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Acyclic list edge coloring of outerplanar graphs

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ABSTRACT

An acyclic list edge coloring of a graph G is a proper list edge coloring such that no bichromatic cycles are produced. In this paper, we prove that an outerplanar graph G with maximum degree $\Delta \geq 5$ has the acyclic list edge chromatic number equal to Δ.

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1. Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper edge k-coloring* is a mapping $c : E(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. We say that G is *edge k-colorable* if G has an edge k-coloring. The *chromatic index* $\chi'(G)$ of G is the smallest integer k such that G is edge k-colorable. A proper edge k-coloring c of G is called *acyclic* if there are no bichromatic cycles in G, i.e., the union of any two color classes induces a subgraph of G that is a forest. The *acyclic chromatic index* of G, denoted by $a'(G)$, is the smallest integer k such that G is acyclically edge k-colorable.

Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and minimum degree of a graph G, respectively. If there is no confusion in the context, we like to denote $\Delta(G)$ by Δ, and $\delta(G)$ by δ. By definition, it is obvious that $a'(G) \geq \chi'(G) \geq \Delta$. The celebrated Vizing's theorem [24] says that $\Delta \leq \chi'(G) \leq \Delta + 1$ for any graph G. Fiamčik [11] and later Alon et al. [2] made the following conjecture independently.

Conjecture 1. *For any graph G, $a'(G) \leq \Delta + 2$.*

Using probabilistic method, Alon et al. [1] proved that $a'(G) \leq 64\Delta$ for any graph G. Molloy and Reed [18] improved this bound to that $a'(G) \leq 16\Delta$. Later, Ndreca et al. [20] showed that $a'(G) \leq 9.62\Delta$. The best general upper bound for $a'(G)$ in terms of Δ is $a'(G) \leq 4\Delta$ obtained by Esperet and Parreau [10]. Alon et al. [2] proved that there is a constant c such that $a'(G) \leq \Delta + 2$ for a graph G whenever $g(G) \geq c\Delta \log \Delta$, where $g(G)$ is the girth of a graph G. They also confirmed Conjecture 1 for almost all Δ-regular graphs. Nešetřil and Wormald [21] gave an improvement to this result by showing that $a'(G) \leq \Delta + 1$ for a random Δ-regular graph G.

The acyclic edge coloring of some special classes of graphs was also considered, including subcubic graphs [3,23], graphs with maximum degree 4 [4], series–parallel graphs [14,25], and planar graphs [6,5,7,9,12,13,22,26,28].

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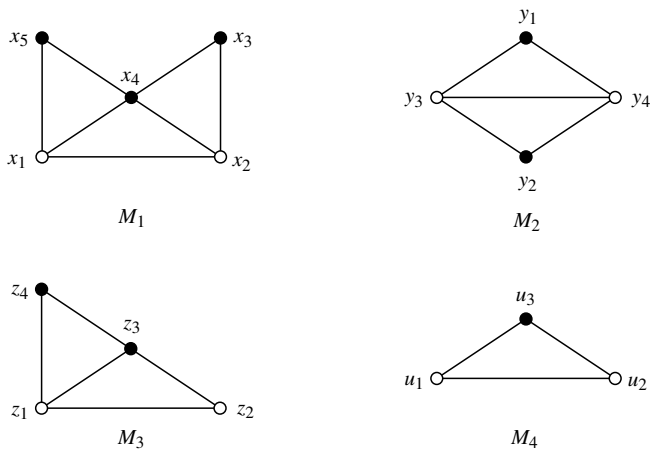

Fig. 1. Four basic configurations M_1–M_4.

Recently, Lai and Lih [16,17] considered the acyclic list edge coloring of graphs. An edge-list L of a graph G is a mapping from $E(G)$ to the set of nonnegative integers. An edge-list L is a k-edge-list if $|L(e)| \geq k$ for each $e \in E(G)$. An acyclic edge coloring ϕ such that $\phi(e) \in L(e)$ for any $e \in E(G)$ is called an acyclic L-edge coloring of G. The graph G is *acyclically k-edge choosable* if it has an acyclic L-edge coloring for any k-edge-list L. The acyclic list chromatic index, denoted $a'_{\text{list}}(G)$, is the least integer k such that G is acyclically k-edge choosable. Obviously, $\Delta \leq \chi'(G) \leq a'(G) \leq a'_{\text{list}}(G)$.

A planar graph is called *outerplanar* if there is an embedding of G into the Euclidean plane such that all the vertices lie on the boundary of the unbounded face. It was shown in [19] that $a'(G) \leq \Delta + 1$ for an outerplanar graph G. Later, Hou et al. [15] improved this result by showing that $a'(G) = \Delta$ for outerplanar graphs G with $\Delta \geq 5$. Lai and Lih [17] further showed that $a'_{\text{list}}(G) \leq \Delta + 1$ for every outerplanar graph G. In this paper, we will prove that $a'_{\text{list}}(G) = \Delta$ for an outerplanar graph G with $\Delta \geq 5$.

2. A structural lemma

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges, and G is called *H-minor free* if G does not have H as a minor.

Lemma 1 ([8]). *A graph G is an outerplanar graph if and only if G is K_4-minor free and $K_{2,3}$-minor free.*

Lemma 2 ([27]). *Every outerplanar graph contains a vertex of degree at most two.*

Given a graph G and a vertex $v \in V(G)$, we denote by $d_G(v)$ (or simply $d(v)$) the degree of v in G. A vertex of degree k (or at least k) is called a *k-vertex* (or k^+-vertex). Let $N_G(v)$ denote the set of neighbors of v in G. A 2-vertex v with neighbors x and y is said to be *type* 1 if $xy \in E(G)$, otherwise it is *type* 2.

Suppose that G is a connected outerplanar graph with $\Delta \geq 5$. We define four basic subgraphs of G, as shown in Fig. 1, which play an important role in the proof of Lemma 3:

Let M_1 be a 5-cycle $x_1x_2 \ldots x_5x_1$ with two chords x_1x_4 and x_2x_4 such that $d_G(x_4) = 4$, and $d_G(x_3) = d_G(x_5) = 2$.

Let M_2 be a subgraph consisting of one edge y_3y_4 and two 2-paths $y_3y_iy_4$ with $d_G(y_i) = 2$ for $i = 1, 2$. So, there is no 2-vertex y other than y_1, y_2 such that $yy_3, yy_4 \in E(G)$ by Lemma 1.

Let M_3 be a 4-cycle $z_1z_2z_3z_4z_1$ with a chord z_1z_3 and $d_G(z_4) = 2$, $d_G(z_3) = 3$.

Let M_4 be a 3-cycle $u_1u_2u_3u_1$ with $d_G(u_3) = 2$ which is not contained in any of M_1, M_2, M_3.

Both x_1 and x_2 are called the *roots* of M_1 and x_1x_2 is called the *root edge* of M_1. Similarly, we can define the roots y_3, y_4 for M_2, the roots u_1, u_2 for M_4, the front root z_2 and the rear root z_1 for M_3, the root edge y_3y_4 for M_2, the root edge z_1z_2 for M_3, and the root edge u_1u_2 for M_4. We say that a vertex x is *incident* to M_i if x belongs to M_i. Moreover, the vertex x_4 in M_1 and the vertex z_3 in M_3 are called *bad* 4-*vertex* and *bad* 3-*vertex* of G, respectively.

In what follows, the maximum degree of G is simply denoted by Δ, whereas the maximum degree of any subgraph H of G is denoted by $\Delta(H)$. For a vertex $x \in V(H) \subseteq V(G)$, where H is any proper subgraph of G, the degree of x in G is simply denoted by $d(x)$, whereas the degree of x in H is denoted by $d_H(x)$ (if any).

Lemma 3. *Let G be a connected outerplanar graph with $\Delta \geq 5$ and $\delta = 2$. Then G contains one of the following configurations* (C1)–(C13), *which are depicted in Fig. 2. (Note that the last figure corresponds to the case $q = 2$ in* (C13).*)*

(C1) *A 2-vertex u has two neighbors x and y such that one of the following conditions holds:*
  (C1.1) *$xy \notin E(G)$ and $d(x) \leq \Delta - 1$; or*
  (C1.2) *$d(y) = 2$ and $xy \in E(G)$.*

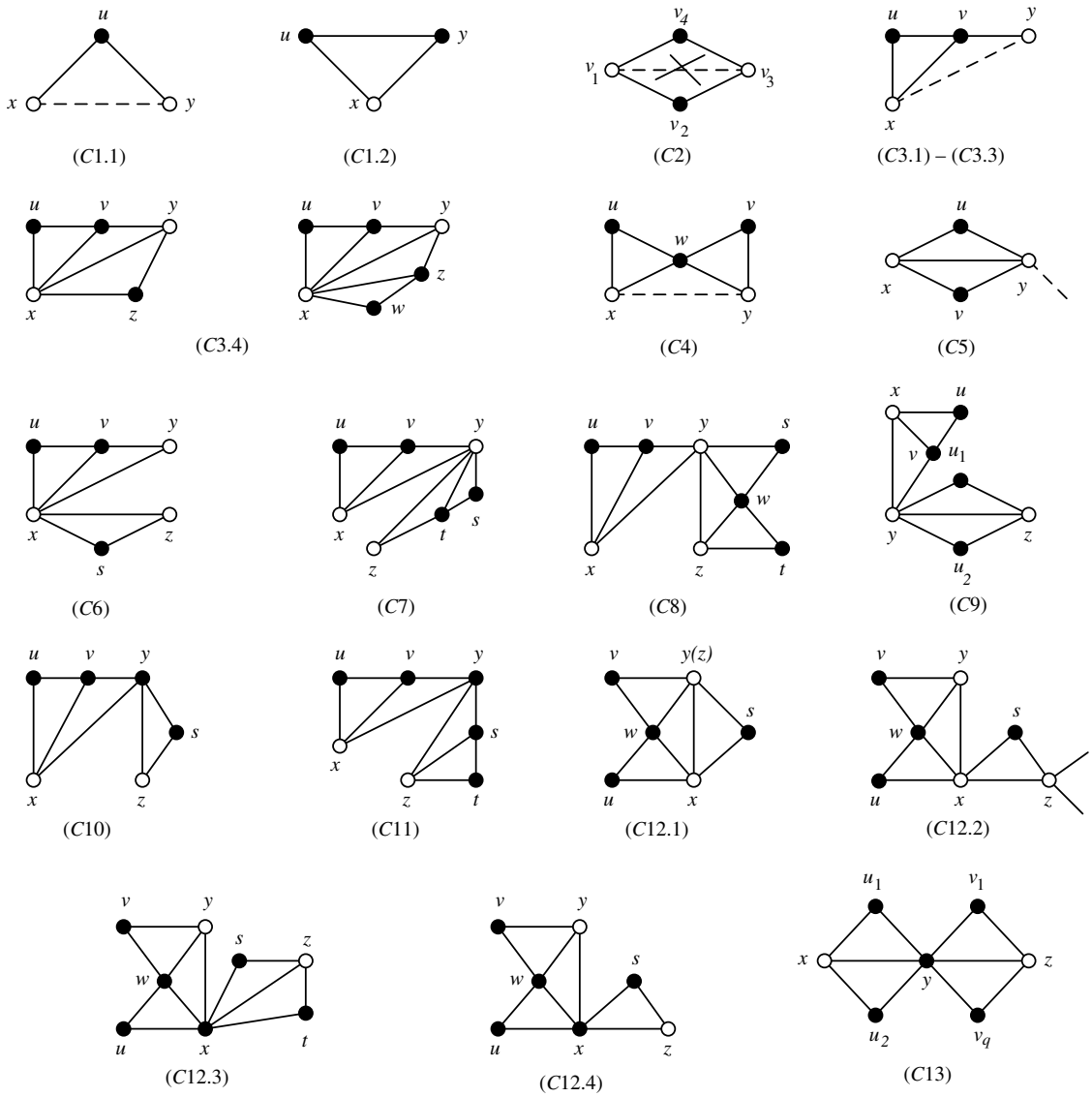

Fig. 2. Configurations (C1)–(C13) in Lemma 3.

