An elastoplastic multi-level damage model for ductile matrix composites considering evolutionary weakened interface

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Abstract

An elastoplastic multi-level damage model considering evolutionary weakened interface is developed in this work to predict the effective elastoplastic behavior and multi-level damage evolution in particle reinforced ductile matrix composites (PRDMCs). The elastoplastic multi-level damage model is micromechanically derived on the basis of the ensemble-volume averaging procedure and the first-order effects of eigenstrains. The Eshelby’s tensor for an ellipsoidal inclusion with slightly weakened interface [Qu, J., 1993a. Eshelby tensor for an elastic inclusion with slightly weakened interfaces. Journal of Applied Mechanics 60 (4), 1048–1050; Qu, J., 1993b. The effect of slightly weakened interfaces on the overall elastic properties of composite materials. Mechanics of Materials, 14, 269–281] is adopted to model particles having mildly or severely weakened interface, and a multi-level damage model [Lee, H.K., Pyo, S.H., in press. Multi-level modeling of effective elastic behavior and progressive weakened interface in particulate composites. Composites Science and Technology] in accordance with the Weibull’s probabilistic function is employed to describe the sequential, progressive weakened interface in the composites. Numerical examples corresponding to uniaxial, biaxial and triaxial tension loadings are solved to illustrate the potential of the proposed micromechanical framework. A series of parametric analysis are carried out to investigate the influence of model parameters on the progression of weakened interface in the composites. Furthermore, the present prediction is compared with available experimental data in the literature to verify the proposed elastoplastic multi-level damage model.

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1. Introduction

Ductile matrix composites (e.g., metal matrix composites, MMCs) reinforced with various shapes of particles, fibers or whiskers have been used in aerospace, electronics and a variety of other engineering applica-
tions. In particular, MMCs combine the properties of high strength and high modulus ceramics with those of high ductile metals or alloys to produce enhanced mechanical properties of the composite over metal or alloys (Ravi et al., 2007).

Particle or fiber reinforced ductile matrix composites are in general subjected to a number of damage modes on the microscale (Drabek and Bohm, 2004). Debonding phenomenon existed in between particles and the matrix is one of the major damage modes in particle reinforced ductile matrix composites (PRDMCs) and its effect on the mechanical behavior of the composites has to be well addressed for an accurate analysis of the composites (Lee and Pyo, in press). Ju and Chen (1994c) developed a micromechanical formulation to predict the effective elastoplastic behavior of two-phase PRDMCs under arbitrary loading histories by considering the first-order stress perturbations of elastic particles on the ductile matrix. Ju and Tseng (1996, 1997) further improved Ju and Chen’s (1994c) work by incorporating second-order stress perturbations due to pairwise particle interactions.

Qu (1993a,b) derived the Eshelby’s tensor for an ellipsoidal inclusion with slightly weakened interface in an elastic matrix of infinite extent where the weakened interface between the inclusion and the matrix was modeled by a spring layer of vanishing thickness. Lee and Pyo (2007) proposed a micromechanics-based elastic damage model to predict the effective elastic behavior and weakened interface evolution in particle composites. The Eshelby’s tensor for an ellipsoidal inclusion with slightly weakened interface (Qu, 1993a,b) was adopted in their formulations to model the weakened interface. Most recently, Lee and Pyo (in press) proposed a multi-level elastic damage model based on a combination of a micromechanical formulation and a multi-level damage model to predict the effective elastic behavior and progressive weakened interface in particulate (brittle) composites. Their multi-level damage model in accordance with the Weibull’s probabilistic function described the sequential, progressive weakened interface in the composites.

The primary objective of the present work is the extension of the framework of Lee and Pyo (in press) to accommodate the elastoplastic behavior and multi-level damage evolution in PRDMCs. Following Lee and Pyo (in press), the four-level elastic damage model, which was illustrated in Fig. 1 of Lee and Pyo (in press), is adopted for a complete description of the sequential progression of weakened interface in the composite: (1) Level 0 of two-phase composite state consisting of a ductile matrix and perfectly bonded particles; (2) Level 1 of three-phase composite state consisting of a ductile matrix, perfectly bonded particles and particles having mildly weakened interface; (3) Level 2 of four-phase composite state consisting of a ductile matrix, perfectly bonded particles, particles having mildly weakened interface and particles having severely weakened interface; (4) Level 3 of five-phase composite state consisting of a ductile matrix, perfectly bonded particles, particles having mildly weakened interface, particles having severely weakened interface and completely debonded particles.

Since two different damage modes of weakened interface (mildly weakened interface and severely weakened interface) are assumed to occur sequentially (Lee and Pyo, in press), weakened interface would be developed and transformed from one mode to another as deformations or loadings continue to increase. Accordingly, there may be two different modes of weakened interface (mildly weakened interface and severely weakened interface) occurring simultaneously at the Level 2 of four-phase composite state. It will be followed by three different modes of weakened interface (mildly weakened interface, severely weakened interface and complete debonding) occurring simultaneously at the Level 3 of five-phase composite state at the next step. The two-parameter Weibull statistics is adopted to estimate the volume fraction of particles with different modes of weakened interface.

Particles are assumed to be non-interacting, randomly dispersed, elastic spheres that are initially embedded firmly in the ductile matrix with perfect interfaces. It is also assumed that the progression of weakened interface is governed by the average internal stresses of perfect bonded particles (Phase 1) as well as the Weibull parameters (Weibull, 1951). Numerical examples corresponding to uniaxial, biaxial and triaxial tension loadings are solved to illustrate the potential of the proposed micromechanical framework. A series of parametric analysis are carried out to investigate the influence of model parameters on the progression of weakened interface in the composites. Furthermore, the present prediction is compared with available experimental data in the literature to further illustrate the elastoplastic damage behavior of the present framework and to verify the proposed elastoplastic multi-level damage model.
The present paper is organized as follows. The effective stiffness tensor for ductile matrix composites with weakened interfaces (Lee and Pyo, in press) is summarized in Section 2. An effective yield criterion is micro-mechanically constructed based on the ensemble-volume averaging procedure and the first-order effects of eigenstrains due to the existence of spherical inclusions in Section 3. In Section 4, a multi-level damage model (Lee and Pyo, in press) for progressive weakened interface in accordance with the Weibull’s probabilistic function is also recapitulated. In Section 5, the proposed elastoplastic multi-level damage formulation is applied to the uniaxial, biaxial and triaxial loading conditions, and corresponding numerical simulations are illustrated. A series of parametric analysis to address the influence of model parameters on the progressive weakened interface in the composites are conducted in Section 6. Finally, the present prediction is compared with experimental data (Papazian and Adler, 1990) in Section 7.

