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Toeplitz operators on Bloch-type spaces in the unit ball of $\mathbf{C}^n$

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ABSTRACT

For $1 \leqslant \alpha < 2$, we consider the Toeplitz operator $T_{\mu,\alpha}$ on Bloch-type space $\mathcal{B}^\alpha(\mathbb{B}_n)$ in the unit ball of $\mathbf{C}^n$, where μ is a positive Borel measure on $\mathbb{B}_n$. We give the necessary and sufficient conditions for $T_{\mu,\alpha}$ to be bounded or compact on $\mathcal{B}^\alpha(\mathbb{B}_n)$. Therefore, positive Borel measure μ on $\mathbb{B}_n$ is completely characterized for which $T_{\mu,\alpha}$ is bounded or compact on the Bloch-type space $\mathcal{B}^\alpha(\mathbb{B}_n)$.

1. Introduction

Let $\mathbb{B}_n$ be the unit ball of $\mathbf{C}^n$. For $\alpha > 0$, let $d\nu_{\alpha-1}(z) = c_{\alpha-1}(1-|z|^2)^{\alpha-1}\,d\nu(z)$, where $d\nu$ is the normalized Lebesgue volume measure on $\mathbb{B}_n$ and $c_{\alpha-1} = \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}$ so that $\nu_{\alpha-1}(\mathbb{B}_n) = 1$. For $\alpha > 0$ and $0 < p < \infty$, the weighted Bergman space $A^p_{\alpha-1}(\mathbb{B}_n)$ consists of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|^p_{A^p_{\alpha-1}(\mathbb{B}_n)} = \int_{\mathbb{B}_n} |f(z)|^p\,d\nu_{\alpha-1}(z) < \infty.$$

When the weight $\alpha = 1$, we simply write $A^p(\mathbb{B}_n)$ for $A^p_0(\mathbb{B}_n)$. In the special case when $p = 2$, $A^2_{\alpha-1}(\mathbb{B}_n)$ is a Hilbert space. It is well known that the Bergman kernel of $A^2_{\alpha-1}(\mathbb{B}_n)$ is given by

$$K^{\alpha-1}(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+\alpha}},$$

where $z, w \in \mathbb{B}_n$. The Bergman projection $P_{\alpha-1}$ is the orthogonal projection from $L^2(\mathbb{B}_n, d\nu_{\alpha-1})$ onto $A^2_{\alpha-1}(\mathbb{B}_n)$ defined by

$$P_{\alpha-1}f(z) = \int_{\mathbb{B}_n} K^{\alpha-1}(z,w)f(w)\,d\nu_{\alpha-1}(w), \quad f \in L^2(\mathbb{B}_n, d\nu_{\alpha-1}).$$

The projection $P_{\alpha-1}$ naturally extends to an integral operator on $L^1(\mathbb{B}_n, d\nu_{\alpha-1})$.

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For $\alpha > 0$ and a complex measure μ, define a Toeplitz operator as follows:

$$T_{\mu,\alpha} f(z) = c_{\alpha-1} \int_{\mathbb{B}_n} \frac{f(w)(1-|w|^2)^{\alpha-1}}{(1-\langle z, w\rangle)^{n+\alpha}} \, d\mu(w),$$

where $z \in \mathbb{B}_n$ and $f \in L^1(\mathbb{B}_n, (1-|z|^2)^{\alpha-1}\, d\mu)$. We also denote $T_{\mu,\alpha}(1) = P_{\alpha-1}(\mu)$. Toeplitz operators have been studied extensively on the Bergman spaces, see [1] and [2]. Recently, in [3], general Toeplitz operators $T_{\mu,\alpha}$ on the α-Bloch spaces have been investigated. Under a prerequisite condition, the authors characterized complex measure μ on the unit disk $\mathbb{D}$ for which $T_{\mu,\alpha}$ is bounded or compact on Bloch-type space $\mathcal{B}^\alpha(\mathbb{D})$ for $0 < \alpha < \infty$. In the present paper, we will extend the Toeplitz operator $T_{\mu,\alpha}$ to $\mathcal{B}^\alpha(\mathbb{B}_n)$ in the unit ball of $\mathbf{C}^n$ and completely characterize the positive Borel measure μ such that $T_{\mu,\alpha}$ is bounded or compact on $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $1 \leqslant \alpha < 2$. The extension requires some different techniques from those used in [3].

For $\alpha > 0$, $\mathcal{B}^\alpha(\mathbb{B}_n)$ is the space of holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_{*,\mathcal{B}^\alpha(\mathbb{B}_n)} = \sup_{z\in\mathbb{B}_n} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right| < \infty,$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z))$. It is easy to show that $\mathcal{B}^\alpha(\mathbb{B}_n)$ is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} = \left|f(0)\right| + \sup_{z\in\mathbb{B}_n} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right|.$$

