Connectivity of generalized prisms over $G$

B.L. Piazza

University of Southern Mississippi, Hattiesburg, MS 38406-5045, USA

R.D. Ringeisen*

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-1907, USA

Received 15 July 1988
Revised 1 March 1989

Abstract

The problem of building larger graphs with a given graph as an induced subgraph is one which can arise in various applications and in particular can be important when constructing large communications networks from smaller ones. What one can conclude from this paper is that generalized prisms over $G$ may provide an important such construction because the connectivity of the newly created graph is larger than that of the original (connected) graph, regardless of the permutation used.

For a graph $G$ with vertices labeled $1, 2, \ldots, n$ and a permutation $\alpha$ in $S_n$, the generalized prisms over $G$, $\alpha(G)$ (also called a permutation graph), consists of two copies of $G$, say $G_1$ and $G_2$, along with the edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. The purpose of this paper is to examine the connectivity of generalized prisms over $G$. In particular, upper and lower bounds are found. Also, the connectivity and edge-connectivity are determined for generalized prisms over trees, cycles, wheels, $n$-cubes, complete graphs, and complete bipartite graphs. Finally, the connectivity of the generalized prism over $G$, $\alpha(G)$, is determined for all $\alpha$ in the automorphism group of $G$.

1. Introduction

The problem of building larger graphs with a given graph as an induced subgraph is one which can arise in various applications and in particular can be important when constructing large communications networks from smaller ones. What one
can conclude from this paper is that generalized prisms over $G$ may provide an important such construction because the connectivity of the newly created graph is larger than that of the original (connected) graph, regardless of the permutation used. This looks particularly promising because new graphs are constructed from old ones where the number of vertices is doubled but the number of edges per vertex remains approximately the same. Furthermore, each such graph contains two copies of the original graph, each existing as an induced subgraph and together spanning the whole graph.

For a labeled graph $G$ with $V(G) = \{1, 2, \ldots, n\}$ and a permutation $\alpha$ in $S_n$, the symmetric group on $\{1, 2, \ldots, n\}$; the $\alpha$-permutation graph of $G$, $\alpha(G)$, consists of two copies of $G$, say $G_x$ and $G_y$ with vertex sets $V(G_x) = \{x_1, x_2, \ldots, x_n\}$ and $V(G_y) = \{y_1, y_2, \ldots, y_n\}$, along with the permutation edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. Permutation graphs were introduced by Chartrand and Harary [3] who were interested in finding those which are planar. Other properties of permutation graphs which have been examined include crossing number [8], chromatic number [1, 2, 5], edge-chromatic number [4, 10], and cut frequency vectors [7].

In [6], Klee studied the Hamiltonian properties of these graphs where the object graph is a cycle and called them generalized prisms. This name seems particularly attractive from a geometric point of view and, given that there are various graphs which have been called "permutation graphs", we introduce the name $\alpha$-generalized prism over $G$, shortened to generalized prism over $G$, to be a synonym for the $\alpha$-permutation graph of $G$, $\alpha(G)$.

The purpose of this paper is to examine the connectivity of generalized prisms over $G$. In particular, upper and lower bounds for the connectivity are found. Also, the connectivity and edge-connectivity are determined for generalized prisms over trees, cycles, wheels, $n$-cubes, complete graphs, and complete bipartite graphs. Finally, the connectivity of $\alpha(G)$ is determined for all $\alpha$ in the automorphism group of the original graph $G$. For the remainder of this paper, $\alpha$ shall represent an arbitrary element of the symmetric group on $\{1, 2, \ldots, |V(G)|\}$ unless otherwise specified.

2. Preliminaries

For a graph $G$, the connectivity, edge-connectivity, and minimum degree shall be denoted $\kappa(G)$, $\lambda(G)$, and $\delta(G)$, respectively.

Remark 2.1 [9]. For any $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

We now define an invariant of $G$ which is useful in obtaining bounds for the connectivity of $\alpha(G)$. The expression $U(G)$ shall denote the minimum value of $|S| + |V(C)|$ taken over all disconnecting sets $S$ of $G$ and all components $C$ of $G - S$,
where $G - S$ is the subgraph induced by $V(G) - S$. (Note: If $G$ is complete, the term disconnecting set shall mean any subset of order $|V(G)| - 1$.)

