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On the quasi-derivation relation for multiple zeta values

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ABSTRACT

Recently, Masanobu Kaneko introduced a conjecture on an extension of the derivation relation for multiple zeta values. The goal of the present paper is to present a proof of the conjecture by reducing it to a class of relations for multiple zeta values studied by Kawashima. In addition, some algebraic aspects of the quasi-derivation operator $\partial_n^{(c)}$ on $\mathbb{Q}\langle x, y\rangle$, which was defined by modeling a Hopf algebra developed by Connes and Moscovici, will be presented.

1. Introduction/main theorem

Let $n \geqslant 1$ be an integer. For each index set $(k_1, k_2, \ldots, k_n)$ of positive integers with $k_1 > 1$, the multiple zeta value (MZV for short) is a real number defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.$$

We call the number $k_1 + \cdots + k_n$ its weight and n its depth.

Through this paper, we employ the algebraic setup introduced by Hoffman [4] to study the quasi-derivation relation for MZV's. Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates x and y, and let $\mathfrak{H}^1$ and $\mathfrak{H}^0$ denote the subalgebras $\mathbb{Q} + \mathfrak{H}y$ and $\mathbb{Q} + x\mathfrak{H}y$, respectively. The $\mathbb{Q}$-linear map $Z : \mathfrak{H}^0 \to \mathbb{R}$ is defined by $Z(1) = 1$ and

$$Z\left(x^{k_1-1} y x^{k_2-1} y \cdots x^{k_n-1} y\right) = \zeta(k_1, k_2, \ldots, k_n).$$

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The degree (resp. degree with respect to y) of a word is the weight (resp. the depth) of the corresponding MZV.

In Zagier's paper [11], it is conjectured that the dimension of the $\mathbb{Q}$-vector space generated by MZV's of weight k is d_k, the number determined by the recursion $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_k = d_{k-2} + d_{k-3}$ for $k \geqslant 3$. Goncharov [3] and Terasoma [10] proved that the number d_k gives the upper bound of the dimension of the $\mathbb{Q}$-vector space generated by MZV's of weight k. The number d_k is far smaller than the total number 2^{k-2} of indices of weight k, hence there should be several relations among MZV's. In the present setup, finding a linear relation among MZV's corresponds to find an element in $\ker Z$.

Before stating the main theorem, the derivation relation for MZV's, which appeared in Ihara, Kaneko, and Zagier [5], is introduced. A derivation ∂ on $\mathfrak{H}$ is a $\mathbb{Q}$-linear endomorphism of $\mathfrak{H}$ satisfying the Leibniz rule $\partial(ww') = \partial(w)w' + w\partial(w')$. Such a derivation is uniquely determined by its images of generators x and y. Let $z = x + y$. For each $n \geqslant 1$, the derivation $\partial_n : \mathfrak{H} \to \mathfrak{H}$ is defined by $\partial_n(x) = xz^{n-1}y$ and $\partial_n(y) = -xz^{n-1}y$. It follows immediately that $\partial_n(\mathfrak{H}) \subset \mathfrak{H}^0$.

Fact 1 *(Derivation relation). (See [5].) For any $n \geqslant 1$, we have $\partial_n(\mathfrak{H}^0) \subset \ker Z$.*

The following extension of the operator ∂_n was first defined by Kaneko [6]. He modified the formula

$$\partial_n = \frac{1}{(n-1)!} \operatorname{ad}(\theta)^{n-1}(\partial_1)$$

in [5], where θ stands for the derivation on $\mathfrak{H}$ defined by $\theta(x) = \frac{1}{2}(xz + zx)$ and $\theta(y) = \frac{1}{2}(yz + zy)$, and $\operatorname{ad}(\theta)(\partial) = [\theta, \partial] := \theta\partial - \partial\theta$.

Definition 2. Let $c \in \mathbb{Q}$ and H the derivation on $\mathfrak{H}$ defined by $H(w) = \deg(w)w$ for any words $w \in \mathfrak{H}$. For each integer $n \geqslant 1$, the $\mathbb{Q}$-linear map $\partial_n^{(c)} : \mathfrak{H} \to \mathfrak{H}$, which we call the quasi-derivation (with respect to n and $\theta^{(c)}$ for the given $c \in \mathbb{Q}$) in the present paper, is defined by

$$\partial_n^{(c)} = \frac{1}{(n-1)!} \operatorname{ad}\bigl(\theta^{(c)}\bigr)^{n-1}(\partial_1),$$

where $\theta^{(c)} : \mathfrak{H} \to \mathfrak{H}$ is the $\mathbb{Q}$-linear map defined by $\theta^{(c)}(x) = \theta(x)$, $\theta^{(c)}(y) = \theta(y)$ and the rule

$$\theta^{(c)}(ww') = \theta^{(c)}(w)w' + w\theta^{(c)}(w') + c\partial_1(w)H(w') \tag{1}$$

for any $w, w' \in \mathfrak{H}$.

If $c = 0$, the quasi-derivation $\partial_n^{(c)}$ is reduced to the ordinary derivation ∂_n. If $c \neq 0$ and $n \geqslant 2$, the operator $\partial_n^{(c)}$ is no longer a derivation. Although the inclusion $\partial_n^{(c)}(\mathfrak{H}) \subset \mathfrak{H}^0$ does not hold in general, we have $\partial_n^{(c)}(\mathfrak{H}^0) \subset \mathfrak{H}^0$ as will be shown in Proposition 11. Then, the main result of the present paper, which we call the class of the quasi-derivation relation, is stated.

Theorem 3. *For any $n \geqslant 1$ and any $c \in \mathbb{Q}$, we have $\partial_n^{(c)}(\mathfrak{H}^0) \subset \ker Z$.*

When c is viewed as a variable, $\partial_n^{(c)}(w)$ ($w \in \mathfrak{H}^0$) is a polynomial in c of degree $n-1$. Then, Theorem 3 implies that each coefficient (which is in $\mathfrak{H}^0$) with respect to c of $\partial_n^{(c)}(w)$, $n \geqslant 1$, $w \in \mathfrak{H}^0$, gives a relation among MZV's. We find that the derivation relation is the constant term of the quasi-derivation relation as a polynomial in c. Hence, we have $V_\partial \subset V_{\partial^{(\bullet)}}$, where $V_\partial = \langle \partial_n(w) \mid n \geqslant 1,$

Table 1

weight k	3	4	5	6	7	8	9	10	11	12	13	14
2^{k-2}	2	4	8	16	32	64	128	256	512	1024	2048	4096
d_k	1	1	2	2	3	4	5	7	9	12	16	21
$\dim_{\mathbb{Q}}(V_\partial)_k$	1	2	5	10	22	44	90	181	363	727	1456	2912
$\dim_{\mathbb{Q}}(V_{\partial^{(\bullet)}})_k$	1	2	5	10	23	46	98	200	410	830	1679	$\cdots$
$\dim_{\mathbb{Q}}(V_O)_k$	1	2	5	10	23	46	98	199	411	830	1691	$\cdots$
$\dim_{\mathbb{Q}}(V_{K^0})_k$	1	2	5	10	23	46	98	200	413	838	1713	$\cdots$
$\dim_{\mathbb{Q}}(V_{K_*})_k$	1	2	5	12	25	55	113	235	480	977	$\cdots$	$\cdots$
$\dim_{\mathbb{Q}}(V_{K_{\text{ш}}})_k$	1	3	6	14	29	60	123	249	503	1012	$\cdots$	$\cdots$

$w \in \mathfrak{H}^0\rangle_{\mathbb{Q}}$ and $V_{\partial^{(\bullet)}} = \langle \partial_n^{(c)}(w) \mid n \geqslant 1,\ w \in \mathfrak{H}^0,\ c \in \mathbb{Q}\rangle_{\mathbb{Q}}$, and the derivation relation is again shown to be a class of relations among MZV's.