It is well known that the norm $\|\cdot\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}$ is equivalent to

$$\left|f(0)\right| + \sup_{z\in\mathbb{B}_n} \left(1-|z|^2\right)^\alpha \left|\Re f(z)\right|,$$

where $\Re f(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z)$. Note that when $\alpha = 1$, $\mathcal{B}^1(\mathbb{B}_n)$ is the classical Bloch space $\mathcal{B}(\mathbb{B}_n)$. Let $\mathcal{B}_0^\alpha(\mathbb{B}_n)$ denote little α-Bloch space which is the closure of the set of polynomials in $\mathcal{B}^\alpha(\mathbb{B}_n)$. It consists exactly of holomorphic functions f on $\mathbb{B}_n$ such that

$$\lim_{|z|\to 1^-} \left(1-|z|^2\right)^\alpha \left|\nabla f(z)\right| = 0.$$

The logarithmic Bloch space $\mathcal{LB}(\mathbb{B}_n)$ is the space of holomorphic functions f such that

$$\sup_{z\in\mathbb{B}_n} \left(1-|z|^2\right) \log \frac{2}{1-|z|^2} \left|\nabla f(z)\right| < \infty.$$

Correspondingly, the little logarithmic Bloch space $\mathcal{LB}_0(\mathbb{B}_n)$ consists of all holomorphic functions f on $\mathbb{B}_n$ such that

$$\lim_{|z|\to 1^-} \left(1-|z|^2\right) \log \frac{2}{1-|z|^2} \left|\nabla f(z)\right| = 0.$$

In [3], the authors have obtained following necessary and sufficient conditions under some restricted conditions for $T_{\mu,\alpha}$ to be bounded or compact on $\mathcal{B}^\alpha(\mathbb{D})$.

Theorem 1.1. *Let μ be a complex measure on $\mathbb{D}$. Suppose μ satisfies the condition that*

$$R_\alpha(\mu)(w) = \alpha\left(1-|w|^2\right) \int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-1}}{\overline{(z-w)}(1-w\bar{z})^{\alpha+1}} \, d\mu(z) \in L^\infty(\mathbb{D}).$$

Then we have:

(i) *If $0 < \alpha < 1$, then $T_{\mu,\alpha}$ is bounded on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}^\alpha(\mathbb{D})$.*
(ii) *If $\alpha = 1$, then $T_{\mu,\alpha}$ is bounded on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{LB}(\mathbb{D})$.*
(iii) *If $\alpha > 1$, then $T_{\mu,\alpha}$ is bounded on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}(\mathbb{D})$.*

Theorem 1.2. *Let μ be a complex measure on $\mathbb{D}$. Suppose μ satisfies the condition that $\lim_{|w|\to 1} R_\alpha(\mu)(w) = 0$. Then we have*:

(i) *If $0 < \alpha < 1$, then $T_{\mu,\alpha}$ is compact on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}^\alpha(\mathbb{D})$.*
(ii) *If $\alpha = 1$, then $T_{\mu,\alpha}$ is compact on $\mathcal{B}(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{LB}_0(\mathbb{D})$.*
(iii) *If $\alpha > 1$, then $T_{\mu,\alpha}$ is compact on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}_0(\mathbb{D})$.*

We will give our main results in Sections 3 and 4. As usual, the letter C will denote a positive constant, possibly different on each occurrence.

2. Preliminaries

In this section, we give some characterization of Bloch-type space $\mathcal{B}^\alpha(\mathbb{B}_n)$ and useful lemmas, which play an important role in the proof of our main results.

Lemma 2.1. *If $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$, then*

$$|f(z)| \leqslant \begin{cases} \frac{1}{1-\alpha}\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}, & 0<\alpha<1; \\ C\log\frac{2}{1-|z|^2}\|f\|_{\mathcal{B}(\mathbb{B}_n)}, & \alpha=1; \\ C(1-|z|^2)^{1-\alpha}\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}, & 1<\alpha<\infty. \end{cases}$$

Proof. Assume that $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$. Then for $z \in \mathbb{B}_n$

$$\begin{aligned} |f(z)-f(0)| &= \left|\int_0^1 \langle \nabla f(tz), \bar{z}\rangle\, dt\right| \leqslant \int_0^1 |\nabla f(tz)||z|\, dt \\ &= \int_0^1 \frac{(1-|tz|^2)^\alpha |\nabla f(tz)||z|}{(1-|tz|^2)^\alpha}\, dt \\ &\leqslant \|f\|_{*,\mathcal{B}^\alpha(\mathbb{B}_n)} \int_0^1 \frac{|z|}{(1-|tz|^2)^\alpha}\, dt. \end{aligned}$$

The stated inequality follows. □

Lemma 2.2. *(See [4].) Let $0<\alpha\leqslant 2$. Let λ be any real number satisfying the following properties: (1) $0\leqslant\lambda\leqslant\alpha$ if $0<\alpha<1$; (2) $0<\lambda<1$ if $\alpha=1$; (3) $\alpha-1\leqslant\lambda\leqslant 1$ if $1<\alpha\leqslant 2$. Then a holomorphic function $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if*

$$S_\lambda(f) = \sup_{\substack{z,w\in\mathbb{B}_n \\ z\neq w}} (1-|z|^2)^\lambda (1-|w|^2)^{\alpha-\lambda} \frac{|f(z)-f(w)|}{|z-w|} < \infty. \tag{2.1}$$

Moreover, for any α and λ satisfying above conditions two seminorms $\sup_{z\in\mathbb{B}_n}(1-|z|^2)^\alpha|\nabla f(z)|$ and $S_\lambda(f)$ are equivalent.

