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The Scalar-Curvature Problem on the Standard Three-Dimensional Sphere

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Let (S^3, c) be the standard 3-sphere, i.e., the 3-sphere equipped with the standard metric. Let K be a C^2 positive function on S^3. The Kazdan–Warner problem [1] is the problem of finding suitable conditions on K such that K is the scalar curvature for a metric g on S^3 conformally equivalent to c. The metric g then reads

$$g = u^4 c$$

and u is a positive function on S^3 satisfying the partial differential equation

$$\begin{aligned} -8\,\Delta u + 6u &= K(x)\, u^5 \\ u &> 0. \end{aligned} \tag{1}$$

Let $L = -8\,\Delta u + 6u$ be the conformal Laplacian. The same problem can be formulated for any compact Riemannian manifold (M^n, g). Since this problem has been formulated, there have been some partial answers (see [3–7, 18]). Obstructions have also been pointed out [1, 2]. The main difficulty, arising when one tries to solve equations of type (4), consists of the failure of the Palais–Smale condition. We show, in this paper, how this difficulty may be overcome in the case Eq. (1). Our method consists of studying the critical points at infinity of the variational problem, in

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computing their total index (as singularities for the gradient flow), and comparing this total index to the Euler–Poincaré characteristic of the space of variations. The equality of both numbers is not automatically satisfied for $n = 3$. The conceptual and analytical framework of this paper originates in [8–10]. Observe, in particular, in this paper the use of the gradient flow, to overcome the noncompactness. This is extensively used in [10] and has displayed the role of Green's function in equations of type (1). See [10] for further details. We state now the theorem that we will prove here.

> Assume K is C^2 and has only nondegenerate critical points $x_1, ..., x_m$ such that $-LK(x_i) \neq 0$ $\forall i = 1, ..., m$. Each x_i is assumed to be of index k_i. (H1)

THEOREM 1. *If* $\sum_{i \text{ such that } -LK(x_i) > 0} (-1)^{k_i} \neq -1$, *then* (1) *has a solution.*

Remark 1. For a corresponding result on (S^4, c), the reader is referred to the end of this paper, Appendix C.

Problem (1) has a variational structure, the functional being

$$J(u) = \frac{1}{3} \frac{1}{(\int_{S^3} K(x)\, u^6\, dv)^{1/2}}, \tag{2}$$

where dv is the volume element of (S^3, c) and where u belongs to the space

$$\Sigma^+ = \left\{ u \in H^1(S^3);\ u \geqslant 0 \text{ s.t. } \int_{S^3} (-Luu)\, dv = 1 \right\}. \tag{3}$$

We will denote

$$|u|^2_{-L} = -\int_{S^3} Luu; \qquad \lambda(u) = \frac{1}{\int K(x)\, u^6\, dv}. \tag{4}$$

The functional J is known not to satisfy the Palais–Smale condition, which leads to the failure of classical existence mechanisms. Although J is lower bounded, the minimum does not need to be achieved, and in fact is not achieved if K is nonconstant. The failure of the Palais–Smale condition has been analyzed throughout the work of [13–17, 19]. The analysis carried out in [11], and [12], in particular, comes out here virtually without any change. Introducing

$$\text{for} \quad a \in S^3,\ \lambda > 0, \qquad \delta(a, \lambda) = c_0 \left(\frac{\lambda}{\lambda^2 + 1 + (\lambda^2 - 1) \cos d(a, x)} \right)^{1/2}, \tag{5}$$

where $d(x, y)$ is the geodesic distance on (S^3, c). $\delta(a, \lambda)$ is a solution of the Yamabe problem on S^3 and therefore satisfies

$$-L\delta(a, \lambda) = \delta(a, \lambda)^5 \tag{6}$$

and for $p \in \mathbb{N}^*$, $\varepsilon > 0$,

$$\begin{aligned} W(p, \varepsilon) = \Big\{ & u \in \Sigma^+ \text{ s.t. } \exists \alpha_1, ..., \alpha_p > 0; \exists a_1, ..., a_p \in S^3, \\ & \exists \lambda_1, ..., \lambda_p; \lambda_i > \frac{1}{\varepsilon} \ \forall i \text{ with} \\ & \Big| u - \sum_{i=1}^{p} \alpha_i \delta_i \Big|_{-L} < \varepsilon; \ |\alpha_i^4 K(x_i)\, \lambda(u)^4 - 1| < \varepsilon \ \forall i; \\ & \varepsilon_{ij}^{-2} = \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j d(a_i, a_j)^2 > \frac{1}{\varepsilon} \ \forall i \neq j \Big\}, \end{aligned} \tag{7}$$

where $\delta_i = \delta(a_i, \lambda_i)$, $\varepsilon_{ij} = (\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i \lambda_j d(a_i, a_j)^2)^{-1/2}$ will be extensively used later.

The failure of the Palais–Smale condition is characterized as follows: Let

$$\partial J(u) \text{ be the gradient of } J. \tag{8}$$

Proposition 1. *Assume* (1) *has no solution. Let* $(u_k) \in \Sigma^+$ *be a sequence such that* $\partial J(u_k) \to 0$ *and* $J(u_k))_k$ *is bounded. Then there exists then an integer* $p \in \mathbb{N}^*$, *a sequence* $\varepsilon_k > 0$, $\varepsilon_k \to 0$, *and an extracted subsequence of the* u_k*'s, again denoted* (u_k) *such that* $u_k \in W(p, \varepsilon_k)$. $(J(u_k))$ *converges then to a limit* l *such that* $l \leqslant p \times (\int \delta^6 / 3\sqrt{m})$ *where* m *is a lower bound for* K *on* S^3.

The proof of Proposition 1 is by now classical (see [15, 16, 10, 11] for instance).

Following a method introduced in [10] and used in [11], we optimize, for $u \in W(p, \varepsilon)$, ε small enough, the approximation of u by $u = \sum_{i=1}^{p} \alpha_i \delta_i$, i.e., we introduce the minimization problem

$$u \in W(p, \varepsilon) \qquad \min_{\substack{\alpha_i > 0 \\ a_i \in S^3 \\ \lambda_i > 0}} \Big| u - \sum_{i=1}^{p} \alpha_i \delta(a_i, \lambda_i) \Big|_{-L}. \tag{9}$$

The following proposition is also, as in [10, 11], available in this framework and its proof, which we omit here, requires only minor modifications.

PROPOSITION 2. *For any* $p \in \mathbb{N}^*$, *there exists* $\varepsilon_p > 0$, $\gamma(p)$, $\delta(p) > 0$ *such that for any u in* $W(p, \varepsilon_p)$, *the problem of minimization* (9) *has a unique solution* $(\alpha_i, a_i, \lambda_i)$, *up to permutations. Denoting* $v = u - u = \sum_{i=1}^{p} \alpha_i \delta_i$, v *satisfies*

$$\begin{aligned} &(v, \delta_i)_{-L} = 0 \\ &\left(v, \frac{\partial \delta_i}{\partial a_i}\right)_{-L} = 0 \qquad \forall i = 1, ..., p. \\ &\left(v, \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} = 0 \end{aligned} \tag{Vo}$$

α_i *satisfies*

$$\delta(p) \leqslant \alpha_i \leqslant \gamma(p) \qquad \forall i = 1, ..., p.$$

In the sequel, we will often split u, a function $W(p, \varepsilon)$, under the form $u = \sum_{i=1}^{p} \alpha_i \delta_i + v$, after solving problem (9). We will refer to this splitting by stating that u is in $W(p, \varepsilon)$ and v satisfies (Vo).

Note that

$$\partial J(u) = \lambda(u) u + \lambda(u)^5 L^{-1} K(x) u^5 \qquad \forall u \in \Sigma^+, \tag{10}$$

where $\lambda(u)$ has been defined in (5). $\partial J(u)$ is the gradient of J with respect tot the $(\ )_{-L}$ scalar product. $\lambda(u)$ is a Lagrange multiplier which is equal to $(3J(u))^2$ and we have the following lemma:

LEMMA 1. 1. *There exists* $\lambda_0 > 0$ *such that* $\lambda(u) \geqslant \lambda_0$ $\forall u \in \Sigma^+$.

2. *For any* $b \in \mathbb{R}^{**}$, *there exists* $C(b) > 0$ *such that* $\forall u, w$ *such that* $J(u), J(w) \leqslant b$, *we have*

$$|\lambda(u) - \lambda(w)| \leqslant C(b)\, |u - w|_{-L}.$$

The proof of Lemma 1 is also very easy and we shall omit it here.

LEMMA 2. *Let* $p \in \mathbb{N}^*$ *be given and* $\varepsilon > 0$ *be small enough. There exists a constant* $C(p)$ *such that for any* $u \in W(p, \varepsilon)$, $u = \sum_{i=1}^{p} \alpha_i \delta(x_i, \lambda_i) + v$, v *satisfying* (Vo), *the following estimate holds*:

$$|v|_{-L} \leqslant C(p) \left(\sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{1/3} + |\partial J(u)| + \sum_{i=1}^{p} \frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right).$$

Proof. We now compute $\partial J(u)\cdot v$.

$$\partial J(u)\cdot v=-\lambda(u)\int_{S^3} Lu\cdot v-\lambda(u)^5\int_{S^3} K(x)\,u^5 v$$

$$=-\lambda(u)\int_{S^3}\sum\alpha_j L\delta_j v-\lambda(u)^5$$

$$\times\int K(x)\left(\sum_{i=1}^{p}\alpha_i\delta_i+v\right)^5 v-\lambda(u)\int_{S^3} Lv\cdot v. \tag{11}$$

Using (Vo), we derive

$$\partial J(u)\cdot v=-\lambda(u)\int_{S^3} Lv\cdot v-\lambda(u)^5\, K(x)\left(\sum_{i=1}^{p}\alpha_i\delta_i+v\right)^5 v$$

$$+\lambda(u)^5\int_{S^3}\sum_{i=1}^{p}\alpha_i^5 K(x_i)\,\delta_i^5 v \tag{12}$$

since $\int_{S^3} -L\,\delta_i v=\int_{S^3}\delta_i^5 v=0$. Then

$$\partial J(u)\cdot v=-\lambda(u)\int_{S^3} Lv\cdot v-5\lambda(u)^5\int_{S^3} K(x)\sum_{i=1}^{p}\alpha_i^4\,\delta_i^4 v^2$$

$$+\lambda(u)^5\sum_{i=1}^{p}\int_{S^3}\alpha_i^5(K(x)-K(x_i))\,\delta_i^5 v+R$$

$$=-\lambda(u)\int_{S^3} Lv\cdot v-5\lambda(u)^5\int_{S^3}\sum_{i=1}^{p} K(x_i)\,\alpha_i^4\,\delta_i^4 v^2$$

$$+\lambda(u)^5\sum_{i=1}^{p}\int_{S^3}\alpha_i^5(K(x)-K(x_i))\,\delta_i^5 v$$

$$-5\lambda(u)^5\sum_{i=1}^{p}\int_{S^3}(K(x)-K(x_i))\,\delta_i^4 v^2+R, \tag{13}$$

where, using the fact that $\delta_i^3\delta_j^2\leqslant(\delta_i^4\delta_j+\delta_i^4\delta_j)$ and $\delta_i^3\delta_j\delta_k\leqslant\delta_i^3(\delta_j^2+\delta_k^2)$,

$$|R|\leqslant C\left(\sum_{i\neq j}\cdot\int_{S^3}\delta_i^4\,\delta_j\,|v|+|v|_{-L}^3\right). \tag{14}$$

By Lemma A5, we have

$$\int_{S^3}\delta_i^4\,\delta_j\,|v|\leqslant|v|_{-L}\left(\int_{S^3}\delta_i^{24/5}\delta_j^{6/5}\right)^{5/6}\leqslant C\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/3}\,|v|_{-L}. \tag{15}$$

By Proposition 1, we have

$$\forall u \in W(p, \varepsilon), \qquad u = \sum_{i=1}^{p} \alpha_i \delta_i + v, \qquad v \text{ satisfying (Vo)}, \tag{16}$$
$$|\lambda(u)^4 K(x_i)\, \alpha_i^4 - 1| < \varphi_p(\varepsilon),$$

where $\varphi_p(\varepsilon)$ satisfies $\lim_{\varepsilon \to 0} \varphi_p(\varepsilon) = 0$ ($\varphi_p(\varepsilon)$ is independent of u in $W_p(\varepsilon)$).
Finally, by Lemma A6, we have the estimates

$$\left| \int (K(x) - K(x_i))\, \delta_i^5 v \right| \leqslant |v|_{-L} \left(\int |K(x) - K(x_i)|^{6/5}\, \delta_i^6 \right)^{5/6}$$
$$\leqslant C \left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) |v|_{-L}. \tag{17}$$
$$\left| \int (K(x) - K(x_i))\, \delta_i^4 v^2 \right| \leqslant \frac{C}{\lambda_i}\, |v|_{-L}^2.$$

Using (14)–(17), we derive from (13)

$$\lambda(u)\, Q(v) \leqslant C \left(|\partial J(u)| + \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{1/3} + \frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) |v|_{-L}$$
$$+ C\, |v|_{-L}^3 + Cp\varphi_p(\varepsilon)\, |v|_{-L}^2, \tag{18}$$

where

$$Q(v) = |v|_{-L}^2 - 5 \int_{S^3} \sum_{i=1}^{p} \delta_i^4 v^2. \tag{19}$$

By Lemma A2, we know that, if ε is small enough,

$$Q(v) \geqslant \alpha_0\, |v|_{-L}^2 \qquad \forall u \in W_p(\varepsilon)$$
$$\alpha_0 > 0. \tag{20}$$

Lastly, by Lemma 1,

$$\lambda(u) \geqslant \lambda_0 > 0 \qquad \forall u \in W_p(\varepsilon). \tag{21}$$

Thus

$$\alpha_0\, |v|_{-L}^2 \leqslant C \left(|\partial J(u)| + \sum_{i \neq j} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{1/3} + \frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \right) |v|_{-L}$$

$$+ C\, |v|_{-L}^3 + Cp\varphi_p(\varepsilon)\, |v|_{-L}^2. \tag{22}$$

Choosing ε small enough, Lemma 2 follows from (22).