Let V_O and V_{K^0} denote the $\mathbb{Q}$-vector spaces generated by Ohno's relation [8] and by the linear part of Kawashima's relation [7], respectively. We also denote the $\mathbb{Q}$-vector space generated by the union of the linear part and the degree-1 part of Kawashima's relation with the products of MZV's expanded linearly according to the harmonic product rule (resp. the iterated integral shuffle product rule) by V_{K_*} (resp. $V_{K_{\text{ш}}}$). For the statement of the degree-1 (or the algebraic) part of Kawashima's relation, see [7, Corollary 5.4]. The harmonic product rule is given in [4] or in the next section of this paper for example. The iterated integral shuffle product rule is introduced in [9] for example. Table 1 gives the dimension of weight-k part of each $\mathbb{Q}$-vector spaces in the left-hand column, together with the numbers d_k, the conjectural dimension of the space generated by MZV's of weight k, and 2^{k-2}, the total number of indices of weight k. Computations were performed using Risa/Asir, an open source general computer algebra system.

In Table 1, $(W)_k$ denotes the weight k part of the vector space W. We see that the sequence of the column '$\dim_{\mathbb{Q}}(V_{K_{\text{ш}}})_k$' equals to $2^{k-2} - d_k$, which suggests that the whole set (or, more precisely, the union of the linear part and the degree-1 part) of Kawashima's relation is enough to reduce the dimensions of the space generated by MZV's to the conjectural ones.

Further experiments using Risa/Asir enable us to find some facts or expectations. For example, the sequence of the column 'V_{K^0}' appears again as the sequence of following three spaces, $V_{\partial^{(\bullet)}} + V_O$, $V_{\partial^{(\bullet)}} + V_{K^0}$, and $V_O + V_{K^0}$, up to weight 13. Hence, Table 1 implies that three spaces $V_{\partial^{(\bullet)}}, V_O$ and V_{K^0} coincide up to weight 9. In addition, from weight 10 to 13 (and probably for higher weights), $V_{\partial^{(\bullet)}}$ and V_O are different spaces but both are contained in V_{K^0} properly.

The $\mathbb{Q}$-vector space V_O is known to be equivalent to $V_\partial + V_\tau$, where V_τ denotes the $\mathbb{Q}$-vector space generated by the well-known duality formula, which is stated in Section 2. A proof of this equivalence was given in [1] and is reviewed in Appendix B. Kawashima showed in [7] that $V_\tau \subset V_{K^0}$. We also have $V_{\partial^{(\bullet)}} \subset V_{K^0}$, which is shown in the present paper. Although Table 1 and further experiments stated above suggest that two spaces $V_{\partial^{(\bullet)}} + V_\tau$ and V_{K^0} are equivalent, we can only show one side inclusion, $V_{\partial^{(\bullet)}} + V_\tau \subset V_{K^0}$, herein.

2. Proof of main result

The main theorem (Theorem 3) is proven by reducing the theorem to the following Kawashima's relation.

Let $z_k = x^{k-1}y$ for $k \geqslant 1$. The harmonic product $* : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ is a $\mathbb{Q}$-bilinear map defined by the following rules:

(i) For any $w \in \mathfrak{H}^1$, $1 * w = w * 1 = w$.
(ii) For any $w, w' \in \mathfrak{H}^1$ and any $k, l \geqslant 1$,

$$z_k w * z_l w' = z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w').$$

This is, as shown in [4], an associative and commutative product on $\mathfrak{H}^1$.

Denote by ε the automorphism of $\mathfrak{H}$ defined by $\varepsilon(x) = x + y$ and $\varepsilon(y) = -y$. For any $w \in \mathfrak{H}$, define the operator L_w on $\mathfrak{H}$ by $L_w(w') = ww'$ ($w' \in \mathfrak{H}$). Next, the linear part of Kawashima's relation [7, Corollary 4.9] is stated using the notation of the present paper.

Fact 4 *(Kawashima's relation).* $L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y) \subset \ker Z$.

Let τ be the anti-automorphism of $\mathfrak{H}$ defined by $\tau(x) = y$ and $\tau(y) = x$. The duality formula states that $(1-\tau)(\mathfrak{H}^0) \subset \ker Z$. To prove Theorem 3, the inclusion

$$\partial_n^{(c)}\big(\mathfrak{H}^0\big) \subset \tau L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y) \tag{2}$$

is shown. In [7], Kawashima proved that Kawashima's relation contains the duality formula:

$$(1-\tau)\big(\mathfrak{H}^0\big) \subset L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y),$$

and hence,

$$\text{RHS of (2)} = \big(1 - (1-\tau)\big)L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y) \subset L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y).$$

Therefore, based on Kawashima's relation, the inclusion (2) gives Theorem 3.

To prove (2), the following key identity, which involves several operators, is established. For any $w \in \mathfrak{H}$, let R_w be the operator defined by $R_w(w') = w'w$ ($w' \in \mathfrak{H}$). The operator $\mathcal{H}_w$ on $\mathfrak{H}^1$ for any $w \in \mathfrak{H}^1$ given by $\mathcal{H}_w(w') = w * w'$ ($w' \in \mathfrak{H}^1$) is also introduced. Put $\chi_x = \tau L_x\varepsilon$.

Key Proposition 5. *For any $n \geqslant 1$ and any $c \in \mathbb{Q}$, there exists an element $w = w(n,c) \in \mathfrak{H}y$ such that $\partial_n^{(c)}\chi_x = \chi_x\mathcal{H}_w$ on $\mathfrak{H}^1$. In other words, the following commutative diagram holds:*

$$\begin{array}{ccc} \mathfrak{H}^1 & \xrightarrow{\ \mathcal{H}_w\ } & \mathfrak{H}^1 \\ {\scriptstyle \chi_x}\big\downarrow & & \big\downarrow{\scriptstyle \chi_x} \\ \mathfrak{H}^0 & \xrightarrow[\ \partial_n^{(c)}\ ]{} & \mathfrak{H}^0 \end{array}$$

The proof of this proposition is the technical core of the present paper and will be carried out in the next two sections. In addition, various beneficial properties of operators, including the commutativity of $\partial_n^{(c)}$'s, are proven.

Assuming Key Proposition 5, the proof of Theorem 3 proceeds as follows. First, note that it is sufficient to prove the inclusion $\partial_n^{(c)}(x\mathfrak{H}y) \subset \chi_x(\mathfrak{H}y * \mathfrak{H}y)$ instead of (2) because $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y$ and $\partial_n^{(c)}(\mathbb{Q}) = \{0\}$. Take any $w_0 \in x\mathfrak{H}y$. Since ε is an automorphism of $\mathfrak{H}$ and $\varepsilon(y) = -y$, we have $\chi_x(\mathfrak{H}y) = x\mathfrak{H}y$, and hence, there is an element $w_1 \in \mathfrak{H}y$ such that $w_0 = \chi_x(w_1)$. By Key Proposition 5, there exists $w_2 \in \mathfrak{H}y$ satisfying $\partial_n^{(c)}\chi_x = \chi_x\mathcal{H}_{w_2}$. Therefore, we have

$$\partial_n^{(c)}(w_0) = \partial_n^{(c)}\chi_x(w_1) = \chi_x\mathcal{H}_{w_2}(w_1) = \chi_x(w_2 * w_1).$$

This proves (2) and Theorem 3 is established.