Lemma 2.3. *If $\alpha > \frac{1}{2}$, then $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if the function*

$$(1-|z|^2)^{\alpha-1}|\widetilde{\nabla} f(z)|$$

is bounded in $\mathbb{B}_n$, where $\widetilde{\nabla} f(z)$ is the Möbius invariant complex gradient of f at z.

Proof. See Theorem 7.2 of [5]. □

For every point $a \in \mathbb{B}_n$, the Möbius transformation $\varphi_a : \mathbb{B}_n \to \mathbb{B}_n$ is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1-\langle z,a\rangle}, \quad z \in \mathbb{B}_n,$$

where $s_a = \sqrt{1-|a|^2}$, $P_a(z) = \frac{\langle z,a\rangle}{|a|^2}a$, $P_0(z)=0$, $Q_a = I - P_a$. The map φ_a has the following properties that

$$\varphi_a(0)=a, \qquad \varphi_a(a)=0, \qquad \varphi_a=\varphi_a^{-1}$$

and

$$1-\langle\varphi_a(z),\varphi_a(w)\rangle = \frac{(1-|a|^2)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)},$$

where z and w are arbitrary points in $\mathbb{B}_n$. In particular,

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}. \tag{2.2}$$

Proposition 2.4. *Suppose $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ and $1 \leqslant \alpha \leqslant 2$. Let λ be any real number satisfying: (1) $0 < \lambda < 1$ if $\alpha = 1$; (2) $\alpha - 1 \leqslant \lambda \leqslant 1$ if $1 < \alpha \leqslant 2$. Then*

$$\sup_{\substack{z,w\in\mathbb{B}_n\\ z\neq w}} \frac{(1-|z|^2)^{2\alpha-\lambda-1}(1-|w|^2)^{\alpha-\lambda}|f(z)-f(w)|}{|1-\langle z,w\rangle|^{2\alpha-2\lambda-1}|z-P_z(w)-s_zQ_z(w)|} \leqslant C\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}.$$

Proof. In (2.1) of Lemma 2.2, let $z = 0$. Then we have

$$\left(1-|w|^2\right)^{\alpha-\lambda}\frac{|f(0)-f(w)|}{|w|} \leqslant C\|f\|_{*,\mathcal{B}^\alpha(\mathbb{B}_n)}, \quad w \in \mathbb{B}_n \setminus \{0\}.$$

Now, replacing f by $f\circ\varphi_z$, we get

$$\left(1-|u|^2\right)^{\alpha-\lambda}\frac{|f\circ\varphi_z(0)-f\circ\varphi_z(u)|}{|u|} \leqslant C\|f\circ\varphi_z\|_{*,\mathcal{B}^\alpha(\mathbb{B}_n)}, \quad u \in \mathbb{B}_n \setminus \{0\}.$$

By Lemma 2.3 and (2.2), we find that

$$\begin{aligned}\|f\circ\varphi_z\|_{*,\mathcal{B}^\alpha(\mathbb{B}_n)} &\approx \sup_{w\in\mathbb{B}_n}\left(1-|w|^2\right)^{\alpha-1}\left|\widetilde{\nabla} f\circ\varphi_z(w)\right| \\ &= \sup_{w\in\mathbb{B}_n}\frac{(1-|w|^2)^{\alpha-1}(1-|\varphi_z(w)|^2)^{\alpha-1}|\widetilde{\nabla} f(\varphi_z(w))|}{(1-|\varphi_z(w)|^2)^{\alpha-1}} \\ &\leqslant \frac{C\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}}{(1-|z|^2)^{\alpha-1}}.\end{aligned}$$

Taking $u = \varphi_z(w)$ and $z \neq w$ we have

$$\begin{aligned}\frac{|f(z)-f(w)|}{|\varphi_z(w)|} &\leqslant \frac{C\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}}{(1-|\varphi_z(w)|^2)^{\alpha-\lambda}(1-|z|^2)^{\alpha-1}} \\ &\leqslant C\frac{|1-\langle z,w\rangle|^{2\alpha-2\lambda}\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}}{(1-|w|^2)^{\alpha-\lambda}(1-|z|^2)^{2\alpha-\lambda-1}}.\end{aligned}$$

Consequently,

$$\frac{(1-|z|^2)^{2\alpha-\lambda-1}(1-|w|^2)^{\alpha-\lambda}|f(z)-f(w)|}{|1-\langle z,w\rangle|^{2\alpha-2\lambda-1}|z-P_z(w)-s_zQ_z(w)|} \leqslant C\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}. \tag{2.3}$$

This completes the proof of Proposition 2.4. □

Let $\beta(\cdot,\cdot)$ be the Bergman metric on $\mathbb{B}_n$. Denote the Bergman metric ball at a, $D(a,r) = \{z \in \mathbb{B}_n:\ \beta(a,z) < r\}$, where $a \in \mathbb{B}_n$ and $r > 0$.