Lemma 2 is not actually the best estimate on $|v|_{-L}$. The following proposition provides us with the best estimate on $|v|_{-L}$. The proof of Proposition 3 is deferred to Appendix A.

PROPOSITION 3. *Let $p \in \mathbb{N}^*$ be given. There exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $u = \sum_{i=1}^{p} \alpha_i \delta_i + v$ in $W(p, \varepsilon_0)$, v satisfying* (Vo), *the following estimate holds*:

$$|v|_{-L} \leqslant C \left(|\partial J(u)| + \sum \varepsilon_{kr} (\log \varepsilon_{kr}^{-1})^{1/3} \right).$$

LEMMA 3. *Let $p \in \mathbb{N}^*$ be given and $\varepsilon > 0$ small enough. For any $u = \sum_{i=1}^{p} \alpha_i \delta_i + v$, v satisfying* (Vo), *in $W_p(\varepsilon)$, the following estimate holds*:

$$\left| \left(\partial J(u) - \partial J \left(\frac{\sum_{i=1}^{p} \alpha_j \delta_j}{|\sum_{i=1}^{p} \alpha_j \delta_j|_{-L}} \right) \right) \cdot \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right|$$
$$\leqslant C\, |v|_{-L}^2 + \left(|v|_{-L} \left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j \neq i} \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{1/3} \right) \right).$$

Proof. We denote

$$w = \frac{\sum_{i=1}^{p} \alpha_j \delta_j}{|\sum_{i=1}^{p} \alpha_j \delta_j|_{-L}}; \qquad w_1 = \sum_{i=1}^{p} \alpha_i \delta_j. \tag{23}$$

Since $|u|_{-L} = 1$ and since $|v|_{-L}$ tends to zero when ε tends to zero, we have

$$|u - w|_{-L} \leqslant c\, |v|_{-L}; \qquad |w_1|_{-L} (1 - C\, |v|_{-L}) \leqslant |w|_{-L} \leqslant |w_1|_{-} (1 + C\, |v|_{-L}) \tag{24}$$

for a suitable constant C. Hence, given $\varepsilon_1 > 0$, $\varepsilon_1 > \varepsilon$, we may choose ε such that $w \in W(p, \varepsilon_1)$. If ε_1 is small enough, J is bounded on $W(p, \varepsilon_1)$ by Proposition 1. Thus, by Lemma 1,

$$\begin{gathered} |(\lambda(u) - \lambda(w))| \leqslant \bar{K}\, |u - w|_{-L} \leqslant \bar{K}_1\, |v|_{-L} \\ \lambda(u) \leqslant \beta(p) \\ \lambda(w) \leqslant \beta(p). \end{gathered} \tag{25}$$

Let us compute now

$$\Delta = (\partial J(u) - \partial J(w)) \cdot \lambda\ i\, \frac{\partial \delta_i}{\partial \lambda_i}. \tag{26}$$

Δ splits in

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3, \tag{27}$$

where

$$\Delta_1 = (\lambda(u) - \lambda(w)) \left(\left(u, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} - \frac{\lambda^5(u) - \lambda^5(w)}{\lambda(u) - \lambda(w)} \int_{S^3} K u^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right) \tag{28}$$

$$\Delta_2 = \lambda(w) \left(u - w_1, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} - \lambda(w)^5 \int_{S^3} K(u^5 - w_1^5) \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \tag{29}$$

$$\Delta_3 = \lambda(w) \left(w_1 - w, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} - \lambda(w)^5 \int_{S^3} K(w_1^5 - w^5) \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}. \tag{30}$$

Clearly, using Lemma A3,

$$\left(u, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} = \left(w_1, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} = \sum_{j \neq i} \left(\alpha_j \delta_j, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} = O\left(\sum_{j \neq i} \varepsilon_{ij} \right); \tag{31}$$

$w_1 - w$ is parallel to w_1, with a coefficient θ satisfying, by (24),

$$w_1 - w = \theta w_1, \qquad |\theta| \leqslant C\, |v|_{-L}. \tag{32}$$

Therefore, if ε is small enough, we also have

$$\left| \left(w_1 - w, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right) \right| = O\left(|v|_{-L} \left(\sum_{j \neq i} \varepsilon_{ij} \right) \right). \tag{33}$$

Using (25), (31), and (33), we therefore have

$$\left| (\lambda(u) - \lambda(w)) \left(u, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} \right| + |\lambda(w)| \left| \left(u - w_1, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} \right|$$
$$+ \left| \lambda(w) \left(w_1 - w, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \right)_{-L} \right| = O\left(|v|_{-L} \left(\sum_{j \neq i} \varepsilon_{ij} \right) \right). \tag{34}$$

We estimate now

$$\int K w_1^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \int K(\alpha_i \delta_i)^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + R, \tag{35}$$

where R is upper bounded by

$$|R| \leqslant C \sum_{j \neq i} \int \delta_j^5 \lambda_i \left| \frac{\partial \delta_i}{\partial \lambda_i} \right|. \tag{36}$$

Observe that we have

$$\lambda_i \left|\frac{\partial \delta_i}{\partial \lambda_i}\right| \leqslant C\delta_i. \tag{37}$$

Therefore by Lemma A3

$$|R| \leqslant C \sum_{j \neq i} \varepsilon_{ij}. \tag{38}$$

On the other hand,

$$\int \delta_i^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = 0. \tag{39}$$

Thus, using (37) and (39)

$$\left|\int K(\alpha_i \delta_i)^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right| = \left|\int (K(x) - K(x_i))\, \alpha_i^5 \delta_i^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right|$$

$$\leqslant C \int |K(x) - K(x_i)|\, \delta_i^6. \tag{40}$$

By Lemma A6, we have

$$\int |(K(x) - K(x_i))|\, \delta_i^6 \leqslant C \left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right). \tag{41}$$

Combining (38), (40), and (41), we derive

$$\int K w_1^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = O\left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum\nolimits_{j \neq i} \varepsilon_{ij}\right). \tag{42}$$

Let us estimate now

$$\int K(u^5 - w_1^5)\, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = 5 \int K w_1^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v + R_1, \tag{43}$$

where, using in particular (37),

$$|R_1| \leqslant C\, |v|_{-L}^2. \tag{44}$$

We have

$$\int K w_1^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v = K(x_i) \int K w_1^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v + \int (K(x) - K(x_i))\, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v w_1^4. \tag{45}$$

Observe that by (37)

$$\left|\int (K(x)-K(x_i))\ \lambda_i \frac{\partial\delta_i}{\partial\lambda_i} vw_1^4\right|$$

$$\leqslant C \int |(K(x)-K(x_i))|\ \delta_i w_1^4\ |v|$$

$$\leqslant C\ |v|_{-L} \times \sum_{j=1}^{p} \left(\int |K(x)-K(x_i)|^{6/5}\ \delta_i^{6/5}\delta_j^{24/5}\right)^{5/6}$$

$$\leqslant C\ |v|_{-L} \left(\left(\int |K(x)-K(x_i)|^{6/5}\ \delta_i^6\right)^{5/6} + C'\left(\sum_{j\neq i} \delta_i^{6/5}\delta_j^{24/5}\right)^{5/6}\right). \quad (46)$$

By Lemma A6, we therefore have

$$\left|\int |(K(x)-K(x_i))|\ \lambda_i \frac{\partial\delta_i}{\partial\lambda_i} vw_1^4\right|$$

$$\leqslant C\ |v|_{-L}\left(\sum_{i\neq j} \varepsilon_{ij}\left(\log\frac{1}{\varepsilon_{ij}}\right)^{1/3} + \left(\frac{|DK(x_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\right)\right). \quad (47)$$

Next we have

$$\int w_1^4\lambda_1 \frac{\partial\delta_i}{\partial\lambda_i} v = \int \alpha_i^4\delta_i^4\lambda_i \frac{\partial\delta_i}{\partial\lambda_i} v + R_2, \quad (48)$$

where

$$|R_2| \leqslant C\int \sum_{j\neq i} \delta_j^4\delta_i\ |v| \leqslant C\ |v|_{-L}\left(\int \delta_i^{6/5}\delta_j^{24/5}\right)^{5/6}$$

$$\leqslant C\ |v|_{-L}\left(\sum_{i\neq j}\varepsilon_{ij}\left(\log\frac{1}{\varepsilon_{ij}}\right)^{1/3}\right). \quad (49)$$

Since

$$-L\frac{\partial\delta_i}{\partial\lambda_i} - 5\delta_i^4\frac{\partial\delta_i}{\partial\lambda_i} = 0 \quad (50)$$

we have

$$\int \delta_i^4\lambda_i \frac{\partial\delta_i}{\partial\lambda_i} v = -\frac{1}{5}\lambda_i\int L\frac{\partial\delta_i}{\partial\lambda_i} v = \frac{\lambda_i}{5}\left(\frac{\partial\delta_i}{\partial\lambda_i}, v\right)_{-L} = 0. \quad (51)$$

Relations (49) and (51) yield

$$\left|\int w_i^4\lambda_i \frac{\partial\delta_i}{\partial\lambda_i} v\right| \leqslant C\ (|v|_{-L}\left(\sum_{i\neq j}\varepsilon_{ij}\left(\log\frac{1}{\varepsilon_{ij}}\right)^{1/3}\right). \quad (52)$$

Thus, by (43), (44), (45), and (47),

$$\left|\int K(u^5 - w_1^5)\,\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i}\right|$$

$$\leqslant C\,|v|^2_{-L} + C\,|v|_{-L}\left(\sum_{i\neq j}\varepsilon_{ij}\left(\log\frac{1}{\varepsilon_{ij}}\right)^{1/3} + \frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right). \quad (53)$$

Combining (25), (32), (42), and (53), we derive

$$\left|(\lambda(u)^5 - \lambda(w)^5)\int Ku^5\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i}\right|$$

$$+\lambda(w)^5\left|\int_{S^3} K(u^5 - w_1^5)\,\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i}\right| + \lambda(w)^5\left|\int_{S^3} K(w_1^5 - w^5)\,\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i}\right|$$

$$\leqslant C\left(|v|^2_{-L} + |v|_{-L}\left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{i\neq j}\varepsilon_{ij}\left(\log\frac{1}{\varepsilon_{ij}}\right)^{1/3}\right)\right). \quad (54)$$

Relations (34) and (54) provide the desired upper bound on Δ. The proof of Lemma 3 is thereby complete.

LEMMA 4. *Let $p\in\mathbb{N}^*$ be given and $\varepsilon>0$ be small enough. For any u in $W(p,\varepsilon)$, $u=\sum_{j=1}^p \alpha_j\delta_j + v$, v satisfying* (Vo), *the following estimate holds*:

$$-\partial J(w)\cdot\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i} = c\sum_{j\neq i}\frac{\lambda_i}{K(x_j)^{1/4}}\,\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right) + o_1\left(\sum_{k\neq r}\varepsilon_{kr}\right),$$

where $w=\sum_{i=1}^p \alpha_j\delta_j/|\sum_{j=1}^p \alpha_i\delta_i|_{-L}$ and where $o_1(\sum_{k\neq r}\varepsilon_{kr})/\sum_{k\neq r}\varepsilon_{kr}$ tends to zero when ε tends to zero; c is a suitable constant.