3. Commutativity of $\partial_n^{(c)}$

To prove Key Proposition 5, several properties of various operators are needed, and that the different $\partial_n^{(c)}$ commute with each other must first be proven.

Proposition 6. *Let $c \in \mathbb{Q}$. For any $n, m \geqslant 1$, we have $[\partial_n^{(c)}, \partial_m^{(c)}] = 0$.*

As mentioned earlier, the operator $\partial_n^{(c)}$ is no longer a derivation if $c \neq 0$ and $n \geqslant 2$ and does not satisfy the Leibniz rule, instead, satisfying the rules such as

$$\partial_2^{(c)}(ww') = \partial_2^{(c)}(w)w' + w\partial_2^{(c)}(w') + c\partial_1^{(c)}(w)\partial_1^{(c)}(w'),$$
$$\partial_3^{(c)}(ww') = \partial_3^{(c)}(w)w' + w\partial_3^{(c)}(w') + \frac{1}{2}c\partial_2^{(c)}(w)\partial_1^{(c)}(w') + \frac{3}{2}c\partial_1^{(c)}(w)\partial_2^{(c)}(w') + \frac{1}{2}c^2\partial_1^{(c)2}(w)\partial_1^{(c)}(w'),$$

for any $w, w' \in \mathfrak{H}$, which can be checked using the definition of the operator $\partial_n^{(c)}$ and Proposition 6. The subalgebra $\mathsf{A}^{(c)}$ of linear endomorphisms of $\mathfrak{H}$ generated by ∂_1, $\theta^{(c)}$ and H (and hence, $\partial_n^{(c)} \in \mathsf{A}^{(c)}$) has the structure of Connes–Moscovici's Hopf algebra introduced in [2], which is helpful to calculate such rule of $\partial_n^{(c)}$.

To prove Proposition 6, the following several operators are needed. Recall the left and right multiplication operators are both additive as well as multiplicative ($L_{ww'} = L_w L_{w'}$) and anti-multiplicative ($R_{ww'} = R_{w'} R_w$), respectively.

Definition 7. Let $c \in \mathbb{Q}$. The operators $\{\phi_n^{(c)}\}_{n=0}^{\infty}$ are defined by $\phi_0^{(c)} = \mathrm{id}_{\mathfrak{H}}$ and the recursive rule:

$$\phi_n^{(c)} = \frac{1}{n}\left(\left[\theta^{(c)}, \phi_{n-1}^{(c)}\right] + \frac{1}{2}\left(R_z\phi_{n-1}^{(c)} + \phi_{n-1}^{(c)}R_z\right) + c\partial_1\phi_{n-1}^{(c)}\right) \tag{3}$$

for $n \geqslant 1$.

Lemma 8. *For $n \geqslant 1$, let $\psi_n^{(c)} = R_y\phi_{n-1}^{(c)}R_x$. The operators $\{\psi_n^{(c)}\}_{n=1}^{\infty}$ satisfy $\psi_1^{(c)} = R_{xy}$ and the recursive rule*

$$\psi_n^{(c)} = \frac{1}{n-1}\left(\left[\theta^{(c)}, \psi_{n-1}^{(c)}\right] - \frac{1}{2}\left(R_z\psi_{n-1}^{(c)} + \psi_{n-1}^{(c)}R_z\right) - c\psi_{n-1}^{(c)}\partial_1\right)$$

for $n \geqslant 2$.

Proof. The lemma is proven by induction on n. The lemma holds for $n = 1$ because $R_{xy} = R_y R_x$. Assume that the lemma is proved for n. Because of the identities $[\theta^{(c)}, R_u] = R_{\theta(u)} + cR_u\partial_1 = \frac{1}{2}(R_zR_u + R_uR_z) + cR_u\partial_1$ for $u = x$ or y, the recursive rule of $\phi_n^{(c)}$ and the induction hypothesis, we have

$$\begin{aligned}\left[\theta^{(c)}, \psi_{n-1}^{(c)}\right] &= \left[\theta^{(c)}, R_y\phi_{n-2}^{(c)}R_x\right] \\ &= R_y\phi_{n-2}^{(c)}\left[\theta^{(c)}, R_x\right] + R_y\left[\theta^{(c)}, \phi_{n-2}^{(c)}\right]R_x + \left[\theta^{(c)}, R_y\right]\phi_{n-2}^{(c)}R_x \\ &= (n-1)\psi_n^{(c)} + \frac{1}{2}\left(R_z\psi_{n-1}^{(c)} + \psi_{n-1}^{(c)}R_z\right) + c\psi_{n-1}^{(c)}\partial_1.\end{aligned}$$

Therefore, the lemma is proven. □

In order to prove Proposition 6, the following general property of a $\mathbb{Q}$-linear map on $\mathfrak{H}$ is needed.

Lemma 9. *A $\mathbb{Q}$-linear map $f : \mathfrak{H} \to \mathfrak{H}$ satisfying $[f, R_x] = [f, R_y] = 0$ and $f(1) = 0$ is necessarily the zero map.*

Proof. Since f is $\mathbb{Q}$-linear, it is only necessary to show $f(w) = 0$ for any words $w \in \mathfrak{H}$. Write $w = u_1 u_2 \cdots u_n$ with $u_1, u_2, \ldots, u_n \in \{x, y\}$. Since $[f, R_{u_i}] = 0$ for any $1 \leqslant i \leqslant n$ by assumption, we have

$$f(w) = f(u_1 u_2 \cdots u_n) = f(u_1 u_2 \cdots u_{n-1}) u_n = \cdots = f(1) u_1 u_2 \cdots u_n = 0. \qquad \square$$

Next, the commutativity property of $\partial_n^{(c)}$ is given. Instead of Proposition 6, the following slightly general statement is shown.

Proposition 10. *For any $n, m \geqslant 1$ and any $c, c' \in \mathbb{Q}$, we have $[\partial_n^{(c)}, \partial_m^{(c')}] = 0$.*

Proof. In the following, (A_n) and (B_n) are shown inductively as $(\mathrm{A}_1), (\mathrm{B}_1) \Rightarrow (\mathrm{A}_2) \Rightarrow (\mathrm{B}_2) \Rightarrow (\mathrm{A}_3) \Rightarrow (\mathrm{B}_3) \Rightarrow (\mathrm{A}_4) \Rightarrow \cdots$.

Let $\mathrm{sgn}(x) = 1$ and $\mathrm{sgn}(y) = -1$.

(A_n) $[\partial_n^{(c)}, R_u] = \mathrm{sgn}(u)\psi_n^{(c)}$ for any $c \in \mathbb{Q}$ and any $u \in \{x, y\}$;
(B_n) $[\partial_n^{(c)}, \partial_i^{(c')}] = 0$ for any $1 \leqslant i \leqslant n$ and any $c, c' \in \mathbb{Q}$.

Note that if (B_n)'s for any $n \geqslant 1$ can be shown, the proposition is shown.

