( P(\mathfrak{p}_+)_1 \). The representation of \( \mathfrak{t}_c \) with highest weight \( \lambda = \gamma_1 \) restricted to \( e_6 \) is dual to the representation of \( e_6 \) on \( P(\mathfrak{p}_+)_1 \) and \( \lambda = \gamma_1 - \gamma_2 - \gamma_3 \) is 0 on \( \mathfrak{h} \cap e_6 \). Since \( \lambda \) and \( \Lambda \) have the correct eigenvalues for the representations of \( \mathfrak{t} \) on \( P(\mathfrak{p}_+)_2 \) and \( P(\mathfrak{p}_+)_3 \) we have Theorem A for this case also.

This completes the proof of Theorem A for all cases.

5. **Final Remarks**

If the bounded symmetric domain \( \Omega \) is not irreducible, it is the product of irreducible bounded symmetric domains. It is an easy exercise to show that the ring \( I \) is then the tensor product of the corresponding rings for the irreducible factors of \( \Omega \).

**References**