J. Math. Anal. Appl. 345 (2008) 566–572

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Estimates on trapped modes in deformed quantum layers

Hynek Kovařík *, Semjon Vugalter

Institute of Analysis, Dynamics and Modeling, Universität Stuttgart, PF 80 11 40, D-70569 Stuttgart, Germany

ARTICLE INFO

Article history:
Received 11 October 2007
Available online 18 April 2008
Submitted by J.A. Goldstein

Keywords:
Schrödinger operator
Eigenvalue estimates
Quantum layers

ABSTRACT

We use the logarithmic Lieb–Thirring inequality for two-dimensional Schrödinger operators and establish estimates on trapped modes in geometrically deformed quantum layers.

1. Introduction

Trapped modes in quantum layers and waveguides have been intensively studied in the last decades, see [1–4,6,8] and references therein. In these papers it has been shown that a suitable geometrical perturbation of a waveguide (or a layer) Ω, such as local enlargement or bending, induces the existence of discrete eigenvalues E_j of the corresponding Laplace operator

$$-\Delta_\Omega \quad \text{in } L^2(\Omega)$$

with Dirichlet boundary conditions. These eigenvalues represent the so-called *trapped modes*, which are the main objects of our interest. For mildly deformed waveguides and layers the corresponding weak coupling behaviour of such eigenvalues has been established in [1–4].

The next step in the analysis of the above mentioned eigenvalues consist of deriving suitable spectral estimates. In other words, one would like to know not only that these eigenvalues exist, but also in which way they are linked to the deformation of Ω, i.e. how the distance of E_j to the essential spectrum of $-\Delta_\Omega$ depends on the perturbation. Such a connection can be formulated in terms of certain Lieb–Thirring type inequalities, which estimate the sums

$$\sum_j |E - E_j|^\gamma, \quad E := \inf \sigma_{\mathrm{ess}}(-\Delta_\Omega), \ \gamma \geqslant 0. \tag{1}$$

In the case in which Ω is a quantum waveguide, these estimates were proved in [7] for potential type perturbations and in [5] for geometrical perturbations and perturbations of the boundary conditions. In the case of a quantum layer with a potential perturbation, the corresponding inequality was recently obtained in [9]. All these estimates have the right order of asymptotics for weak perturbations, i.e. the respective upper bounds on the sum (1) reflect the correct weak coupling behaviour established in [1–3,10].

The aim of the present paper is to extend these results also to the case of a geometrical deformation of a quantum layer. We note that in the case of quantum waveguides the key ingredient of the proof of an estimate, which has the correct asymptotical behaviour, was the Lieb–Thirring inequality for one-dimensional Schrödinger operators with the critical power $\gamma = \frac{1}{2}$ proved in [11]. Since a layer might be considered as a two-dimensional analog of a waveguide, the key ingredient of our proof will be the corresponding logarithmic critical Lieb–Thirring inequality for two-dimensional Schrödinger operators,

* Corresponding author.
E-mail address: kovarik@mathematik.uni-stuttgart.de (H. Kovařík).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.04.022

which was recently established in [9]. Therefore we first briefly recall the result of [9]; see Theorem 1. In Section 3 we then show how the problem can be reduced to the spectral analysis of certain two-dimensional Schrödinger operator with the effective potential induced by the geometrical deformation of the layer.

The following notation will be adopted in the text. Given a Hilbert space $\mathcal{H}$ and a self-adjoint operator T in $\mathcal{H}$ we denote by $N_{\mathcal{H}}(T)$ the number of negative eigenvalues of T, counting their geometrical multiplicities. When necessary we will use the symbols $\Delta_{x,y}$, $\nabla_{x,y}$ etc. in order to specify in which variables the respective operators act.