(C2) *A 4-cycle $v_1v_2v_3v_4v_1$ with $d(v_2) = d(v_4) = 2$ and $v_1v_3 \notin E(G)$.*

(C3) *A 3-vertex v has neighbors u, x, y such that $d(u) = 2$, $ux \in E(G)$, and one of the following conditions holds:*

- (C3.1) *$xy \notin E(G)$; or*
- (C3.2) *$xy \in E(G)$ and $d(x) \le \Delta - 1$; or*
- (C3.3) *$xy \in E(G)$ and $d(y) \le \Delta - 2$; or*
- (C3.4) *$xy \in E(G)$ and there is $z \in (N_G(x) \cap N_G(y)) \setminus \{v\}$ such that either $d(z) = 2$, or $d(z) = 3$ and x, z are adjacent to a common 2-vertex w.*

(C4) *Two 3-cycles $wuxw$ and $wvyw$ share the unique vertex w such that $d(u) = d(v) = 2$, $d(w) = 4$ and satisfies one of the following conditions:*

- (C4.1) *$xy \notin E(G)$; or*
- (C4.2) *$xy \in E(G)$ and $d(x) \le \Delta - 1$.*

(C5) *An edge xy such that both x and y have 2 common neighbors u, v of degree 2 and $d(y) \le 4$.*

(C6) *A 4-cycle $xuvyx$ with a chord xv and a 3-cycle $xszx$ share the unique vertex x such that $d(u) = d(s) = 2$ and $d(v) = 3$.*

(C7) *A 4-cycle $yxuvy$ with a chord vx and another 4-cycle $ystzy$ with a chord yt share the vertex y such that $d(u) = d(s) = 2$ and $d(v) = d(t) = 3$, and x may be identical to z.*

(C8) *A 5-cycle $yswtzy$ with two chords wy, wz and a 4-cycle $yxuvy$ with a chord vx share the unique vertex y such that $d(u) = d(s) = d(t) = 2$, $d(v) = 3$ and $d(w) = 4$.*

(C9) *A 4-cycle $yxuvy$ with a chord vx and another 4-cycle yu_1zu_2y with a chord yz share the unique vertex y such that $d(u) = d(u_1) = d(u_2) = 2$ and $d(v) = 3$.*
(C10) *A 4-cycle $yxuvy$ with a chord vx and a 3-cycle $yszy$ share the unique vertex y such that $d(u) = d(s) = 2$, $d(v) = 3$, and $d(y) = 4$, where $\Delta = 5$.*
(C11) *A 4-cycle $yxuvy$ with a chord vx and another 4-cycle $ystzy$ with a chord sz share the unique vertex y such that $d(u) = d(t) = 2$, $d(v) = d(s) = 3$, and $d(y) = 4$, where $\Delta = 5$.*
(C12) *A 5-cycle $xuwvyx$ with chords wx, wy and a 3-cycle $xszx$ share the vertex x such that $d(u) = d(v) = d(s) = 2$ and $d(w) = 4$, and one of the following holds:*
(C12.1) *y is identical to z; or*
(C12.2) *$z \neq y$ and $d(z) \leq 4$; or*
(C12.3) *$z \neq y$ and x, z lie on another 3-cycle xtz such that $d(t) = 2$, and $d(x) = 6$, where $\Delta = 6$; or*
(C12.4) *$z \neq y$ and $d(x) = 5$, where $\Delta = 5$.*
(C13) *A path xyz such that x and y have two common neighbors u_1, u_2 of degree 2, y and z have q $(1 \leq q \leq 2)$ common neighbors v_1, v_q of degree 2, and $d(y) = 4 + q$.*

Proof. Assume to the contrary that G contains none of the configurations (C1)–(C13). Let H be the graph obtained from G by carrying out the following operations:

(OP1) Removing all 2-vertices and joining any two nonadjacent vertices that had a 2-vertex as a common neighbor.
(OP2) Removing all bad 3-vertices and bad 4-vertices.

It is easy to see that H is still an outerplanar graph. We shall show that $\delta(H) \geq 3$, which contradicts Lemma 2.

Since G is connected and G contains no (C1), there are no two adjacent 2-vertices in G. Let v be a 2-vertex of type 2 with neighbors x and y. Since G contains no (C2), there is no other 2-vertex, distinct from v, adjacent to both x and y. By (OP1), we remove v and add the edge xy. This means that the removal of v does not change the degrees of x and y.

Let $\mathscr{M}$ denote the set of subgraphs in G which are isomorphic to M_i for some $i \in \{1, 2, 3, 4\}$. It is not difficult to inspect the following Claim 1:

Claim 1. *Every 2-vertex of type 1 is contained in a subgraph in $\mathscr{M}$.*

Claim 2. *No root edge is removed when (OP1) and (OP2) are carried out on G.*

Proof. Note that no 2-vertices of type 2 are contained in $M_i \setminus \{x, y\}$, where xy is the root edge of M_i for any $i \in \{1, 2, 3, 4\}$. Hence, in $\mathscr{M}$, if an edge xy is removed, then x or y cannot be a 2-vertex of type 2, and at least one of x and y is a 2-vertex of type 1, or a bad 3-vertex, or a bad 4-vertex. Let e be an arbitrary root edge in G.

If e belongs to M_1, i.e., $e = x_1x_2$, then since G contains no (C4.2), it follows that $d(x_1) = d(x_2) = \Delta \geq 5$. Thus, e cannot be removed under (OP1) and (OP2).

Assume that e belongs to M_2, i.e., $e = y_3y_4$, as shown in Fig. 1. Since G contains no (C5), we have $d(y_3), d(y_4) \geq 5$. Thus, e is not a removable edge.

Assume that e belongs to M_3, i.e., $e = z_1z_2$. Since G contains neither (C3.2) nor (C3.3), it follows that $d(z_1) = \Delta \geq 5$ and $d(z_2) \geq \Delta - 1 \geq 4$. If $d(z_2) \geq 5$, then e is not a removable edge. If $d(z_2) = 4$, then z_2 cannot be a bad 4-vertex because it is adjacent to a 3-vertex z_3. Thus, e cannot be removed.

Assume that e belongs to M_4, i.e., $e = u_1u_2$. Note that $d(u_i) \geq 3$ for $i = 1, 2$ since G contains no adjacent 2-vertices. Since M_4 is not contained in M_i for all $i = 1, 2, 3$ by its definition, both u_1 and u_2 are not bad 3- and 4-vertices. Thus, e is not a removable edge. Claim 2 is proved. □

Claim 3. *Any two subgraphs in $\mathscr{M}$ cannot share a root edge.*

Proof. Assume to the contrary that there exist $M_i, M_j \in \mathscr{M}$ with $1 \leq i \leq j \leq 4$ that share a root edge. Assume that $i = 1$ and $j = 1$, i.e., there exist two M_1 that share a root edge. Let M_1 be a 5-cycle $x_1x_2x_3x_4x_5x_1$ with two chords x_1x_4 and x_2x_4 such that $d_G(x_4) = 4$, and $d_G(x_3) = d_G(x_5) = 2$, M_1' be a 5-cycle $x_1x_2y_3y_4y_5x_1$ with two chords x_1y_4 and x_2y_4 such that $d_G(y_4) = 4$, and $d_G(y_3) = d_G(y_5) = 2$. Then, (C12.2) is obtained in G with $d_G(y_5) = 2$ and $d_G(y_4) = 4$. If $i = 1$ and $j \in \{2, 3, 4\}$, then it is easy to inspect that a subgraph (C12.1) or (C12.2) occurs in G similarly, a contradiction. If $i = 2$ and $j \in \{2, 3, 4\}$, then we will get a contradiction to Lemma 1. If $i = 3$ and $j \in \{3, 4\}$, we get (C3.4) or (C7). If $i = j = 4$, we get an M_2, contradicting the definition of M_4. This proves Claim 3. □

Let $v \in V(H)$. Then $v \in V(G)$ and $d(v) \geq 3$ by the construction of H. Let r denote the number of root edges incident v. Let s denote the number of 2-vertices of type 2 in $N_G(v)$. Let t denote the number of 3^+-vertices u in $N_G(v)$ which are not bad 3- and 4-vertices and uv are not root edges. It easily follows from Claims 2 and 3 that $d_H(v) = s + t + r$ and it therefore suffices to show that $s + t + r \geq 3$. If $r = 0$, then it is easy to see that $d_H(v) = d(v) = s + t \geq 3$. So assume that $r \geq 1$. If $r \geq 3$, then it is immediate to obtain that $d_H(v) \geq r \geq 3$. Thus assume that $r \leq 2$.

The argument is split into the following two cases:

Case 1. $r = 1$.

If $s + t \geq 2$, we are done. So assume that $s + t \leq 1$. If v is incident to an M_1, then $d(v) = s + t + d_{M_1}(v) = s + t + 3 \leq 4 \leq \Delta - 1$. Hence a subgraph (C4) occurs in G, which is a contradiction.

If v is incident to an M_2, then $d(v) = s + t + d_{M_2}(v) = s + t + 2 + 1 \leq 4$ and hence (C5) occurs in G, also a contradiction.