Lemma 2.5. *(See [1,5].) For fixed $r > 0$, there is a sequence $\{w_j\}$ in $\mathbb{B}_n$ such that*

(1) $\bigcup_{j=1}^\infty D(w_j,r) = \mathbb{B}_n$;
(2) *there is a positive integer N such that each $z \in \mathbb{B}_n$ is contained in at most N of the sets $D(w_j,2r)$.*

A positive Borel measure μ on the unit ball $\mathbb{B}_n$ is said to be a Carleson measure for the Bergman space $A^p(\mathbb{B}_n)$ if

$$\int_{\mathbb{B}_n}\left|f(z)\right|^p d\mu(z) \leqslant C\|f\|^p_{A^p(\mathbb{B}_n)}, \quad \forall f \in A^p(\mathbb{B}_n).$$

It is well known that a positive Borel measure μ is a Carleson measure for $A^p(\mathbb{B}_n)$ if and only if

$$\sup_{w_j\in\mathbb{B}_n}\frac{\mu(D(w_j,r))}{\nu(D(w_j,r))} < \infty,$$

where $\{w_j\}$ is the sequence in Lemma 2.5. If μ satisfies that

$$\lim_{j\to\infty}\frac{\mu(D(w_j,r))}{\nu(D(w_j,r))} = 0,$$

then μ is called vanishing Carleson measure for $A^p(\mathbb{B}_n)$. See [1] and [5].

3. Bounded Toeplitz operators

In this section, we will provide a complete characterization of bounded Toeplitz operator $T_{\mu,\alpha}$ on $\mathcal{B}^\alpha(\mathbb{B}_n)$ for $1 \leqslant \alpha < 2$. In order to have the operator $T_{\mu,\alpha}$ well defined, we will assume that the positive measure μ is such that $\int_{\mathbb{B}_n} \log \frac{1}{1-|z|^2}\, d\mu(z) < \infty$ for $T_{\mu,1}$ and $\int_{\mathbb{B}_n} d\mu(z) < \infty$ for $T_{\mu,\alpha}$ with $1 < \alpha < 2$.

Theorem 3.1. *Let μ be a positive Borel measure on $\mathbb{B}_n$. Then we have*

(1) *if $\alpha = 1$, then $T_{\mu,\alpha}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{LB}(\mathbb{B}_n)$ and μ is a Carleson measure*;
(2) *if $1 < \alpha < 2$, then $T_{\mu,\alpha}$ is bounded on $\mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}(\mathbb{B}_n)$ and μ is a Carleson measure.*

Proof. It is well known that $A^1(\mathbb{B}_n)^* = \mathcal{B}^\alpha(\mathbb{B}_n)$ under the integral pairing

$$\langle h, g\rangle_{\alpha-1} = c_{\alpha-1} \int_{\mathbb{B}_n} h(z)\overline{g(z)}\big(1-|z|^2\big)^{\alpha-1}\, d\nu(z), \quad h \in A^1(\mathbb{B}_n),\ g \in \mathcal{B}^\alpha(\mathbb{B}_n).$$

To prove the boundedness of $T_{\mu,\alpha}$, it suffices to show that

$$\big|\langle h, T_{\mu,\alpha} g\rangle_{\alpha-1}\big| \leqslant C\|h\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}$$

for all $h \in A^1(\mathbb{B}_n)$ and $g \in \mathcal{B}^\alpha(\mathbb{B}_n)$.

By Fubini's Theorem we have

$$\begin{aligned}
\langle h, T_{\mu,\alpha} g\rangle_{\alpha-1} &= c_{\alpha-1} \int_{\mathbb{B}_n} h(z)\overline{T_{\mu,\alpha} g(z)}\big(1-|z|^2\big)^{\alpha-1}\, d\nu(z) \\
&= c_{\alpha-1} \int_{\mathbb{B}_n} h(z)\overline{g(z)}\big(1-|z|^2\big)^{\alpha-1}\, d\mu(z) \\
&= c_{\alpha-1} \int_{\mathbb{B}_n} P_\alpha(h\overline{g})(z)\big(1-|z|^2\big)^{\alpha-1}\, d\mu(z) + c_{\alpha-1} \int_{\mathbb{B}_n} (I-P_\alpha)(h\overline{g})(z)\big(1-|z|^2\big)^{\alpha-1}\, d\mu(z) \\
&= I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
(I-P_\alpha)(h\overline{g})(z) &= h(z)\overline{g(z)} - c_\alpha \int_{\mathbb{B}_n} \frac{h(w)\overline{g(w)}(1-|w|^2)^\alpha}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, d\nu(w) \\
&= c_\alpha \int_{\mathbb{B}_n} \frac{(\overline{g(z)}-\overline{g(w)})h(w)(1-|w|^2)^\alpha}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, d\nu(w).
\end{aligned}$$