Note the following three considerations. First, the statement (A_n) means that, for any $w \in \mathfrak{H}$ and any $u \in \{x, y\}$,

$$\partial_n^{(c)}(wu) = \partial_n^{(c)}(w)u + \mathrm{sgn}(u)\psi_n^{(c)}(w)$$

and implies

(α_n) $[\partial_n^{(c)}, R_z] = 0$ for any $c \in \mathbb{Q}$,

where $z = x + y$.

Second, let

$(\mathrm{B}_{n,i})$ $[\partial_n^{(c)}, \partial_i^{(c')}] = 0$ for a fixed $1 \leqslant i \leqslant n$ and any $c, c' \in \mathbb{Q}$.

Clearly, the statement (B_n) is equivalent to the union of $(\mathrm{B}_{n,i})$'s for $1 \leqslant i \leqslant n$. Because of Lemma 9 and $[\partial_n^{(c)}, \partial_i^{(c')}](1) = 0$ by $\partial_n^{(c)}(\mathbb{Q}) = 0$, each $(\mathrm{B}_{n,i})$ is equivalent to the statement

$(\mathrm{B}'_{n,i})$ $[[\partial_n^{(c)}, \partial_i^{(c')}], R_u] = 0$ for a fixed $1 \leqslant i \leqslant n$, any $c, c' \in \mathbb{Q}$, and any $u \in \{x, y\}$.

Instead of (B_{n+1}), $(\mathrm{B}'_{n+1,i})$'s for $1 \leqslant i \leqslant n+1$ are shown by induction on i.

Third, note that we can consider $\mathbb{Q}[R_z, \partial_1^{(c)}, \ldots, \partial_n^{(c)}]$ as a commutative polynomial ring if (A_i) (hence (α_i)) and (B_i) hold for all $1 \leqslant i \leqslant n$. Let $\mathbb{Q}[R_z, \partial_1^{(c)}, \ldots, \partial_n^{(c)}]_{(i)}$ denote the degree-i homogenous part with $\deg(R_z) = 1$ and $\deg(\partial_d^{(c)}) = d$. These assumptions together with the recursive rule (3) give us the fact

(β_n) $\phi_n^{(c)} \in \mathbb{Q}[R_z, \partial_1^{(c)}, \ldots, \partial_n^{(c)}]_{(n)}$ for any $c \in \mathbb{Q}$.

Based on the above considerations, the proof of (A_n) and (B_n) is now given. Since $[\partial_1^{(c)}, R_u](w) = \partial_1^{(c)}(wu) - \partial_1^{(c)}(w)u = w\partial_1^{(c)}(u) = R_{\partial_1^{(c)}(u)}(w)$ for $w \in \mathfrak{H}$ and $\operatorname{sgn}(u)\psi_1^{(c)} = \operatorname{sgn}(u)R_{xy} = R_{\partial_1^{(c)}(u)}$ for any $u \in \{x, y\}$, the statement (A_1) holds. The statement (B_1) is trivial because $\partial_1^{(c)} = \partial_1^{(c')} = \partial_1$ for any $c, c' \in \mathbb{Q}$.

Assume that (A_n) (hence (α_n)) and (B_n) are proven. By the definition of $\partial_{n+1}^{(c)}$,

$$n\big[\partial_{n+1}^{(c)}, R_u\big] = \big[\big[\theta^{(c)}, \partial_n^{(c)}\big], R_u\big].$$

Using Jacobi's identity, the right-hand side equals

$$-\big[\big[\partial_n^{(c)}, R_u\big], \theta^{(c)}\big] - \big[\big[R_u, \theta^{(c)}\big], \partial_n^{(c)}\big].$$

By (A_n) and $[\theta^{(c)}, R_u] = R_{\theta(u)} + cR_u\partial_1$ for $u \in \{x, y\}$, this yields

$$-\operatorname{sgn}(u)\big[\psi_n^{(c)}, \theta^{(c)}\big] + \big[R_{\theta(u)} + cR_u\partial_1, \partial_n^{(c)}\big].$$

Using $R_{\theta(u)} = \frac{1}{2}(R_zR_u + R_uR_z)$, ($\alpha_n$), and ($B_n$),

$$\big[R_{\theta(u)} + cR_u\partial_1, \partial_n^{(c)}\big] = \frac{1}{2}\big(R_z\big[R_u, \partial_n^{(c)}\big] + \big[R_u, \partial_n^{(c)}\big]R_z\big) + c\big[R_u, \partial_n^{(c)}\big]\partial_1.$$

Hence, using (A_n), we have

$$\big[\partial_{n+1}^{(c)}, R_u\big] = \frac{\operatorname{sgn}(u)}{n}\left(\big[\theta^{(c)}, \psi_n^{(c)}\big] - \frac{1}{2}\big(R_z\psi_n^{(c)} + \psi_n^{(c)}R_z\big) - c\psi_n^{(c)}\partial_1\right) = \operatorname{sgn}(u)\psi_{n+1}^{(c)},$$

and therefore (A_{n+1}) (as well as (α_{n+1})) is proven.

In order to prove (B_{n+1}), assume that all (A_j)'s (hence (α_j)'s) for $1 \leqslant j \leqslant n+1$ and all (B_j)'s (hence (β_j)'s) for $1 \leqslant j \leqslant n$ are proven. As mentioned above, ($B'_{n+1,i}$)'s for $1 \leqslant i \leqslant n+1$ are proven instead of (B_{n+1}). Using Jacobi's identity, we have

$$\big[\big[\partial_{n+1}^{(c)}, \partial_i^{(c')}\big], R_u\big] = -\big[\big[\partial_i^{(c')}, R_u\big], \partial_{n+1}^{(c)}\big] - \big[\big[R_u, \partial_{n+1}^{(c)}\big], \partial_i^{(c')}\big] \tag{4}$$

for every $1 \leqslant i \leqslant n+1$. By ($A_i$) and Lemma 8,

$$\big[\partial_i^{(c)}, R_u\big] = \operatorname{sgn}(u)\psi_i^{(c)} = \operatorname{sgn}(u)R_y\phi_{i-1}^{(c)}R_x$$

for any $1 \leqslant i \leqslant n+1$, any $c \in \mathbb{Q}$, and any $u \in \{x, y\}$, and hence,

$$-\operatorname{sgn}(u)\big(\text{RHS of (4)}\big) = \big[R_y\phi_{i-1}^{(c')}R_x, \partial_{n+1}^{(c)}\big] - \big[R_y\phi_n^{(c)}R_x, \partial_i^{(c')}\big].$$

The right-hand side is equal to the sum

$$\begin{aligned} &R_y\phi_{i-1}^{(c')}\big[R_x, \partial_{n+1}^{(c)}\big] + R_y\big[\phi_{i-1}^{(c')}, \partial_{n+1}^{(c)}\big]R_x + \big[R_y, \partial_{n+1}^{(c)}\big]\phi_{i-1}^{(c')}R_x \\ &\quad - R_y\phi_n^{(c)}\big[R_x, \partial_i^{(c')}\big] - R_y\big[\phi_n^{(c)}, \partial_i^{(c')}\big]R_x - \big[R_y, \partial_i^{(c')}\big]\phi_n^{(c)}R_x. \end{aligned} \tag{5}$$