Assume that v is incident to an M_3. If v is the front root of M_3, then $d(v) = s + t + d_{M_3}(v) \leq 1 + 2 = 3 \leq \Delta - 2$. If v is the rear root of M_3, then $d(v) \leq 4 \leq \Delta - 1$ similarly. We get a subgraph (C3.2) or (C3.3), which is impossible.

Assume that v is incident to an M_4, say $v = u_1$ (see Fig. 1). Let $u \neq u_2, u_3$ be the third neighbor of u_1. If $uu_2 \notin E(G)$, we get a subgraph (C3.1). Assume that $uu_2 \in E(G)$. If $d(u_2) \leq \Delta - 1$ or $d(u) \leq \Delta - 2$, we get (C3.2) or (C3.3). Otherwise, $d(u_2) = \Delta$ or $d(u) \geq \Delta - 1$. Then, we get an M_3 with M_4 as a subgraph, which contradicts to the definition of M_4.

Case 2. $r = 2$.

If $s + t \geq 1$, we are done. So assume that $s + t = 0$. This means that v is just incident to two elements M_i, M_j of $\mathscr{M}$. Let $1 \leq i \leq j \leq 4$.

Case 2.1. $i = 1$.

If $j \in \{1, 2, 4\}$, it is easy to see that G contains a subgraph (C12.2), (C12.3), or (C12.4). Assume that $j = 3$. If v is the front root of M_3, we get (C8). If v is the rear root of M_3, we get again (C12.2).

Case 2.2. $i = 2$ and $2 \leq j \leq 4$.

If $j \in \{2, 4\}$, we get (C13). If $j = 3$ and v is the front root of M_3, we get (C9). If $j = 3$ and v is the rear root of M_3, we get (C6).

Case 2.3. $i = 3$ and $3 \leq j \leq 4$.

First, assume that $j = 3$. If v is the front root of some M_3 and the rear root of another M_3, we get (C7). If v is the rear roots of two distinct M_3, we get (C6). Assume that v is the front roots of two distinct M_3, which implies that $d(v) = 2 + 2 = 4$. When $\Delta \geq 6$, we get (C3.3). When $\Delta = 5$, we get (C11).

Next, assume that $j = 4$. If v is the rear root of M_3, we again get (C6). Assume that v is the front root of M_3. It follows that $d(v) = 2 + 2 = 4$. If $\Delta \geq 6$, we get (C3.3). If $\Delta = 5$, we get (C10).

Case 2.4. $i = j = 4$.

In this case, $d(v) = 4$ and we may assume that v lies on two 3-cycles vx_1x_2v and vy_1y_2v such that $d(x_1) = d(y_1) = 2$. If either $x_2y_2 \notin E(G)$, or $x_2y_2 \in E(G)$ and $\min\{d(x_2), d(y_2)\} \leq \Delta - 1$, we get (C4). Otherwise, we get an M_1 with M_4 as a subgraph, contradicting the definition of M_4.

This completes the proof of the lemma. □

3. The main result

In this section, we shall state and prove the main result of this paper. We need to use the following known result, which appeared in [17]:

Lemma 4. *If G is an outerplanar graph, then $a'_{\text{list}}(G) \leq \Delta + 1$.*

Assume that ϕ is an acyclic L'-edge coloring of a graph H. For a vertex $v \in V(H) \subseteq V(G)$, we use $C(v)$ to denote the set of colors assigned to the edges in $E(G)$ incident to v. If the edges of a cycle $C = uu_1 \cdots u_kvu$ are alternatively colored with colors i and j, then we call such cycle an $(i, j)_{(u,v)}$-*cycle* or (i, j)-*cycle* if there is no confusion. Similarly, if the edges of a path $P = uu_1 \cdots u_kv$ are alternatively colored with colors i and j, then we call such path an $(i, j)_{(u,v)}$-*path*.

Theorem 1. *If G is an outerplanar graph with $\Delta \geq 5$, then $a'_{\text{list}}(G) = \Delta$.*

Proof. It is obvious that $a'_{\text{list}}(G) \geq \Delta$. It suffices to show that $a'_{\text{list}}(G) \leq \Delta$ by induction on the edge number $|E(G)|$. If $|E(G)| \leq \Delta$, then G is clearly acyclically Δ-edge choosable. So assume that G is an outerplanar graph with $|E(G)| \geq \Delta+1 \geq 6$. Obviously, we may assume that G is connected. Let L be an arbitrary Δ-edge-list of G. By Lemma 4 and the induction assumption, for any outerplanar graph H with $|E(H)| < |E(G)|$, H is acyclically L'-edge choosable for any Δ-edge-list L'.

If G has a leaf u adjacent to v, then $H = G - u$ is an outerplanar graph with $\Delta(H) \leq \Delta$ and $|E(H)| \leq |E(G)| - 1$. By the induction assumption or Lemma 4, H has an acyclic L-edge coloring ϕ. We may color uv with a color $a \in L(uv) \setminus C(v)$ since $|L(uv) \setminus C(v)| = \Delta - |C(v)| \geq \Delta - (\Delta - 1) = 1$. Hence, we assume that $\delta = 2$. By Lemma 3, G contains one of the configurations (C1)–(C13). We deal separately with each of these cases.

(C1) A 2-vertex u is adjacent to x and y such that (C1.1) or (C1.2) holds.

• Assume that (C1.1) holds, i.e., $xy \notin E(G)$ and $d(x) \leq \Delta - 1$.

Let $H = G - u + xy$. Then H is an outerplanar graph with $\Delta(H) \leq \Delta$ and $|E(H)| < |E(G)|$. Let $L'(xy) = L(uy)$ and $L'(e) = L(e)$ for any $e \in E(H) \setminus \{xy\}$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ with $\phi(xy) = 1$. We color uy with 1 and xu with a color in $L(xu) \setminus (C(x) \cup \{1\})$. Since $|L(xu) \setminus (C(x) \cup \{1\})| \geq |L(xu)| - |C(x)| - 1 \geq \Delta - d(x) \geq 1$, the coloring for xu is available.

• Assume that (C1.2) holds, i.e., $d(y) = 2$ and $xy \in E(G)$.

Let $H = G - uy$. Then H has an acyclic L-edge coloring ϕ. We color uy with a color in $L(uy) \setminus \{\phi(ux), \phi(xy)\}$.

(C2) A 4-cycle $v_1v_2v_3v_4v_1$ with $d(v_2) = d(v_4) = 2$ and $v_1v_3 \notin E(G)$.

Let $H = G - v_2 + v_1v_3$. Then H is an outerplanar graph with $\Delta(H) \leq \Delta$ and $|E(H)| \leq |E(G)| - 1$. Let $L'(v_1v_3) = L(v_1v_2)$ and $L'(e) = L(e)$ for any $e \in E(H) \setminus \{v_1v_3\}$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ with $\phi(v_1v_3) = 1$, $\phi(v_1v_4) = 2$, and $\phi(v_3v_4) = 3$.

If $L(v_2v_3) \setminus (C(v_3) \cup \{1\}) \neq \emptyset$, we color v_1v_2 with 1, v_2v_3 with a color in $L(v_2v_3) \setminus (C(v_3) \cup \{1\})$. Otherwise, $L(v_2v_3) \setminus (C(v_3) \cup \{1\}) = \emptyset$, i.e., $L(v_2v_3) = C(v_3) \cup \{1\}$.

If $L(v_1v_2) \setminus (C(v_1) \cup \{1\}) \neq \emptyset$, then we color v_1v_2 with a color in $L(v_1v_2) \setminus (C(v_1) \cup \{1\})$, and v_2v_3 with 1. Otherwise, $L(v_1v_2) \setminus (C(v_1) \cup \{1\}) = \emptyset$, i.e., $L(v_1v_2) = C(v_1) \cup \{1\}$.

If $1 \in L(v_1v_4)$, we recolor v_1v_4 with 1, color v_1v_2 with 2 and v_2v_3 with 1. So assume that $1 \notin L(v_1v_4)$. If $L(v_1v_4) \setminus (C(v_1) \cup \{3\}) \neq \emptyset$, we color v_1v_4 with a color in $L(v_1v_4) \setminus (C(v_1) \cup \{3\})$, color v_1v_2 with 2 and v_2v_3 with 1. By Lemma 1, there is no path of length at least 2 from v_1 to v_3 in $G - \{v_2, v_4\}$ and no bichromatic cycles are produced. Otherwise, $L(v_1v_4) \setminus (C(v_1) \cup \{3\}) = \emptyset$, i.e., $L(v_1v_4) = C(v_1) \cup \{3\}$.

If $1 \in L(v_3v_4)$, we recolor v_3v_4 with 1, color v_1v_2 with 1 and v_2v_3 with 3. So we assume that $1 \notin L(v_3v_4)$. If $L(v_3v_4) \setminus (C(v_3) \cup \{2\}) \neq \emptyset$, we color v_3v_4 with a color in $L(v_3v_4) \setminus (C(v_3) \cup \{2\})$, color v_1v_2 with 1 and v_2v_3 with 3. By Lemma 1, there is no path of length at least 2 from v_1 to v_3 in $G - \{v_2, v_4\}$ and no bichromatic cycles are produced. Otherwise, $L(v_3v_4) \setminus (C(v_3) \cup \{2\}) = \emptyset$, i.e., $L(v_3v_4) = C(v_3) \cup \{2\}$.

Thus, we can recolor or color v_1v_2, v_2v_3, v_3v_4, v_4v_1 with 1, 3, 2, 3, respectively.

(C3) A 3-vertex v has neighbors u, x, y such that $d(u) = 2$, $ux \in E(G)$, and one of (C3.1)–(C3.4) is true.

• Assume that (C3.1) holds, i.e., $xy \notin E(G)$.

Let $H = G - \{u, v\} + xy$. Set $L'(xy) = L(vy)$ and $L'(e) = L(e)$ for all $e \in E(H) \setminus \{xy\}$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ with $\phi(xy) = 1$ and $1 \notin C(x)$. We color vy with 1, xv with a color $a \in L(xv) \setminus (C(x) \cup \{1\})$, xu with a color $b \in L(xu) \setminus (C(x) \cup \{a\})$ and color uv with a color in $L(uv) \setminus \{1, a, b\}$.

So assume that (C3.2), (C3.3), or (C3.4) holds. Let $H = G - uv$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ. If $\phi(ux) \neq \phi(vy)$, we color uv with a color in $L(uv) \setminus \{\phi(ux), \phi(vy), \phi(xv)\}$. Otherwise, assume that $\phi(ux) = \phi(vy) = 1$.