From Proposition 2.4, we have

$$\begin{aligned}
|I_2| &= c_{\alpha-1}c_\alpha \left| \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{(\overline{g(z)}-\overline{g(w)})h(w)(1-|w|^2)^\alpha(1-|z|^2)^{\alpha-1}}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, d\nu(w)\, d\mu(z)\right| \\
&= c_{\alpha-1}c_\alpha \left| \int_{\mathbb{B}_n} h(w)\big(1-|w|^2\big)^\alpha \int_{\mathbb{B}_n} \frac{(\overline{g(z)}-\overline{g(w)})(1-|z|^2)^{\alpha-1}}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, d\mu(z)\, d\nu(w)\right| \\
&\leqslant c_{\alpha-1}c_\alpha \int_{\mathbb{B}_n} \big|h(w)\big|\big(1-|w|^2\big)^\lambda \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\alpha-\lambda}(1-|z|^2)^{2\alpha-\lambda-1}|g(z)-g(w)|}{|1-\langle z, w\rangle|^{2\alpha-2\lambda-1}|z-P_z(w)-s_zQ_z(w)|} \\
&\quad \cdot \frac{|z-P_z(w)-s_zQ_z(w)|(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z, w\rangle|^{n+2\lambda-\alpha+2}}\, d\mu(z)\, d\nu(w) \\
&\leqslant C \int_{\mathbb{B}_n} \big|h(w)\big|\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}\big(1-|w|^2\big)^\lambda \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z, w\rangle|^{n+2\lambda-\alpha+1}}\, d\mu(z)\, d\nu(w).
\end{aligned}$$

Since μ is a Carleson measure, taking $\lambda-\alpha>-1$, then by Proposition 1.4.10 of [6] we get

$$\begin{aligned}
&\left(1-|w|^2\right)^{\lambda}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\mu(z)\\
&\quad\leqslant\left(1-|w|^2\right)^{\lambda}\sum_{j=1}^{\infty}\int_{D(z_j,r)}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\mu(z)\\
&\quad\leqslant\left(1-|w|^2\right)^{\lambda}\sum_{j=1}^{\infty}\mu\big(D(z_j,r)\big)\sup_{z\in D(z_j,r)}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\\
&\quad\leqslant C\left(1-|w|^2\right)^{\lambda}\sum_{j=1}^{\infty}\frac{\mu(D(z_j,r))}{\nu(D(z_j,r))}\int_{D(z_j,2r)}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\nu(z)\\
&\quad\leqslant C\left(1-|w|^2\right)^{\lambda}\sup_{z_j\in\mathbb{B}_n}\frac{\mu(D(z_j,r))}{\nu(D(z_j,r))}\sum_{j=1}^{\infty}\int_{D(z_j,2r)}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\nu(z)\\
&\quad\leqslant C\left(1-|w|^2\right)^{\lambda}N\sup_{z_j\in\mathbb{B}_n}\frac{\mu(D(z_j,r))}{\nu(D(z_j,r))}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\nu(z)\leqslant C.
\end{aligned}$$

Therefore

$$|I_2|\leqslant C\|h\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^{\alpha}(\mathbb{B}_n)}.$$

Next consider I_1. By Fubini's Theorem it follows that

$$\begin{aligned}
I_1&=c_{\alpha-1}c_{\alpha}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{h(w)\overline{g(w)}(1-|w|^2)^{\alpha}(1-|z|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,d\nu(w)\,d\mu(z)\\
&=c_{\alpha-1}c_{\alpha}\int_{\mathbb{B}_n}h(w)\overline{g(w)\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\mu(z)}\left(1-|w|^2\right)^{\alpha}d\nu(w).
\end{aligned}$$

Let

$$Q_{\alpha}\mu(w)=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\mu(z).$$

Thus

$$I_1=c_{\alpha}\int_{\mathbb{B}_n}h(w)\overline{g(w)Q_{\alpha}(\mu)(w)}\left(1-|w|^2\right)^{\alpha}d\nu(w).$$

By simple calculation, we have

$$\begin{aligned}
Q_{\alpha}\mu(w)&=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}(1-\langle w,z\rangle)}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\mu(z)+c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}\langle w,z\rangle}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\mu(z)\\
&=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}\,d\mu(z)+c_{\alpha-1}\sum_{k=1}^{n}w_k\int_{\mathbb{B}_n}\frac{\overline{z_k}(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+1+\alpha}}\,d\mu(z)\\
&=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}\,d\mu(z)+c_{\alpha-1}\sum_{k=1}^{n}\frac{w_k}{n+\alpha}\frac{\partial}{\partial w_k}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}\,d\mu(z)\\
&=c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}\,d\mu(z)+\frac{1}{n+\alpha}\Re\left(c_{\alpha-1}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}}{(1-\langle w,z\rangle)^{n+\alpha}}\,d\mu(z)\right)\\
&=P_{\alpha-1}(\mu)(w)+\frac{1}{n+\alpha}\Re P_{\alpha-1}(\mu)(w).
\end{aligned}\tag{3.1}$$

It is easy to see that

(1) if $\alpha = 1$ and $P_{\alpha-1}(\mu) \in \mathcal{LB}(\mathbb{B}_n)$, then $Q_\alpha(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \in L^\infty(\mathbb{B}_n)$;
(2) if $1 < \alpha < 2$ and $P_{\alpha-1}(\mu) \in \mathcal{B}(\mathbb{B}_n)$, then $Q_\alpha(\mu)(w)(1-|w|^2) \in L^\infty(\mathbb{B}_n)$.

This implies that $|I_1| \leqslant C\|h\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}$. Hence, $T_{\mu,\alpha}$ is a bounded operator on $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $1 \leqslant \alpha < 2$.