**Remark 2.2.** Since the neighborhood of any vertex is a disconnecting set, $U(G) \leq \delta(G) + 1 \leq |V(G)|$.

**Remark 2.3.** If $\kappa(G) = \delta(G)$, $U(G) = \delta(G) + 1$.

### 3. Connectivity bounds

First, we obtain bounds for $\kappa(\alpha(G))$ which are independent of the permutation $\alpha$.

**Theorem 3.1.** $\min\{2\kappa(G), U(G)\} \leq \kappa(\alpha(G)) \leq U(G)$.

**Proof.** (Upper bound) Let $S$ be a disconnecting set of $G$ and let $C$ be a component of $G - S$ such that $U(G) = |S| + |V(C)|$. In $\alpha(G)$, let $S_x$ and $C_x$ be the corresponding items in $G_x$ and let $S_y$ be those vertices in $G_y$ which are adjacent to some vertex in $C_x$. Then $T = S_x \cup S_y$ is a disconnecting set of $\alpha(G)$ of order $U(G)$. Thus, $\kappa(\alpha(G)) \leq U(G)$.

(Lower bound) Let $S$ be a minimum disconnecting set of $\alpha(G)$ and let $S_x$ and $S_y$ be those vertices of $S$ which are also in $G_x$ and $G_y$, respectively. Note that $\kappa(\alpha(G)) = |S_x| + |S_y|$.

We proceed by examining the structure of the disconnected graph $\alpha(G) - S$. There are four cases which must be considered.

**Case 1:** Suppose exactly one of the graphs, $G_x - S_x$ or $G_y - S_y$, is the empty graph. Without loss of generality assume $G_x - S_x$ is the empty graph. Then $S_x = V(G_x)$ and $\kappa(\alpha(G)) \geq |S_x| = |V(G_x)| \geq U(G)$.

**Case 2:** Suppose $G_x - S_x$ and $G_y - S_y$ are both connected. Then there are no edges between vertices in $G_x - S_x$ and vertices in $G_y - S_y$ in the graph $\alpha(G)$. Thus if $x_i$ is a vertex in $G_x - S_x$, $y_{\alpha(i)}$ is a vertex in $S_y$ which implies $|S_y| \geq |V(G_x - S_x)|$ and $\kappa(\alpha(G)) \geq |S_x| + |V(G_x - S_x)| \geq U(G)$.

**Case 3:** Suppose $G_x - S_x$ and $G_y - S_y$ are both disconnected. Then $S_x$ and $S_y$ each correspond to disconnecting sets in $G$ and $\kappa(\alpha(G)) = |S_x| + |S_y| \geq 2\kappa(G)$.

**Case 4:** Suppose exactly one of the graphs, $G_x - S_x$ or $G_y - S_y$, is disconnected. Without loss of generality assume $G_x - S_x$ is disconnected. If, for every connected component $C_x$ of $G_x - S_x$ there exists a vertex $x_i$ in $C_x$ such that $y_{\alpha(i)} \in G_y - S_y$, then $\alpha(G) - S$ is (clearly) connected, a contradiction. Thus, there exists a component $C_x$ of $G_x - S_x$ such that, for each $x_i$ in $C_x$, $y_{\alpha(i)}$ is in $S_y$. So, $|S_y| \geq |V(C_x)|$ and $\kappa(\alpha(G)) \geq |S_x| + |V(C_x)| \geq U(G)$.

In each case, $\kappa(\alpha(G))$ has been shown to be at least as large as $2\kappa(G)$ or $U(G)$; thus, $\kappa(\alpha(G)) \geq \min\{2\kappa(G), U(G)\}$. □
4. Connectivity of some classes of generalized prisms over $G$

Since the upper bound is a parameter in the lower bound, the connectivity of $\alpha(G)$ can be determined exactly for some graphs $G$. As an immediate consequence of Theorem 3.1, we obtain the following result.