Proof. By Proposition 1, we know that

$$\lambda^4(u)\,K(x_j)\,\alpha_j^4 \xrightarrow[\varepsilon\to 0]{} 1. \quad (55)$$

By (25), we know that

$$|(\lambda(u) - \lambda(w)| \leqslant C\,|v|_{-L} \xrightarrow[\varepsilon\to 0]{} 0. \quad (56)$$

Then

$$\lambda^4(w)\,K(x_j)\,\alpha_j^4 \xrightarrow[\varepsilon\to 0]{} 1. \quad (57)$$

Let us compute $-\partial J(w)\cdot\lambda_i(\partial\delta_i/\partial\lambda_i)$.

$$-\partial J(w)\cdot\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}=\lambda(w)\left(w,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}+\lambda(w)^5\int Kw^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}. \tag{58}$$

As in the previous lemma, we denote w_1 by the function $\sum_{j=1}^{p}\alpha_j\delta_j$ and we have

$$w=(1-\theta)\,w_1;\qquad |\theta|\leqslant C\,|v|_{-L}\xrightarrow[\varepsilon\to 0]{}0. \tag{59}$$

Clearly, by Lemmas A3 and A4, (37), and (55),

$$\left(\alpha_i\delta_i,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}=0$$

$$\lambda(w)\left(\alpha_i\delta_i,\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}=\lambda(w)\,\alpha_j\int\delta_j^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$

$$=\frac{1}{K(x_j)^{1/4}}\int\lambda_j^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+o(\varepsilon_{ij})\text{ for } i\neq j. \tag{60}$$

Next, we estimate $\lambda(w)^5\int Kw^5\lambda_i(\partial\delta_i/\partial\lambda_i)$.

$$\lambda(w)^5\int Kw^5\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}$$

$$=\lambda(w)^5\left\{\int K\left(\sum_{j=1}^{p}\alpha_j^5\delta_j^5\right)\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}+5\sum_{j\neq i}\int K\alpha_j\alpha_i^4\delta_i^4\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\delta_j\right\}+R_3, \tag{61}$$

where

$$|R_3|\leqslant C\sum_{\substack{j\neq k\\ j\neq i}}\int\delta_j^4\delta_k\lambda_i\left|\frac{\partial\delta_i}{\partial\lambda_i}\right|. \tag{62}$$

Observe that $\delta_j^3\delta_k^2\leqslant\delta_j^4\delta_k+\delta_k^4\delta_j$; hence, by (37)

$$|R_3|\leqslant C\sum_{\substack{j\neq k\\ j\neq i}}\int\delta_j^4\delta_k\delta_i. \tag{63}$$

Then, by Lemma A5,

$$|R_3|=o\left(\sum_{k\neq r}\varepsilon_{kr}\right). \tag{64}$$

We consider now

$$\lambda^5 \int K\left(\sum_{j=1}^{p} \alpha_j^5 \delta_j^5\right) \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \lambda^5 \left\{\int K\alpha_i^5 \delta_i^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + \sum_{j \neq i} \int K\alpha_j^5 \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right\}$$

$$= \lambda^5 \alpha_i^5 \int (K(x) - K(x_i))\, \delta_i^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}$$

$$+ \lambda^5 \sum_{j \neq i} \alpha_j^5 \int (K(x) - K(x_j))\, \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}$$

$$+ \lambda^5 \sum_{j \neq i} \alpha_j^5 K(x_j) \int \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}. \tag{65}$$

We have, using Lemma A6

$$\left|\lambda^5 \alpha_i^5 \int (K(x) - K(x_i))\, \delta_i^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right| \leqslant \frac{C}{\lambda_i^2}. \tag{66}$$

$$\left|\lambda^5 \sum_{j \neq i} \alpha_j^5 \int (K(x) - K(x_j))\, \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right| \leqslant \lambda^5 \sum_{j \neq i} \alpha_j^5 \int |(K(x) - K(x_j))|\, \delta_j^5 \delta_i$$

$$\leqslant o\left(\sum \varepsilon_{ij}\right). \tag{67}$$

Using (55), we derive

$$\lambda^5 \sum_{j \neq i} \alpha_j^5 K(x_j) \int \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \sum_{j \neq i} \frac{1}{K(x_j)^{1/4}} \int \alpha_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + o\left(\sum \varepsilon_{ij}\right). \tag{68}$$

Relations (66), (67), and (68) imply

$$\lambda^5 \int K\left(\sum_{j=1}^{p} \alpha_j^5 \delta_j^5\right) \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \sum_{j \neq i} \frac{1}{K(x_j)^{1/4}} \int \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^2}\right) + o\left(\sum_{j \neq 0} \varepsilon_{ij}\right). \tag{69}$$

Finally, we estimate

$$5\lambda(w)^5 \sum_{j \neq i} \int K\alpha_i^4 \alpha_j \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j$$

$$= 5\lambda(w)^5 \sum_{j \neq i} \alpha_i^4 K(x_i) \int \alpha_j \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j$$

$$+ 5\lambda(w)^5 \sum_{j \neq i} \alpha_i^4 \alpha_j \int (K(x) - K(x_i))\, \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j. \tag{70}$$

Using (55), we derive

$$5\lambda(w)^5 \sum_{j\neq i} \alpha_i^4 K(x_i) \int \alpha_j \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j$$
$$= 5 \sum_{j\neq i} \frac{1}{K(x_j)^{1/4}} \int \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j + o\left(\sum \varepsilon_{ij}\right). \tag{71}$$

Using Lemma A6 and (37), we derive

$$5\lambda(w)^5 \sum_{j\neq i} \alpha_i^4 \alpha_j \left| \int K(x) - K(x_i)) \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j \right|$$
$$\leqslant C \sum_{j\neq i} \int |K(x) - K(x_i)| \, \delta_i^5 \delta_j \leqslant o\left(\sum \varepsilon_{ij}\right). \tag{72}$$

Combining (71) and (72) yields

$$5\lambda(w)^5 \sum_{j\neq i} \int K\alpha_i^4 \alpha_j \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j$$
$$= 5 \sum_{j\neq i} \frac{1}{K(x_j)^{1/4}} \int \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j + o\left(\sum \varepsilon_{ij}\right). \tag{73}$$

Observe now that, by Lemma A4 and (50)

$$5 \int \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j = \int \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = c\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij}). \tag{74}$$

Combining (58), (60), (61), (64), (69), (71), (74), and the fact that $\theta = o(1)$ (see (59)), see we derive

$$-\partial J(w) \cdot \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = c \sum_{j\neq i} \lambda_i \frac{\lambda_i}{K(x_j)^{1/4}} \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\left(\frac{1}{\lambda_i^2}\right) + o\left(\sum_{k\neq r} \varepsilon_{kr}\right). \tag{75}$$

The proof of Lemma 4 is thereby complete.

LEMMA 5. *Let $p \geqslant 2$ be given and $\varepsilon_0 > 0$ be given small enough. Then, for any solution $u(s, u_0)$ of the differential equation*

$$\begin{cases} \dfrac{\partial u}{\partial s} = -\partial J(u) \\ u(0, u_0) = u_0 \end{cases} \qquad s \geqslant 0 \tag{E1}$$

starting at u_0 *in* $W(p, \varepsilon_0)$, *there exists* $s_1 > 0$ *such that* $u(s, u_0) \notin W(p, \varepsilon_0/2)$ *for any* $s \geqslant s_1$.

Remark 2. As states in [9, 10], the Palais–Smale condition is satisfied on the flow-lines of the gradient for $p \geqslant 2$. See [9, 10] for further precisions, results, and conjectures.

Remark 3. The fact that along the gradient flow u cannot "concentrate" at several points is due to an "interaction" between the δ_i's: any δ_i leads δ_j with $j \neq i$ to deconcentrate (see, e.g., (102)). This interaction is strong enough on (S^3, c). It is no longer the case for S^n, with $n \geqslant 4$; but still, in that case, the analysis of the gradient lines can be carried out (see Appendix C for $n = 4$; also [10]).

Remark 4. In Siu and Yau [21] and Taubes [17], interaction between two "bubbles" has been used, in a very different way, in order to establish the existence of harmonic maps [21] and for the Yang–Mills equations [17]. The result of Lemma 5, involving direct computations on gradient lines, with no restriction on the number of "bubbles," is of a new type.

Proof. Let us consider a solution of

$$\frac{\partial u}{\partial s} = -\partial J(u)$$

$$u(s) = u_0.$$

We first claim that

$$\int_0^{+\infty} |\partial J(u)|^2 < +\infty; \qquad \lim_{s \to +\infty} |\partial J(u)| = 0. \tag{76}$$

Relation (76) is proven in Appendix A in Lemma A1. Taking ε_0 small enough, let us suppose that $u(s_1, u_0) \in W(p, \varepsilon_0/2)$. Let s_2 be the largest time larger than s_1 such that $u(s, u_0) \in W(p, \varepsilon_0)$, for $s \in [s_1, s_2)$. Since ε_0 is small, we may solve problem (9) for $u(s, u_0)$ and write

$$u(s, u_0) = \sum_{i=1}^{p} \alpha_i(s)\, \delta(x_i(s), \lambda_i(s)) + v(s), \tag{77}$$

where v satisfies (Vo). The uniqueness of $(\alpha_i, x_i, \lambda_i)$ as solutions of (9) implies that $\alpha_i(s)$, $x_i(s)$, $\lambda_i(s)$, and also $v(s)$ are differentiable functions of s. We now successively complete the scalar product of Eq. (E1) with $\delta_i(s)$, $\lambda_i(\partial \delta_i / \partial \lambda_i)(s)$, $(1/\lambda_i)(\partial \delta_i / \partial x_i)(s)$. We then derive

$$\left(\frac{\partial}{\partial s}\left(\sum_{i=1}^{p} \alpha_j\delta_j+v\right), \delta_i\right)_{-L} = -(\partial J(u), \delta_i)_{-L} \tag{78}$$

$$\left(\frac{\partial}{\partial s}\left(\sum_{i=1}^{p} \alpha_j\delta_j+v\right), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\right)_{-L} = -\left(\partial J(u), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\right)_{-L} \tag{79}$$

$$\left(\frac{\partial}{\partial s}\left(\sum_{i=1}^{p} \alpha_j\delta_j+v\right), \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L} = -\left(\partial J(u), \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}. \tag{80}$$

Differentiating (Vo), we obtain

$$\left(\frac{\partial}{\partial s} v, \varphi\right)_{-L} = -\left(v, \frac{\partial\varphi}{\partial s}\right)_{-L}; \qquad \varphi=\delta_i, \frac{\partial\delta_i}{\partial x_i}, \frac{\partial\delta_i}{\partial\lambda_i}. \tag{81}$$

Thus, Eqs. (78)–(80) may be rewritten as

$$\begin{aligned}&\sum_{j=1}^{p} \dot\alpha_j(\delta_j, \delta_i)_{-L} + \sum_{j=1}^{p} \alpha_j\left(\frac{\partial\delta_j}{\partial x_j}(\dot x_j), \delta_i\right)_{-L}\\ &\qquad+\sum_{j=1}^{p} \alpha_j\left(\frac{\partial\delta_j}{\partial\lambda_j}, \delta_i\right)_{-L}\dot\lambda_j - \left(v, \frac{\partial\delta_i}{\partial x_i}(\dot x_i)\right) - \left(v, \frac{\partial\delta_i}{\partial\lambda_i}\dot\lambda_i\right)_{-L}\\ &= (-\partial J(u), \delta_i)_{-L},\end{aligned} \tag{82}$$

and the analogs of (82) with δ_i replaced by $\partial\delta_i/\partial x_i$ and $\partial\delta_i/\partial\lambda_i$. Observe that

$$\left(\frac{\partial\delta_i}{\partial x_i}(w), \delta_i\right)_{-L} = 0 \qquad \forall w \in T_{x_i}(S^3). \tag{83}$$

($T_{x_i}(S^3)$ is the tangent space to S^3 at x_i).

$$\left(\frac{\partial\delta_i}{\partial\lambda_i}, \delta_i\right)_{-L} = 0 \tag{84}$$

since $(\delta_i, \delta_i)_{-L}$ is independent of x_i and λ_i. Observe also that

$$(\delta_i, \delta_i)_{-L} = C_0 > 0 \tag{85}$$

$$\left(\frac{\partial\delta_i}{\partial x_i}, \frac{\partial\delta_i}{\partial x_i}\right)_{-L} = C_1\lambda_i^2; \qquad C_1 > 0 \tag{86}$$

$$\left(\frac{\partial\delta_i}{\partial\lambda_i}, \frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L} = C_2/\lambda_i^2; \qquad C_2 > 0 \tag{87}$$

$$(\delta_i, \delta_j)_{-L} \leqslant C\varepsilon_{ij} \qquad \text{for} \quad i \neq j \tag{88}$$

$$\left(\frac{\partial\delta_i}{\partial x_i}, \delta_j\right)_{-L} \leqslant C\lambda_i\varepsilon_{ij} \qquad \text{for} \quad i \neq j \tag{89}$$

$$\left(\frac{\partial\delta_i}{\partial\lambda_i}, \delta_j\right)_{-L} \leqslant C\varepsilon_{ij}/\lambda_i \qquad \text{for} \quad i \neq j \tag{90}$$

$$\left|\left(\frac{\partial\delta_i}{\partial x_i}, \frac{\partial\delta_j}{\partial x_j}\right)_{-L}\right| \leqslant C\lambda_i\lambda_j\varepsilon_{ij} \qquad \text{for} \quad i \neq j \tag{91}$$

$$\left|\left(\frac{\partial\delta_i}{\partial x_i}, \frac{\partial\delta_j}{\partial\lambda_j}\right)_{-L}\right| \leqslant C\frac{\lambda_i}{\lambda_j}\varepsilon_{ij} \qquad \text{for} \quad i \neq j \tag{92}$$

$$\left|\frac{\partial^2\delta_i}{\partial x_i^2}\right|_{-L} = C_3\lambda_i^2 \tag{93}$$

$$\left|\frac{\partial^2\delta_i}{\partial x_i^2}\right|_{-L} = C_4/\lambda_i^2 \tag{94}$$

$$\left|\frac{\partial^2\delta_i}{\partial\lambda_i\,\partial x_i}\right|_{-L} = C_5. \tag{95}$$