Conversely, suppose that $T_{\mu,\alpha}$ is a bounded operator on $\mathcal{B}^\alpha(\mathbb{B}_n)$. Take

$$h_w(z) = \frac{(1-|w|^2)^t}{(1-\langle z, w\rangle)^{n+1+t}},$$

for $t > 0$. It is clear that $\|h_w\|_{A^1(\mathbb{B}_n)} \leqslant C$. On the other hand, take

$$g_w(z) = \frac{(1-|w|^2)^{n+2+t-\alpha}}{(1-\langle z, w\rangle)^{n+1+t}}.$$

We have $\|g_w\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \leqslant C$. Therefore

$$c_{\alpha-1}\big(1-|w|^2\big)^{n+2+2t-\alpha}\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\alpha-1}\,d\mu(z)}{|1-\langle z, w\rangle|^{2n+2+2t}} = \big|\langle h_w, T_{\mu,\alpha}g_w\rangle_{\alpha-1}\big| \leqslant C\|T_{\mu,\alpha}\|\|h_w\|_{A^1(\mathbb{B}_n)}\|g_w\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \leqslant C.$$

Thus

$$\big(1-|w|^2\big)^{n+2+2t-\alpha}\int_{D(w,r)}\frac{(1-|z|^2)^{\alpha-1}\,d\mu(z)}{|1-\langle z, w\rangle|^{2n+2+2t}} \leqslant C,$$

for every $w \in \mathbb{B}_n$. This implies that

$$\sup_{w\in\mathbb{B}_n}\frac{\mu(D(w,r))}{\nu(D(w,r))} < \infty.$$

Hence μ is a Carleson measure on $\mathbb{B}_n$.

From the proof of the sufficient condition, we find that there exists a constant C such that

$$|I_1| = \left|c_\alpha\int_{\mathbb{B}_n} h(w)\overline{g(w)Q_\alpha(\mu)(w)}\big(1-|w|^2\big)^\alpha\,d\nu(w)\right| \leqslant C\|h\|_{A^1(\mathbb{B}_n)}\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}.$$

This implies that

$$\big\|g(w)Q_\alpha(\mu)(w)\big\|_{\mathcal{B}^{\alpha+1}(\mathbb{B}_n)} \leqslant C\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}.$$

As $\alpha = 1$, we have $|g(w)Q_\alpha(\mu)(w)(1-|w|^2)| \leqslant C\|g\|_{\mathcal{B}(\mathbb{B}_n)}$. Take $g_w(z) = \log\frac{2}{1-\langle z, w\rangle}$. It is clear that $\|g_w\|_{\mathcal{B}(\mathbb{B}_n)} \leqslant C$. Taking $z = w$, then

$$\big|Q_\alpha(\mu)(w)\big|\big(1-|w|^2\big)\log\frac{2}{1-|w|^2} \leqslant C.$$

Notice that $P_{\alpha-1}(\mu) = T_{\mu,\alpha}(1) \in \mathcal{B}^\alpha(\mathbb{B}_n)$. From (3.1) we have $P_{\alpha-1}(\mu) \in \mathcal{LB}(\mathbb{B}_n)$.

When $1 < \alpha < 2$, taking $g_w(z) = (1-\langle z, w\rangle)^{1-\alpha}$, we have $\|g_w\|_{\mathcal{B}_\alpha(\mathbb{B}_n)} \leqslant C$. From Lemma 2.1 we get

$$Q_\alpha(\mu)(w)\big(1-|w|^2\big) \leqslant C,$$

for $w \in \mathbb{B}_n$. By (3.1) it is obvious that $P_{\alpha-1}(\mu) \in \mathcal{B}(\mathbb{B}_n)$.

This completes the proof of Theorem 3.1. □

Remark. For $n = 1$, notice that $|\varphi_z(w)| = |\frac{z-w}{1-\bar{z}w}| < 1$ for $z \in \mathbb{B}_n$ and $w \in \mathbb{B}_n$. Using Lemma 2.2 and the same method as above, we can show that for $0 < \alpha < 1$, $T_{\mu,\alpha}$ is bounded on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}^\alpha(\mathbb{D})$ and μ is a Carleson measure on $\mathbb{D}$.

Since $P(L^\infty(\mathbb{B}_n)) = \mathcal{B}(\mathbb{B}_n)$, as a consequence of Theorem 3.1 we can know that not every $f \in L^\infty(\mathbb{B}_n)$ induced a bounded Toeplitz operator on $\mathcal{B}(\mathbb{B}_n)$. When $f \in H^\infty$, Toeplitz operator induced by f is multiplication operator on $\mathcal{B}(\mathbb{B}_n)$. In this case, our results are correspondent with the multipliers on $\mathcal{B}(\mathbb{B}_n)$ in [7]. These are the same as in the disk $\mathbb{D}$ presented in [3].

4. Compact Toeplitz operators

For the proof of following theorem related to compact of $T_{\mu,\alpha}$ we need the following lemma.