• If (C3.2) holds, i.e., $d(x) \leq \Delta - 1$, we can color uv with a color in $L(uv) \setminus C(x)$ since $|C(x)| = d(x) \leq \Delta - 1$.

• If (C3.3) holds, i.e., $d(y) \leq \Delta - 2$, we color uv with a color in $L(uv) \setminus (C(y) \cup \{\phi(xv)\})$ since $|C(y) \cup \{\phi(xv)\}| \leq d(y) + 1 \leq \Delta - 2 + 1 = \Delta - 1$.

• If (C3.4) holds, then there is $z \in (N_G(x) \cap N_G(y)) \setminus \{v\}$ such that either $d(z) = 2$, or $d(z) = 3$ and x, z are adjacent to a common 2-vertex w. If $\phi(xz) \in L(uv)$, we color uv with $\phi(xz)$. Otherwise, $\phi(xz) \notin L(uv)$ and $L(uv) \setminus C(x) \neq \emptyset$. Then, we color uv with a color in $L(uv) \setminus C(x)$ and no bichromatic cycles are produced.

(C4) Two 3-cycles $wuxw$ and $wvyw$ share the unique vertex y such that $d(u) = d(v) = 2$, $d(w) = 4$ and (C4.1) or (C4.2) holds.

• Assume that (C4.1) holds, i.e., $xy \notin E(G)$.

Let $H = G - \{u, v\} + xy$. Set $L'(xy) = L(vy)$ and $L'(e) = L(e)$ for all $e \in E(H) \setminus \{xy\}$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ with $\phi(xy) = 1$, $\phi(wx) = 2$ and $\phi(wy) = 3$. We color vy with 1.

If $L(ux) \setminus (C(x) \cup \{3\}) \neq \emptyset$, we color xu with $a \in L(ux) \setminus (C(x) \cup \{3\})$, uw with $b \in L(uw) \setminus \{1, 2, 3, a\}$ and wv with a color in $L(wv) \setminus \{1, 2, 3, b\}$. Otherwise, $L(ux) \setminus (C(x) \cup \{3\}) = \emptyset$, i.e., $L(ux) = C(x) \cup \{3\}$.

If $1 \in L(uw)$, we color xu with 3, uw with 1 and wv with a color in $L(wv) \setminus \{1, 2, 3\}$. So we assume that $1 \notin L(uw)$. If $L(uw) \setminus (\{2, 3\} \cup ((C(x) \setminus \{2\}) \cap (C(y) \setminus \{3\}))) \neq \emptyset$, we color xu with 3, uw with $s \in L(uw) \setminus (\{2, 3\} \cup ((C(x) \setminus \{2\}) \cap (C(y) \setminus \{3\})))$ and wv with a color in $L(wv) \setminus \{1, 2, 3, s\}$. Otherwise, $L(uw) \setminus (\{2, 3\} \cup ((C(x) \setminus \{2\}) \cap (C(y) \setminus \{3\}))) = \emptyset$, which implies that $C(x) \setminus \{2\} = C(y) \setminus \{3\}$ and $L(uw) = C(x) \cup \{3\}$.

If $1 \in L(wx)$, we recolor wx with 1, color ux with 3, uw with 2, and wv with a color in $L(wv) \setminus \{1, 2, 3\}$. Since ϕ is an acyclic edge coloring of H with $\phi(xy) = 1$, no bichromatic cycles are produced. So assume that $1 \notin L(wx)$. If $L(wx) \setminus (C(x) \cup \{3\}) \neq \emptyset$, we recolor wx with $t \in L(wx) \setminus (C(x) \cup \{3\})$, color ux with 3, uw with 2, and wv with a color in $L(wv) \setminus \{1, 2, 3, t\}$. Otherwise, $L(wx) \setminus (C(x) \cup \{3\}) = \emptyset$, i.e., $L(wx) = C(x) \cup \{3\}$.

Assume that there exists $h \in L(wy) \setminus (C(y) \cup \{2\})$. First, we recolor wy with h. If $h \neq 1$, we color xu with 3, uw with $h_1 \in L(uw) \setminus \{2, 3, 1, h\}$ and wv with a color in $L(wv) \setminus \{1, 2, h, h_1\}$. Otherwise, $h = 1$. Since $|L(vy)| = \Delta$, there exists $g \in L(vy) \setminus (C(y) \cup \{1\})$ and we recolor vy with g. If $g \neq 2$, we color xu with 3, uw with $g_1 \in L(uw) \setminus \{1, 2, 3, g\}$ and wv with a color in $L(wv) \setminus \{1, 2, g, g_1\}$. If $g = 2$, we recolor xw with 3, and color xu with 2, uw with $g_2 \in L(uw) \setminus \{1, 2, 3\}$, and wv with a color in $L(wv) \setminus \{1, 2, 3, g_2\}$. So assume that $L(wy) \setminus (C(y) \cup \{2\}) = \emptyset$, i.e., $L(wy) = C(y) \cup \{2\} = C(x) \cup \{3\}$.

By the previous discussion, it follows that $L(ux) = L(uw) = L(xw) = L(wy) = C(x) \cup \{3\}$ and $1 \in L(vy)$, i.e., $L(wy) \neq L(vy)$.

Next, let $H' = G - \{u, v\} + xy$. Set $L''(xy) = L(xu)$ and $L''(e) = L(e)$ for all $e \in E(H') \setminus \{xy\}$. By the induction assumption or Lemma 4, H' has an acyclic L''-edge coloring ϕ' with $\phi'(xy) \in L'(xy) = L(xu)$. Based on ϕ', we can establish an acyclic L-edge coloring of G. Otherwise, similar to the previous discussion, we derive that $L(vy) = L(vw) = L(wy) = L(xw)$, a contradiction.

• Assume that (C4.2) holds, i.e., $xy \in E(G)$ and $d(x) \leq \Delta - 1$.

Let $H = G - u$. By the induction assumption or Lemma 4, H has an acyclic L'-edge coloring ϕ with $\phi(xw) = 1$, $\phi(wv) = 2$ and $\phi(wy) = 3$. If $L(xu) \setminus (C(x) \cup \{2, 3\}) \neq \emptyset$, we color xu with $q \in L(xu) \setminus (C(x) \cup \{2, 3\})$, and uw with a color in $L(uw) \setminus \{1, 2, 3, q\}$. Otherwise, $L(xu) \setminus (C(x) \cup \{2, 3\}) = \emptyset$, which implies that $d(x) = \Delta - 1$ and $L(xu) = C(x) \cup \{2, 3\}$. Then, we color xu with 2, and uw with a color in $L(uw) \setminus \{1, 2, 3, \phi(vy)\}$.

(C5) An edge xy such that both x and y have two common neighbors u, v of degree 2 and $d(y) \leq 4$.

Let $H = G - uy$, and let ϕ be an acyclic L-edge coloring of H. If $\phi(xu) = \phi(vy)$, we color uy with a color in $L(uy) \setminus (C(y) \cup \{\phi(xv)\})$. Otherwise, we color uy with a color in $L(uy) \setminus (C(y) \cup \{\phi(xu)\})$. By Lemma 1, there is no path in $G - \{u, v\} - xy$ from x to y and no bichromatic cycles are produced even if $\phi(xu) \in C(y) \setminus \{\phi(vy)\}$.

Now assume that one of (C6)–(C11) holds.

Let $H = G - uv$, which admits an acyclic L-edge coloring ϕ with $\phi(xu) = 2$, $\phi(xv) = 1$, and $\phi(xy) = 3$. If $\phi(vy) \neq 2$, then we color uv with a color in $L(uv) \setminus \{1, 2, \phi(vy)\}$. Otherwise, $\phi(vy) = 2$. If $L(ux) \setminus C(x) \neq \emptyset$, we recolor ux with a color $k \in L(ux) \setminus C(x)$ and color uv with a color in $L(uv) \setminus \{1, 2, k\}$. Otherwise, $L(ux) \setminus C(x) = \emptyset$, i.e., $L(ux) = C(x)$. Without loss of generality, assume that $C(x) = \{1, 2, \ldots, \Delta\}$.

Let $S_2 = \{i|$ there is a $(2, i)$-cycle if uv is colored with $i, i \neq 1, 2\}$. If $L(uv) \setminus (\{1, 2\} \cup S_2) \neq \emptyset$, then we color uv with a color in $L(uv) \setminus (\{1, 2\} \cup S_2)$. Otherwise, assume that $L(uv) \setminus (\{1, 2\} \cup S_2) = \emptyset$, which implies that $S_2 = C(x) \setminus \{1, 2\} = \{3, 4, \ldots, \Delta\}$, $L(uv) = C(x) = \{1, 2, \ldots, \Delta\}$ and $C(x) \setminus \{1\} = \{2, 3, \ldots, \Delta\} \subseteq C(y)$.

If $L(xv) \setminus C(x) \neq \emptyset$, we recolor xv with a color in $L(xv) \setminus C(x)$ and color uv with 1. Otherwise, $L(xv) \setminus C(x) \neq \emptyset$, i.e., $L(xv) = C(x) = \{1, 2, \ldots, \Delta\}$.

(C6) A 4-cycle $xuvyx$ with a chord xv and a 3-cycle $xszx$ share the unique vertex x such that $d(u) = d(s) = 2$ and $d(v) = 3$.

Since $\{\phi(xz), \phi(xs)\} \setminus S_2 \neq \emptyset$, $S_2 \neq \{3, 4, \ldots, \Delta\}$.

(C7) A 4-cycle $yxuvy$ with a chord vx and another 4-cycle $ystzy$ with a chord yt share the vertex y such that $d(u) = d(s) = 2$ and $d(v) = d(t) = 3$, and x may be identical to z.

If x coincides with z, then we color uv with a color in $L(uv) \setminus \{1, 2, 3, \phi(ys)\}$. By Lemma 1, there is no path from x to y in $G - \{u, v, t, s\} - xy$ and no bichromatic cycles are produced. Otherwise, $x \neq z$. If $\phi(yt) \in S_2$, then $\phi(zt) = 2$ and $\phi(ys), \phi(yz) \notin S_2$, which implies that $|S_2| \leq \Delta - 3$, a contradiction. Otherwise, $\phi(yt) \notin S_2$. Since $|S_2| = \Delta - 2$, $C(y) \setminus \{\phi(yt)\} = \{2, 3, \ldots, \Delta\}$ and $\phi(ys) = 4$, $\phi(yz) = 5$, $\phi(st) = 2$, and $\phi(zt) = 4$. So it is suffice to consider the following subcases.