2. Recapitulation of the effective stiffness tensor for composites with weakened interfaces

When small strain formulations are considered, the total macroscopic strain \( \varepsilon \) of ductile matrix composites can be decomposed into two parts according to the theory of continuum plasticity:

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]  

where \( \varepsilon^e \) is the effective (macroscopic) elastic strain and \( \varepsilon^p \) signifies the effective (macroscopic) plastic strain of the composites. In addition, the macroscopic elastic strain \( \varepsilon^e \) is related with the macroscopic stress \( \sigma \) via the following constitutive relation:

\[ \sigma = C^* : \varepsilon^e \]  

where the effective stiffness \( C^* \) of ductile matrix composites with weakened interfaces were explicitly derived by Lee and Pyo (in press) and are briefly repeated here for completeness of the proposed model.

First, an initially perfectly bonded two-phase composite consisting of a ductile matrix (Phase 0) with bulk modulus \( k_0 \) and shear modulus \( \mu_0 \), and randomly dispersed elastic spherical particles (Phase 1) with bulk modulus \( k_1 \) and shear modulus \( \mu_1 \). As loads or deformations continue to increase, some initially perfectly bonded particles are transformed to particles with mildly weakened interface, some particles with mildly weakened interface are then transformed to particles with severely weakened interface, and all particles are transformed to completely debonded particles (voids) asymptotically. The Eshelby’s tensor for perfectly bonded spherical particles and completely debonded spherical particles in the matrix was driven in Ju and Chen (1994a,b), and the Eshelby’s tensor for a spherical particle with weakened interface in the matrix was driven in our preceding work (Lee and Pyo, 2007, in press) following Qu (1993a,b).

With the help of the Eshelby’s tensors for perfectly bonded spherical particles, completely debonded spherical particles, spherical particles with mildly weakened interface and spherical particles with severely weakened interface, the effective stiffness tensor \( C^* \) for multi-level, ductile matrix composites was explicitly derived in our previous work (Lee and Pyo, in press) based on the governing equations for linear elastic composites containing arbitrarily non-aligned and/or dissimilar inclusions (Ju and Chen, 1994a) as

\[ C^* = \lambda^* \delta_{ij} \delta_{kl} + \mu^* (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \]  

where the effective Lame constants \( \lambda^* \) and \( \mu^* \) in Eq. (3) for the two-phase composite state (Level 0), three-phase composite state (Level 1), and four-phase composite state (Level 2) were given in Eqs. (26) and (27); (28) and (29); and (30) and (31), respectively, of Lee and Pyo (in press). In addition, the effective Lame constants \( \lambda^* \) and \( \mu^* \) in Eq. (3) for the five-phase composite state (Level 3) can be expressed as (Lee and Pyo, in press)

\[ \lambda^* = (3\lambda_0 + 2\mu_0)(A_1 + A_3 + A_5 + A_7) + 2\lambda_0 \left( \frac{1}{2} + A_2 + A_4 + A_6 + A_8 \right) \]  

\[ \mu^* = 2\mu_0 \left( \frac{1}{2} + A_2 + A_4 + A_6 + A_8 \right) \]  

where the components \( A_1, \ldots, A_8 \) were listed in Appendix of Lee and Pyo (in press).
3. Ensemble-volume-average homogenization process

3.1. Overview

The homogenized elastoplastic damage responses of ductile matrix composites with weakened interfaces are formulated in this section. Specifically, five-phase composites consisting of a ductile matrix (Phase 0), perfectly bonded particles (Phase 1), particles having mildly weakened interface (Phase 2), particles having severely weakened interface (Phase 3) and completely debonded particles (voids) (Phase 4) are considered. The von-Mises yield criterion with isotropic hardening law is adopted here for the ductile matrix; thus, the stress \( \sigma \) and the equivalent plastic strain \( \varepsilon^p \) at any matrix point satisfy the following yield function (see, e.g., Ju and Lee, 2000; Ju and Ko, in press):

\[
F(\sigma, \varepsilon^p) = H(\sigma) - K^2(\varepsilon^p) \leq 0
\]

(6)

where \( K(\varepsilon^p) \) denotes the isotropic hardening function of the ductile matrix and \( H(\sigma) \equiv \sigma : \mathbf{I}_d : \sigma \) signifies the square of the deviatoric stress norm in which \( \mathbf{I}_d \) denotes the deviatoric part of the fourth-rank identity tensor.

Following the work of Ju and Chen (1994c), Ju and Lee (2000, 2001), Sun et al. (2003), Ju et al. (2006) and Ju and Ko (in press), a micromechanical framework, where an ensemble-averaged yield criterion is constructed for the entire composite and only the first-order effects are considered in the formulation of effective plastic response, is employed.

3.2. A first-order formulation of the current stress norm of multi-phase ductile matrix composites

Following Ju and Chen (1994c), Ju and Tseng (1996) and Ju and Lee (2000), the square of the “current stress norm” at the local point \( x \), which determines the plastic strain in ductile matrix composites for a given phase configuration \( D \), can be written as

\[
H(x|D) = \left\{ \begin{array}{ll}
\sigma(x|D) : \mathbf{I}_d : \sigma(x|D), & \text{if } x \text{ in the matrix;} \\
0, & \text{otherwise.}
\end{array} \right.
\]

(7)

The ensemble average of \( H(x|D) \) over all possible realizations, denoted by \( \langle H \rangle_m(x) \), can be obtained by integrating \( H \) over all possible configurations (cf. Ju and Lee, 2000)

\[
\langle H \rangle_m(x) = H^o + \int_{D_1} \{H(x|D_1) - H^o\} P(D_1) \, dD + \int_{D_2} \{H(x|D_2) - H^o\} P(D_2) \, dD \\
+ \int_{D_3} \{H(x|D_3) - H^o\} P(D_3) \, dD + \int_{D_4} \{H(x|D_4) - H^o\} P(D_4) \, dD
\]

(8)

where \( P(D_q) \) is the probability density function for finding the \( q \)-phase (\( q = 1, 2, 3, 4 \)) configuration \( D_q \) in the composites and \( H^o = \sigma^o : \mathbf{I}_d : \sigma^o \) is the square of the far-field stress norm in the matrix.