2. Preliminaries

2.1. Quantum layers

A quantum layer may be represented by an open domain $\Omega = \mathbb{R}^2 \times (0,d)$, more precisely $\Omega := \{x, y, z \in \mathbb{R}^3\colon\ 0 < z < d\}$, where d is the width of Ω. It will be convenient to work with the shifted Laplace operator

$$A = -\Delta_\Omega - \frac{\pi^2}{d^2} \quad \text{in } L^2(\Omega) \tag{2}$$

with the Dirichlet boundary conditions at $\partial\Omega$. The operator A is associated with closed quadratic form

$$Q[u] = \int_\Omega \left(|\nabla u|^2 - \frac{\pi^2}{d^2}|u|^2 \right) dx\, dy\, dz \tag{3}$$

with the form domain $H_0^1(\Omega)$. It can be easily verified that

$$\sigma_{\text{ess}}(A) = [0,\infty), \qquad \sigma_d(A) = \emptyset.$$

As noted in [1], a local enlargement of the width of the layer will not affect the essential spectrum of A, but will lead to the existence of negative discrete eigenvalues of A. To find a suitable spectral estimate on these eigenvalues we need the two-dimensional logarithmic Lieb–Thirring inequality, which we formulate in the next section.

2.2. Two-dimensional Lieb–Thirring inequality

Consider the Schrödinger operator

$$-\Delta - V \quad \text{in } L^2(\mathbb{R}^2), \tag{4}$$

where V is a potential function decaying at infinity such that $\sigma_{\text{ess}}(-\Delta - V) = [0,\infty)$. Denote by $-\lambda_j$ the negative eigenvalues of $-\Delta - V$ and introduce the family of functions $F_s : (0,\infty) \to (0,1]$ defined by

$$\forall s > 0 \quad F_s(t) := \begin{cases} |\ln ts^2|^{-1}, & 0 < t \leqslant e^{-1}s^{-2}, \\ 1, & t > e^{-1}s^{-2}. \end{cases} \tag{5}$$

An upper bound on the sum

$$\sum_j F_s(\lambda_j)$$

in terms of integrals of V has been recently found in [9]. Its formulation requires some additional notation. The space $L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))$ is defined as the space of functions f such that

$$\|f\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))} := \int_0^\infty \left(\int_0^{2\pi} |f(r,\theta)|^p \, d\theta \right)^{1/p} r\, dr < \infty, \tag{6}$$

where (r,θ) are the polar coordinates in $\mathbb{R}^2$. Moreover, given an $s > 0$ we introduce $B(s) := \{x \in \mathbb{R}^2\colon\ |x| < s\}$. The result of [9] then reads as follows.

Theorem 1. *Let $V \geqslant 0$ and $V \in L^1_{\text{loc}}(\mathbb{R}^2, |\ln|x||\,dx)$. Assume that $V \in L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))$ for some $p > 1$. Then the eigenvalues $-\lambda_j$ satisfy the inequality*

$$\sum_j F_s(\lambda_j) \leqslant c_1 \big\| V \ln(|x|/s) \big\|_{L^1(B(s))} + c_p \|V\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))} \tag{7}$$

for all $s \in \mathbb{R}_+$. The constants c_1 and c_p are independent of s and V.

In particular, if $V(x) = V(|x|)$, then there exists a constant c_4, such that

$$\sum_j F_s(\lambda_j) \leqslant c_1 \big\| V \ln(|x|/s) \big\|_{L^1(B(s))} + c_4 \|V\|_{L^1(\mathbb{R}^2)} \tag{8}$$

holds true for all $s \in \mathbb{R}_+$.

Note that for weak potentials V the estimate (7) reflects the exponential asymptotical behaviour of the lowest eigenvalue of $-\Delta - V$ established in [10]. Since the behaviour of weakly coupled eigenvalues in a layer is essentially two-dimensional, the corresponding asymptotics for weakly deformed layers is again of the exponential type, see [1]. Our goal thus is to find a similar upper bound for geometrical induced eigenvalues in quantum layers.