(i) $\phi(yt) = 1$. If $L(vy) \setminus C(y) \neq \emptyset$, we recolor vy with a color in $L(vy) \setminus C(y)$ and color uv with 3. Otherwise, $L(vy) \setminus C(y) = \emptyset$, i.e., $L(vy) = C(y)$. If $L(yt) \setminus C(y) \neq \emptyset$, we recolor yt with $k_2 \in L(yt) \setminus C(y)$, $\{vy, xu\}$ with 1, xv with 2, and color uv with 4. Otherwise, $L(yt) \setminus C(y) = \emptyset$, i.e., $L(yt) = C(y)$. If $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with a color in $L(ys) \setminus C(y)$, and color uv with 4. Otherwise, $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor ys, yt, vy with 1, 2, 4, ts with a color in $L(ts) \setminus \{1, 2, 4\}$ and color uv with 3.

(ii) $\phi(yt) = \Delta + 1$. If $1 \in L(vy)$, we recolor $\{xu, vy\}$ with 1, xv with 2 and color uv with 4. Otherwise, $1 \notin L(vy)$. If $L(vy) \setminus C(y) \neq \emptyset$, we recolor vy with a color in $L(vy) \setminus C(y)$ and color uv with 3. Otherwise, $L(vy) \setminus C(y) = \emptyset$, i.e., $L(vy) = C(y) = \{2, 3, \ldots, \Delta + 1\}$. If $L(yt) \setminus C(y) \neq \emptyset$, we recolor yt with a color in $L(yt) \setminus C(y)$, vy with $\Delta + 1$ and color uv with 3. Otherwise, $L(yt) \setminus C(y) = \emptyset$, i.e., $L(yt) = C(y)$. If $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with a color in $L(ys) \setminus C(y)$, vy with 4 and color uv with 3. Otherwise, $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor yt, ys, vy with 2, $\Delta + 1$, 4, and color uv with 3, st with a color in $L(ts) \setminus \{2, 4, \Delta + 1\}$.

(C8) A 5-cycle $yswtzy$ with two chords wy, wz and a 4-cycle $yxuvy$ with a chord vx share the unique vertex y such that $d(u) = d(s) = d(t) = 2$, $d(v) = 3$ and $d(w) = 4$.

Since $|S_2| = \Delta - 2$, $|\{\phi(ys), \phi(yw)\} \cap S_2| = 1$ and $C(y) = \{2, 3, \ldots, \Delta\} \cup \{\phi(ys), \phi(yw)\}$ with $\phi(yz) = 5$. So we need to consider the following subcases.

• Assume that $1 \in \{\phi(ys), \phi(yw)\}$ and $C(y) = C(x)$.

If $L(vy) \setminus C(y) \neq \emptyset$, we recolor vy with a color in $L(vy) \setminus C(y)$ and color uv with 3. Otherwise, $L(vy) \setminus C(y) = \emptyset$, i.e., $L(vy) = C(y)$.

(i) $\phi(yw) = 1$, $\phi(ys) = \phi(wz) = 4$ and $\phi(sw) = 2 \in C(z) \setminus \{\phi(zt)\}$. If $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with a color in $L(ys) \setminus C(y)$ and color uv with 4. Otherwise, $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor ys with 2, vy with 4, sw with a color in $L(sw) \setminus \{1, 2, 4, \phi(wt)\}$ and color uv with 3.

(ii) $\phi(ys) = 1$, $\phi(yw) = \phi(zt) = 4$ and $\phi(wt) = 2 \in C(z) \setminus \{\phi(zw)\}$. Assume that $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with $q \in L(ys) \setminus C(y)$, $\{xu, vy\}$ with 1, xv with 2 and color uv with 4. If $\phi(ws) = q$, we recolor ws with a color in $L(ws) \setminus \{2, 4, q, \phi(zw)\}$. If $\phi(zw) = q$ and there is a $(q, \phi(sw))$-cycle in G, we recolor ws with a color $q_1 \in L(ws) \setminus ((C(y) \setminus \{1, 3\}) \cup \{\phi(sw)\})$. Since $q \notin C(x)$, no bichromatic cycles are produced even if $q_1 \in \{1, 3\}$. So we assume that $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor $\{xv, ys\}$ with 2, $\{xu, vy\}$ with 1, sw with a color in $L(sw) \setminus \{2, 4, \phi(zw)\}$ and color uv with 4.

• Assume that $\Delta + 1 \in \{\phi(ys), \phi(yw)\}$ and $C(y) = \{2, 3, \ldots, \Delta + 1\}$.

If $L(vy) \setminus (C(y) \cup \{1\}) \neq \emptyset$, we recolor vy with a color in $L(vy) \setminus (C(y) \cup \{1\})$ and color uv with 3.

(i) $\phi(yw) = \Delta + 1$, $\phi(ys) = \phi(wz) = 4$ and $\phi(sw) = 2 \in C(z) \setminus \{\phi(zt)\}$. If $1 \in L(vy)$, we recolor $\{xu, vy\}$ with 1, xv with 2 and color uv with 4. So we assume that $L(vy) = C(y)$. If $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with a color $q_2 \in L(ys) \setminus C(y)$ and color uv with 4. Then, no bichromatic cycles are produced even if $q_2 = 1$. Otherwise, $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor ys with 2, vy with 4, sw with a color in $L(sw) \setminus \{2, 4, \Delta + 1, \phi(wt)\}$ and color uv with 3.

(ii) $\phi(ys) = \Delta + 1$, $\phi(yw) = \phi(zt) = 4$ and $\phi(wt) = 2 \in C(z) \setminus \{\phi(zw)\}$. If $1 \in L(vy)$, we recolor $\{xu, vy\}$ with 1, xv with 2 and color uv with 4. So assume that $L(vy) = C(y)$. Assume that $L(ys) \setminus C(y) \neq \emptyset$, we recolor ys with $q_3 \in L(ys) \setminus C(y)$, vy with $\Delta + 1$ and color uv with 3. If $\phi(ws) = q_3$, we recolor ws with a color in $L(ws) \setminus \{2, 4, q_3, \phi(zw)\}$. If $\phi(zw) = q_3$ and there is a $(q_3, \phi(sw))$-cycle in G, we recolor ws with a color $q_4 \in L(ws) \setminus ((C(y) \setminus \{\Delta + 1, 3\}) \cup \{2\})$. Since $q_3 = 1$ or $q_3 \notin C(x)$, no bichromatic cycles are produced even if $q_4 \in \{\Delta + 1, 3\}$. So assume that $L(ys) \setminus C(y) = \emptyset$, i.e., $L(ys) = C(y)$. Then, we recolor ys with 2, vy with $\Delta + 1$ and color uv with 3.

(C9) A 4-cycle $yxuvy$ with a chord vx and another 4-cycle yu_1zu_2y with a chord yz share the unique vertex y such that $d(u) = d(u_1) = d(u_2) = 2$ and $d(v) = 3$.

Since $|\{\phi(yz), \phi(yu_1), \phi(yu_2)\} \cap S_2| \leq 1$, $|S_2| \leq \Delta - 3$.

(C10) A 4-cycle $yxuvy$ with a chord vx and a 3-cycle $yszy$ share the unique vertex y such that $d(u) = d(s) = 2$, $d(v) = 3$, and $d(y) = 4$, where $\Delta = 5$.

Since $|\{\phi(ys), \phi(yz)\} \cap S_2| \leq 1$, $|S_2| \leq 2 = 5 - 3$.

(C11) A 4-cycle $yxuvy$ with a chord vx and another 4-cycle $ystzy$ with a chord sz share the unique vertex y such that $d(u) = d(t) = 2$, $d(v) = d(s) = 3$, and $d(y) = 4$, where $\Delta = 5$.

Recall that $|S_2| = \Delta - 2 = 3$. We may assume that $\phi(ys) = \phi(tz) = 4$, $\phi(yz) = 5$ and $\phi(st) = 2$. If $L(vy) \setminus \{1, 2, 3, 4, 5\} \neq \emptyset$, we recolor vy with a color in $L(vy) \setminus \{1, 2, 3, 4, 5\}$ and color uv with 3. Otherwise, $L(vy) = \{1, 2, 3, 4, 5\}$. Recall $L(ux) = L(xv) = L(uv) = C(x) = \{1, 2, 3, 4, 5\}$. Then, we recolor $\{xu, vy\}$ with 1, xv with 2 and uv with 4.

(C12) A 5-cycle $xuwvyx$ with chords wx, wy and a 3-cycle $xszx$ share the vertex x such that $d(u) = d(v) = d(s) = 2$ and $d(w) = 4$, at least one of (C12.1)–(C12.4) holds.

Let $H = G - uw$. Then H has an acyclic L-edge coloring ϕ with $\phi(xu) = 4$, $\phi(xw) = 1$, $\phi(xy) = 2$ and $\phi(xs) = 3$. If $4 \notin \{\phi(wv), \phi(wy)\}$, we color uw with a color in $L(uw) \setminus \{1, 4, \phi(wv), \phi(wy)\}$. So assume that $4 \in \{\phi(wv), \phi(wy)\}$. If $\phi(wv) = 4$, we color uw with a color in $L(uw) \setminus \{1, 4, \phi(wy), \phi(vy)\}$. Next, assume that $\phi(wy) = 4$. If $L(ux) \setminus C(x) \neq \emptyset$, we recolor ux with a color $\alpha \in L(ux) \setminus C(x)$, and then color uw with a color in $L(uw) \setminus \{1, 4, \alpha, \phi(wv)\}$ if $\alpha \neq \phi(wv)$ or with a color in $L(uw) \setminus \{1, 4, \alpha, \phi(vy)\}$ if $\alpha = \phi(wv)$. Otherwise, $L(ux) \setminus C(x) \neq \emptyset$, which implies that $L(ux) = C(x)$. Without loss of generality, assume that $C(x) = \{1, 2, \ldots, \Delta\}$.

If (C12.1) holds, i.e., y is identical to z, then color uw with a color in $L(uw) \setminus \{1, 2, 4, \phi(wv)\}$. By Lemma 1, there is no path from x to y in $G - \{u, v, w, s\} - xy$ and no bichromatic cycles are produced.

If (C12.3) holds, i.e., $z \neq y$ and x, z lie on another 3-cycle $xtzx$ such that $d(t) = 2$, and $d(x) = 6$, where $\Delta = 6$, then we color uw with a color in $L(uw) \setminus \{1, 2, 4, \phi(wv)\}$. By Lemma 1, there is no path from z to y in $G - \{u, v, w, s, t\} - \{xy, xz\}$ and no bichromatic cycles are produced.