Lemma 4.1. *Let $0<\alpha<\infty$ and $T_{\mu,\alpha}$ be a bounded linear operator from $\mathcal{B}^\alpha(\mathbb{B}_n)$ into $\mathcal{B}^\alpha(\mathbb{B}_n)$. When $0<\alpha<1$, $T_{\mu,\alpha}$ is compact if and only if $\|T_{\mu,\alpha}(f_j)\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}\to 0$ as $j\to\infty$ whenever $\{f_j\}$ is a bounded sequence in $\mathcal{B}^\alpha(\mathbb{B}_n)$ that converges to 0 uniformly on $\overline{\mathbb{B}}_n$. When $1\leqslant\alpha<\infty$, T^α_μ is compact if and only if $\|T_{\mu,\alpha}(f_j)\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}\to 0$ as $j\to\infty$ whenever $\{f_j\}$ is a bounded sequence in $\mathcal{B}^\alpha(\mathbb{B}_n)$ that converges to 0 uniformly on compact subset of $\mathbb{B}_n$.*

Proof. This lemma can be proved by Montel Theorem and Lemma 2.1. □

Theorem 4.2. *Let μ be a positive Borel measure on $\mathbb{B}_n$. Then we have*

(1) *if $\alpha=1$, then $T_{\mu,\alpha}$ is compact on $\mathcal{B}(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu)\in\mathcal{LB}_0(\mathbb{B}_n)$ and μ is a vanishing Carleson measure*;
(2) *if $1<\alpha<2$, then $T_{\mu,\alpha}$ is compact on $\mathcal{B}^\alpha(\mathbb{B}_n)$ if and only if $P_{\alpha-1}(\mu)\in\mathcal{B}_0(\mathbb{B}_n)$ and μ is a vanishing Carleson measure.*

Proof. For $1\leqslant\alpha<2$, let $\{g_j\}$ be a sequence in $\mathcal{B}^\alpha(\mathbb{B}_n)$ such that $\|g_j\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}\leqslant 1$ and $g_j\to 0$ uniformly on compact of $\mathbb{B}_n$. Suppose $h\in A^1(\mathbb{B}_n)$. Then

$$\begin{aligned}\langle h, T_{\mu,\alpha}g_j\rangle_{\alpha-1} &= c_{\alpha-1}\int_{\mathbb{B}_n} h(z)\overline{g_j(z)}\big(1-|z|^2\big)^{\alpha-1}\,d\mu(z)\\ &= c_{\alpha-1}\int_{\mathbb{B}_n} P_\alpha(h\overline{g}_j)(z)\big(1-|z|^2\big)^{\alpha-1}\,d\mu(z)+c_{\alpha-1}\int_{\mathbb{B}_n}(I-P_\alpha)(h\overline{g}_j)(z)\big(1-|z|^2\big)^{\alpha-1}\,d\mu(z)\\ &= I_{1,j}+I_{2,j}.\end{aligned}$$

For fixed $0<\varepsilon<1$, since μ is vanishing measure, there exists $0<\eta<1$ such that

$$\big(1-|w|^2\big)^\lambda\int_{\mathbb{B}_n\setminus\eta\mathbb{B}_n}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\mu(z)<\varepsilon,$$

where $\eta\mathbb{B}_n=\{z\in\mathbb{B}_n\colon |z|<\eta\}$ and $\lambda-\alpha>-1$. Taking a constant $\delta>0$ such that $1-(\varepsilon(1-\eta)^{n+1+\lambda})^{\frac{1}{\lambda}}\leqslant\delta<1$, as in the proof of Theorem 3.1, by Proposition 2.4 we have

$$\begin{aligned}\lim_{j\to\infty}|I_{2,j}| &= \lim_{j\to\infty}c_{\alpha-1}c_\alpha\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{(\overline{g_j(z)}-\overline{g_j(w)})h(w)(1-|w|^2)^\alpha(1-|z|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,d\nu(w)\,d\mu(z)\right|\\ &= \lim_{j\to\infty}c_{\alpha-1}c_\alpha\left|\int_{\mathbb{B}_n}h(w)\big(1-|w|^2\big)^\alpha\int_{\mathbb{B}_n}\frac{(\overline{g_j(z)}-\overline{g_j(w)})(1-|z|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+1+\alpha}}\,d\mu(z)\,d\nu(w)\right|\\ &\leqslant \lim_{j\to\infty}C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\big|h(w)\big|\|g_j\|_{\mathcal{B}^\alpha(\mathbb{B}_n)}\big(1-|w|^2\big)^\lambda\int_{\mathbb{B}_n}\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\mu(z)\,d\nu(w)\\ &\quad+\lim_{j\to\infty}c_{\alpha-1}c_\alpha\int_{\delta\mathbb{B}_n}\big|h(w)\big|\big(1-|w|^2\big)^\alpha\int_{\mathbb{B}_n}\frac{|g_j(z)-g_j(w)|(1-|z|^2)^{\alpha-1}}{|1-\langle z,w\rangle|^{n+1+\alpha}}\,d\mu(z)\,d\nu(w)\\ &\leqslant C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\big|h(w)\big|\big(1-|w|^2\big)^\lambda\bigg(\int_{\mathbb{B}_n\setminus\eta\mathbb{B}_n}+\int_{\eta\mathbb{B}_n}\bigg)\frac{(1-|z|^2)^{\lambda-\alpha}}{|1-\langle z,w\rangle|^{n+2\lambda-\alpha+1}}\,d\mu(z)\,d\nu(w)\\ &\leqslant C\varepsilon\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\big|h(w)\big|\,d\nu(w)+C\int_{\mathbb{B}_n\setminus\delta\mathbb{B}_n}\big|h(w)\big|\frac{(1-\delta)^\lambda}{(1-\eta)^{n+1+\lambda}}\,d\nu(w)\leqslant C\varepsilon\|h\|_{A^1(\mathbb{B}_n)}.\end{aligned}$$