3. A layer with a geometrical perturbation

Here we apply Theorem 1 to obtain the estimates on the discrete eigenvalues of the Dirichlet Laplacian in a layer whose width is locally enlarged

$$\Omega_f := \{x, y, z \in \mathbb{R}^3:\ 0 < z < d + f(x, y)\},$$

where $f : \mathbb{R}^2 \to [0, \infty)$. We consider the shifted Laplace operator

$$A_f = -\Delta_{\Omega_f} - \frac{\pi^2}{d^2} \quad \text{in } L^2(\Omega_f) \tag{9}$$

with the Dirichlet boundary conditions at $\partial\Omega_f$ which is associated with the closed quadratic form

$$Q_f[u] = \int_{\Omega_f} \left(|\nabla u|^2 - \frac{\pi^2}{d^2}|u|^2\right) dx \tag{10}$$

with the form domain $H_0^1(\Omega_f)$. From the assumptions on f it follows that

$$\sigma_{\mathrm{ess}}(A_f) = [0, \infty).$$

Let us denote by $-\mu_j$ the non-decreasing sequence of negative eigenvalues of A_f taking into account their multiplicities. We shall estimate the total number of $-\mu_j$ by the number of negative eigenvalues of a certain two-dimensional Schrödinger operator $-\Delta - V_f$ with V_f depending on the deformation function f.

Theorem 2. *Assume that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is in $C^2(\mathbb{R}^2)$ and such that* $\operatorname{supp} f \subset B(R)$ *for some $R > 0$, and $\|f\|_\infty < d$. For any $t \geqslant 0$ we have*

$$N_{L^2(\Omega_f)}(A_f - t) \leqslant N_{L^2(\mathbb{R}^2)}(-\Delta + 3V_f - 3t), \tag{11}$$

where

$$V_f = \frac{\pi^2}{(d+f)^2} - \frac{\pi^2}{d^2} - b_1|\nabla f|^2 - b_2(R)|\Delta f|^2 - b_3(R)|\nabla f|^4,$$

with b_1, $b_2(R)$ and $b_3(R)$ satisfying (19).

Proof. We write a given trial function $\psi \in H_0^1(\Omega_f)$ as

$$\psi(x, y, z) = \varphi(x, y, z)g(x, y) + h(x, y, z), \tag{12}$$

where

$$\varphi(x, y, z) = \sqrt{\frac{2}{d + f(x, y)}}\sin\left(\frac{\pi z}{d + f(x, y)}\right), \quad g \in H^1(\mathbb{R}^2)$$

and

$$\int_0^{d+f(x,y)} \varphi(x, y, z)h(x, y, z)\, dz = 0 \quad \forall (x, y) \in \mathbb{R}^2. \tag{13}$$

Hence

$$\begin{aligned}\int_{\Omega_f} \left(|\nabla\psi|^2 - \frac{\pi^2}{d^2}|\psi|^2\right) dx\, dy\, dz = \int_{\Omega_f} \Big(&|\nabla\varphi|^2|g|^2 + |\nabla_{x,y} g|^2|\varphi|^2 + |\nabla h|^2 - \frac{\pi^2}{d^2}\left(|\varphi g|^2 + |h|^2\right) + 2gg'_x\varphi'_x\varphi + 2g\varphi'_y h'_y \\ &+ 2gg'_y\varphi'_y\varphi + 2g\varphi'_x h'_x + 2g\varphi'_z h'_z + 2\varphi g'_x h'_x + 2\varphi g'_y h'_y\Big)\, dx\, dy\, dz. \end{aligned} \tag{14}$$

Here and in the sequel we will use the shorthand $u'_x = \frac{\partial u}{\partial x}$ and analogously for other partial derivatives. We estimate all the mixed terms in (14), except for the last three, point-wise in the following way:

$$\begin{aligned}
&2gg'_x\varphi'_x\varphi \leqslant a_1^{-1}|\varphi g'_x|^2 + a_1|g\varphi'_x|^2,\\
&2gg'_y\varphi'_y\varphi \leqslant a_1^{-1}|\varphi g'_y|^2 + a_1|g\varphi'_y|^2,\\
&2g\varphi'_x h'_x \leqslant a_2^{-1}|h'_x|^2 + a_2|g\varphi'_x|^2,\\
&2g\varphi'_y h'_y \leqslant a_2^{-1}|h'_y|^2 + a_2|g\varphi'_y|^2,
\end{aligned} \tag{15}$$

where a_1 and a_2 are real positive numbers whose values will be specified later. Furthermore, from integration by parts and (13) it follows that

$$\int_{\Omega_f} g\varphi'_z h'_z \, dx\, dy\, dz = -\int_{\Omega_f} g\varphi''_z h \, dx\, dy\, dz = 0.$$