Now assume that (C12.2) or (C12.4) holds. Since $y \neq z$, we assume that $\phi(xz) = 5$. Let $S_4 = \{i|$ there is a $(4, i)$-cycle if uw is colored with $i, i \notin \{1, 4\}\}$. If $|S_4| \leq \Delta - 4$, we color uv with a color in $L(uw) \setminus (C(w) \cup S_4)$. Otherwise, $|S_4| \geq \Delta - 3$. On the other hand, since $1, 4 \in C(x)$ and $\{3, 5\} \setminus S_4 \neq \emptyset$, $|S_4| \leq \Delta - 3$. Hence, $|S_4| = \Delta - 3$, $\{3, 5\} \cap S_4 \neq \emptyset$ and $C(x) = S_4 \cup \{1, 4\} \cup (\{3, 5\} \setminus S_4)$, which implies that $\phi(wv) \notin S_4$, and there is no $(4, i)$-cycle if uw is colored with $i, i \in \{3, 5\} \setminus S_4$. If $L(uw) \setminus (S_4 \cup \{1, 4, \phi(wv)\}) \neq \emptyset$, then we color uw with a color in $L(uw) \setminus (S_4 \cup \{1, 4, \phi(wv)\})$. Otherwise, $L(uw) \setminus (S_4 \cup \{1, 4, \phi(wv)\}) = \emptyset$, which implies that $L(uw) = S_4 \cup \{1, 4, \phi(wv)\}$. So we assume that:

$(*_{12.1})$ $|S_4| = \Delta - 3$, $C(x) = S_4 \cup \{1, 4\} \cup (\{3, 5\} \setminus S_4)$, $\phi(wv) \notin S_4$, $L(uw) = S_4 \cup \{1, 4, \phi(wv)\}$ and there is no $(4, i)$-cycle if uw is colored with $\phi(wv)$ even if $\phi(wv) \in \{3, 5\}$.

Let $a = \phi(wv)$. If $\phi(vy) \neq 1$, we recolor wv with a color in $L(wv) \setminus \{1, 4, \phi(vy)\}$ and color uw with a. Otherwise, we assume that:

$(*_{12.2})$ $\phi(vy) = 1$.

Let $S_1 = \{i|$ there is a $(1, i)$-cycle if wv is recolored with $i, i \neq 1, a, 4\}$. If $L(wv) \setminus (S_1 \cup \{1, a, 4\}) \neq \emptyset$, we recolor wv with a color in $L(vv) \setminus (S_1 \cup \{1, a, 4\})$ and uw with a. Otherwise, $L(wv) \setminus (S_1 \cup \{1, a, 4\}) = \emptyset$, which implies that $|S_1| \geq \Delta - 3$. Since $\{3, 5\} \setminus S_1 \neq \emptyset$, $|S_1| \leq \Delta - 3$. Hence, we may assume that:

$(*_{12.3})$ $|S_1| = \Delta - 3$ and $L(wv) = S_1 \cup \{1, a, 4\}$.

If $L(vy) \setminus C(y) \neq \emptyset$, we recolor vy with $d_1 \in L(vy) \setminus C(y)$, wv with a color in $L(wv) \setminus \{1, a, 4, d_1\}$ and color uw with a. Otherwise, $L(vy) = C(y)$. If $L(wy) \setminus C(y) \neq \emptyset$, we recolor wy with $d_2 \in L(wy) \setminus C(y)$, vw with 4 and color uw with a color in $L(uw) \setminus \{1, a, 4, d_2\}$. Otherwise, $L(wy) = C(y)$. If $L(wx) \setminus C(x) \neq \emptyset$, we recolor wx with $d_3 \in L(wx) \setminus C(x)$, vw with a color in $L(vw) \setminus \{1, 4, a, d_3\}$ and color uw with 1. Since $\phi(vy) = 1 \neq d_3$, no bichromatic cycles are produced even if $\phi(wv) \in C(x)$. Otherwise, $L(wx) = C(x)$. We need to consider the following subcases.

• $\phi(wv) = 3$.

By $(*_{12.1})$, $(*_{12.2})$ and $(*_{12.3})$, $\phi(vy) = 1$, $5 \in S_4 \cap S_1$ and $1, 4 \in C(z) \setminus \{\phi(sz)\}$.

If $4 \in L(xs)$, we recolor xs with 4, ux with 3 and color uw with 5. Otherwise, $4 \notin L(xs)$. If $1 \in L(xs)$, we recolor xs with 1, xw with 3, vw with 2 and color uw with 1. Otherwise, $1 \notin L(xs)$. Hence, $|L(xs) \setminus C(x)| \geq 2$, assume that $\Delta + 1, \Delta + 2 \in L(xs)$.

If $\phi(sz) \notin C(x)$, we recolor xs with a color in $\{\Delta+1, \Delta+2\} \setminus \{\phi(sz)\}$, xu with 3 and color uw with 5. If $\phi(sz) = 2$, we recolor xs with a color in $\{\Delta + 1, \Delta + 2\} \setminus C(y)$, xu with 3, and color uw with 5. Otherwise, assume that $\phi(sz) \in C(x) \setminus \{1, 2, 3, 4, 5\}$. Then $\Delta \geq 6$ and (C12.3) holds, i.e., $d(z) \leq 4$. We recolor xs with $\Delta + 1$, ux with 3 and color uw with 5. Since $\Delta + 1 \notin C(z)$, no bichromatic cycles are produced.

• $\phi(wv) = 5$.

By $(*_{12.1})$, $3 \in S_4$ and $\phi(xs) = 4$. It suffices to assume that $\phi(vy) = 1$. Since $3 \notin S_1$ and $|S_1| \leq \Delta - 4$, we recolor wv with a color in $L(wv) \setminus (S_1 \cup \{1, 4, 5\})$ and color uw with 5.

• $\phi(wv) = \Delta + 1$.

By $(*_{12.1})$ and $(*_{12.2})$, $|S_4| = \Delta - 3$, $\{3, 5\} \cap S_4 \neq \emptyset$, $L(uw) = S_4 \cup \{1, 4, 6\}$ and $\phi(vy) = 1$. We may assume that $3 \in S_4$, $\phi(sz) = 4$ and $3 \in C(z)$ or $5 \in S_4$, $4 \in C(z) \setminus \{\phi(sz)\}$.

(i) Assume that $3 \in S_4$.

Then $\phi(sz) = 4$ and $3 \in C(z)$. By $(*_{12.3})$, $5 \in S_1 \cap L(wv)$ and $1 \in C(z)$. If $1 \in L(xs)$, then recolor xs with 1, wx with 3 and color uw with 1. Otherwise, $1 \notin L(xs)$ and $L(xs) \setminus C(x) \neq \emptyset$. Then, recolor xs with a color in $L(xs) \setminus C(x)$, wx with 3 and color uw with 1.

(ii) Assume that $5 \in S_4$.

Then $4 \in C(z) \setminus \{\phi(sz)\}$. By $(*_{12.3})$, $\{3, 5\} \cap S_1 \neq \emptyset$. First assume $3 \in S_1$, then $\phi(sz) = 1$ and $3 \in C(z) \cap L(wv)$. If $4 \in L(xs)$, we recolor xs with 4, ux with 3 and color uw with 5. Otherwise, $4 \notin L(xs)$ and $L(xs) \setminus C(x) \neq \emptyset$. Then, we recolor xs with a color in $L(xs) \setminus C(x)$, wx with 3 and color uw with 1. Next assume $5 \in S_1$, then $1 \in C(z) \setminus \{\phi(sz)\}$ and $5 \in L(wv)$. If $1 \in L(xs)$, we recolor xs with 1, xw with 3 and color uw with 1. Otherwise, $1 \notin L(xs)$. If $4 \in L(xs)$, we recolor xs with 4, ux with 3 and color uw with 5. Otherwise, $4 \notin L(xs)$. Hence, $|L(xs) \setminus C(x)| \geq 2$, assume that $k_1, k_2 \in L(xs)$. First, we recolor ux with 3 and color uw with 5. Then, if $\phi(sz) \notin C(x)$, we recolor xs with a color in $\{k_1, k_2\} \setminus \phi(sz)$; if $\phi(sz) = 2$, we recolor xs with a color in $\{k_1, k_2\} \setminus C(y)$. Otherwise, assume that $\phi(sz) \in C(x) \setminus \{1, 2, 3, 4, 5\}$. Then $\Delta \geq 6$ and (C12.3) holds, i.e., $d(z) \leq 4$. Assume that $\phi(sz) = 6$. We recolor xs with k_1. Since $k_1 \notin C(z)$ and no bichromatic cycles are produced.

(C13) A path xyz such that x and y have two common neighbors u_1, u_2 of degree 2, y and z have q (≥ 1) common neighbors v_1, v_q of degree 2, and $d(y) = 4 + q$.

Let $H = G - yv_1$, which has an acyclic L-edge coloring ϕ with $\phi(yu_1) = 1$, $\phi(yx) = 2$, $\phi(yu_2) = 3$, and $\phi(yz) = 4$. Since $q \in \{1, 2\}$, we need to consider the following two subcases.

(I) $q = 1$ and $d(y) = 5$.

• Assume that $\phi(zv_1) \in \{1, 2, 3\}$. Then color yv_1 with a color in $L(yv_1) \setminus \{1, 2, 3, 4\}$. By Lemma 1, no bichromatic are produced.

• Assume that $\phi(zv_1) = 5$. If there exists $\alpha_1 \in L(yv_1) \setminus \{1, 2, 3, 4, 5\}$, then we color yv_1 with α_1. Otherwise, $L(yv_1) = \{1, 2, 3, 4, 5\}$ and $\Delta = 5$. Let $a = \phi(xu_1)$ and $b = \phi(xu_2)$.

(i) $a = 3$. If there exists $\alpha_2 \in L(yu_1) \setminus \{1, 2, 3, 4, b\}$, then recolor yu_1 with α_2 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, b\} = \emptyset$, which implies that $b \neq 4$ and $L(yu_1) = \{1, 2, 3, 4, b\}$. If there exists $\alpha_3 \in L(yu_2) \setminus \{1, 2, 3, 4, b\}$, then recolor yu_2 with α_3 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, b\} = \emptyset$, which implies that $L(yu_2) = \{1, 2, 3, 4, b\}$. Then, we recolor yu_1 with b, yu_2 with 1 and color yv_1 with 3.