If $i = 1$, we have $\phi_{i-1}^{(c')} = \phi_0^{(c')} = \mathrm{id}_{\mathfrak{H}}$, and hence,

$$\big[\phi_{i-1}^{(c')}, \partial_{n+1}^{(c)}\big] = \big[\phi_0^{(c')}, \partial_{n+1}^{(c)}\big] = 0.$$

Thanks to (β_n) and the identity $\partial_1^{(c')} = \partial_1^{(c)}$ $(= \partial_1)$, we also have

$$\bigl[\phi_n^{(c)}, \partial_i^{(c')}\bigr] = \bigl[\phi_n^{(c)}, \partial_1^{(c')}\bigr] = 0.$$

Thus, in this case, the entire expression (5) turns into

$$-R_y\psi_{n+1}^{(c)} + \psi_{n+1}^{(c)}R_x + R_y\phi_n^{(c)}\psi_1^{(c')} - \psi_1^{(c')}\phi_n^{(c)}R_x \tag{6}$$

by (A_1) and (A_{n+1}). Using Lemma 8, $\psi_1^{(c')} = R_yR_x$ and $R_z = R_x + R_y$, we obtain the expression (6) equals $-R_yR_z\phi_n^{(c)}R_x + R_y\phi_n^{(c)}R_zR_x$. The right-hand side becomes zero because $[R_z, \phi_n^{(c)}] = 0$ by (β_n). Thus, $(\mathrm{B}'_{n+1,1})$ (as well as $(\mathrm{B}_{n+1,1})$) is proven.

In order to conclude the expression (5) equals zero for i with $1 < i \leqslant n+1$, assume that $(\mathrm{B}_{n+1,i-1})$ (hence $(\mathrm{B}'_{n+1,i-1})$) is proven. We then obtain

$$\bigl[\phi_{i-1}^{(c')}, \partial_{n+1}^{(c)}\bigr] = 0$$

based on (β_{i-1}), $(B_{n+1,i-1})$, and (α_{n+1}). In addition, we obtain

$$\bigl[\phi_n^{(c)}, \partial_i^{(c')}\bigr] = 0 \quad \text{for } 1 < i \leqslant n+1,$$

by (β_n), (B_n), and (α_i) (when $1 < i < n+1$) or by (β_n), $(\mathrm{B}_{n+1,n})$, and (α_{n+1}) (when $i = n+1$). Thus, in this case, the entire expression (5) turns into

$$-R_y\phi_{i-1}^{(c')}\psi_{n+1}^{(c)} + \psi_{n+1}^{(c)}\phi_{i-1}^{(c')}R_x + R_y\phi_n^{(c)}\psi_i^{(c')} - \psi_i^{(c')}\phi_n^{(c)}R_x \tag{7}$$

by (A_i) and (A_{n+1}). Using Lemma 8 and $R_z = R_x + R_y$, we obtain the expression (7) equals $-R_y\phi_{i-1}^{(c')}R_z\phi_n^{(c)}R_x + R_y\phi_n^{(c)}R_z\phi_{i-1}^{(c')}R_x$. The right-hand side becomes zero because the operators $\phi_{i-1}^{(c')}$, $\phi_n^{(c)}$ and R_z commute with one another. Thus, $(\mathrm{B}'_{n+1,i})$ (as well as $(\mathrm{B}_{n+1,i})$) holds, and by induction, we obtain (B_{n+1}). This concludes the proof of the proposition. □

According to (β_n), $\phi_n^{(c)}$ commutes with R_z, and so the recursive rule (3) is simplified as

$$\phi_n^{(c)} = \frac{1}{n}\bigl(\bigl[\theta^{(c)}, \phi_{n-1}^{(c)}\bigr] + (R_z + c\partial_1)\phi_{n-1}^{(c)}\bigr). \tag{8}$$

Masanobu Kaneko pointed out a formula for $\phi_n^{(c)}$,

$$R_z\phi_n^{(c)} = \frac{1}{n!}\,\mathrm{ad}\bigl(\theta^{(c)}\bigr)^n(R_z).$$

This is shown by using $[\theta^{(c)}, R_z] = R_{\theta(z)} + cR_z\partial_1$ and the recursive formula (8).

Using Proposition 10, we also obtain:

Proposition 11. *We have $\partial_n^{(c)}(\mathbb{Q}\cdot x + \mathbb{Q}\cdot y + \mathfrak{H}^0) \subset \mathfrak{H}^0$ for any integer $n \geqslant 1$ and any $c \in \mathbb{Q}$.*

Proof. By Lemma 8 and (A_n) in the proof of Proposition 10, we have

$$\partial_n^{(c)}(wu) = \partial_n^{(c)}(w)u + \mathrm{sgn}(u)\phi_{n-1}^{(c)}(wx)y \quad \bigl(w \in \mathfrak{H},\ u \in \{x, y\}\bigr). \tag{9}$$

This implies

$$\partial_n^{(c)}\big(\mathbb{Q}\cdot x+\mathfrak{H}^1\big)\subset\mathfrak{H}^1. \tag{10}$$

Next, the proposition is shown by induction on n. The proposition holds for $n=1$ because $\partial_1^{(c)}=\partial_1$. Assume that the proposition is proven for $n-1$. Using Eq. (9), (β_{n-1}), and $\partial_n^{(c)}(1)=0$, by induction on the degree of a word, we find that both $\partial_n^{(c)}(x)$ and $\partial_n^{(c)}(xwy)$ for any words $w\in\mathfrak{H}$ begin with the letter x (hence, using (10), $\partial_n^{(c)}(x), \partial_n^{(c)}(xwy)\in\mathfrak{H}^0$). In addition, because of (α_n), we have $\partial_n^{(c)}(z)=\partial_n^{(c)}R_z(1)=R_z\partial_n^{(c)}(1)=0$ where $z=x+y$, and hence, we have $\partial_n^{(c)}(y)=-\partial_n^{(c)}(x)\in\mathfrak{H}^0$. Therefore, the proposition is proven for n. □

4. Proof of Key Proposition

In this section, the proof of Key Proposition 5 is given.

Denote by $\mathfrak{H}_n^1$ the weight n homogenous part of $\mathfrak{H}^1$. Recall that $z_k=x^{k-1}y$ for $k\geqslant 1$ as defined in Section 2. Let $\mathfrak{W}$ be the $\mathbb{Q}$-vector space generated by $\{\mathcal{H}_w \mid w\in\mathfrak{H}^1\}$, and $\mathfrak{W}_n$ the vector subspace of $\mathfrak{W}$ generated by $\{\mathcal{H}_w \mid w\in\mathfrak{H}_n^1\}$. Let $\mathfrak{W}'$ be the $\mathbb{Q}$-vector space generated by $\{L_{z_k}\mathcal{H}_w \mid k\geqslant 1,\ w\in\mathfrak{H}^1\}$, and $\mathfrak{W}'_n$ the vector subspace of $\mathfrak{W}'$ generated by $\{L_{z_k}\mathcal{H}_w \mid 1\leqslant k\leqslant n,\ w\in\mathfrak{H}_{n-k}^1\}$. The $\mathbb{Q}$-linear map $\lambda:\mathfrak{W}'\to\mathfrak{W}$ is defined by $\lambda(L_{z_k}\mathcal{H}_w)=\mathcal{H}_{z_k w}$.