For $I_{1,j}$, we have

$$I_{1,j}=c_\alpha\int_{\mathbb{B}_n}h(w)\overline{g_j(w)Q_\alpha(\mu)(w)}\big(1-|w|^2\big)^\alpha\,d\nu(w).$$

From (3.1) it is easy to see that

(1) if $\alpha = 1$ and $P_{\alpha-1}(\mu) \in \mathcal{LB}_0(\mathbb{B}_n)$, then $Q_\alpha(\mu)(w)(1-|w|^2)\log\frac{2}{1-|w|^2} \to 0$ as $|w| \to 1$;
(2) if $1 < \alpha < 2$ and $P_{\alpha-1}(\mu) \in \mathcal{B}_0(\mathbb{B}_n)$, then $Q_\alpha(\mu)(w)(1-|w|^2) \to 0$ as $|w| \to 1$.

Combined with $\{g_j\}$ converges to 0 on compact of $\mathbb{B}_n$, we have $\lim_{j\to\infty} I_{1,j} = 0$. Therefore, $\|T_{\mu,\alpha} g_j\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \to 0$ as $j \to \infty$, which implies that $T_{\mu,\alpha}$ is compact operator.

Next assume that $T_{\mu,\alpha}$ is compact on $\mathcal{B}^\alpha(\mathbb{B}_n)$. Take

$$h_w(z) = \frac{(1-|w|^2)^t}{(1-\langle z, w\rangle)^{n+1+t}}$$

for $t > 0$. We know that $\|h_w\|_{A^1(\mathbb{B}_n)} \leqslant C$. Take

$$g_w(z) = \frac{(1-|w|^2)^{n+2+t-\alpha}}{(1-\langle z, w\rangle)^{n+1+t}}.$$

Then $\|g\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \leqslant C$ and $g_w \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $|w| \to 1$. From Lemma 4.1 we have

$$\begin{aligned} &c_{\alpha-1}\big(1-|w|^2\big)^{n+2+2t-\alpha} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha-1}\, d\mu(z)}{|1-\langle z, w\rangle|^{2n+2+2t}} \\ &\quad = \big|\langle h_w, T_{\mu,\alpha} g_w\rangle_{\alpha-1}\big| \leqslant C\|h_w\|_{A^1(\mathbb{B}_n)} \|T_{\mu,\alpha} g_w\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} \to 0, \quad |w| \to 1. \end{aligned}$$

This implies that μ is a vanishing Carleson measure.

Next let $h_w(z) = \frac{(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+1+\alpha}}$. We have $\|h_w\|_{A^1(\mathbb{B}_n)} \leqslant C$. Let $\{g_j\}$ be a bounded sequence in $\mathcal{B}^\alpha(\mathbb{B}_n)$ that converges to 0 uniformly on compact subset of $\mathbb{B}_n$. By the compactness of $T_{\mu,\alpha}$, we have

$$\begin{aligned} 0 &= \lim_{j\to\infty} I_{1,j} \\ &= \lim_{j\to\infty} c_\alpha \int_{\mathbb{B}_n} h_w(z)\overline{g_j(z) Q_\alpha(\mu)(z)}\big(1-|z|^2\big)^\alpha\, d\nu(z) \\ &= \lim_{j\to\infty} c_\alpha\big(1-|w|^2\big)^\alpha \int_{\mathbb{B}_n} \frac{\overline{g_j(z) Q_\alpha(\mu)(z)}(1-|z|^2)^\alpha}{(1-\langle z, w\rangle)^{n+1+\alpha}}\, d\nu(z) \\ &= \lim_{j\to\infty} \big(1-|w|^2\big)^\alpha \overline{g_j(w) Q_\alpha(\mu)(w)}. \end{aligned}$$

When $\alpha = 1$, taking $g_w(z) = (\log\frac{1}{1-|w|^2})^{-1}(\log\frac{2}{1-\langle z,w\rangle})^2$ with $|w| \geqslant \frac{1}{2}$, we have $P_{\alpha-1}(\mu) \in \mathcal{LB}_0(\mathbb{B}_n)$. When $\alpha > 1$, taking $g_w(z) = \frac{1-|w|^2}{(1-\langle z,w\rangle)^\alpha}$, then we have $P_\alpha(\mu) \in \mathcal{B}_0(\mathbb{B}_n)$. The proof of the theorem is completed. □

Remark. For $n = 1$, we also can show that for $0 < \alpha < 1$, $T_{\mu,\alpha}$ is compact on $\mathcal{B}^\alpha(\mathbb{D})$ if and only if $P_{\alpha-1}(\mu) \in \mathcal{B}^\alpha(\mathbb{D})$ and μ is a vanishing Carleson measure on $\mathbb{D}$.

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