Relation (82) and the corresponding formulas for $\partial\delta/\partial\lambda_i$ and $\partial\delta_j/\partial x_j$ may therefore be rewritten as

$$\sum_{j\neq i}\left(\dot{\alpha}_j O(\varepsilon_{ij}) + \alpha_j\lambda_j\dot{x}_j\cdot O(\varepsilon_{ij}) + \alpha_j\frac{\dot{\lambda}_j}{\lambda_j}O(\varepsilon_{ij})\right)$$
$$+ C_0\dot{\alpha}_i + O(|v|_{-L})\cdot\lambda_i\dot{x}_i + O(|v|_{-L})\cdot\frac{\dot{\lambda}_j}{\lambda_i} = -(\partial J(u), \delta_i)_{-L} \tag{96}$$

$$\sum_{j\neq i}\left(\dot{\alpha}_j O(\varepsilon_{ij}) + \alpha_j\lambda_j\dot{x}_j\cdot O(\varepsilon_{ij}) + \alpha_j\frac{\dot{\lambda}_j}{\lambda_j}O(\varepsilon_{ij})\right)$$
$$+ (C_1\alpha_i + O(|v|_{-L}))\,\lambda_i\dot{x}_i + O(|v|_{-L})\cdot\frac{\dot{\lambda}_i}{\lambda_i} = -\left(\partial J(u), \frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\right) \tag{97}$$

$$\sum_{j\neq i}\left(\dot{\alpha}_j O(\varepsilon_{ij}) + \alpha_j\lambda_j\dot{x}_j\cdot O(\varepsilon_{ij}) + \alpha_j\frac{\dot{\lambda}_j}{\lambda_j}O(\varepsilon_{ij})\right)$$
$$+ O(|v|)\,\lambda_i\dot{x}_i + (C_2\alpha_i + O(|v|_{-L}))\,\frac{\dot{\lambda}_i}{\lambda_i} = \left(-\partial J(u), \lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}. \tag{98}$$

These three equations may be rewritten in a $3p\times 3p$ matrical form

$$
\begin{array}{c}
\qquad\qquad i \qquad\qquad\qquad i+p \qquad\qquad\qquad i+2p \\
\begin{array}{r}
\\ j \\ \\ \\ j+p \\ \\ \\ j+2p \\ \\
\end{array}
\left[\begin{array}{ccccccc}
C_0 & & & & & & \\
\cdots & O(\varepsilon_{ij}) & \cdots & O(|v|_{-L}+\varepsilon_{ij}) & \cdots & O(|v|_{-L}+\varepsilon_{ij}) & \\
 & C_0 & & & & & \\
 & & C_1(\alpha_1+O(|v|)) & & & & \\
\cdots & & & O(\varepsilon_{ij}) & \cdots & O(|v|_{-L}+\varepsilon_{ij}) & \\
 & & & C_1\alpha_p+O(|v|)) & & & \\
 & & & C_2(\alpha_1+O(|v|)) & & & \\
\cdots & & O(|v|_{-L}+\varepsilon_{ij}) & \cdots & & O(\varepsilon_{ij}) & \cdots \\
 & & & & & C_2(\alpha_p+O\,|v|_{-L}) &
\end{array}\right]
\left[\begin{array}{c}
\dot{\alpha}_i \\ \\ \lambda_i\dot{x}_i \\ \\ \dfrac{\dot{\lambda}_i}{\lambda_i}
\end{array}\right]
\end{array}
$$

$$
=\left[\begin{array}{c}
-(\partial J(u),\delta_i)_{-L} \\
-\left(\partial J(u),\dfrac{1}{\lambda_i}\dfrac{\partial\delta_i}{\partial x_i}\right)_{-L} \\
\left(-\partial J(u),\lambda_i\dfrac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}
\end{array}\right]. \tag{99}
$$

Remark. The term $O(|v|_{-L}+\varepsilon_{ij})$ is $O(\varepsilon_{ij})$ at row (i, j) for $j-i\not\equiv O(p)$. It is $O(|v|_{-L})$ at row (i, j) for $j-i\equiv O(p)$.

Observe that $|(\partial J(u),\delta_i)_{-L}|$, $|(\partial J(u),(1/\lambda_i)(\partial\delta_i/\partial x_i))_{-L}|$, and $|(\partial J(u), \lambda_i(\partial\delta_i/\partial\lambda_i))_{-L}|$ are upper bounded by $C\,|\partial J(u)|$ with a suitable constant C. Thus

$$
\begin{aligned}
\dot{\alpha}_i &= -\frac{1}{C_0}(\partial J(u),\delta_i)_{-L}+O\left(|\partial J(u)|\left(\sum_{k\neq r}\varepsilon_{kr}+|v|_{-L}\right)\right)\\
\lambda_i\dot{x}_i &= -\frac{1}{C_1\alpha_i}\left(\partial J(u),\lambda_i\frac{\partial\delta_i}{\partial x_i}\right)_{-L}+O\left(|\partial J(u)|\left(\sum_{k\neq r}\varepsilon_{kr}+|v|_{-L}\right)\right).\\
\frac{\dot{\lambda}_i}{\lambda_i} &= -\frac{1}{C_2\alpha_i}\left(\partial J(u),\lambda_i\frac{\partial\delta_i}{\partial\lambda_i}\right)_{-L}+O\left(|\partial J(u)|\left(\sum_{k\neq r}\varepsilon_{kr}+|v|_{-L}\right)\right).
\end{aligned}
\tag{100}
$$

Relation (100) holds for $s\in[s_1, s_2)$. In view of the statement of Lemma 5, we may choose s_1 as large as we wish. Therefore, by (76), the terms $O(|\partial J(u)|\,(\sum_{k\neq r}\varepsilon_{kr}))$ may be considered $o_2(\sum_{k\neq r}\varepsilon_{kr})$, where $o_2(\ )$, here, refers to s_1 tending to $+\infty$.

Using Lemmas 3 and 4, the third equation in (100) yields

$$\frac{\dot{\lambda}_i}{\lambda_i} = -\frac{1}{C_2\alpha_i} \sum_{j\neq i} \frac{\lambda_i}{K(x_j)^{1/4}} \frac{\partial\varepsilon_{ij}}{\partial\lambda_j} + O\left(\frac{1}{\lambda_i^2}\right) + o_1\left(\sum_{k\neq r} \varepsilon_{kr}\right) + o_2\left(\sum_{k\neq r} \varepsilon_{kr}\right)$$
$$+ O\left(|v|_{-L}^2 + |v|_{-L}\left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{j\neq i} \varepsilon_{ij}(\log \varepsilon_{ij}^{-1})^{1/3}\right) + |\partial J(u)|^2\right). \tag{101}$$

Thus, using Proposition 3, we derive

$$\frac{\dot{\lambda}_i}{\lambda_i} = \frac{1}{C_2\alpha_i} \sum_{j\neq i} \frac{\lambda_i}{K(x_j)^{1/4}} \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o_1\left(\sum_{k\neq r} \varepsilon_{kr}\right) + o_2\left(\sum_{k\neq r} \varepsilon_{kr}\right)$$
$$+ O\left(\frac{1}{\lambda_i^2}\right) + O\left(\sum_{k\neq r} \varepsilon_{kr}^2(\log \varepsilon_{ij}^{-1})^{2/3} + |\partial J(u)|^2\right). \tag{102}$$

By (55), $\alpha_i^4 \lambda(u)^4 K(x_i)$ tends to 1 when ε_0 tends to zero. Observe that $\lambda_i(\partial\varepsilon_{ij}/\partial\lambda_i) = O(\varepsilon_{ij})$. Thus

$$\frac{\dot{\lambda}_i}{\lambda_i} = \frac{K(x_i)^{1/4}}{C_2\lambda(u)} \sum_{j\neq i} \frac{\lambda_i}{K(x_j)^{1/4}} \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + o\left(\sum_{k\neq r} \varepsilon_{kr}\right) + O\left(\frac{1}{\lambda_i^2}\right) + O(|\partial J(u)|^2). \tag{103}$$

In this last formula, $o(\sum_{k\neq r} \varepsilon_{kr})$ means that, when ε_0 is small and s_1 is large, $o(\sum_{k\neq r} \varepsilon_{kr})/(\sum_{k\neq r} \varepsilon_{kr})$ is small. We observe now the following facts:

$$\lambda_i \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \lambda_j \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = -\frac{\lambda_i\lambda_j d(x_i, x_j)^2}{(\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j d(x_i, x_j)^2)^{3/2}} < 0. \tag{104}$$

Assuming $\lambda_i/\lambda_j \geqslant 1$ and ε_0 is small enough, then

$$-\lambda_i \frac{\partial\varepsilon_{ij}}{\partial\lambda_i} \geqslant \frac{1}{4}\,\varepsilon_{ij} \tag{105}$$

($\varepsilon_{ij} < \varepsilon_0$ here; thus λ_i/λ_j or $\lambda_i\lambda_j d(x_i, x_j)^2$ is very large, if ε_0 is small). Considering the functions $\lambda_1(s), ..., \lambda_p(s)$, we may order them in an increasing order at each time s:

$$\lambda_{i_1} \leqslant \text{——} \leqslant \lambda_{i_p}(s). \tag{106}$$

Let m and M be such that

$$\sqrt{m} \leqslant \frac{1}{K(x)^{1/4}} \leqslant \sqrt{M}. \tag{107}$$

We introduce the function

$$\phi(s) = \log \lambda_{i_2}(s) + \frac{2M}{m} \log \lambda_{i_3}(s) + \text{——} + \left(\frac{2M}{m}\right)^{p-2} \log \lambda_{i_p}(s). \tag{108}$$

ϕ is a continuous function, which may be not differentiable at times s s.t. $\lambda_{i_k}(s)=\lambda_{i_{k+1}}(s)$ for a certain index k. However, at those points, ϕ has at most a finite number of derivatives, i.e.;

$$\lim_{h\to 0}\frac{\phi(s+h)-\phi(s)}{h} \tag{109}$$

takes at most a finite number of values. We compute one of these values:

$$\omega(s)=\sum_{j=2}^{p}\left(\frac{2M}{m}\right)^{j-2}\dot{\lambda}_{i_j}(s)/\lambda_{i_j}(s)\in\{\dot{\phi}(s)\} \tag{110}$$

if (105) holds. Observe that, for any x_1, x_2, x_1', $x_2'\in S^3$, $(M/m)(K(x_1')^{1/4}/K(x_1)^{1/4})\geqslant K(x_2')^{1/4}/K(x_2)^{1/4}$. Using (103) and (107), we derive

$$\begin{aligned}
\omega(s)\leqslant\frac{1}{C_2\lambda(u)}&\left\{\sum_{j\geqslant 2}\left(\lambda_{i_j}\frac{\partial\varepsilon_{i_1i_j}}{\partial\lambda_{i_j}}\right)\left(\frac{2M}{m}\right)^{j-2}\cdot m\right.\\
&+\sum_{\substack{j\neq i_2\\ j\neq i_1}}\frac{K(x_{i_2})^{1/4}}{K(x_{i_j})^{1/4}}\left(\lambda_{i_2}\frac{\partial\varepsilon_{i_2i_j}}{\partial\lambda_{i_2}}+\lambda_{i_j}\frac{\partial\varepsilon_{i_2i_j}}{\partial\lambda_{i_j}}\right)\\
&+\frac{2M}{m}\sum_{\substack{j\neq i_3\\ j\neq i_2\\ j\neq i_1}}\frac{K(x_{i_2})^{1/4}}{K(x_{i_j})^{1/4}}\left(\lambda_{i_3}\frac{\partial\varepsilon_{i_3i_j}}{\partial\lambda_{i_3}}+\lambda_{i_j}\frac{\partial\varepsilon_{i_3i_j}}{\partial\lambda_{i_j}}\right)\\
&+\frac{M}{m}\lambda_{i_3}\frac{\partial\varepsilon_{i_2i_3}}{\partial\lambda_{i_3}}+\cdots+\left[\left(\left(\frac{2M}{m}\right)^{k-2}-\frac{M}{m}\right)\lambda_{i_k}\frac{\partial\varepsilon_{i_2i_k}}{\partial\lambda_{i_k}}+\cdots\right.\\
&+\left(\left(\frac{2M}{m}\right)^{k-2}-\left(\frac{2M}{m}\right)^{k-3}\frac{M}{m}\right)\lambda_{i_k}\frac{\partial\varepsilon_{i_ki_{k-1}}}{\partial\lambda_{i_k}}\\
&\left.+\left(\frac{2M}{m}\right)^{k-2}\sum_{j>k}\frac{K(x_{i_k})^{1/4}}{K(x_{i_j})^{1/4}}\left(\lambda_{i_k}\frac{\partial\varepsilon_{i_ki_j}}{\partial\lambda_{i_k}}+\lambda_{i_j}\frac{\partial\varepsilon_{i_ki_j}}{\partial\lambda_{i_j}}\right)\right]\\
&+\left[\left(\left(\frac{2M}{m}\right)^{p-2}-\frac{M}{m}\right)\lambda_{i_p}\frac{\partial\varepsilon_{i_2i_p}}{\partial\lambda_{i_p}}+\cdots\right.\\
&+\left(\left(\frac{2M}{m}\right)^{p-2}-\left(\frac{2M}{m}\right)^{k-2}\frac{M}{m}\right)\lambda_{i_p}\frac{\partial\varepsilon_{i_ki_p}}{\partial\lambda_{i_p}}+\cdots\\
&\left.\left.+\left(\left(\frac{2M}{m}\right)^{p-2}-\left(\frac{2M}{m}\right)^{p-3}\frac{M}{m}\right)\lambda_{i_p}\frac{\partial\varepsilon_{i_pi_{p-1}}}{\partial\lambda_{i_p}}\right]\right\}\\
&+o\left(\sum_{k\neq r}\varepsilon_{kr}\right)+O\left(\sum_{j=2}^{p}\frac{1}{\lambda_{i_j}^2}\right)+O(|\partial J(u)|^2).
\end{aligned} \tag{111}$$