Remark 12. Here, we show the well-definedness of the map λ. Assume that

$$\sum_{(z_k,w)} C_{(z_k,w)}L_{z_k}\mathcal{H}_w=0 \quad (\in\mathfrak{W}), \tag{11}$$

where the sum is over different pairs of words (z_k,w). Applying (11) to $1\in\mathfrak{H}$, we have

$$\sum_{(z_k,w)} C_{(z_k,w)}z_k w=0.$$

Then, for each z_k, we have

$$\sum_{w} C_{(z_k,w)}w=0$$

where the sum is over different words w. Therefore, each coefficient $C_{(z_k,w)}$ becomes zero, and hence, $L_{z_k}\mathcal{H}_w$'s are linearly independent.

Recall that $\varepsilon\in\operatorname{Aut}(\mathfrak{H})$ has been defined by $\varepsilon(x)=x+y$, $\varepsilon(y)=-y$, the anti-automorphism τ on $\mathfrak{H}$ by $\tau(x)=y$, $\tau(y)=x$, and $\chi_x=\tau L_x\varepsilon$. Then, we have:

Proposition 13. *Let $n\geqslant 1$. Then the following two statements,* (C_n) *and* (D_n) *hold*:

(C_n) $\varepsilon\tau\phi_{n-1}^{(c)}R_x\tau\varepsilon\in\mathfrak{W}'_n$;
(D_n) $\chi_x^{-1}\partial_n^{(c)}\chi_x=-\lambda(\varepsilon\tau\phi_{n-1}^{(c)}R_x\tau\varepsilon)\in\mathfrak{W}_n$ *on* $\mathfrak{H}^1$.

By (10), the expression $\chi_x^{-1}=\varepsilon\tau R_y^{-1}$ in (D_n) has a well-defined meaning. According to (D_n), there exists an element $w\in\mathfrak{H}y$ such that

$$\chi_x^{-1}\partial_n^{(c)}\chi_x=\mathcal{H}_w, \tag{12}$$

which is equivalent to Key Proposition 5 in Section 2. Therefore, Proposition 13 is proven instead of Key Proposition 5.

Remark 14. Note that $w = w(n,c)$ in (12) can be determined as follows. Equation (12) holds on $\mathfrak{H}^1$, and hence on $\mathbb{Q}$. Since $\partial_n^{(c)}(y) \in \mathfrak{H}^0$ by Proposition 11, $R_y^{-1}\partial_n^{(c)}(y) \in x\mathfrak{H}$. Hence,

$$\chi_x^{-1}\partial_n^{(c)}\chi_x(1) = \chi_x^{-1}\partial_n^{(c)}(y) \in \varepsilon\tau(x\mathfrak{H}) = \varepsilon(\mathfrak{H}y) = \mathfrak{H}y.$$

Since $\mathcal{H}_w(1) = w$, we have $w = \chi_x^{-1}\partial_n^{(c)}(y)$ $(\in \mathfrak{H}y)$ by (12).

For the proof of Proposition 13, following lemmata are needed.

Lemma 15. *For any $X \in \mathfrak{W}'$ and any $l \geqslant 1$, we have $[\lambda(X), L_{z_l}] = XL_{z_l} + L_{x^l}X$.*

Proof. It is sufficient to show the case in which $X = L_{z_k}\mathcal{H}_w$, which follows directly from

$$[\mathcal{H}_{z_k w}, L_{z_l}] = L_{z_k}\mathcal{H}_w L_{z_l} + L_{z_{k+l}}\mathcal{H}_w, \tag{13}$$

the harmonic product rule. □

Lemma 16. *For any $k, l \geqslant 1$, we have $(\lambda - 1)(\mathfrak{W}'_k)L_{z_l} \subset \mathfrak{W}'_{k+l}$.*

Proof. The proof follows directly from (13). □

Lemma 17. *We have $(\lambda - 1)(\mathfrak{W}'_k) \cdot (\lambda - 1)(\mathfrak{W}'_l) \subset (\lambda - 1)(\mathfrak{W}'_{k+l})$ for any $k, l \geqslant 1$.*

Proof. Let d and d' be the weights of words w and w', respectively. The assertion $(\lambda - 1)(L_{z_k}\mathcal{H}_w) \cdot (\lambda - 1)(L_{z_l}\mathcal{H}_{w'}) \in (\lambda - 1)(\mathfrak{W}'_{k+l+d+d'})$ is only necessary to show.

$$\begin{aligned}
\text{LHS} &= (\mathcal{H}_{z_k w} - L_{z_k}\mathcal{H}_w)(\mathcal{H}_{z_l w'} - L_{z_l}\mathcal{H}_{w'}) \\
&= \mathcal{H}_{z_k w * z_l w'} - \mathcal{H}_{z_k w}L_{z_l}\mathcal{H}_{w'} - L_{z_k}\mathcal{H}_{w * z_l w'} + L_{z_k}\mathcal{H}_w L_{z_l}\mathcal{H}_{w'} \\
&= \mathcal{H}_{z_k(w * z_l w') + z_l(z_k w * w') + z_{k+l}(w * w')} - (L_{z_k}\mathcal{H}_w L_{z_l} + L_{z_l}\mathcal{H}_{z_k w} + L_{z_{k+l}}\mathcal{H}_w)\mathcal{H}_{w'} \\
&\quad - L_{z_k}\mathcal{H}_{w * z_l w'} + L_{z_k}\mathcal{H}_w L_{z_l}\mathcal{H}_{w'} \\
&= \mathcal{H}_{z_k(w * z_l w')} - L_{z_k}\mathcal{H}_{w * z_l w'} + \mathcal{H}_{z_l(z_k w * w')} - L_{z_l}\mathcal{H}_{z_k w * w'} + \mathcal{H}_{z_{k+l}(w * w')} - L_{z_{k+l}}\mathcal{H}_{w * w'} \\
&= (\lambda - 1)(L_{z_k}\mathcal{H}_{w * z_l w'} + L_{z_l}\mathcal{H}_{z_k w * w'} + L_{z_{k+l}}\mathcal{H}_{w * w'}). \\
&\in \text{RHS}.
\end{aligned}$$

Hence, the lemma is proven. □

Lemma 18. *For any $X \in \mathfrak{W}'$, we have $\lambda(X)(1) = X(1)$.*

Proof. $(\lambda - 1)(L_{z_k}\mathcal{H}_w)(1) = \mathcal{H}_{z_k w}(1) - L_{z_k}\mathcal{H}_w(1) = z_k w - z_k w = 0.$ □

Lemma 19. *Let $X \in \mathfrak{W}$. If $X(1) = 0$ and $[X, L_{z_k}] = 0$ for any $k \geqslant 1$, we have $X = 0$.*

Proof. If $[X, L_{z_k}] = 0$ for any $k \geqslant 1$,

$$X(z_{k_1} \cdots z_{k_n}) = z_{k_1} X(z_{k_2} \cdots z_{k_n}) = \cdots = z_{k_1} \cdots z_{k_n} X(1) = 0. \qquad \square$$

Using their validity and various properties obtained in the proof of Proposition 10, Proposition 13 can be shown as follows.

Proof. In the following, (C_n) and (D_n) are proven inductively as $(\mathrm{C}_1) \Rightarrow (\mathrm{D}_1) \Rightarrow (\mathrm{C}_2) \Rightarrow (\mathrm{D}_2) \Rightarrow (\mathrm{C}_3) \Rightarrow \cdots$.