Moreover, the total stress at any point \( x \) in the matrix can be decomposed into two parts: the far-field stress \( \sigma^o \) and the perturbed stress \( \sigma^p \) due to the presence of the particles and voids as:

\[
\sigma(x) = \sigma^o + \sigma^p(x)
\]

(9)

with \( \sigma^o \equiv C_0 : \varepsilon^o \) where \( C_0 \) denotes the fourth-rank elasticity tensor of the matrix and \( \varepsilon^o \) is the elastic strain field induced by the far-field loading. Following Ju and Chen (1994c) and Ju and Lee (2000), the perturbed stress for any matrix point \( x \) due to a typical isolated \( q \)-phase inhomogeneity centered at \( x^{(1)}_q \) reads

\[
\sigma^p(x|x^{(1)}_q) = [C_0 \cdot \mathbf{G}(x - x^{(1)}_q)] : \varepsilon_q^{ao}
\]

(10)

where \( \varepsilon^{ao}_q \) is the solution of the eigenstrain \( \varepsilon_q^e \) for the single inclusion problem of the \( q \)-phase, and

\[
\mathbf{G}(\tilde{r}_q) = \frac{1}{30(1 - v_0)} \left( \rho_q^1 \mathbf{H}^1 + \rho_q^5 \mathbf{H}^2 \right)
\]

(11)

The components of \( \mathbf{H}^1 \) and \( \mathbf{H}^2 \) are given by
\[
\mathbf{H}^{1}_{ijk\ell}(\mathbf{r}_q) \equiv 5 \tilde{\mathbf{F}}_{ijk\ell}(-15, 3v_0, 3, 3 - 6v_0, -1 + 2v_0, 1 - 2v_0) \\
\mathbf{H}^{2}_{ijk\ell}(\mathbf{r}_q) \equiv 3 \tilde{\mathbf{F}}_{ijk\ell}(35, -5, -5, -5, 1, 1)
\]

where \( \mathbf{r}_q = \mathbf{x} - \mathbf{x}_q^{(1)} \), \( \tilde{\mathbf{r}}_q \equiv \| \tilde{\mathbf{r}}_q \| \), \( \rho_{q} = \alpha / \tilde{r}_q \), \( \alpha \) is the radius of a spherical particle or void, and \( v_0 \) is the Poisson’s ratio of the matrix material. The components of the fourth-rank tensor \( \tilde{\mathbf{F}} \)—which depends on six scalar quantities \( B_1, B_2, B_3, B_4, B_5, B_6 \)—were previously defined by Ju and Lee (2000):

\[
\tilde{\mathbf{F}}_{ijk\ell}(B_m) \equiv B_1 \delta_{ik} \mathbf{n}_i \mathbf{n}_j \mathbf{n}_k + B_2 (\delta_{ik} \mathbf{n}_i \mathbf{n}_k + \delta_{ij} \mathbf{n}_k \mathbf{n}_i + \delta_{jk} \mathbf{n}_i \mathbf{n}_i) + B_3 \delta_{ij} \mathbf{n}_i \mathbf{n}_j + B_4 \delta_{ij} \mathbf{n}_i \mathbf{n}_j
\]

with the unit outer normal vector \( \mathbf{n} \equiv \mathbf{n}_q/\tilde{r}_q \) and index \( m = 1–6 \).

In addition, the noninteracting eigenstrain \( \mathbf{e}^{\mathrm{so}}_q \) in Eq. (10) is given by (see, e.g., Ju and Chen, 1994a)

\[
\mathbf{e}^{\mathrm{so}}_q = - (\mathbf{A}_q + \mathbf{S}_q)^{-1} : \mathbf{e}^o, \quad q = 1, 2, 3, 4
\]

where \( \mathbf{S}_q \) is the Eshelby’s tensor for the \( q \)-phase inclusion and the fourth-rank tensor \( \mathbf{A}_q \) is defined as \( \mathbf{A}_q \equiv (\mathbf{C}_q - \mathbf{C}_0)^{-1} \). \( \mathbf{C}_0 \) in which \( \mathbf{C}_q \) signifies the elasticity tensor of the \( q \)-phase.

3.3. A first-order formulation of effective elastoplastic behavior of multi-phase ductile matrix composites

Since a matrix point receives the perturbations from particles and voids, the ensemble-average stress norm for any matrix point \( \mathbf{x} \) can be evaluated by collecting and summing up all the current stress norm perturbations produced by any typical particle or void centered at \( \mathbf{x}_q^{(1)} \) (cf. Ju and Lee, 2000). Accordingly, \( \langle H \rangle_m(\mathbf{x}) \) in Eq. (8) can be rephrased as

\[
\langle H \rangle_m(\mathbf{x}) \cong \mathbf{H}^o + \frac{N_1}{V} \int_{A(\tilde{r}_1)} dA \int_{A(\tilde{r}_1)} \langle H(\tilde{r}_1) - \mathbf{H}^o \rangle dA + \frac{N_2}{V} \int_{A(\tilde{r}_2)} dA \int_{A(\tilde{r}_2)} \langle H(\tilde{r}_2) - \mathbf{H}^o \rangle dA
\]

where \( P(\mathbf{x}_q^{(1)}) \) signifies the probability density functions for finding a particle or a void centered at \( \mathbf{x}_q^{(1)} \). It is assumed that the probability density functions take the form \( P(\mathbf{x}_q^{(1)}) = N_q/V \), where \( N_q \) is the total numbers of particles or voids dispersed in a representative volume \( V \). Further, Eq. (16) can be recast into a more simplified form:

\[
\langle H \rangle_m(\mathbf{x}) \cong \mathbf{H}^o + \frac{N_1}{V} \int_{A(\tilde{r}_1)} dA \int_{A(\tilde{r}_1)} \langle H(\tilde{r}_1) - \mathbf{H}^o \rangle dA + \frac{N_2}{V} \int_{A(\tilde{r}_2)} dA \int_{A(\tilde{r}_2)} \langle H(\tilde{r}_2) - \mathbf{H}^o \rangle dA
\]

where \( A(\tilde{r}_q) \) is a spherical surface of radius \( \tilde{r}_q (q = 1, 2, 3, 4) \).