Using then (111), (104), and (105), we derive

$$\omega(s) \leqslant -C \sum_{i \neq j} \varepsilon_{ij} + o\left(\sum_{i \neq j} \varepsilon_{ij}\right) + O\left(\sum_{j=2}^{p} \left(\frac{1}{\lambda_{i_j}^2}\right)\right) + O(|\partial J(u)|^2), \quad (112)$$

where C is suitable positive constant. Observe now that

$$\frac{1}{\lambda_{i_j}^2} = o(\varepsilon_{i_1 i_j}) \qquad \text{for} \quad j \geqslant 2. \quad (113)$$

Thus, if ε_0 is small enough and s_1 is large enough

$$\omega(s) \leqslant -C \sum_{i \neq j} \varepsilon_{ij} + O(|\partial J(u)|^2). \quad (114)$$

From (109), (110), and (114), we derive

$$\phi(s) \leqslant \phi(s_1) + C \int_{s_1}^{s} |\partial J(u)|^2; \quad (115)$$

(115) holds if $s \in [s_1, s_2)$. We claim that

$$s_2 < +\infty. \quad (116)$$

Indeed, if s_2 is equal to $+\infty$, then by (76) and Proposition 1, $\phi(s)$ tends to $+\infty$ when $s \to +\infty$, since $u(s) \in W(p, \varepsilon_0)$ for $s \geqslant s_1$ (ε_0 small enough); this contradicts (115).

Thus, for any s_1 large enough such that

$$u(s_1, u_0) \in W(p, \varepsilon_{0/2}), \quad (117)$$

there exists $s_2 < +\infty$ such that $u(s_2, u_0) \notin W(p, \varepsilon_0)$. Any flow-line entering $W(p, \varepsilon_{0/2})$ must therefore leave $W(p, \varepsilon_0)$ at a later time. For $s \in [s_1, s_2]$, $u(s) \in W(p, \varepsilon_0) \setminus W(p, \varepsilon_{0/2})$. By Proposition 1,

$$\exists \alpha > 0 \text{ such that } |\partial J(u)| \geqslant \alpha \qquad \forall u \in W(p, \varepsilon_0) \setminus W(p, \varepsilon_{0/2}). \quad (118)$$

Moreover, there exists $\beta > 0$ such that

$$\beta \leqslant d(W(p, \varepsilon_0)^c, W(p, \varepsilon_{0/2})). \quad (119)$$

Thus

$$\int_{s_1}^{s_2} |\partial J(u)|^2 \geqslant \alpha \int_{s_1}^{s_2} |\partial J(u)| \geqslant \alpha\beta > 0. \quad (120)$$

Relation (120) implies that there are only finitely many of these intervals $[s_1, s_2)$. Lemma 5 follows.

LEMMA 6. *Let $\varepsilon > 0$ be small enough. For any $u_0 = \alpha_1 \delta_1 + v$, v satisfying* (Vo), *in $W(1, \varepsilon)$, the following estimates hold*:

$$\frac{\dot{\lambda}_1}{\lambda_1}(0) = -c_1(1+o(1)) \frac{\Delta K(x_1)}{\lambda_1^2 K(x_1)^{5/4}} + O\left(|\partial J(u_0)|^2 + \frac{|DK(x_1)|^2}{\lambda_1^2}\right) + o\left(\frac{1}{\lambda_1^2}\right) \tag{121}$$

$$\dot{x}_1(0) = -c_2(1+o(1)) \frac{DK(x_1)}{\lambda_1^2 K(x_1)^{5/4}} + O\left(\frac{|\partial J(u_0)|^2}{\lambda_1} + \frac{1}{\lambda_1^3}\right), \tag{122}$$

where c_1 and c_2 are positive constants and where $o(1)$ tends to zero when ε tends to zero.

Proof. We apply (100) with a single function δ_1. We obtain

$$\begin{aligned} \dot{\alpha}_1 &= \frac{1}{C_0} (\partial J(u_0), \delta_1)_{-L} + O(|v|_{-L} |\partial J(u_0)|) \\ \lambda_1 \dot{x}_1 &= -\frac{1}{c_1 \alpha_1} \left(\partial J(u_0), \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1}\right)_{-L} + O(|v|_{-L} |\partial J(u_0)|) \\ \frac{\dot{\lambda}_1}{\lambda_1} &= -\frac{1}{C_2 \alpha_1} \left(\partial J(u_0), \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1}\right)_{-L} + O(|v|_{-L} |\partial J(u_0)|). \end{aligned} \tag{123}$$

Since $(\delta_1, \partial\delta_1/\partial x_1)_{-L} = (\delta_1, \partial\delta_1/\partial\lambda_1)_{-L} = (v, \partial\delta_1/\partial x_1)_{-L} = (v, \partial\delta_1/\partial\lambda_1)_{-L} = 0$, we have

$$\begin{aligned} \dot{x}_1 &= \frac{\lambda^5(u_0)}{C_1 \lambda_1 \alpha_1} \int K(x)(\alpha_1 \delta_1 + v)^5 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} + O\left(|v|_{-L} \frac{|\partial J(u_0)|}{\lambda_1}\right) \\ \frac{\dot{\lambda}_1}{\lambda_1} &= \frac{\lambda(u_0)^5}{C_2 \alpha_1} \int K(x)(\alpha_1 \delta_1 + v)^5 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} + O(|v|_{-L} |\partial J(u_0)|). \end{aligned} \tag{124}$$

Expanding, we obtain

$$\begin{aligned} \dot{x}_1 &= \frac{\lambda^5(u_0)\, \alpha_1^5}{C_1 \lambda_1 \alpha_1} \int K(x)\, \delta_1^5 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} + \frac{5\lambda(u_0)^5}{C_1 \lambda_1} \alpha_1^4 \\ &\quad \times \int K(x)\, \delta_1^4 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} v + \frac{1}{\lambda_1} O(|v|_{-L}^2) + O\left(|v|_{-L} \frac{|\partial J(u_0)|}{\lambda_1}\right) \\ \frac{\dot{\lambda}_1}{\lambda_1} &= \frac{\lambda(u_0)^5\, \alpha_1^5}{C_2 \alpha_1} \int K(x)\, \delta_1^5 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} + \frac{5\lambda(u_0)^5}{C_2} \alpha_1^4 \\ &\quad \times \int K(x)\, \delta_1^4 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} v + O(|v|_{-L}^2) + O(|v|_{-L} |\partial J(u_0)|). \end{aligned} \tag{125}$$

Observe that

$$\int \delta_1^5 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} = \int \delta_1^5 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} = \int \delta_1^4 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} v$$
$$= \int \delta_1^4 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} v = 0. \tag{126}$$

Setting

$$\varphi = \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} \quad \text{or} \quad \varphi = \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1} \tag{127}$$

we are led to estimate, in both cases,

$$\int K(x)\, \delta_1^5 \varphi = \int (K(x) - K(x_1))\, \delta_1^5 \varphi \tag{128}$$

$$\int K(x)\, \delta_1^4 v\varphi = \int (K(x) - K(x_1))\, \delta_1^4 v\varphi. \tag{129}$$

In both cases φ is upper bounded by $C\delta_1$. Thus, by Lemma A6,

$$\left| \int K(x)\, \delta_1^4 v\varphi \right| \leqslant C \int |K(x) - K(x_1)|\, |\delta_1^5|\, |v|$$
$$\leqslant C\, |v|_{-L} \left(\int |K(x) - K(x_1)|^{6/5}\, \delta_1^6 \right)^{5/6}$$
$$\leqslant C\, |v|_{-L} \left(\frac{DK(x_1)}{\lambda_1} + \frac{1}{\lambda_1^2} \right). \tag{130}$$

This, together with Lemma 2, yields

$$\lambda(u_0)^5\, \alpha_1^4 \left| \int K(x)\, \delta_1^4 v\varphi \right| \leqslant C \left(|\partial J(u_0)|^2 + \frac{|DK(x_1)|^2}{\lambda_1^2} + \frac{1}{\lambda_1^4} \right). \tag{131}$$

Expanding K around x_1, we obtain (see Lemma A7)

$$\int K(x)\, \delta_1^5 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1} = \int (K(x) - K(x_1))\, \delta_1^5 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial x_1}$$
$$= -c\, \frac{DK(x_1)}{\lambda_1} + O\left(\frac{1}{\lambda_1^2} \right) \tag{132}$$

for a suitable constant $c > 0$.

We also obtain, using the symmetry of $\partial\delta_1/\partial\lambda_1$ around x_1 and its symmetry with respect to the permutation of coordinates (see Lemma A7)

$$\int K(x)\,\delta_1^5\lambda_1\,\frac{\partial\delta_1}{\partial\lambda_1}=\int (K(x)-K(x_1))\,\delta_1^5\lambda_1\,\frac{\partial\delta_1}{\partial\lambda_1}$$
$$= -c'\,\frac{\Delta K(x_1)}{\lambda_1^2}+o\left(\frac{1}{\lambda_1^2}\right). \tag{133}$$

Observe lastly that, by (55),

$$\lambda(u_0)^5\,\alpha_1^5=\frac{1}{K(x_1)^{5/4}}\,(1+o(1));\qquad \alpha_1=\frac{1}{(\int|\nabla\delta_1|^2)^{1/2}}\,(1+o(1)). \tag{134}$$

Relations (125), (131), (132), (133), (134), and Lemma 2 imply then Lemma 6.

LEMMA 7. *There exists* $\varepsilon_0>0$ *such that for any* $u(s,u_0)=\alpha_1(s)\,\delta_1+v$ *satisfying*

$$\frac{\partial u}{\partial s}=-\partial J(u)$$

$$u(0,u_0)=u_0\in W(1,\varepsilon_0);\qquad u(s,u_0)\in W(1,\varepsilon_0)\ \forall s\geqslant 0$$

(v *satisfying* (Vo)), *we have*:

(1) $x_1(s)$ *converges to a critical point* y_j *of* K *s.t.* $\Delta K(y_j)<0$

(2) $\lambda(u)\,\alpha_1(s)\to 1/K(y_j)^{1/4}$; $J(u)(s)\to[\int\delta^6]^2/K(y_j)$

(3) $\lambda_1(s)\sim_{s\to+\infty} c_3\sqrt{-\Delta K(y_j)}\,\sqrt{s}$; $c_3>0$.

Proof. As a consequence of Lemma 6, the expansions provided in (121) and (122) hold if ε_0 is chosen small enough, for any $s\geqslant 0$. Thus

$$\dot{x}_1(s)=-c_2(1+o(1))\,\frac{DK(x_1)}{\lambda_1^2K(x_1)^{5/4}}+O\left(\frac{|\partial J(u)|^2}{\lambda_1}+\frac{1}{\lambda_1^3}\right) \tag{135}$$

$$\frac{\dot\lambda_1}{\lambda_1}(s)=-c_1(1+o(1))\,\frac{\Delta K(x_1)}{\lambda_1^2K(x_1)^{5/4}}+o\left(\frac{1}{\lambda_1^2}\right)$$
$$+O\left(|\partial J(u)|^2+\frac{|DK(x_1)|^2}{\lambda_1^2}\right). \tag{136}$$

Observe that, as already pointed out (see (76) in particular),

$$\int_0^{+\infty}|\partial J(u)|^2<+\infty. \tag{137}$$

Relation (136) then implies the existence of a constant C such that

$$\lambda_1^2 \leqslant C(s+1). \tag{138}$$

Indeed, we have $|2\lambda_1 \dot{\lambda}_1 + \lambda_1^2 O(|\partial J(u)|^2)| \leqslant C_1$, where C_1 is a suitable constant. Thus

$$\lambda_1^2 e^{\int_0^s O(|\partial J(u)|^2)} \leqslant C_1 s + \lambda_1^2(0). \tag{139}$$

Using (137), (138) follows. Relation (138) implies that

$$\int_0^2 \frac{1}{\lambda_1^2} \geqslant c \log s - 1 \tag{140}$$

for a suitable $c > 0$.