Since $\varepsilon\tau\phi_0^{(c)} R_x \tau\varepsilon = -L_y \in \mathfrak{W}'_1$, the claim (C_1) holds.

Assume that (C_n) is proven. Note that we have the equality

$$R_y^{-1} \partial_n^{(c)} R_y = \partial_n^{(c)} - \phi_{n-1}^{(c)} R_x \tag{14}$$

based on (A_n), Lemma 8, and Proposition 11. Then, we obtain

$$\begin{aligned} \left[\chi_x^{-1} \partial_n^{(c)} \chi_x, L_{z_k}\right] &= \chi_x^{-1} \partial_n^{(c)} \chi_x L_{z_k} - L_{z_k} \chi_x^{-1} \partial_n^{(c)} \chi_x \\ &= \varepsilon\tau\partial_n^{(c)} \tau\varepsilon L_{z_k} - \varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon L_{z_k} - L_{z_k} \varepsilon\tau\partial_n^{(c)} \tau\varepsilon + L_{z_k} \varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon. \end{aligned} \tag{15}$$

Note that

$$\varepsilon L_x = L_z \varepsilon, \qquad \varepsilon L_y = -L_y \varepsilon, \qquad \tau L_x = R_y \tau, \qquad \tau L_y = R_x \tau. \tag{16}$$

Using (16), the first term of the expression (15) turns into $-\varepsilon\tau\partial_n^{(c)} R_{z^{k-1}} R_x \tau\varepsilon$. According to (A_n), (α_n), and Lemma 8,

$$-\varepsilon\tau\partial_n^{(c)} R_{z^{k-1}} R_x \tau\varepsilon = -\varepsilon\tau R_{z^{k-1}} \left(R_x \partial_n^{(c)} + R_y \phi_{n-1}^{(c)} R_x\right) \tau\varepsilon.$$

Again apply (16). Then, two terms cancel and two others combine to the second term on the right in the statement below it.

$$\left[\chi_x^{-1} \partial_n^{(c)} \chi_x, L_{z_k}\right] = -\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon L_{z_k} - L_{x^k} \varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon.$$

This is equal to $[\lambda(-\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon), L_{z_k}]$ by Lemma 15 and (C_n). Moreover,

$$\chi_x^{-1} \partial_n^{(c)} \chi_x(1) = \varepsilon\left(\tau\partial_n^{(c)} \tau - \tau\phi_{n-1}^{(c)} R_x \tau\right) \varepsilon(1) = -\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon(1)$$

because of (14) and $\partial_n^{(c)}(1) = 0$. By Lemma 18, this equals $-\lambda(\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon)(1)$. Hence, by Lemma 19, we have (D_n): $\chi_x^{-1} \partial_n^{(c)} \chi_x = -\lambda(\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon)$ on $\mathfrak{H}^1$.

Next, assume that (D_n) is proven. Using (14) and (D_n), we obtain

$$\varepsilon\tau\partial_n^{(c)} \tau\varepsilon = \chi_x^{-1} \partial_n^{(c)} \chi_x + \varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon = (\lambda - 1)\left(-\varepsilon\tau\phi_{n-1}^{(c)} R_x \tau\varepsilon\right).$$

According to (B_n), we have the expression

$$\phi_n^{(c)} = \sum_{i=0}^{n} f_i^{(c)} R_{z^{n-i}} \quad \left(f_i^{(c)} \in \mathbb{Q}\left[\partial_1^{(c)}, \ldots, \partial_i^{(c)}\right]_{(i)}\right).$$

Hence,

$$\varepsilon\tau\phi_n^{(c)}R_x\tau\varepsilon = \varepsilon\tau\sum_{i=0}^{n} f_i^{(c)}R_{z^{n-i}}R_x\tau\varepsilon = -\sum_{i=0}^{n}\varepsilon\tau f_i^{(c)}\tau\varepsilon L_{z_{n+1-i}}.$$

By Lemma 17, this is an element of $\sum_{i=0}^{n}(\lambda-1)(\mathfrak{W}'_i)L_{z_{n+1-i}}$. Then, by Lemma 16, this is a subset of $\mathfrak{W}'_{n+1}$. Hence, (C_{n+1}) is proven. □

5. Alternative extension of ∂_n

In this section, an alternative operator $\hat{\partial}_n^{(c)}$ is defined instead of $\partial_n^{(c)}$ in Definition 2. Several properties of $\hat{\partial}_n^{(c)}$'s are discussed here. In particular, $\partial_n^{(c)}$ and $\hat{\partial}_n^{(c)}$ give the same class of relations for MZV's.

Definition 20. Let $c \in \mathbb{Q}$ and H the same operator as in Definition 2. For each integer $n \geqslant 1$, the $\mathbb{Q}$-linear map $\hat{\partial}_n^{(c)} : \mathfrak{H} \to \mathfrak{H}$ is defined by

$$\hat{\partial}_n^{(c)} = \frac{1}{(n-1)!}\,\mathrm{ad}\big(\hat{\theta}^{(c)}\big)^{n-1}(\partial_1)$$

where $\hat{\theta}^{(c)}$ is the $\mathbb{Q}$-linear map defined by $\hat{\theta}^{(c)}(x) = \theta(x)$, $\hat{\theta}^{(c)}(y) = \theta(y)$ and the rule

$$\hat{\theta}^{(c)}(ww') = \hat{\theta}^{(c)}(w)w' + w\hat{\theta}^{(c)}(w') + cH(w)\partial_1(w') \tag{17}$$

for any $w, w' \in \mathfrak{H}$.

The operator $\hat{\partial}_n^{(c)}$ gives another quasi-derivation operator (with respect to n and $\hat{\theta}^{(c)}$ for the given $c \in \mathbb{Q}$). The only difference between $\theta^{(c)}$ and $\hat{\theta}^{(c)}$ is the order of H and ∂_1 appearing in the right-hand side of (1) and (17).

Lemma 21. *For any $c \in \mathbb{Q}$, we have $\hat{\theta}^{(c)} = \theta^{(-c)} + c\partial_1(H-1)$.*

Proof. Calculate the recursive rules for both sides. □

Proposition 22. *For any $n \geqslant 1$ and any $c \in \mathbb{Q}$, we have $\hat{\partial}_n^{(c)} \in \mathbb{Q}[\partial_1^{(-c)}, \dots, \partial_n^{(-c)}]$.*

Proof. The proposition holds for $n = 1$ because $\hat{\partial}_1^{(c)} = \partial_1^{(-c)} = \partial_1$. Assume that the proposition is proven for n. Using Lemma 21, we obtain

$$n\hat{\partial}_{n+1}^{(c)} = \big[\hat{\theta}^{(c)}, \hat{\partial}_{n+1}^{(c)}\big] = \big[\theta^{(-c)} + c\partial_1(H-1), \hat{\partial}_n^{(c)}\big] = \big[\theta^{(-c)}, \hat{\partial}_n^{(c)}\big] + c(n-1)\partial_1\hat{\partial}_n^{(c)}.$$

Hence, by induction, the proposition holds for $n+1$. □

Example 23. The polynomials in Proposition 22 can be constructed explicitly. For example,

$$\begin{aligned}
\hat{\partial}_2^{(c)} &= \partial_2^{(-c)} + c\partial_1^2,\\
\hat{\partial}_3^{(c)} &= \partial_3^{(-c)} + 2c\partial_1\partial_2^{(-c)} + c^2\partial_1^3,\\
\hat{\partial}_4^{(c)} &= \partial_4^{(-c)} + \frac{7}{3}c\partial_1\partial_3^{(-c)} + \frac{2}{3}c\partial_2^{(-c)2} + 3c^2\partial_1^2\partial_2^{(-c)} + c^3\partial_1^4.
\end{aligned}$$

Corollary 24. *For any rational numbers c, c', and any positive integers n, m, we have* $[\partial_n^{(c)}, \hat{\partial}_m^{(c')}] = 0$.