Using the two identities Eqs. (28) and (29) in Ju and Chen (1994c) and the perturbed stress given in Eq. (10), the ensemble-averaged current stress norm at any matrix point can be obtained as:

\[
\langle H \rangle_m(\mathbf{x}) = \mathbf{e}^o : T^{(q)} : \mathbf{e}^o
\]

The components of the positive definite fourth-rank tensor \( T^{(q)} \) for the “\( q \)”-phase composite state read

\[
T^{(q)}_{ijkl} = T^{(q)}_1 \delta_{ij} \delta_{kl} + T^{(q)}_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

where \( T^{(5)}_1 \) and \( T^{(5)}_2 \) for the five-phase composite state are given by
3T_1^{(5)} + 2T_2^{(5)} = 50(1 - 2\nu_0)^2 \left[ \frac{\phi_1}{(3\omega_1 + 2\psi_1)^2} + \frac{80^2\alpha^2(1 - \nu_0)^2\phi_2}{(3\omega_2 + 2\psi_2)^2} + \frac{80^2\alpha^2(1 - \nu_0)^2\phi_3}{(3\omega_3 + 2\psi_3)^2} + \frac{\phi_4}{(3\omega_4 + 2\psi_4)^2} \right] \quad (20)

T_2^{(5)} = \frac{1}{2} + (23 - 50\nu_0 + 35\nu_0^2) \left[ \frac{\phi_1}{4\psi_1} + \frac{40^2\alpha^2(1 - \nu_0)^2\phi_2}{\psi_2^2} + \frac{40^2\alpha^2(1 - \nu_0)^2\phi_3}{\psi_3^2} + \frac{\phi_4}{4\psi_4} \right] \quad (21)

where the volume fraction for the q-phase is defined as \phi_q \equiv \frac{4\pi^3}{3}\frac{N_q}{V}, and the parameters \omega_1, \ldots, \omega_4 and \psi_1, \ldots, \psi_4 are listed in Appendix.

Similarly, \( T_1^{(2)} \) and \( T_2^{(2)} \), \( T_1^{(3)} \), \( T_2^{(3)} \), \( T_1^{(4)} \) and \( T_2^{(4)} \) for the two-, three-, four-phase composite states, respectively, read:

\[
3T_1^{(2)} + 2T_2^{(2)} = 50(1 - 2\nu_0)^2 \frac{\phi_1}{(3\omega_1 + 2\psi_1)^2}
\]

\[
T_2^{(2)} = \frac{1}{2} + (23 - 50\nu_0 + 35\nu_0^2) \frac{\phi_1}{4\psi_1}
\]

\[
3T_1^{(3)} + 2T_2^{(3)} = 50(1 - 2\nu_0)^2 \left[ \frac{\phi_1}{(3\omega_1 + 2\psi_1)^2} + \frac{80^2\alpha^2(1 - \nu_0)^2\phi_2}{(3\omega_2 + 2\psi_2)^2} \right]
\]

\[
T_2^{(3)} = \frac{1}{2} + (23 - 50\nu_0 + 35\nu_0^2) \left[ \frac{\phi_1}{4\psi_1} + \frac{40^2\alpha^2(1 - \nu_0)^2\phi_2}{\psi_2^2} \right]
\]

\[
3T_1^{(4)} + 2T_2^{(4)} = 50(1 - 2\nu_0)^2 \left[ \frac{\phi_1}{(3\omega_1 + 2\psi_1)^2} + \frac{80^2\alpha^2(1 - \nu_0)^2\phi_2}{(3\omega_2 + 2\psi_2)^2} + \frac{80^2\alpha^2(1 - \nu_0)^2\phi_3}{(3\omega_3 + 2\psi_3)^2} \right]
\]

\[
T_2^{(4)} = \frac{1}{2} + (23 - 50\nu_0 + 35\nu_0^2) \left[ \frac{\phi_1}{4\psi_1} + \frac{40^2\alpha^2(1 - \nu_0)^2\phi_2}{\psi_2^2} + \frac{40^2\alpha^2(1 - \nu_0)^2\phi_3}{\psi_3^2} \right]
\]

Following Ju and Chen (1994c), the relation between the far-field stress \( \sigma^q \) and the macroscopic stress \( \bar{\sigma} \) takes the form

\[
\sigma^q = P^{(q)} : \bar{\sigma}
\]

where the components of \( P^{(q)} \) for the “q”-phase composite state are

\[
P^{(q)}_{ijkl} = P_1^{(q)} \delta_{ij} \delta_{kl} + P_2^{(q)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

with

\[
3P_1^{(q)} + 2P_2^{(q)} = \frac{1}{1 + \eta_1^{(q)}}
\]

\[
P_2^{(q)} = \frac{1}{2 \left( 1 + \eta_2^{(q)} \right)}
\]

and the coefficients \( \eta_1^{(5)} \) and \( \eta_2^{(5)} \) for the five-phase composite state read:

\[
\eta_1^{(5)} = \frac{10\phi_1(1 - 2\nu_0)}{3\omega_1 + 2\psi_1} + \frac{\phi_2(-3\chi_1 - 2\chi_2 + 1200\alpha(1 - \nu_0)^2)}{3\omega_2 + 2\psi_2} + \frac{\phi_3(-3\chi_3 - 2\chi_4 + 1200\alpha(1 - \nu_0)^2)}{3\omega_3 + 2\psi_3}
\]

\[
+ \frac{10\phi_4(1 - 2\nu_0)}{3\omega_4 + 2\psi_4}
\]

\[
\eta_2^{(5)} = \frac{\phi_1(7 - 5\nu_0)}{2\psi_1} + \frac{\phi_2(600\alpha(1 - \nu_0)^2 - \chi_2)}{\psi_2} + \frac{\phi_3(600\alpha(1 - \nu_0)^2 - \chi_4)}{\psi_3} + \frac{\phi_4(7 - 5\nu_0)}{2\psi_4}
\]
where the parameters $\chi_1, \ldots, \chi_4$ are listed in Appendix. Similarly, the coefficients $\eta_1^{(2)}$, $\eta_2^{(2)}$, $\eta_1^{(3)}$, $\eta_2^{(3)}$, $\eta_1^{(4)}$, and $\eta_2^{(4)}$ for the two-, three-, four-phase composite states, respectively, read:

$$
\eta_1^{(2)} = \frac{10\phi_1(1-2v_0)}{3\omega_1 + 2\psi_1},
$$

$$
\eta_2^{(2)} = \frac{\phi_1(7-5v_0)}{2\psi_1},
$$

$$
\eta_1^{(3)} = \frac{10\phi_1(1-2v_0)}{3\omega_1 + 2\psi_1} + \phi_2\left\{ -3\chi_1 - 2\chi_2 + 1200a(1-v_0)^2 \right\}
$$

$$
\eta_2^{(3)} = \frac{\phi_1(7-5v_0)}{2\psi_1} + \phi_2\left\{ 600a(1-v_0)^2 - \chi_2 \right\},
$$

$$
\eta_1^{(4)} = \frac{10\phi_1(1-2v_0)}{3\omega_1 + 2\psi_1} + \phi_2\left\{ -3\chi_1 - 2\chi_2 + 1200a(1-v_0)^2 \right\} + \phi_3\left\{ -3\chi_3 - 2\chi_4 + 1200a(1-v_0)^2 \right\}
$$

$$
\eta_2^{(4)} = \frac{\phi_1(7-5v_0)}{2\psi_1} + \phi_2\left\{ 600a(1-v_0)^2 - \chi_2 \right\} + \phi_3\left\{ 600a(1-v_0)^2 - \chi_4 \right\}. 
$$

By combining Eqs. (18) and (28), the ensemble-averaged current stress norm in a matrix point can be rephrased as:

$$
\langle H \rangle_m(x) = \mathbf{\sigma} : \mathbf{T}^{(q)} : \mathbf{\sigma}
$$

where the positive definite fourth-rank tensor $\mathbf{T}^{(q)}$ for the “$q$”-phase composite state is defined as

$$
\mathbf{T}^{(q)} = [\mathbf{p}^{(q)}]^T : \mathbf{T}^{(q)} : \mathbf{p}^{(q)}
$$

and can be shown to be

$$
\mathbf{T}^{(q)}_{ijkl} = T^{(q)}_{ijkl} \delta_{ij} \delta_{kl} + T^{(q)}_{ijkl} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
$$

where

$$
3T^{(q)}_{11} + 2T^{(q)}_{22} = \frac{3T^{(q)}_{11} + 2T^{(q)}_{22}}{(1 + \eta_1^{(q)})^2}
$$

$$
T^{(q)}_{22} = \frac{T^{(q)}_{22}}{(1 + \eta_2^{(q)})^2}
$$

Following Ju and Lee (2000), the effective yield function for the multi-phase ductile matrix composites can be proposed as

$$
F = (1 - \phi_1)\mathbf{\sigma} : \mathbf{T} : \mathbf{\sigma} - K^2(\mathbf{\varepsilon}^p)
$$

where $\phi_1$ is the current (perfectly bonded) particle volume fraction. The effective equivalent plastic strain rate for the composites is defined as (Ju and Lee, 2000)

$$
\dot{\mathbf{\varepsilon}}^p = \sqrt{\frac{2}{3}} \dot{\mathbf{\varepsilon}}^p : \mathbf{T}^{-1} : \dot{\mathbf{\varepsilon}}^p = 2(1 - \phi_1)^2 \lambda \sqrt{\frac{2}{3}} \mathbf{\sigma} : \mathbf{T} : \mathbf{\sigma}
$$

in which $\lambda$ denotes the plastic consistency parameter. The $\lambda$ together with the yield function $F$ must obey the Kuhn-Tucker loading/unloading conditions. In our derivations, the simple power-law type isotropic hardening function is employed (see also Ju and Lee, 2000):

$$
K(\mathbf{\varepsilon}^p) = \sqrt{\frac{2}{3}} [\mathbf{\sigma}_y + h(\mathbf{\varepsilon}^p)^q]
$$
where \(\sigma_y\) denotes the initial yield stress, and \(h\) and \(\tilde{q}\) are the linear and exponential isotropic hardening parameters, respectively, for the multi-phase composites.

4. Multi-level damage modeling

A four-level damage model developed by Lee and Pyo (in press) in the order of sequence of progressive weakened interface as illustrated in Fig. 1 of Lee and Pyo (in press) is employed here to model the sequential, progressive weakened interface in ductile matrix composites. The model is briefly recapitulated in this section.

Following Tohgo and Weng (1994) and Zhao and Weng (1995, 1996, 1997), the probability of weakened interface is modeled as a two-parameter Weibull process and the average internal stresses of perfect bonded particles (Phase 1) are the controlling factor of the Weibull function. Assuming that (1) the Weibull (1951) statistics governs and (2) some initially perfectly bonded particles are transformed to particles with mildly weakened interface as illustrated in Fig. 1 of Lee and Pyo (in press) is employed here to model the sequential, progressive weakened interface in ductile matrix composites. The model is briefly recapitulated in this section.

Following Tohgo and Weng (1994) and Zhao and Weng (1995, 1996, 1997), the probability of weakened interface is modeled as a two-parameter Weibull process and the average internal stresses of perfect bonded particles (Phase 1) are the controlling factor of the Weibull function. Assuming that (1) the Weibull (1951) statistics governs and (2) some initially perfectly bonded particles are transformed to particles with mildly weakened interface, some particles with mildly weakened interface are then transformed to particles with severely weakened interface, and all particles are transformed to completely debonded particles (voids) asymptotically, the current volume fractions of completely debonded particles \(\phi_4\), particles having severely weakened interface \(\phi_3\), particles having mildly weakened interface \(\phi_2\) and perfectly bonded particles \(\phi_1\) in the five-phase composite at a given level of \((\bar{\sigma}_{11})_1\) can be derived through the following three-step Weibull approach

\[
\bar{\phi}_2 = \phi \left\{ 1 - \exp \left[ - \left( \frac{(\bar{\sigma}_{11})_1}{S_0} \right)^M \right] \right\} \\
\bar{\phi}_3 = \bar{\phi}_2 \left\{ 1 - \exp \left[ - \left( \frac{(\bar{\sigma}_{11})_1}{S_0} \right)^M \right] \right\} \\
\phi_4 = \bar{\phi}_3 \left\{ 1 - \exp \left[ - \left( \frac{(\bar{\sigma}_{11})_1}{S_0} \right)^M \right] \right\} \\
\phi_3 = \bar{\phi}_3 - \phi_4 \\
\phi_2 = \bar{\phi}_2 - \bar{\phi}_3 \\
\phi_1 = \phi - \bar{\phi}_2
\]

(48) (49) (50) (51) (52) (53)

where \(\phi\) is the original particle volume fraction, \((\bar{\sigma}_{11})_1\) is the internal stress of particles (Phase 1) in the 1-direction, the subscript \((\cdot)_1\) denotes the particle phase, and \(S_0\) and \(M\) are the Weibull parameters. For the biaxial and triaxial loading cases, \((\bar{\sigma}_{11})_1\) is replaced by \([(\bar{\sigma}_{22})_1 + (\bar{\sigma}_{33})_1]/2\) and \([(\bar{\sigma}_{11})_1 + (\bar{\sigma}_{22})_1 + (\bar{\sigma}_{33})_1]/3\), respectively (see also Ju and Lee, 2000).