Assuming (1), (3) follows then immediately, since $\Delta K(y_j) < 0$ and $\int_0^{+\infty} |\partial J(u)|^2 < +\infty$. Part (2) also follows immediately from (1) and Proposition 1.

We prove now (1). Let s_0 be a large positive time. Considering the $O(|\partial J(u)|^2/\lambda_1)$ in (135), we set, after completing a suitable stereographic projection, in order to be able to work linearly,

$$x_1'(s) = x_1(s) - \int_{s_0}^s O\left(\frac{|\partial J(u)|^2}{\lambda_1}\right). \tag{141}$$

Then x_1' satisfies using the mean value theorem

$$\dot{x}_1'(x) = -c_2(1+o(1)) \frac{DK(x_1')}{\lambda_1^2 K(x_1')^{5/4}} + O\left(\frac{1}{\lambda_1^2} \int_{s_0}^s \frac{|\partial J(u)|^2}{\lambda_1} + \frac{1}{\lambda_1^3}\right). \tag{142}$$

Let

$$d\tau = \frac{1}{\lambda_1^2}\, ds; \qquad \tau = \int_0^s \frac{1}{\lambda_1^2}\, ds. \tag{143}$$

Relation (140) implies that τ runs from zero to $+\infty$ when s runs from zero to $+\infty$. We complete this change of variables in (135). This yields

$$\dot{x}_1'(\tau) = -c_2(1+o(1)) \frac{DK(x_1')}{K(x_1')^{5/4}} + O\left(\int_{s_0}^{s(\tau)} \frac{|\partial J(u)|^2}{\lambda_1} + \frac{1}{\lambda_1(\tau)}\right) \tag{144}$$

$$\tau \in (0, +\infty).$$

Clearly, for any $\rho > 0$, we may find, using the fact that

$\int_0^{+\infty} (|\partial J(u)|^2/\lambda_1) < +\infty$, an $s_0(\rho)$, $N_\rho(y_1), ..., N_\rho(y_m)$, ρ-neighbourhoods of the y_i's, and $\gamma(\rho) > 0$ such that

$$\frac{d}{d\tau} K(x_1'(\tau)) \leqslant \frac{-c_2}{2} \frac{|DK(x_1')|^2}{K(x_1')^{5/4}} \leqslant -\gamma(\rho) \qquad \text{for any } \tau \text{ such that } s(\tau) \geqslant s_0(\rho) \tag{145}$$

and such that

$$x_1'(\tau) \notin N_\rho(y_j) \qquad \forall j = 1, ..., m.$$

We claim now that, for τ large enough, $x_1'(\tau)$ stabilizes in one of the $N_\rho(y_j)$'s. This statement implies the convergence of $x_1'(\tau)$ to one of the y_j's, after considering a sequence $\rho_k \to 0$. Taking $s_0(\rho_k)$ s.t. $s_0(\rho_k) \to +\infty$, (141) implies that $x_1(\tau)$ converges to the same y_j.

Since $u(s, u_0)$ remains in $W(1, \varepsilon_0)$ $\forall s \geqslant 0$, $\lambda_1(s)$ remains large (depending on ε_0 small). This, together with (136) and (138), implies that $-\Delta K(y_j) > 0$ as stated. Thus, the proof of (1) relies on the proof of our claim, which we establish now. Relation (145) implies that, for any τ_1, there exists $\tau_2 > \tau_1$ such that $x_1'(\tau_2)$ belongs to one of the $N_\rho(y_j)$'s. Let us assume now that during the time $[\tau_1, \tau_2]$, $x_1'(\tau)$ travels from one $N_\rho(y_j)$ to another $N_\rho(y_k)$. If ρ is chosen small enough, then, with a suitable constant C,

$$C(\tau_1 - \tau_2) \geqslant \int_{\tau_1}^{\tau_2} |\dot{x}_1'(\tau)| \, d\tau \geqslant \frac{1}{2} d(y_k, y_j) > 0. \tag{146}$$

We then have

$$\begin{aligned} K(x_1(\tau_2)) - K(x_1(\tau_1)) &= \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} K(x_1'(\tau)) \leqslant \gamma(\rho)(\tau_1 - \tau_2) \\ &\leqslant \frac{-\gamma(\rho)}{2C} d(y_j, y_k) \leqslant -\gamma_1(\rho) \\ &= \frac{-\gamma(\rho)}{2C} \inf_{j \neq k} d(y_j, y_k) < 0. \end{aligned} \tag{147}$$

Therefore

$$K(y_k) - K(y_j) \leqslant C\rho^2. \tag{148}$$

Taking ρ small enough, (148) implies

$$K(y_k) \leqslant K(y_j). \tag{149}$$

Since y_k is distinct from y_j, should $K(y_k)$ be equal to $K(y_j)$, there would be no trajectory of $-DK(x)$ from y_j to y_k. Thus, for ρ small enough and

$s_0(\rho)$ large enough, there would be no trajectory of (144) from $N_\rho(y_j)$ to $N_\rho(y_k)$. Thus

$$K(y_k) < K(y_j). \tag{150}$$

Clearly, $x_1'(\tau)$, because of (150), can travel between the $N_\rho(y_j)$'s only a finite number of times. This proves our claim and (1). The proof of Lemma 7 is thereby complete.

Let

$$y_1, ..., y_s \text{ be the critical points of } K \text{ such that } \Delta K(y_j) < 0, \tag{151}$$

y_j has k_j as Morse index.

We denote

$$c_j = \frac{1}{3}\left(\int \delta^6\right)\Big/ K(y_j)^{1/2} \leqslant c_{j+1} = \frac{1}{3}\left(\int \delta^6\right)\Big/ K(y_{j+1})^{1/2}. \tag{152}$$

For the sake of simplicity in the presentation, we will assume in the remainder of this paper that

$$c_j < c_{j+1}. \tag{153}$$

Our arguments adapt to the case where equalities occur with only minor modifications. Let $\rho > 0$ be a small number and let

$$\omega_\rho : S^3 \to [0, 1] \tag{154}$$

be a C^∞ function such that

$$\begin{aligned} &\omega_\rho(x) = 1 \quad \text{if } x \text{ belongs a } \rho/2\text{-neighbourhood } N_{\rho/2} \text{ of } y_j \\ &\omega_\rho(x) = 0 \quad \text{if } x \text{ belongs to the complement} \\ &\qquad\qquad\quad \text{of a } \rho\text{-neighbourhood } N_\rho \text{ of } y_j \\ &|\omega_\rho'(x)| \leqslant 4/\rho. \end{aligned} \tag{155}$$

Let

$$0 < \mu < \eta \text{ be two small numbers such that } \mu/\eta \text{ is small.} \tag{156}$$

Let

$$\omega: \mathbb{R}^+ \to [0, \mu] \text{ be a } C^\infty \text{ function such that } \omega(x) = \mu \text{ if } |x| \leqslant \eta/2;$$
$$\omega(x) = 0 \text{ if } x \geqslant \eta; \qquad |\omega'(x)| \leqslant 4\mu/\eta. \tag{157}$$

For ε small enough, the function

$$\omega_\rho(x_1)\,\omega(|\partial J(u)|^2) = g(u) \qquad (u = \alpha_1 \delta_1 + v) \tag{158}$$

is well defined on $W(1, \varepsilon)$. Choosing $\varepsilon > 0$ small enough such that $W(1, \varepsilon) \cap W(p, \varepsilon) = \phi\ \forall p \geqslant 2$, we then choose η such that

$$0 < \eta < \inf_{u \in W(1,\varepsilon) - W(1, \varepsilon/2)} |\partial J(u)|^2. \tag{159}$$

g may then be extended by zero to all of Σ^+. Let

$$F(u) = J(u) - g(u). \tag{160}$$

We have:

LEMMA 8. *Let $b > 0$ be given. There exists $c > 0$ such that if ρ, η, and ε are given small enough, η subject to* (159), *and if $\mu < c \inf(\rho^2, \rho^3/\varepsilon, \eta)$ then $\partial F(u) \cdot \partial J(u) > 0$ for any u such that $J(u) \leqslant b$.*

Proof. In the proof of Lemma 8, we will consider functions u, u_0 in $W(1, \varepsilon)$, which can be split, by Proposition 2, as $u = \alpha_1 \delta_1 + v$, δ_1 being $\delta(x_1, \lambda_1)$, and v satisfying (Vo). For the sake of simplicity in the presentation, we will omit stating this splitting and we will refer directly to x_1 and v.

Relations (122) and (123) imply

There exist constants c, C such that

$$\forall u_0 \in W(1, \varepsilon),\ |\partial J(u_0)| \geqslant \frac{\lambda_1\, |\dot{x}_1(0)|}{C}$$
$$\geqslant c\, \frac{DK(x_1)}{\lambda_1} - C\left(|\partial J(u_0)|^2 + \frac{1}{\lambda_1^2}\right). \tag{161}$$

Taking ε small enough, we may upper bound $C\,|\partial J(u_0)|^2$ by $|\partial J(u_0)|$. Thus (161) implies

$$\forall u \in W(1, \varepsilon), \qquad |\partial J(u)| \geqslant c'\, \frac{|DK(x_1)|}{\lambda_1} - \frac{C'}{\lambda_1^2}. \tag{162}$$

Relations (121) and (123) imply

$$\forall u \in W(1, \varepsilon), \qquad |\partial J(u)| \geqslant c\left(\frac{|\Delta K(x_1)|}{\lambda_1^2}\right)$$
$$- C\left(|\partial J(u)|^2 + \frac{|DK(x_1)|^2}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right)\right). \tag{163}$$

Using the fact that y_j is nondegenerate, we derive the existence of $\alpha > 0$ such that, if ρ is small enough,

$$|DK(x_1)| \geqslant \alpha\rho > 0 \qquad \forall x_1 \text{ such that } 0 < \omega_\rho(x_1) < 1 \text{ (i.e., } x_1 \in N_\rho - N_{\rho/2}). \tag{164}$$

Since $\Delta K(y_j)$ is nonzero, we derive the existence of $\beta > 0$ such that, if ρ is small enough,

$$|\Delta K(x_1)| \geqslant \beta > 0 \qquad \forall x_1 \text{ such that } \omega_\rho(x_1) > 0 \text{ (i.e., } x_1 \in N_\rho). \tag{165}$$

Therefore, if ρ is small enough, we have by (163) and (165)

$$\forall u \in W(1, \varepsilon) \text{ such that } x_1 \in N_\rho,\ |\partial J(u)| \geqslant c_3 \beta / \lambda_1^2. \tag{166}$$

Relations (162) and (166) imply

$$\forall u \in W(1, \varepsilon) \text{ such that } x_1 \in N_\rho,\ |\partial J(u)| \geqslant c_4 \left(\frac{|DK(x_1)|}{\lambda_1} + \frac{1}{\lambda_1^2} \right). \tag{167}$$

Relations (164) and (167) imply then

$$\forall u \in W(1, \varepsilon) \text{ such that } x_1 \in N_\rho - N_{\rho/2},\ |\partial J(u)| \geqslant \frac{c_4 \alpha \rho}{\lambda_1} \tag{168}$$

under no other constraints than the smallness of ε and ρ. C_4 and α are uniform when ε and ρ are small enough. Observe that, for any $u \in W(1, \varepsilon)$, we have, by (122) and (155),

$$|\omega_\rho'(x_1) \cdot \dot{x}_1| \leqslant \frac{C_5}{\rho} \left(\frac{|DK(x_1)|}{\lambda_1^2} + \frac{|\partial J(u)|^2}{\lambda_1} + \frac{1}{\lambda_1^3} \right), \tag{169}$$

where C_5 is uniform for ε small and is independent of ρ. We now impose on μ the two following constraints. Let γ be such that

$$|DK(x_1)| \leqslant \gamma\rho \qquad \forall x_1 \in N_\rho, \tag{170}$$

$$\frac{\mu C_5}{\rho} \left(\frac{\gamma\rho}{\lambda_1^2} + \frac{|\partial J(u)|^2}{\lambda_1} + \frac{1}{\lambda_1^3} \right) \leqslant \frac{1}{2} |\partial J(u)|^2 \qquad \forall u \in W(1, \varepsilon) \text{ such that } x_1 \in N_\rho - N_{\rho/2} \tag{171}$$

and

$$|\omega'(|\partial J(u)|^2)(|\partial J(u)|^2)'| \leqslant \tfrac{1}{4} |\partial J(u)| \qquad \forall u \text{ such that } J(u) \leqslant b. \tag{172}$$