**Corollary 4.1.** If $U(G) \leq 2\kappa(G)$, $\kappa(\alpha(G)) = U(G)$.

**Theorem 4.2.** For a connected graph $G$ with $\kappa(G) = \delta(G)$, $\kappa(\alpha(G)) = \lambda(\alpha(G)) = \delta(\alpha(G)) = \delta(G) + 1$.

**Proof.** We need only show that $\kappa(\alpha(G)) = \delta(G) + 1$. If $G = K_1$, then $\alpha(G) = K_2$ and the proposition clearly holds. For $G \neq K_1$, $\kappa(G) \geq 1$ since $G$ is connected. By Remark 2.3, $U(G) = \delta(G) + 1$. Thus $U(G) = \kappa(G) + 1 \leq 2\kappa(G)$ and the result follows from Corollary 4.1.

If a graph $G$ satisfies the hypotheses of Theorem 4.2, the operation of taking a generalized prism of $G$ will result in a graph with larger connectivity than $G$ that also satisfies the hypotheses of the theorem. Thus repeated applications of the operation will achieve any desired connectivity, without introducing too many edges.

Furthermore, Theorem 4.2 gives the edge-connectivity as well as the connectivity of $\alpha(G)$ in this case and allows us to determine these parameters for generalized prisms over many well-known classes of graphs. Table 1 gives the connectivity and edge-connectivity for the generalized prisms over $G$ whenever $G$ is a tree, a cycle, a wheel, an $n$-cube, a complete graph, or a complete bipartite graph.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\kappa(\alpha(G)) = \lambda(\alpha(G))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nontrivial tree</td>
<td>2</td>
</tr>
<tr>
<td>$n$-cycle $C_n$</td>
<td>3</td>
</tr>
<tr>
<td>Wheel $W_n$</td>
<td>4</td>
</tr>
<tr>
<td>$n$-cube $Q_n$</td>
<td>$n + 1$</td>
</tr>
<tr>
<td>Complete graph $K_n$</td>
<td>$n$</td>
</tr>
<tr>
<td>Complete bipartite graph $K_{m,n}$</td>
<td>$\min{m,n} + 1$</td>
</tr>
</tbody>
</table>

Note that the converse of Theorem 4.2 is false. As a counterexample, let $G$ be the lexicographic product $C_n[K_m]$, where $n \geq 4$ and $m \geq 2$. Then $\kappa(G) = 2m$, $\delta(G) = 3m - 1$, and $U(G) = 3m$. By Corollary 4.1, $\kappa(\alpha(G)) = U(G) = \delta(G) + 1 = 3m$, but clearly $\kappa(G) \neq \delta(G)$. 

5. Connectivity of generalized prisms over $G$ for $\alpha$ in $\text{Aut}(G)$

Finally, we show that the lower bound in Theorem 3.1 is sharp since for any non-trivial graph $G$ there exists a permutation $\alpha$ such that $\kappa(\alpha(G)) = \min\{2\kappa(G), U(G)\}$.

**Theorem 5.1.** Let $G$ be a nontrivial graph. If $\alpha$ is in $\text{Aut}(G)$, the automorphism group of $G$, then $\kappa(\alpha(G)) = \min\{2\kappa(G), U(G)\}$.

**Proof.** If $\alpha$ is in $\text{Aut}(G)$, then $\alpha(G) \cong \beta(G) \cong G \times K_2$, where $\beta$ is the identity permutation and $G \times K_2$ is the Cartesian product of $G$ and $K_2$. So, without loss of generality assume $\alpha$ is the identity permutation. If $U(G) \leq 2\kappa(G)$, the result follows from Corollary 4.1.

Therefore, assume $2\kappa(G) < U(G)$. By Theorem 3.1, $\kappa(\alpha(G)) \geq 2\kappa(G)$. Let $S$ be a minimum disconnecting set of $G$ and let $S_x$ and $S_y$ be the corresponding sets in $G_x$ and $G_y$, respectively. Then $S_x \cup S_y$ is a disconnecting set of $\alpha(G)$ of order $2\kappa(G)$ and so $\kappa(\alpha(G)) = 2\kappa(G)$. \(\square\)

**Acknowledgement**

The authors are grateful to the referee for the counterexample in Section 4 which shows that the converse of Theorem 4.2 is false.

**References**