The internal stresses of particles required for the initiation of the weakened interface for the multi-phase composite state were explicitly given by (Lee and Pyo, in press)

\[
\bar{\sigma}_1 \equiv U : \bar{\varepsilon}
\]

with

\[
U_{ijkl} = U_1 \delta_{ij} \delta_{kl} + U_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]

(54) (55)

where \(U_1\) and \(U_2\) for the five-phase composite state in Eq. (55) can be expressed as (Lee and Pyo, in press)

\[
U_1 = \frac{(1 - 2\bar{\zeta}_2 - 2\bar{\zeta}_4 - 2\bar{\zeta}_6 - 2\bar{\zeta}_8)\{ -\bar{\zeta}_1(3\lambda_1 + 2\mu_1) + \lambda_1(\phi_1 - 2\bar{\zeta}_2) \} + 2\mu_1(\bar{\zeta}_1 + \bar{\zeta}_3 + \bar{\zeta}_5 + \bar{\zeta}_7)(\phi_1 - 2\bar{\zeta}_2)}{\phi_1(1 - 2\bar{\zeta}_2 - 2\bar{\zeta}_4 - 2\bar{\zeta}_6 - 2\bar{\zeta}_8)(1 - 3\bar{\zeta}_1 - 2\bar{\zeta}_2 - 3\bar{\zeta}_3 - 2\bar{\zeta}_4 - 3\bar{\zeta}_5 - 2\bar{\zeta}_6 - 3\bar{\zeta}_7 - 2\bar{\zeta}_8)}
\]

(56)

\[
U_2 = \frac{\mu_1(\phi_1 - 2\bar{\zeta}_2)}{\phi_1(1 - 2\bar{\zeta}_2 - 2\bar{\zeta}_4 - 2\bar{\zeta}_6 - 2\bar{\zeta}_8)}
\]

(57)
where the components $\xi_1, \ldots, \xi_8$ were listed in Appendix of Lee and Pyo (in press). The damage modeling for the three-phase and four-phase composite states can also be found in Lee and Pyo (in press).

5. Numerical simulations

To show the potential of the proposed micromechanical framework, we first predict the behavior of silicon-carbide particle reinforced 6061-T6 aluminum alloy matrix composites under uniaxial tension loading ($\bar{\sigma}_{11} \neq 0$ and $\bar{\sigma}_{ij} = 0$ for all other stress components). The elastic properties of the composites are adopted according to Zhao and Weng (1995), Ju and Lee (2001) and Lee and Pyo (2007) as: $E_0 = 68.3$ GPa, $\nu_0 = 0.33$, $E_1 = 490$ GPa, $\nu_1 = 0.17$, $\phi = 0.2$. Following the parametric analysis on the compliance parameters $\alpha$ and $\beta$ reported in preceding papers (Lee and Pyo, 2007, in press), the compliance parameters of the mildly weakened interface, denoted by $\alpha_1$ and $\beta_1$, and the severely weakened interface, denoted by $\alpha_2$ and $\beta_2$, in Eq. (6) of Lee and Pyo (in press) are chosen to be $\alpha_1 = 2.0 \times 10^{-5}$, $\beta_1 = 3.0 \times 10^{-5}$; $\alpha_2 = 2.0$, $\beta_2 = 3.0$. Moreover, the plastic parameters and Weibull parameters are assumed to be $\sigma_y = 400$ MPa, $h = 220$ MPa, $\bar{\sigma} = 0.45$; $S_0 = 380$ MPa, $M = 5$.

Four different phase composite states as illustrated in Fig. 1 of Lee and Pyo (in press) are considered in this simulation to examine the effect of the level of damage on the elastoplastic stress–strain response of the composites. The present predicted elastoplastic stress–strain responses of the particulate composites under uniaxial tension considering the four different phase composite states are plotted in Fig. 1. Fig. 2 exhibits the predicted evolution of volume fractions of perfectly bonded particles and particles with mildly weakened interface of the composites at the three-phase composite state versus uniaxial strains, which is corresponding to Fig. 1. The predicted evolution of volume fractions of perfectly bonded particles and various types of damaged particles at the four- and five-phase composite states versus uniaxial strains corresponding to Fig. 1 are exhibited in Figs. 3 and 4, respectively.

In Figs. 1–4, the elastoplastic stress–strain responses corresponding to various phase composite states are shown to be bounded by the responses of the two- and five-phase composite states. It is observed that the responses with the higher phase composite states are lower than those with the lower phase composite states.
It is also noted from the figures that a smooth and gradual transition from the perfectly bonded interface to the perfectly debonded interface occurs within the present framework.

In the present numerical simulations corresponding to Figs. 2–4, the lower value of the Weibull parameters, which relates to the interfacial strength, \( S_0 = 380 \text{ MPa} \) and \( M = 5.0 \) is used to clearly show the effect of

---

**Fig. 2.** The predicted evolution of volume fractions of perfectly bonded particles and particles with mildly weakened interface of the composites at the three-phase composite state corresponding to Fig. 1.

**Fig. 3.** The predicted evolution of volume fractions of perfectly bonded particles, particles with mildly weakened interface and particles with severely weakened interface of the composites at the four-phase composite state corresponding to Fig. 1.
weakened interface on the stress–strain behavior of weakened particulate composites. Experimental characterization of the parameters is, however, required for more accurate prediction of weakened interface evolution. Details of the relationship between the parameter and interfacial strength can be found in Lee (2001).

The proposed multi-level elastic damage model is further exercised to predict the behavior of the composites under biaxial tension ($\sigma_{22} = \sigma_{33} \neq 0$ and $\sigma_{ij} = 0$ for all other components) and triaxial tension.