We assume $\mu C_5/\rho < 1/4$; $\lambda_1 \geqslant 1$. By (168), (171) will then be satisfied as soon as

$$\left(\gamma + \frac{1}{\lambda_1 \rho}\right) \frac{\mu}{\lambda_1^2} < C_6 \frac{\rho^2}{\lambda_1^2}. \tag{173}$$

Since $\lambda_1 > 1/\varepsilon$ on $W(1, \varepsilon)$, (173) will be satisfied if

$$\mu < C_7 \inf(\rho^2, \rho^3/\varepsilon); \qquad C_7 \text{ a uniform constant.} \tag{174}$$

In order to ensure (172), we observe that $|((|\partial J(u)|^2)'|$ is bounded by $C(b)\, |\partial J(u)|$ on the set of u's such that $J(u) \leqslant b$. Hence, (172) is satisfied if

$$4\mu/\eta \cdot C(b)\, |\partial J(u)| \leqslant \tfrac{1}{4}\, |\partial J(u)|, \tag{175}$$

i.e.,

$$\mu \leqslant \frac{\eta}{16 C(b)}. \tag{176}$$

Relations (174) and (176) provide the constraint on μ as stated in Lemma 8. Let us compute

$$\partial F(u) \cdot \partial J(u) = |\partial J(u)|^2 - g'(u) \cdot \partial J(u). \tag{177}$$

By (171) and (172),

$$\begin{aligned} |g'(u) \cdot \partial J(u)| &\leqslant |\omega_\rho'(x_1) \cdot \dot{x}_1|\, \omega(|\partial J(u)|^2) + |\partial J(u)|\, |\omega'(|\partial J(u)|^2)(|\partial J(u)|^2)'| \\ &\leqslant \mu \frac{C_5}{\rho} \left(\frac{\gamma\rho}{\lambda_1^2} + \frac{|\partial J(u)|^2}{\lambda_1} + \frac{1}{\lambda_1^3}\right) + \frac{1}{4}\, |\partial J(u)|^2 \leqslant \frac{3}{4}\, |\partial J(u)|^2, \end{aligned} \tag{178}$$

$\forall u \in W(1, \varepsilon)$ such that $x_1 \in N_\rho - N_{\rho/2}$ and $J(u) \leqslant b$. Since3 $\omega_\rho = 1$ outside N_ρ and in $N_{\rho/2}$, we also have by (172)

$$\begin{aligned} |g'(u) \cdot \partial J(u)| &\leqslant |\partial J(u)|\, |\omega'(|\partial J(u)|^2)(|\partial J(u)|^2)'| \\ &\leqslant \tfrac{1}{4}\, |\partial J(u)|^2 \end{aligned} \tag{179}$$

$\forall u \in W(1, \varepsilon)$ s.t. $x_1 \notin N_\rho - N_{\rho/2}$ and $J(u) \leqslant b$. Lastly, if $u \notin W(1, \varepsilon)$, then $g(u) = 0$ and $g'(u) = 0$. This, together with (178) and (179), implies

$$\partial F(u) \cdot \partial J(u) \geqslant \tfrac{1}{4}\, |\partial J(u)|^2 > 0. \tag{180}$$

Lemmas 5 and 8 imply the following deformation lemma (here ε is given small enough so that Lemma 8 applies):

LEMMA 9. *Let $c_j < b < c_{j+1}$ be given. There exists $\theta_0(\varepsilon) > 0$, $\bar{c}$ and $c > 0$ s.t. for any $0 < \theta < \theta_0(\varepsilon)$, the set $J_b = \{u \in \Sigma^+$ s.t. $J(u) \leqslant b\}$ retracts by deformation onto $J_{c_j - \theta} \cup A$, where*

$$A \subset \left\{ \begin{matrix} u \in W(1, \varepsilon),\ u = \alpha_1 \delta_1 + v \ \text{ such that } |\partial J(u)| < \bar{c}\theta, \\ x_1 \in N_{\varphi(\theta/c)}(y_j) \end{matrix} \right\}$$

and where $\varphi(x) = \sqrt{x}$ if $x \geqslant \varepsilon^2$; $\varphi(x) = \sqrt[3]{\varepsilon x}$ if $x \leqslant \varepsilon^2$.

Proof. Consider the flow $\eta(s, \cdot)$ of the differential equation

$$\begin{aligned} \frac{\partial u}{\partial s} &= -\partial J(u) \\ u(0) = u_0; &\qquad J(u_0) \leqslant b. \end{aligned} \tag{181}$$

Let $\eta > 0$ be subject to (159), i.e.,

$$0 < \eta < \eta_0(\varepsilon). \tag{182}$$

Let

$$\inf(\rho^3/\varepsilon, \rho^2) = \eta; \qquad \mu = \frac{c}{2}\,\eta. \tag{183}$$

Then μ satisfies the constraint of Lemma 8; $\eta_0(\varepsilon)$ may be chosen small enough so that this lemma applies with such choices of ε, ρ, μ, η.

Setting

$$\theta_0 = \frac{c}{4}\,\eta_0(\varepsilon), \tag{184}$$

any θ less than θ_0 may be written as

$$\theta = \mu/2 \tag{185}$$

with

$$\mu = \frac{c}{2}\,\eta; \qquad 0 < \eta < \eta_0(\varepsilon). \tag{186}$$

We are then given $F(u)$ with $\rho = \varphi(\eta)$ for each θ less than θ_0. (φ is the converse function of $\varphi^{-1}(x) = \inf(x^3/\varepsilon, x^2)$; $x \geqslant 0$.)

Let $u_0 \in J_b$. We introduce

$$s(u_0) = \inf\left\{s \text{ such that } F(\eta(s, \mu_0)) < c_j - \frac{\mu}{2}\right\}. \tag{187}$$

We claim that $s(u_0) < +\infty$. Indeed, by Lemma 5, $\eta(s, u_0)$ remains outside $W(p, \varepsilon)$, $p \geqslant 2$, for s large. If $\eta(s, u_0)$ does not remain in $W(1, \varepsilon)$ for s large, then standard deformation arguments show that $\eta(s, u_0)$ has to enter $J_{c_j - \mu/2} = \{u \text{ such that } J(u) \leqslant c_j - 3\mu/4\}$. Thus, $s(u_0) < +\infty$ in this case. If $\eta(s, u_0)$ remains in $W(1, \varepsilon)$ for s large, then, by Lemma 7, $u_1(s)$ converges to y_j, $J(\eta(s, u_0))$ converges to c_j, while, by Lemma A1, $|\partial J(\eta(s, u_0))| \to_{s \to +\infty} 0$. Thus, $\omega_\rho(x_1)\, \omega(|\partial J(\eta(s, u_0))|^2)$ converges to μ and $F(\eta(s, u_0))$ to $c_j - \mu < c_j - \mu/2$. Again, $s(u_0) < +\infty$ and our claim is proved.

Due to Lemma 8, the function

$$\begin{aligned} u_0 &\to s(u_0) \\ J_b &\to \mathbb{R}^+ \end{aligned} \tag{188}$$

is then continuous. The map

$$\begin{aligned} [0, 1] \times J_b &\to J_b \\ (t, u_0) &\to \eta(ts(u_0), u_0) \end{aligned} \tag{189}$$

retracts by deformation J_b onto $F_{c_j - \mu/2} = \{u \in \Sigma^+ \text{ such that } F(u) \leqslant c_j - \mu/2\}$. If u belongs to $F_{c_j - \mu/2}$ and does not belong to $J_{c_j - \theta} = J_{c_j - \mu/2}$, then $g(u)$ is strictly positive. Therefore, $u \in W(1, \varepsilon)$, $x_1 \in N_\rho(y_j)$, $|\partial J(u)|^2$ is less than η. By (185)–(186), η is equal to $(4/c)\theta$ and ρ is $\varphi(\eta)$; this completes the proof of Lemma 9.

The last step in the proof of Theorem 1 is provided by the following lemma, which provides an expansion of J, holding on functions which are not necessarily positive.

LEMMA 10. *Let $\varepsilon_0 > 0$, $\rho_0 > 0$ be small; let (h_1, h_2, h_3) being local coordinates on S^3 around y_j, representing x_1. Let $u = \alpha_1 \delta_1 + v$, with $|\alpha_1 \delta_1(x_1, \lambda_1) + v|_{-L} = 1$, v satisfying* (Vo). *We make the following assumptions on v, x_1, ρ, $\alpha_1, \lambda_1 : |v|_{-L} < \varepsilon_0$; $x_1 \in N_\rho(y_j)$ with $\rho < \rho_0$; $|\alpha_1 - 1/|\delta|_{-L}| < \varepsilon_0$; $1/\lambda_1 < \varepsilon_0$.*

Under these assumptions, the following expansions hold:

$$\begin{aligned} &\frac{1}{3} \frac{1}{[\int_{S^3} K(x)(\alpha_1 \delta_1 + v)^6]^{1/2}} \\ &\quad = \frac{1}{3} \frac{\int \delta^6}{(K(y_j))^{1/2}} \left(1 + \frac{1}{2}\left(\sum_{i=1}^{k_j} |h_i|^2 - \sum_{k_{j+1}}^{3} |h_i|^2\right.\right. \\ &\qquad \left.\left. - \frac{c}{K(y_j)} \int_{S^3} (K(x) - K(x_1))\, \delta_1^6 + \frac{1}{\int \delta^6} \frac{3}{\alpha_1^2} \left(|v|_{-L}^2 - 5 \int_{S^3} \delta_1^4 v^2\right)\right)\right. \\ &\qquad \left. + O\left(|v|_{-L}^3 + \frac{1}{\lambda_1^3}\right) + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2}\right). \end{aligned}$$

The quantity $-\int_{S^3} (K(x)-K(x_1))\,\delta_1^6$ *decreases for* $\lambda_1 \geqslant A_1$, A_1 *uniform on* x_1 *in* $N_{\rho_0}(y_j)$, *and has the expansion*

$$-\int_{S^3} (K(x)-K(x_1))\,\delta_1^6 = -c'\,\frac{\Delta K(y_j)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right).$$

Lemma 10 has the following corollary, which we prove now:

COROLLARY 1. *Let* $c_{j-1} < a < c_j < b < c_{j+1}$. *For any coefficient group G,* $H_q(J_b, J_a) = 0$ *for* $q \neq 3-k_j$; $H_{3-k_j}(J_b, J_a) = G$.

Proof. We first derive, assuming the expansion of Lemma 10, the homological conclusion. Let, for ε, μ, ρ, η given satisfying (159), (183),

$$B = \left\{\begin{matrix} u \in W(1,\varepsilon)u = \alpha_1\delta_1 + v,\ v \text{ satisfying (Vo),} \\ \text{such that } x_1 \in N_\rho(y_j) \end{matrix}\right\}. \tag{190}$$

By Lemma 9, we know that the pair $(J_b, J_{c_j-\mu/2})$ retracts by deformation onto $(J_{c_j-\mu/2} \cup A, J_{c_j-\mu/2})$, when $A \subset B$. Therefore

$$H(J_b, J_{c_j-\mu/2}) = H(A \cap B, J_{c_j-\mu/2} \cap B). \tag{191}$$

Let $u = \alpha_1\delta_1 + v$, v satisfying (Vo), belong to $A \cap B$. Since $F(u)$ is less than or equal to $c_j - \mu/2$ on A,

$$J(u) = J(\alpha_1\delta_1 + v) < c_j + \mu \qquad \forall u \in A \cap B \tag{192}$$

while

$$J(u) = J(\alpha_1\delta_1 + v) < c_j - \mu/2 \qquad \forall u \in J_{c_j-\mu/2} \cap B. \tag{193}$$

Using then the expansion of Lemma 10, we see that

$$\sum_{i=1}^{k_j} |h_i|^2 - \sum_{k_{j+1}}^{3} |h_i|^2 < 2\mu/c_j \qquad \forall u \in A \cap B \tag{194}$$

$$\sum_{i=1}^{k_j} |h_i|^2 - \sum_{k_{j+1}}^{3} |h_i|^2 < -\mu/c_j \qquad \forall u \in J_{c_j-\mu/2} \cap B \tag{195}$$

since the remainder term in the expansion is strictly positive. We have therefore defined a map

$$(A \cap B, J_{c_j-\mu/2} \cap B) \xrightarrow{r_\mu} (X_\mu, X_{-\mu/2}), \tag{196}$$

where

$$X_\mu = \{x_1 \in N_\rho(y_j) \text{ such that (194) holds}\}. \tag{197}$$

$$X_{-\mu/2} = \{x_1 \in N_\rho(y_j) \text{ such that (195) holds}\}. \tag{198}$$

Let

$$X_{-\mu/2} = \left\{x_1 \in N_\rho(y_j) \text{ such that } \sum_{i=1}^{k_j} |h_i|^2 - \sum_{k_{j+1}}^{3} |h_i|^2 < -2\mu/c_j\right\}. \tag{199}$$

We define

$$\begin{aligned} (X_\mu, X_{-\mu}) &\xrightarrow{s_\mu} (A \cap B, J_{c_j - \mu/4} \cap B) \\ x_1 &\to \frac{\delta_1(x_1, \lambda_1(x_1))}{|\delta_1(x_1, \lambda_1(x_1))|_{-L}}, \end{aligned} \tag{200}$$

where $\lambda_1(x_1)$ is chosen continuously dependent of x_1 and subject to the two following constraints:

$$\frac{\delta_1(x_1, \lambda_1(x_1))}{|\delta_1(x_1, \lambda_1(x_1))|_{-L}} \in W(1, \varepsilon) \tag{201}$$

(hence $\lambda_1(x_1)$ must be large enough).