Integrating by parts again and using (13) we can rewrite the last two terms in (14) as

$$\int_{\Omega_f} \varphi h'_x g'_x \, dx\, dy\, dz = -\int_{\Omega_f} \varphi'_x h g'_x \, dx\, dy\, dz = \int_{\Omega_f} g(\varphi''_x h + \varphi'_x h'_x)\, dx\, dy\, dz,$$

$$\int_{\Omega_f} \varphi h'_y g'_y \, dx\, dy\, dz = -\int_{\Omega_f} \varphi'_y h g'_y \, dx\, dy\, dz = \int_{\Omega_f} g(\varphi''_y h + \varphi'_y h'_y)\, dx\, dy\, dz.$$

The terms $2g\varphi'_x h'_x$ and $2g\varphi'_y h'_y$ will be estimated in the same way as in (15). For the rest we use the following point-wise inequalities

$$\begin{aligned}
&2g\varphi''_x h \leqslant a_3 g^2|\varphi''_x|^2 + a_3^{-1}h^2\chi_f,\\
&2g\varphi''_y h \leqslant a_3 g^2|\varphi''_y|^2 + a_3^{-1}h^2\chi_f,
\end{aligned}$$

where χ_f denotes the characteristic function of the support of f. Now we put $a_1 = a_2 = 3$ and arrive at

$$\begin{aligned}
\int_{\Omega_f} \left(|\nabla\psi|^2 - \frac{\pi^2}{d^2}|\psi|^2\right) dx\, dy\, dz \geqslant{}& \int_{\mathbb{R}^2} \left(\frac{1}{3}|\nabla_{x,y} g|^2 + \tilde{V}_f(x,y)|g|^2\right) dx\, dy\\
&+ \int_{\Omega_f} \left(\frac{1}{3}|\nabla_{x,y} h|^2 + |h'_z|^2 - \frac{\pi^2}{d^2}|h|^2 - a_3^{-1}|h|^2\chi_f\right) dx\, dy\, dz
\end{aligned} \tag{16}$$

with

$$\tilde{V}_f = \frac{\pi^2}{(d+f)^2} - \frac{\pi^2}{d^2} - \int_0^{d+f} \left(8\left(|\varphi'_x|^2 + |\varphi'_y|^2\right) + a_3\left(|\varphi''_x|^2 + |\varphi''_y|^2\right)\right) dz.$$

Since h satisfies Dirichlet boundary conditions at $\partial\Omega_f$ and $f < d$, we deduce from (13) that

$$\begin{aligned}
\int_{\Omega_f} \left(\frac{1}{3}|\nabla_{x,y} h|^2 + |h'_z|^2 - \frac{\pi^2}{d^2}|h|^2 - a_3^{-1}|h|^2\chi_f\right) dx\, dy\, dz \geqslant{}& \int_{\Omega_f} \left(\frac{1}{3}|\nabla_{x,y} h|^2 + \left(\frac{4\pi^2}{(d+f)^2} - \frac{\pi^2}{d^2}\right)|h|^2 - a_3^{-1}|h|^2\chi_f\right) dx\, dy\, dz\\
\geqslant{}& \int_0^d \int_{\mathbb{R}^2} \left(\frac{1}{3}|\nabla_{x,y} h|^2 + \frac{3\pi^2}{d^2}|h|^2 - \left(a_3^{-1} + \frac{3\pi^2}{d^2}\right)|h|^2\chi_f\right) dx\, dy\, dz\\
&+ \int_d^{2d} \int_{\operatorname{supp} f} \left(\frac{1}{3}|\nabla_{x,y} h|^2 - a_3^{-1}|h|^2\right) dx\, dy\, dz.
\end{aligned} \tag{17}$$