(ii) $a \neq 3$, $b \neq 1$. If there exists $\alpha_4 \in L(yu_1) \setminus \{1, 2, 3, 4, a\}$, then recolor yu_1 with α_4 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, a\} = \emptyset$, which implies that $a \neq 4$ and $L(yu_1) = \{1, 2, 3, 4, a\}$. If there exists $\alpha_5 \in L(yu_2) \setminus \{1, 2, 3, 4, b\}$, then recolor yu_2 with α_5 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, b\} = \emptyset$, which implies that $b \neq 4$ and $L(yu_2) = \{1, 2, 3, 4, b\}$. If there exists $\alpha_6 \in L(xu_1) \setminus C(x)$, then recolor xu_1 with α_6, yu_1 with a and color yv_1 with 1. Otherwise, $L(xu_1) = C(x)$. Similarly, we can assume that $L(xu_2) = C(x)$. Then, we recolor $\{xu_2, yu_1\}$ with a, xu_1 with b and color yv_1 with 1.

(II) $q = 2$, $d(y) = 6$ and assume that $\phi(yv_2) = 5$.

• Assume that $\phi(zv_1) \in \{1, 2, 3\}$. Then color yv_1 with a color in $L(yv_1) \setminus \{1, 2, 3, 4, 5\}$.

• Assume that $\phi(zv_1) = 5$. If $|L(yv_1)| \geq 7$, then color yv_1 with a color in $L(yv_1) \setminus \{1, 2, 3, 4, 5, \phi(zv_2)\}$. Otherwise, $|L(yv_1)| = \Delta = 6$ and $L(yv_1) \setminus \{1, 2, 3, 4, 5, \phi(zv_2)\} = \emptyset$, i.e., $L(yv_1) = \{1, 2, 3, 4, 5, \phi(zv_2)\}$, assume that $\phi(zv_2) = 6$. If there exists $\beta_1 \in L(yv_2) \setminus \{1, 2, 3, 4, 5, 6\}$, then recolor yv_2 with β_1 and color yv_1 with 6. Otherwise, $L(yv_2) = \{1, 2, 3, 4, 5, 6\}$. Let $a = \phi(xu_1)$ and $b = \phi(xu_2)$.

(i) $a = 3$. If there exists $\beta_2 \in L(yu_1) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_1 with β_2 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $b \notin \{4, 5\}$ and $L(yu_1) = \{1, 2, 3, 4, 5, b\}$. If there exists $\beta_3 \in L(yu_2) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_2 with β_3 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $L(yu_2) = \{1, 2, 3, 4, 5, b\}$. Then, we recolor yu_1 with b, yu_2 with 1 and color yv_1 with 3.

(ii) $a \neq 3$, $b \neq 1$.

If there exists $\beta_4 \in L(yu_1) \setminus \{1, 2, 3, 4, 5, a\}$, then recolor yu_1 with β_4 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, 5, a\} = \emptyset$, which implies that $a \notin \{4, 5\}$ and $L(yu_1) = \{1, 2, 3, 4, 5, a\}$. If there exists $\beta_5 \in L(yu_2) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_2 with β_5 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $b \notin \{4, 5\}$ and $L(yu_2) = \{1, 2, 3, 4, 5, b\}$. If there exists $\beta_6 \in L(xu_1) \setminus C(x)$, then recolor xu_1 with β_6 with a and color yv_1 with 1. Otherwise, $L(xu_1) = C(x)$. Similarly, we can assume that $L(xu_2) = C(x)$. Then, we recolor $\{xu_2, yu_1\}$ with a, xu_1 with b and color yv_1 with 1.

• Assume that $\phi(zv_1) \notin C(y)$ and $\phi(zv_1) = 6$. If there exists $\gamma_1 \in L(yv_1) \setminus \{1, 2, 3, 4, 5, 6\}$, then color yv_1 with γ_1. Otherwise, $L(yv_1) = \{1, 2, 3, 4, 5, 6\}$ and $\Delta = 6$.

Let $a = \phi(xu_1)$ and $b = \phi(xu_2)$.

(i) $a = 3$. If there exists $\gamma_2 \in L(yu_1) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_1 with γ_2 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $b \notin \{4, 5\}$ and $L(yu_1) = \{1, 2, 3, 4, 5, b\}$. If there exists $\gamma_3 \in L(yu_2) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_2 with γ_3 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $L(yu_2) = \{1, 2, 3, 4, 5, b\}$. Then, we recolor yu_1 with b, yu_2 with 1 and color yv_1 with 3.

(ii) $a \neq 3$, $b \neq 1$. If there exists $\gamma_4 \in L(yu_1) \setminus \{1, 2, 3, 4, 5, a\}$, then recolor yu_1 with γ_4 and color yv_1 with 1. Otherwise, $L(yu_1) \setminus \{1, 2, 3, 4, 5, a\} = \emptyset$, which implies that $a \notin \{4, 5\}$ and $L(yu_1) = \{1, 2, 3, 4, 5, a\}$. If there exists $\gamma_5 \in L(yu_2) \setminus \{1, 2, 3, 4, 5, b\}$, then recolor yu_2 with γ_5 and color yv_1 with 3. Otherwise, $L(yu_2) \setminus \{1, 2, 3, 4, 5, b\} = \emptyset$, which implies that $b \notin \{4, 5\}$ and $L(yu_2) = \{1, 2, 3, 4, 5, b\}$. If there exists $\gamma_6 \in L(xu_1) \setminus C(x)$, then recolor xu_1 with γ_6, yu_1 with a and color yv_1 with 1. Otherwise, $L(xu_1) = C(x)$. Similarly, we can assume that $L(xu_2) = C(x)$. Then, we recolor $\{xu_2, yu_1\}$ with a, xu_1 with b and color yv_1 with 1.

By Lemma 1, no bichromatic cycles are produced in G under the constructed coloring. □

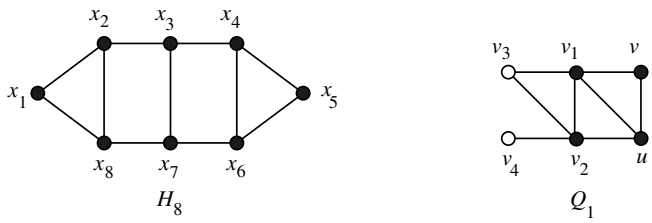

Fig. 3. Graphs H_8 and Q_1.

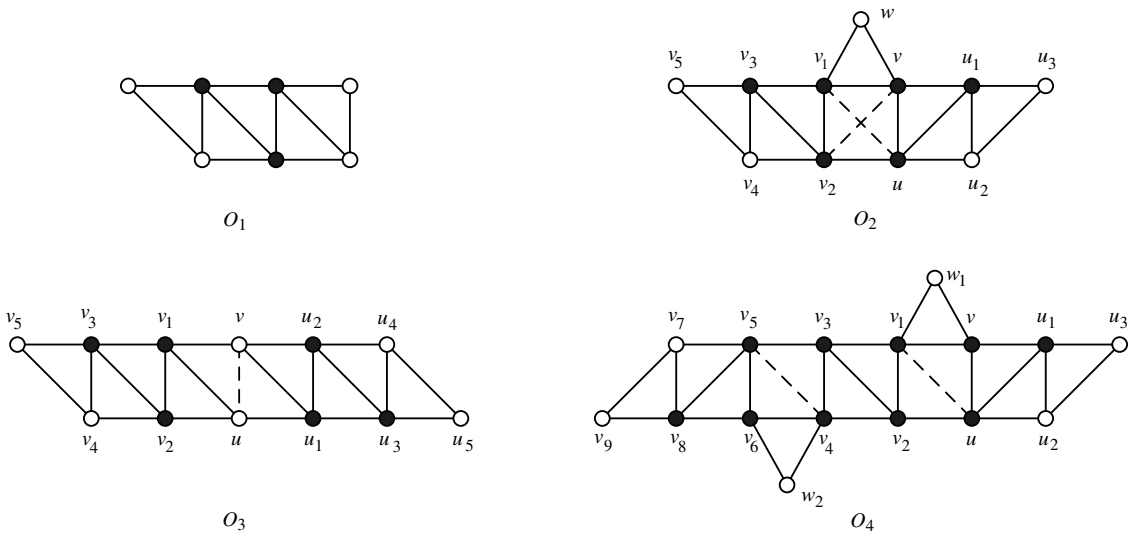

Fig. 4. Outerplanar graphs O_1–O_4 with $\Delta = 4$ and $a'(G) = 5$.

4. Concluding remark

Theorem 1 shows that $a'_{\text{list}}(G) = \Delta$ if G is an outerplanar graph with $\Delta \geq 5$. This result is best possible in the sense that Δ cannot be reduced to 3 or 4.

Let H_{2n}, $n \geq 2$, be a graph obtained by adding $n-1$ chords $x_2x_{2n}, x_3x_{2n-1}, \ldots, x_nx_{n+2}$ to a $2n$-cycle $x_1x_2 \ldots x_{2n}x_1$ and Q_1 be an outerplanar graph on six vertices, where H_8 and Q_1 are depicted in Fig. 3.

Lemma 5 ([15]). *$a'(H_{2n}) = 4$ and $a'(O_1) = 5$.*

Combining Lemmas 4 and 5, we immediately derive that $a'_{\text{list}}(H_{2n}) = 4 = \Delta + 1$ and $a'_{\text{list}}(O_1) = 5 = \Delta + 1$.

In fact, there exist many outerplanar graphs G with $\Delta = 4$ and $a'_{\text{list}}(G) = a'(G) = 5 = \Delta + 1$. Let us observe the graphs O_i for $2 \leq i \leq 4$ in Fig. 4.

Lemma 6. *Let G be an outerplanar graph with Q_1 as a subgraph such that $\Delta = 4$ and $a'(G) = 5$.*

(1) If $G - v$ has an acyclic edge 4-coloring ϕ with $\phi(uv_2) = 1$ and $\phi(uv_1) = 2$, then $1 \notin C(v_1)$ and there exist a $(1, 3)_{(u,v_1)}$-path and a $(1, 4)_{(u,v_1)}$-path in $G - v$.

(2) If $G - \{v, uv_1\}$ has an acyclic edge 4-coloring ϕ with $\phi(v_2v_4) = 3$, $\phi(v_2u) = 4$, $\phi(v_2v_3) = 2$ and $\phi(v_1v_2) = 1$, then there exists a $(3, 4)_{(u,v_1)}$-path in $G - \{v, uv_1\}$. Furthermore, there exists a $(4, 2)_{(u,v_1)}$-path in $G - \{v, uv_1\}$ if $\phi(v_1v_3) = 4$.

Proof. Let $C = \{1, 2, 3, 4\}$. To prove (1), assume to the contrary that $1 \in C(v_1)$, that is $\phi(v_1v_3) = 1$. Then we color vv_1 with $k_1 \in C \setminus C(v_1)$ and uv with a color in $C \setminus \{1, k_1, 2\}$ to obtain an acyclic edge 4-coloring of G, a contradiction. This shows that $1 \notin C(v_1)$. Further, if there is no $(1, i_1)_{(u,v_1)}$-path for some $i_1 \in \{3, 4\}$, then we can color vv_1 with 1 and uv with i_1, which implies that $a'(G) = 4$, a contradiction.