Fig. 4. The predicted evolution of volume fractions of perfectly bonded particles, particles with mildly weakened interface, particles with severely weakened interface and completely debonded particles of the composites at the five-phase composite state corresponding to Fig. 1.

Fig. 5. The present predicted elastoplastic stress–strain responses of particulate composites under biaxial tension.
(\(\sigma_{22} = \sigma_{33} = \Gamma \sigma_{11} \neq 0\) and \(\sigma_{ij} = 0\) for all other components) loading cases. The proportional loading ratio \(\Gamma\) is chosen to be 0.6 for the axisymmetric triaxial tension in this simulation. The predicted stress–strain responses of the composites considering the four different phase composite states under biaxial tension and triaxial tension loading cases are exhibited in Figs. 5 and Fig. 7, respectively. The predicted evolution of volume fractions of perfectly bonded particles and various types of damaged particles under biaxial tension loading correspond-

Fig. 6. The predicted evolution of volume fractions of perfectly bonded particles and various types of damaged particles corresponding to Fig. 5.

Fig. 7. The present predicted stress–strain responses of particulate composites under triaxial tension (\(\Gamma = 0.6\)).
Fig. 8. The predicted evolution of volume fractions of perfectly bonded particles and various types of damaged particles corresponding to Fig. 7.

Fig. 9. The present predicted elastoplastic stress–strain responses of PRDMCs under uniaxial tension at the four-phase composite state with various $S_0$ values.
Fig. 7. It is observed from Figs. 1, 5 and 7 that triaxial tension results in a higher elastoplastic stress–strain behavior, which corresponds to findings from Ju and Lee’s (2000) and Lee and Pyo’s (2007) observations.

6. Parametric analysis

A series of parametric analysis are conducted to examine the influence of the Weibull parameters $S_0$ and $M$ and the compliance parameters $\alpha$ and $\beta$ on the elastoplastic behavior and progression of weakened interface in PRDMCs. First, following Lee’s (2001) investigation on the Weibull parameters, three sets of Weibull parameters that are closely related to the strength at the particle-matrix interface in the PRDMCs are used: $S_0 = 1.09 \times \sigma_y, M = 5; S_0 = 2.18 \times \sigma_y, M = 5; S_0 = 3.27 \times \sigma_y, M = 5$. For simplicity, a uniaxial tensile simulation is conducted using the same elastic properties as used in Section 5. The compliance parameters and the plastic parameters are assume to be $\alpha_1 = 2.0 \times 10^{-5}, \beta_1 = 3.0 \times 10^{-5}, \alpha_2 = 2.0, \beta_2 = 3.0; \sigma_y = 250$ MPa, $h = 220$ MPa, $\bar{q} = 0.45$.

Fig. 9 shows the predicted elastoplastic stress–strain responses of the PRDMCs under uniaxial tension at the four-phase composite state with various $S_0$ values. It is clear from the figure that higher interfacial strength parameter $S_0$ leads to higher elastoplastic stress–strain response. It is also found from this parametric analysis that the influence of the Weibull parameter $S_0$ on the elastoplastic response is quite influential. Thus, the Weibull parameter $S_0$ needs to be experimentally characterized via the measurement of the interfacial strength of PRDMCs. The methodology for determining model parameters including the Weibull parameters can be found in our preceding work (Lee and Pyo, in press). A parametric analysis conducted to determine lower and upper limits of the Weibull parameters can also be found in Lee (2001).

To evaluate the proposed elastoplastic multi-level damage model sensitivity to the compliance parameters $\alpha$ and $\beta$, a parametric analysis of $\alpha$ and $\beta$ is also carried out. Following Lee and Pyo (in press), four sets of the compliance parameters for the severely weakened interface are used: $\alpha_2 = 2.0, \beta_2 = 3.0; \alpha_2 = 2.0 \times 10^{-4}, \beta_2 = 3.0 \times 10^{-4}; \alpha_2 = 6.0 \times 10^{-5}, \beta_2 = 7.0 \times 10^{-5}; \alpha_2 = 2.0 \times 10^{-5}, \beta_2 = 3.0 \times 10^{-5}$. The compliance parameters for the mildly weakened interface are fixed as follows: $\alpha_1 = 2.0 \times 10^{-5}, \beta_1 = 3.0 \times 10^{-5}$. Moreover, the Weibull
parameters and plastic parameters are assumed to be $S_0 = 1.09 \times \sigma_y$, $M = 5$; $\sigma_y = 250$ MPa, $h = 220$ MPa and $\bar{q} = 0.45$. The predicted stress–strain responses of PRDMCs with severely weakened interfaces under uniaxial tension with various values of $\alpha_2$ and $\beta_2$ are shown in Fig. 10. It is seen from the figure that the effect of the severely weakened interface on the elastoplastic stress–strain behavior of the composites become more noticeable as $\alpha_2$ and $\beta_2$ become higher.

7. Experimental comparison

To assess the predictive capability of the proposed elastoplastic multi-level damage model, the present prediction is compared with the experimental data on SiC particulate-reinforced 5456 aluminum alloy matrix composites reported by Papazian and Adler (1990). We adopt the material properties of the composites as follows (Papazian and Adler, 1990; Bonfoh and Lipinski, 2007): $E_0 = 73$ GPa, $\nu_0 = 0.33$, $E_1 = 485$ GPa, $\nu_1 = 0.2$, $\phi = 0.2$, $\sigma_y = 230$ MPa. Since the model parameters of the proposed model were not reported by Papazian and Adler (1990), the model parameters are estimated by fitting the experimentally obtained stress–strain curve (Papazian and Adler, 1990) with the prediction. The fitted model parameters are: $\alpha_1 = 2.0 \times 10^{-5}$, $\beta_1 = 3.0 \times 10^{-5}$, $\alpha_2 = 2.0$, $\beta_2 = 3.0$; $h = 1,350$ MPa and $\bar{q} = 0.24$; $S_0 = 530$ MPa, $M = 5$. Fig. 11 shows the comparison between the present prediction and the experimental data on the overall elastoplastic stress–strain response of the composites at the five-phase composite state on the based of the above material properties and parameters. Overall, the present prediction and the experimental data match well. The predicted evolution of volume fractions of particles corresponding to Fig. 11 is shown in Fig. 12.