From the fact that the support of f is compact it follows that the last term in (17) is non-negative for all $a_3 \geqslant \lambda^{-1}(R)$, where $\lambda(R)$ is the lowest eigenvalue of $-\Delta_{x,y}$ on the disc $B(R)$ with Dirichlet boundary conditions. Moreover, the expression on the second line of (17) can be bounded from below as follows

$$\int_0^d \int_{\mathbb{R}^2} \left(\frac{1}{3}|\nabla_{x,y} h|^2 + \frac{3\pi^2}{d^2}|h|^2 - \left(a_3^{-1} + \frac{3\pi^2}{d^2}\right)|h|^2 \chi_f\right) dx\,dy\,dz$$

$$\geqslant \int_0^d \left(\int_{\mathbb{R}^2} \left(\frac{1}{3}|\nabla_{r,\theta} h|^2 + \frac{3\pi^2}{d^2}|h|^2 \chi_{[R,\infty)} - a_3^{-1}|h|^2 \chi_{[0,R]}\right) r\,dr\,d\theta\right) dz, \tag{18}$$

where we have used the polar coordinates (r,θ) in $\mathbb{R}^2$. In view of Lemma 1, see Appendix A, (18) is positive for $a_3 \geqslant \max\{8R^2, \frac{d^2}{3\pi^2}\}$. Therefore we choose

$$a_3(R) = \max\left\{\frac{d^2}{3\pi^2}, 8R^2, \lambda^{-1}(R)\right\}.$$

Now it remains to estimate the first term on the right-hand side of (16). By a direct calculation we arrive at

$$\int_0^{d+f} \left(5\left(|\varphi_x'|^2 + |\varphi_y'|^2\right) + a_3(R)\left(|\varphi_x''|^2 + |\varphi_y''|^2\right)\right) dz \leqslant b_1 |\nabla f|^2 + b_2(R)|\Delta f|^2 + b_3(R)|\nabla f|^4,$$

where b_1, $b_2(R)$, $b_3(R)$ are positive numbers which satisfy

$$b_1 \leqslant \frac{4\pi^2}{5d^2}, \qquad b_2(R) \leqslant \frac{a_3(R)\pi^2}{d^2}, \qquad b_3(R) \leqslant 4a_3(R)\left(\frac{\pi^2}{d^4} + \frac{\pi^4}{5d^2}\right). \tag{19}$$

Finally, combining (16) and (13) we obtain that

$$\int_{\Omega_f} \left(|\nabla\psi|^2 - \frac{\pi^2}{d^2}|\psi|^2 - t|\psi|^2\right) dx\,dy\,dz \geqslant \frac{1}{3}\int_{\mathbb{R}^2} \left(|\nabla_{x,y} g|^2 + 3V_f(x,y)|g|^2 - 3t|g|^2\right) dx\,dy \tag{20}$$

holds true for any $t \leqslant 0$.

Let us show that (20) implies (11). We introduce the subspace $\mathcal{M}_t \subset L^2(\mathbb{R}^2)$ spanned by the eigenvectors associated with the negative eigenvalues of the operator

$$\frac{1}{3}(-\Delta + 3V_f - 3\tau) \quad \text{in } L^2(\mathbb{R}^2),$$

and define $M_t \subset L^2(\Omega_f)$ by

$$M_t = \{g\varphi\colon\ g \in \mathcal{M}_t\}.$$

Obviously

$$\dim \mathcal{M}_t \leqslant N_{L^2(\mathbb{R}^2)}\left(\frac{1}{3}(-\Delta + 3V_f - 3\tau)\right) = N_{L^2(\mathbb{R}^2)}(-\Delta + 3V_f - 3t).$$

Assume that $\psi \perp M_t$ and write $\psi = \tilde g\varphi + h$. Then $\tilde g\varphi \perp M_t$ and since $\int_0^{d+f} |\varphi(x,y,z)|^2\,dz = 1$ for all $(x,y) \in \mathbb{R}^2$, this means that $\tilde g \perp \mathcal{M}_t$. In view of (20) this implies

$$\int_{\Omega_f} \left(|\nabla\psi|^2 - \frac{\pi^2}{d^2}|\psi|^2 - t|\psi|^2\right) dx\,dy\,dz \geqslant 0.$$