To prove (2), first assume that $\phi(v_1v_3) = 4$. Clearly, there is a $(4, 2)_{(u,v_1)}$-path. If there is no $(4, 3)_{(u,v_1)}$-path, we color uv_1 with 3, vv_1 with 2 and uv with 1, which implies that $a'(G) = 4$, a contradiction. Next, assume that $\phi(v_1v_3) = 3$. If there is no $(4, 3)_{(u,v_1)}$-path, we color uv_1 with 2, vv_1 with 4 and uv with 3, also a contradiction. □

Theorem 2. $a'_{\text{list}}(O_i) = a'(O_i) = 5,\ i = 2, 3, 4.$

Proof. Otherwise, for $i \in \{2, 3, 4\}$, we may assume that $a'(O_i) = 4$ and O_i has an acyclic edge 4-coloring ϕ using the color set $C = \{1, 2, 3, 4\}$.

• $i = 2$. Let $G_1 = G - \{u_1, u_2, u_3, w\} + uv_1$ and $G_2 = G - \{v_3, v_4, v_5, w\} + vv_2$. Obviously, $G_1 = G_2 = O_1$. Without loss of generality, we assume that $\phi(uv_2) = 1$ and $\phi(v_2v_1) = 2$. It suffices to consider the following two subcases:

(a) Assume that $c(v_1v_3) = 1$. Since $a'(G_1) = 5$, there exist a $(1, 3)_{(u,v_1)}$-path and a $(1, 4)_{(u,v_1)}$-path in $G_1 - \{v, uv_1\}$ by Lemma 6. Without loss of generality, we may assume that $\phi(v_1w) = 3$ and $\phi(v_1v) = 4$. Note that $\phi(wv) \in \{1, 2\}$.

(a1) Assume that $\phi(wv) = 1$. Since there is no $(1, 3)$-cycle in G, it follows that $\phi(uv) = 2$, and then $\phi(vu_1) = 3$, $\phi(uu_1) = 4$, and $\phi(uu_2) = 3$. Since $a'(G_2) = 5$, there exists a $(1, 3)$-path in $G_2 - \{v_1, vv_2\}$ by Lemma 6, which implies that there exists a $(1, 3)$-cycle in G, a contradiction.

(a2) Assume that $\phi(vw) = 2$. Then $\phi(vu) = 3$ and $\phi(vu_1) = 1$. Since there is no $(1, 4)$-cycle in G, we get that $\phi(uu_1) = 2$, and then $\phi(uu_2) = 4$. By Lemma 6, there exists a $(1, 4)_{(v,v_2)}$-path in $G_2 - \{v_1, vv_2\}$, which implies that there exists a $(1, 4)$-cycle in G, a contradiction.

(b) Assume that $\phi(v_1v_3) = 3$. Then there exists a $(1, 3)_{(u,v_1)}$-path in $G_1 - \{v, uv_1\}$ by Lemma 6.

(b1) Assume that $\phi(vv_1) = 1$ and $c(v_1w) = 4$. Since there are neither $(1, 3)$-cycle nor $(1, 2)$-cycle in G, we see that $\phi(uv) = 4$, and then $\phi(vw) = \phi(uu_1) = a$, $\phi(vu_1) = \phi(uu_2) = b$, $\{a, b\} = \{2, 3\}$. By Lemma 6, there exists a $(1, b)_{(v,v_2)}$-path in $G_2 - \{v_1, vv_2\}$, which implies that there exists a $(1, b)$-cycle in G, a contradiction.

(b2) Assume that $\phi(vv_1) = 4$ and $\phi(v_1w) = 1$. We assume that $\phi(vw) = a$, $\phi(vu) = b$, and $\phi(vu_1) = 1$, $\{a, b\} = \{2, 3\}$. Since there is no $(1, a)$-cycle in G, $\phi(uu_1) = 4$, and $\phi(uu_2) = a$. By Lemma 6, there exists a $(1, a)_{(v,v_2)}$-path in $G_2 - \{v_1, vv_2\}$, which implies that there exists a $(1, a)$-cycle in G, a contradiction.

• $i = 3$. Let $G_1 = G - \{u_1, u_2, u_3, u_4, u_5\} + uv$ and $G_2 = G - \{v_1, v_2, v_3, v_4, v_5\} + uv$. Then $G_1 = G_2 = O_1$.

Without loss of generality, we assume that $\phi(uv_2) = 1$, $\phi(uv_1) = 2$ and $\phi(uu_1) = 3$. By Lemma 6, we may assume that $\phi(vv_1) = 1$, $\phi(vu_2) = 3$, and there exists a $(1, 3)_{(u,v)}$-path in $G_1 - v$ and a $(1, 3)_{(u,v)}$-path in $G_2 - u$, which implies that there exists a $(1, 3)$-cycle in G, a contradiction.

• $i = 4$. Let $G_1 = G - \{u_1, u_2, u_3, v, w_1\} + uv_1$ and $G_2 = G - \{v_6, v_7, v_8, v_9, w_2\} + v_4v_5$. Then $G_1 = G_2 = O_2$.

Without loss of generality, assume that $\phi(v_3v_1) = 1$, $\phi(v_3v_2) = 2$, $\phi(v_3v_4) = 3$ and $\phi(v_3v_5) = 4$. By Lemma 6, we may assume that $\phi(v_2u) = \phi(v_3v_1) = 1$, and there exists a $(1, 4)_{(u,v_1)}$-path in $G_1 - u$ and a $(1, 4)_{(v_4,v_5)}$-path in $G_2 - v_5$, which implies that there exists a $(1, 4)$-cycle in G, a contradiction. □

We like to conclude this paper by raising the following problem:

Problem 1. Characterize outerplanar graphs G with $3 \le \Delta \le 4$ according to their acyclic list chromatic indices.

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References

[1] N. Alon, C. McDiarmid, B. Reed, Acyclic coloring of graphs, Random Structures Algorithms 2 (1991) 277–288.
[2] N. Alon, B. Sudakov, A. Zaks, Acyclic edge colorings of graphs, J. Graph Theory 37 (2001) 157–167.
[3] M. Basavaraju, L.S. Chandran, Acyclic edge coloring of subcubic graphs, Discrete Math. 308 (2008) 6650–6653.
[4] M. Basavaraju, L.S. Chandran, Acyclic edge coloring of graphs with maximum degree 4, J. Graph Theory 61 (2009) 192–209.
[5] M. Basavaraju, L.S. Chandran, Acyclic edge coloring of triangle free planar graphs, J. Graph Theory 71 (2012) 365–385.
[6] M. Basavaraju, L.S. Chandran, N. Cohen, F. Havet, T. Müller, Acyclic edge-coloring of planar graphs, SIAM J. Discrete Math. 25 (2011) 463–478.
[7] M. Borowiecki, A. Fiedorowicz, Acyclic edge coloring of planar graphs without short cycles, Discrete Math. 310 (2010) 1445–1455.
[8] G. Chartrand, F. Harary, Planar permutation graphs, Ann. Inst. Henri Poincare Sec. B 3 (1967) 433–438.
[9] W. Dong, B. Xu, Some results on acyclic edge coloring of plane graphs, Inform. Process. Lett. 110 (2010) 887–892.
[10] L. Esperet, A. Parreau, Acyclic edge-coloring using entropy compression, http://arxiv.org/abs/1206.1535v1.
[11] J. Fiamčik, The acyclic chromatic class of a graph, Math. Slovaca 28 (1978) 139–145 (in Russian).
[12] A. Fiedorowicz, M. Haluszczak, N. Narayanan, About acyclic edge colourings of planar graphs, Inform. Process. Lett. 108 (2008) 412–417.
[13] J. Hou, G. Liu, J. Wu, Acyclic edge coloring of planar graphs without small cycles, Graphs Combin. 28 (2012) 215–226.
[14] J. Hou, J. Wu, G. Liu, B. Liu, Acyclic edge colorings of planar graphs and series–parallel graphs, Sci. China Ser. A 52 (2009) 605–616.
[15] J. Hou, J. Wu, G. Liu, B. Liu, Acyclic edge chromatic number of outerplanar graphs, J. Graph Theory 64 (2010) 22–36.
[16] H. Lai, K. Lih, Acyclic list edge coloring of planar graphs, Bull. Inst. Math. Acad. Sin. (N.S.) 5 (2010) 413–436.
[17] H. Lai, K. Lih, Acyclic list edge coloring of graphs, J. Graph Theory. Article first published online: 18 MAY 2012, http://dx.doi.org/10.1002/jgt.21641.
[18] M. Molloy, B. Reed, Further algorithmic aspects of Lovász local lemma, in: Proceedings of the 30th Annual ACM Symposium on Theory of Computing, 1998, pp. 524–529.
[19] R. Muthu, N. Narayanan, C.R. Subramanian, Acyclic edge colouring of outerplanar graphs, Lecture Notes in Comput. Sci. 4508 (2007) 144–152.
[20] S. Ndreca, A. Procacci, B. Scoppola, Improved bounds on coloring of graphs, European J. Combin. 33 (2012) 592–609.
[21] J. Nešetřil, N.C. Wormald, The acyclic edge chromatic number of a random d-regular graph is $d + 1$, J. Graph Theory 49 (2005) 69–74.
[22] Q. Shu, W. Wang, Q. Wang, Acyclic edge coloring of planar graphs without 5-cycles, Discrete Appl. Math. 160 (2012) 1211–1223.
[23] S. Skulrattanakulchai, Acyclic colorings of subcubic graphs, Inform. Process. Lett. 92 (2004) 161–167.
[24] V. Vizing, On an estimate of the chromatic index of a p-graph, Diskret. Anal. 3 (1964) 25–30.
[25] W. Wang, Q. Shu, Acyclic chromatic indices of K_4-minor free graphs, Sci. Sin. Math. 41 (2011) 733–744 (in Chinese).
[26] W. Wang, Q. Shu, W. Kai, P. Wang, Acyclic chromatic indices of planar graphs with large girth, Discrete Appl. Math. 159 (2011) 1239–1253.
[27] D.B. West, Introduction to Graph Theory, Prentice Hall, India, 2001.
[28] D. Yu, J. Hou, G. Liu, B. Liu, L. Xu, Acyclic Acyclic edge coloring of planar graphs with large girth, Theoret. Comput. Sci. 410 (2009) 5196–5200.