8. Concluding remarks

An elastoplastic multi-level damage model considering evolutionary weakened interface to predict the effective elastoplastic behavior and multi-level damage evolution in particle reinforced ductile matrix composites (PRDMCs) has been presented. The Eshelby’s tensor for an ellipsoidal inclusion with slightly weakened interface (Qu, 1993a,b) is adopted to model particles having mildly or severely weakened interface, and a multi-level damage model (Lee and Pyo, in press) in accordance with the Weibull’s probabilistic function is
employed to describe a sequential, progressive weakened interface in the composites. The proposed micromechanical elastoplastic damage model is applied to the uniaxial, biaxial and triaxial tension loadings to predict the corresponding elastoplastic stress–strain responses. A series of parametric analysis are carried out to investigate the influence of model parameters on the elastoplastic behavior and progression of weakened interface in the composites. Furthermore, the present prediction is compared with available experimental data in the literature to verify the proposed elastoplastic multi-level damage model. The observations and findings of this numerical study can be summarized as:

(1) The elastoplastic stress–strain responses corresponding to various phase composite states are bounded by the responses of the two- and five-phase composite states. The higher phase composite states are lower than those with the lower phase composite states.

(2) A smooth and gradual transition from the perfectly bonded interface to the perfectly debonded interface occurs within the present framework, resulting in a nonlinear stress–strain response within the elastic range.

(3) The influence of the Weibull parameters $S_0$ and $M$ and the compliance parameters $a$ and $b$ on the elastoplastic response and progression of weakened interface in the composites is shown to be significant.

(4) Most particles are mildly weakened, severely weakened or completely debonded in their interfaces even in early stage of loading with the lower $S_0$. As the values of the compliance parameters $a_2$ and $b_2$ become higher, the effect of the severely weakened interface on the elastoplastic stress–strain behavior of the composites is more pronounced.

(5) The predicted elastoplastic stress–strain behavior of PRDMCs is shown to have a very good correlation with the experimental data (Papazian and Adler, 1990).

It should be noted that particles having different levels of weakened interface (different levels of homogeneously distributed interfacial damage), giving rise to isotropic behavior, under general loading conditions are a certainly modeling assumption made in this framework. However, as deformations proceed under a uniaxial or biaxial loading, the composites progressively become transversely isotropic after evolutionary partial interfacial debonding as stated in Ju and Lee (2001). This issue is beyond the scope of the present work; nevertheless, it should be subject of future research for more accurate prediction of particulate composites behavior.
It is obvious that the probabilities of perfectly bonded particles transforming to mildly weakened particles or mildly weakened particles transforming to severely weakened particles are dependent upon the two Weibull parameters $S_0$ and $M$ and the internal stress of those particles. For simplicity, the internal stress of perfectly bonded particles has been considered as the controlling factor of the Weibull statistical function for the initiation of various stages of weakened interface of the multi-phase composite state in the present study. The above assumption would, however, tend to lead to a slight overestimation of the volume fraction of particles that have reached Levels 2 and 3.

In a forthcoming paper, the internal stress of weakened particles (Phase 2 or 3) will be chosen as the controlling factors of the Weibull statistical function and corresponding appropriate values of the Weibull parameters will also be chosen for more realistic prediction of weakened interface evolution in the multi-phase composites.

Moreover, a characterization test on PRDMCs needs to be conducted for the calibration of the model parameters of the proposed model. As an extension of the present study, the proposed elastoplastic multi-level damage model will be implemented into a finite element code to solve boundary value problems of particulate ductile matrix composites.

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Appendix

Parameters $\omega_1, \ldots, \omega_4$ and $\psi_1, \ldots, \psi_4$ in Eqs. (20) and (21) and $\chi_1, \ldots, \chi_4$ in Eqs. (32) and (33)

\[
\begin{align*}
\omega_1 &= 15(1 - v_0) - \frac{\mu_1 \lambda_0 - \mu_0 \lambda_1}{(\mu_1 - \mu_0)\{3(\lambda_1 - \lambda_0) + 2(\mu_1 - \mu_0)\}} + 5v_0 - 1 \\
\omega_2 &= 1200\alpha(1 - v_0)^2 - \frac{\mu_2 \lambda_0 - \mu_0 \lambda_2}{(\mu_2 - \mu_0)\{3(\lambda_2 - \lambda_0) + 2(\mu_2 - \mu_0)\}} + \chi_1 \\
\omega_3 &= 1200\alpha(1 - v_0)^2 - \frac{\mu_3 \lambda_0 - \mu_0 \lambda_3}{(\mu_3 - \mu_0)\{3(\lambda_3 - \lambda_0) + 2(\mu_3 - \mu_0)\}} + \chi_3 \\
\omega_4 &= 5v_0 - 1 \\
\psi_1 &= \frac{(7 - 5v_0)\mu_0 + (8 - 10v_0)\mu_1}{2(\mu_1 - \mu_0)} \\
\psi_2 &= 600\alpha(1 - v_0)^2 - \frac{\mu_2}{\mu_2 - \mu_0} + \chi_2 \\
\psi_3 &= 600\alpha(1 - v_0)^2 - \frac{\mu_3}{\mu_3 - \mu_0} + \chi_4 \\
\psi_4 &= \frac{5v_0 - 7}{2}
\end{align*}
\]

with

\[
\begin{align*}
\chi_1 &= (-80 + 480v_0 - 400v_0^2)\alpha + 500(1 - 2v_0)^2\lambda_0\beta_1 + (-98 + 140v_0 - 50v_0^2)\mu_0\chi_1 \\
&\quad + (268 - 1240v_0 + 1300v_0^2)\mu_0\beta_1 \\
\chi_2 &= (320 - 720v_0 + 400v_0^2)\alpha + \mu_0(3\chi_1 + 2\beta_1)(7 - 5v_0)^2 \\
\chi_3 &= (-80 + 480v_0 - 400v_0^2)\alpha + 500(1 - 2v_0)^2\lambda_0\beta_2 + (-98 + 140v_0 - 50v_0^2)\mu_0\chi_2 \\
&\quad + (268 - 1240v_0 + 1300v_0^2)\mu_0\beta_2 \\
\chi_4 &= (320 - 720v_0 + 400v_0^2)\alpha + \mu_0(3\chi_2 + 2\beta_2)(7 - 5v_0)^2.
\end{align*}
\]
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