Proof. The proof follows immediately from Propositions 10 and 22. □

Lemma 25. *For any rational number c, we have* $\hat{\theta}^{(c)} = \tau\theta^{(-c)}\tau$.

Proof. By direct calculations, each image of x and y of $\mathfrak{H}$ coincides. Write $w = w_1 w_2$ where w_1 and w_2 are words of $\mathfrak{H}$ with $\deg(w_i) \geqslant 1$, $i = 1, 2$. Then,

$$\begin{aligned}\tau\theta^{(-c)}\tau(w) &= \tau\theta^{(-c)}\big(\tau(w_2)\tau(w_1)\big)\\ &= \tau\big(\theta^{(-c)}\tau(w_2)\tau(w_1) + \tau(w_2)\theta^{(-c)}\tau(w_1) - c\partial_1\tau(w_2)H\tau(w_1)\big)\\ &= w_1\tau\theta^{(-c)}\tau(w_2) + \tau\theta^{(-c)}\tau(w_1)w_2 - c\tau H\tau(w_1)\tau\partial_1\tau(w_2).\end{aligned}$$

Use $\tau H\tau = H$, $\tau\partial_1\tau = -\partial_1$ to complete the proof. □

Proposition 26. *For any integer $n \geqslant 1$ and any rational number c, we have* $\hat{\partial}_n^{(c)} = -\tau\partial_n^{(-c)}\tau$.

Proof. The proof is given by induction on n. The proposition holds for $n = 1$. Assume that the proposition is proven for n. Using Lemma 25, we have

$$(n+1)\hat{\partial}_{n+1}^{(c)} = \big[\hat{\theta}^{(c)}, \hat{\partial}_n^{(c)}\big] = -\big[\tau\theta^{(-c)}\tau, \tau\partial_n^{(-c)}\tau\big] = -\tau\big[\theta^{(-c)}, \partial_n^{(-c)}\big]\tau = -n\tau\partial_{n+1}^{(-c)}\tau.$$

Thus, the proposition holds for $n + 1$. □

By Proposition 26, we have $\hat{\partial}_n^{(c)}(\mathfrak{H}^0) \subset \ker Z$ for any $n \geqslant 1$ and any $c \in \mathbb{Q}$, which assigns the same space to Theorem 3 because of Proposition 22.

Appendix A. A new proof of the derivation relation

In the case of $c = 0$ in Theorem 3, we have an alternative proof of the derivation relation for MZV's, reducing to Kawashima's relation. For this, we introduce certain automorphisms on the algebra $\hat{\mathfrak{H}} = \mathbb{Q}\langle\langle x, y\rangle\rangle$. (See [5] for details.) Let Φ be the automorphism on $\hat{\mathfrak{H}}$ defined by $\Phi(x) = x$ and $\Phi(z) = z(1+y)^{-1}$. The automorphism Φ satisfies

$$\frac{1}{1+y} * w = \frac{1}{1+y}\Phi(w)$$

for $w \in \mathfrak{H}^1$ [5, Proposition 6]. Let Δ be $\exp(\sum_{n\geqslant 1}\frac{\partial_n}{n})$ which is the automorphism on $\hat{\mathfrak{H}}$ characterized by $\Delta(x) = x(1-y)^{-1}$ and $\Delta(z) = z$. Then, we have $\Phi = \varepsilon\Delta\varepsilon$ on $\hat{\mathfrak{H}}$. This implies that $\mathcal{H}_{\frac{1}{1+y}} = \varepsilon L_x^{-1}\Delta L_x\varepsilon$ on $\hat{\mathfrak{H}}^1$, the completion of $\mathfrak{H}^1$. Hence, $(\Delta - 1)(\mathfrak{H}^0) \subset L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y)$. Expanding the exponential map, each degree-i part of $\Delta - 1$ sends $\mathfrak{H}^0$ to $L_x\varepsilon(\mathfrak{H}y * \mathfrak{H}y)$, and, therefore, the derivation relation is shown again to be a class of relations of MZV's according to Kawashima's relation in Fact 4.

Appendix B. Ohno's relation and the derivation relation

For $n \geqslant 1$, the derivation D_n on $\mathfrak{H}$ is defined by $D_n(x) = 0$, $D_n(y) = x^n y$. The map $\bar{D}_n = \tau D_n\tau$ is another derivation on $\mathfrak{H}$ such that $\bar{D}_n(x) = xy^n$, $\bar{D}_n(y) = 0$. Let

$$\sigma = \sum_{l=0}^{\infty}\sigma_l = \exp\left(\sum_{n=1}^{\infty}\frac{D_n}{n}\right), \qquad \bar{\sigma} = \sum_{l=0}^{\infty}\bar{\sigma}_l = \exp\left(\sum_{n=1}^{\infty}\frac{\bar{D}_n}{n}\right).$$

The maps σ, $\bar{\sigma}$ are automorphisms on $\mathfrak{H}$. Putting $D = \sum_{n=1}^{\infty} \frac{D_n}{n}$, we find $D^m(x) = 0$, $D^m(y) = (-\log(1-x))^m y$ for $m \geqslant 1$, and hence,

$$\sigma(x) = x, \qquad \sigma(y) = \frac{1}{1-x} y.$$

Since the map σ is an automorphism,

$$\begin{aligned}\sigma\left(x^{k_1-1} y \cdots x^{k_n-1} y\right) &= x^{k_1-1} \frac{1}{1-x} y \cdots x^{k_n-1} \frac{1}{1-x} y \\ &= \sum_{l=0}^{\infty} \sum_{\substack{e_1+\cdots+e_n=l \\ e_1,\ldots,e_n \geqslant 0}} x^{k_1+e_1-1} y \cdots x^{k_n+e_n-1} y,\end{aligned}$$

and hence,

$$\sigma_l\left(x^{k_1-1} y \cdots x^{k_n-1} y\right) = \sum_{\substack{e_1+\cdots+e_n=l \\ e_1,\ldots,e_n \geqslant 0}} x^{k_1+e_1-1} y \cdots x^{k_n+e_n-1} y.$$

Thus, Ohno's relation introduced in [8] can be stated as $\sigma_l(1-\tau)(\mathfrak{H}^0) \subset \ker Z$ for any $l \geqslant 0$. If $l = 0$, Ohno's relation is reduced to the duality formula.

The automorphisms σ, $\bar{\sigma}$ and Δ, which has been defined in Appendix A, have a property as follows. (See [5, Theorem 4(ii)].)

Proposition 27. $\Delta = \bar{\sigma}\sigma^{-1}$.

According to this proposition, we have $\sigma - \bar{\sigma} = (1-\Delta)\sigma$. Since $\bar{\sigma}_l = \tau\sigma_l\tau$ and the duality formula is included in Ohno's relation, this identity implies that Ohno's relation is equivalent to the union of the duality formula and the derivation relation.

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