By the variational principle we conclude that

$$N_{L^2(\Omega_f)}(A_f - t) \leqslant \dim M_t = \dim \mathcal{M}_t \leqslant N_{L^2(\mathbb{R}^2)}(-\Delta + 3V_f - 3t). \qquad \square$$

Remark 1. From the assumption $f < d$ it follows that all negative eigenvalues of A_f come from the first channel only. However, we would like to mention that this assumption is purely technical and could be replaced by $f < nd$, $n \in \mathbb{N}$. In that case we would have to use another decomposition of a test function ψ, analogous to (12), taking into account also the functions associated with higher transversal modes in z. For the sake of simplicity we therefore suppose $f < d$.

Corollary 1. *For any $p > 1$ there exist positive constants C_1 and C_p such that*

$$\sum_j F_s(\mu_j) \leqslant C_1 \big\| V_f \ln\big(\sqrt{x^2+y^2}/s\big)\big\|_{L^1(B(s))} + C_p \|V_f\|_{L^1(\mathbb{R}_+, L^p(\mathbb{S}^1))} \tag{21}$$

holds for all $s > 0$.

Proof. Since F_s' is non-negative we have

$$\begin{aligned}\sum_j F_s(\mu_j) &= \int_0^\infty F_s'(t) N_{L^2(\Omega_f)}(A_f - t)\,dt \\ &\leqslant \int_0^\infty F_s'(t) N_{L^2(\mathbb{R}^2)}(-\Delta + 3V_f - 3t)\,dt \\ &\leqslant 3\int_0^\infty F_s'(t) N_{L^2(\mathbb{R}^2)}(-\Delta + 3V_f - t)\,dt = 3\sum_j F_s(\lambda_j)\end{aligned}$$

and the statement follows from Theorem 1. □

The disadvantage of estimate (21) is the presence of the terms in V_f which contain the derivatives of f. Firstly, small oscillations of f will lead to the unnecessary growth of the right-hand side in (21). Secondly, the deformation function f in general need not be C^2-smooth. This can remedied using the monotonicity property of eigenvalues of Laplace operators in domains with Dirichlet boundary conditions. Namely, for any $\tilde f \geqslant f$ we have

$$N_{L^2(\Omega_f)}(A_f - t) \leqslant N_{L^2(\Omega_{\tilde f})}(A_{\tilde f} - t) \quad \forall t \geqslant 0.$$

As an immediate consequence of Theorem 1 and Corollary 1 we thus get

Theorem 3. *Let $0 \leqslant f < d$ be a continuous function with support in $B(R)$. Then there exist constants C_3 and C_4 such that*

$$\sum_j F_s(\mu_j) \leqslant \inf_{\tilde f \geqslant f}\left(C_3 \left\| V_{\tilde f} \ln\frac{r}{s}\right\|_{L^1(B(s))} + C_4 \|V_{\tilde f}\|_{L^1(\mathbb{R}^2)}\right), \tag{22}$$

where the infimum is taken over all radially symmetric functions $\tilde f \in C_0^2(B(R))$.

Remark 2. Let us consider the behaviour of the estimate (21) for weakly deformed layers. This means replacing f by αf and letting α go to zero. Theorem 2 and the result of [9] yield the following upper bound on the number of negative eigenvalues of $A_{\alpha f}$:

$$N_{L^2(\Omega_f)}(A_{\alpha f}) \leqslant 1 + \text{const}\left(\left\| V_{\alpha \tilde f} \ln\frac{r}{s}\right\|_{L^1(\mathbb{R}^2)} + \|V_{\alpha \tilde f}\|_{L^1(\mathbb{R}^2)}\right).$$

From the explicit form of $V_{\alpha\tilde f}$ it follows that $A_{\alpha f}$ has only one negative eigenvalue, $-\mu_1(\alpha)$, for α small enough. Moreover, inequality (21) implies

$$\big|\mu_1(\alpha)\big| \leqslant \exp\left(-\frac{C(f,d)}{w(\alpha)}\right), \tag{23}$$

where $C(f,d)$ is a positive factor independent of α and

$$w(\alpha) = \alpha + \mathcal{O}\big(\alpha^2\big), \quad \alpha \to 0. \tag{24}$$

This agrees, in order of α, with the asymptotics found in [1].

Acknowledgment

The support from the DFG grant WE 1964/2 is gratefully acknowledged. The authors are grateful to the referee for important corrections of the original text.

Appendix A

Lemma 1. *Let $u \in H^1(\mathbb{R}_+, r\,dr)$. Then for any $R > 0$ the inequality*

$$\int_0^R |u|^2 r\,dr \leqslant \int_R^{2R} |u|^2 r\,dr + \frac{8}{3} R^2 \int_0^{2R} |u'|^2 r\,dr \tag{A.1}$$

holds true.

Proof. Let us define the function $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(r) = \begin{cases} 1, & 0 < r \leqslant R, \\ 1 - \frac{r-R}{R}, & R < r < 2R, \\ 0, & 2R \leqslant r. \end{cases}$$

For any $r \in (0, R)$ we then have

$$u(r) = h(r)u(r) = -\int_r^{2R} (hu)'(t)\,dt = \frac{1}{R}\int_R^{2R} u\,dt - \int_r^{2R} hu'\,dt. \tag{A.2}$$

The Cauchy–Schwarz inequality thus implies

$$\left|u(r)\right|^2 \leqslant \frac{2}{R}\int_R^{2R} |u|^2\,dt + 2\|h\|^2 \int_r^{2R} |u'|^2\,dt.$$

Multiplying by r and integrating over $(0, R)$ we get

$$\int_0^R |u|^2 r\,dr \leqslant \int_R^{2R} |u|^2 r\,dr + 2R\|h\|^2 \int_0^{2R} |u'|^2 r\,dr.$$

To conclude the proof we note that $\|h\|^2 = \frac{4}{3}R$. □

References

[1] D. Borisov, P. Exner, R. Gadyl'shin, D. Krejčiřík, Bound states in weakly deformed strips and layers, Ann. Henri Poincaré 2 (2001) 553–572.
[2] W. Bulla, F. Gesztesy, W. Renger, B. Simon, Weakly coupled bound states in quantum waveguides, Proc. Amer. Math. Soc. 125 (1997) 1487–1495.
[3] P. Duclos, P. Exner, Curvature-induced bound states in quantum waveguides in two and three dimensions, Rev. Math. Phys. 7 (1995) 73–102.
[4] P. Duclos, P. Exner, D. Krejčiřík, Bound states in curved quantum layers, Comm. Math. Phys. 223 (2001) 13–28.
[5] P. Exner, H. Linde, T. Weidl, Lieb–Thirring inequalities for geometrically induced bound states, Lett. Math. Phys. 70 (2004) 83–95.
[6] P. Exner, P. Šeba, Bound states in curved quantum waveguides, J. Math. Phys. 30 (1989) 2574–2580.
[7] P. Exner, T. Weidl, Lieb–Thirring inequalities on trapped modes in quantum wires, in: XIIIth International Congress on Mathematical Physics, London, 2000, Int. Press, Boston, MA, 2001, pp. 437–443.
[8] J. Goldstone, R.L. Jaffe, Bound states in twisting tubes, Phys. Rev. B 45 (1992) 14100–14107.
[9] H. Kovařík, S. Vugalter, T. Weidl, Spectral estimates for two-dimensional Schrödinger operators with application to quantum layers, Comm. Math. Phys. 275 (2007) 827–838.
[10] B. Simon, The bound state of weakly coupled Schrödinger operators in one and two dimensions, Ann. Phys. 97 (1976) 279–288.
[11] T. Weidl, On the Lieb–Thirring constants $L_{\gamma,1}$ for $\gamma \geqslant 1/2$, Comm. Math. Phys. 178 (1996) 135–146.