Considering $-c\, \Delta K(y_j)/\lambda_1(x_1)^2 + o(1/\lambda_1(x_1)^2)$ from the expansion of Lemma 10,

$$\forall x_1 \in X_{-\mu}, \qquad -c\, \frac{\Delta K(y_j)}{\lambda_1(x_1)^2} + o\left(\frac{1}{\lambda_1(x_1^2)}\right) \leqslant \mu/2c_j. \tag{202}$$

Observe that $r_\mu \circ s_\mu$ is the injection of $(X_\mu, X_{-\mu})$ in $(X_\mu, X_{-\mu/2})$. Observe also that, with $N_\rho(y_j)$ chosen to be disc of radius ρ around y_j,

$$\begin{aligned} H_*(X_\mu, X_{-\mu}) &\overset{(r_\mu \circ s_\mu)_*}{\simeq} H_*(X_\mu, X_{-\mu/2}) \\ &\simeq G \qquad \text{if } * = 3 - k_j \\ &\simeq 0 \qquad \text{if } * \neq 3 - k_j. \end{aligned} \tag{203}$$

Therefore $H_*(A \cap B, J_{c_j - \mu/2} \cap B) \simeq H_*(J_b, J_{c_j - \mu/2}) \simeq H_*(J_b, J_a)$ has $H_*(X_\mu, X_{-\mu})$ as a direct factor and contains thus G in dimension $3 - k_j$.

Conversely, let

$$B_1 = \left\{ \begin{array}{l} u \in W(1, \varepsilon) \text{ such that } u = \alpha_1 \delta_1 + v,\ v \text{ satisfying (Vo)}, \\ \\ |\partial J(u)|^2 \leqslant \bar{c}\, \dfrac{\mu}{2};\ x_1 \in N_{\varphi(\mu/2c)}(y_j) \end{array} \right\}. \tag{204}$$

By Lemma 9, $(J_b, J_{c_j-\mu/2})$ retracts by deformation onto $(J_{c_j-\mu/2} \cup A, J_{c_j-\mu/2})$, when $A \subset B_1$. Therefore

$$H_*(J_b, J_{c_j-\mu/2}) \simeq H_*(A \cap B_1, J_{c_j-\mu/2} \cap B_1). \tag{205}$$

Clearly

$$H_*(J_b, J_{c_j-\mu/2}) \simeq H_*(J_b, J_{c_j-\mu/4}). \tag{206}$$

Therefore, the injection

$$i_\mu: (A \cap B_1, J_{c_j-\mu/2} \cap B_1) \to (J_b, J_{c_j-\mu/4}) \tag{207}$$

provides us with a homological isomorphism.

Let us consider the following homotopy of i_μ:

$$U(t, \alpha_1\delta_1 + v) = \frac{|\alpha_1\delta_1 + (1-t)v|}{|\alpha_1\delta_1 + (1-t)v|_{-L}}, \qquad t \in [0, 1]. \tag{208}$$

Setting

$$\bar\lambda_1 = \sqrt{-\frac{16cc'\,\Delta K(y_j)\,c_j}{\mu K(y_j)}}, \tag{209}$$

$$U(t, \alpha_1\delta_1(x_1, \lambda_1) + v) = \frac{\delta_1(x_1, (2-t)\,\lambda_1 + (t-1)\,\bar\lambda_1)}{|\delta_1(x_1, (2-t)\,\lambda_1 + (t-1)\,\bar\lambda_1)|_{-L}}$$
$$\text{for} \quad t \in [1, 2]. \tag{210}$$

We will denote

$$\lambda_1(t) = (2-t)\lambda + (t-1)\,\bar\lambda_1 \qquad \text{for} \quad t \in [1, 2]. \tag{211}$$

Clearly $U(t, \cdot)$ is continuous. $U(0, \cdot) = i_\mu$; $U(2, \alpha_1\delta_1(x_1, \lambda_1) + v) = \delta_1(x_1, \bar\lambda_1)/|\delta_1(x_1, \bar\lambda_1)|_{-L}$. We will prove later that $U(t\,\cdot)$ is a homotopy of i_μ is a map of pairs from $(A \cap B_1, J_{c_j-\mu/2} \cap B_1)$ in $(J_b, J_{c_j-\mu/4})$. Assuming this, then $(i_\mu)_*$ from $H_*(A \cap B_1, J_{c_j-\mu/2} \cap B_1)$ into $H_*(J_b, J_{c_j-\mu/4})$ is equal to $(s_\mu)_*$ where

$$s_\mu(\alpha_1\delta_1(x_1, \lambda_1) + v) = \frac{\delta_1(x_1, \bar\lambda_1)}{|\delta_1(x_1, \bar\lambda_1)|_{-L}}. \tag{212}$$

To any $u = \alpha_1\delta_1(x_1, \lambda_1) + v$ in $A \cap B_1$, we may associate

$$\begin{aligned} &A \cap B_1 \xrightarrow{\mathscr{X}} N_\rho \\ &\alpha_1\delta_1 + v \to (h_1, h_2, h_3) \text{ local coordinates of } x_1. \end{aligned} \tag{213}$$

If $\alpha_1 \delta_1 + v \in J_{c_j - \mu/2} \cap B_1$, then the expansion of Lemma 10 implies

$$-\sum_{k_j+1}^{3} |h_i|^2 + \sum_{i=1}^{k_j} |h_i|^2 + O(|h|^3) + Q(v) + O(|v|^3_{-L})$$
$$- c\, \frac{\Delta K(y_j)}{K(y_j)\, \lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right) \leqslant -\frac{\mu}{c_j}. \tag{214}$$

Since v satisfies (Vo),

$$Q(v) \geqslant \alpha_0\, |v|^2_{-L}, \qquad \alpha_0 > 0. \tag{215}$$

Therefore, if ε is small enough,

$$Q(v) + O(|v|^3_{-L}) - c\, \frac{\Delta K(y_j)}{K(y_j)\, \lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right) > 0. \tag{216}$$

On the other hand, if ε and μ are small enough, with $\rho = \varphi(\mu/2c)$

$$O(|h|^3) = O(\rho^3) = o(\inf(\rho^2, \rho^3/\varepsilon)) = o(\mu). \tag{217}$$

Relations (214), (216), and (217) imply

$$\sum_{i=1}^{k_j} |h_i|^2 - \sum_{k_j+1}^{3} |h_i|^2 < -\frac{\mu}{2c_j} \qquad \forall \alpha_1 \delta_1 + v \in J_{c_j - \mu/2} \cap B_1. \tag{218}$$

Therefore, the map $\mathscr{X}$ maps

$$\mathscr{X}\colon (A \cap B1, J_{c_j - \mu/2} \cap B_1) \to (N_\rho, X_{-\mu/4}). \tag{219}$$

Let

$$\begin{aligned} \mathscr{T}\colon N_\rho &\to \Sigma^+ \\ (h_1, h_2, h_3) &\to \frac{\delta_1(x_1, \bar{\lambda})}{|\delta_1(x_1, \bar{\lambda}_1)|_{-L}}, \end{aligned} \tag{220}$$

where $\bar{\lambda}_1$ is defined in (209). Using the expansion of Lemma 10, we see that if ε, μ are small enough (hence ρ small, $\rho = \varphi(\mu/2c)$)

$$J(\mathscr{T}(h_1, h_2, h_3)) \leqslant c_j \left(1 + 4\rho^2 + \frac{\mu}{4c_j}\right) < b \qquad \forall (h_1, h_2, h_3) \in N_\rho. \tag{221}$$

Furthermore,

$$J(\mathcal{T}(h_1, h_2, h_3)) \leqslant c_j\left(1 - \frac{\mu}{4c_j} + O(\rho^3) + \frac{\mu}{16c_j} + o(\mu)\right)$$
$$\leqslant c_j\left(1 - \frac{3\mu}{16c_j} + o(\mu)\right) \leqslant c_j\left(1 - \frac{\mu}{8c_j}\right) \tag{222}$$

if $(h_1, h_2, h_3) \in X_{-\mu/4}$ and μ is small enough. (Again $O(\rho^4) = o(\inf(\rho^2, \rho^3/c)) = o(\mu)$.)

Therefore

$$\mathcal{T} : (N_\rho, X_{-\mu/4}) \to (J_b, J_{c_j - \mu/8}). \tag{223}$$

Observe that

$$\mathcal{T} \circ \mathcal{X}(\alpha_1 \delta_1 + v) = \frac{\delta_1(x_1, \lambda_1)}{|\delta_1(x_1, \lambda_1)|} = ios_\mu, \tag{224}$$

where

$$i: (J_b, J_{c_j - \mu/4}) \to (J_b, J_{c_j - \mu/8}) \tag{225}$$

is the inclusion. i_* is of course an isomorphism. Therefore $(ios_\mu)_* = i_* o(s_\mu)_*$ is an isomorphism. Thus $\mathcal{T}_* \circ \mathcal{X}_*$ is an isomorphism; $\mathcal{X}_*$ is therefore an injection. Observe that for $\gamma > 0$,

$$H_q(N_\rho, X_{-\gamma}) = 0 \qquad \text{if} \quad q \neq 3 - k_j \tag{226}$$

and

$$\begin{aligned} H_{3-k_j}(N_\rho, X_{-\gamma}) &= G \qquad \text{if} \quad \gamma/c_j < \rho^2 \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{227}$$

We already know that $H_{3-k_j}(J_b, J_{c-\mu/2})$ has $H_{3-k_j}(X_\mu, X_{-\mu}) = G$ as a direct factor. Therefore, $\mu/4c_j$ is less than ρ^2; and since $\mathcal{X}_*$ is injective, $H_{3-k_j}(J_b, J_{c_j-\mu/2}) = H_{3-k_j}(N_\rho, X_{-\mu/4}) = G$. The injectivity of $\mathcal{X}_*$ implies also that $H_q(J_b, J_{c_j-\mu/2}) = 0$ for $q \neq 3 - k_j$ since $H_q(N_\rho, X_{-\mu/4}) = 0$. Corollary 1 follows.

We prove now that $U(t, \cdot)$ takes its values in $(J_b, J_{c_j-\mu/4})$.

We first observe that, if $u = \alpha_1 \delta_1 + v \in B_1$, then

$$|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} = o(\mu). \tag{228}$$

Indeed, by (167), $|\partial J(u)|$ is lower bounded by $c_4(|DK(x_1)|/\lambda_1 + 1/\lambda_1^2)$

for u in B_1. By Lemma 2, $|\partial J(u)|$ is lower bounded by $c\,|v|_{-L} - C(|DK(x_1)|/\lambda_1 + 1/\lambda_1^2)$. Therefore

$$|\partial J(u)|^2 \geqslant c_8\left(\frac{|DK(x_1)|}{\lambda_1^2} + \frac{1}{\lambda_1^4} + |v|^2_{-L}\right) \qquad \forall u \in B_1 \tag{229}$$

since

$$|\partial J(u)|^2 < \frac{\bar{c}\mu}{2} \qquad \forall u \in B_1, \tag{230}$$

and we have

$$|v|_L^2 = O(\mu); \qquad |v|^3_{-L} = o(\mu) \qquad \forall u \in B_1. \tag{231}$$

Next, we have, if ε is small enough,

$$\left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} \leqslant \left[\varphi\left(\frac{\mu}{2C}\right)\right]^3 = \rho^3 = o(\inf(\rho^2, \rho^3/\varepsilon)) = o(\mu). \tag{232}$$

Relations (231) and (232) imply (228). If furthermore, $u \in B_1 \cap J_{c_j - \mu/2}$, then

$$|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} + \frac{1}{\lambda_1^3} = o(\mu). \tag{233}$$

Indeed, using the expansion of Lemma 10, we see that

$$-c\,\frac{\Delta K(y_j)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right) \leqslant C\left(\sum_{i=1}^{3} |h_i| + |v|^2\right). \tag{234}$$

Relations (228) and (234) imply (233), since $-c\,\Delta K(y_j)$ is strictly positive. Thus, if ε is small enough, we have

$$C\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2}\right) < \frac{\mu}{8} \qquad \forall u \in B_1 \tag{235}$$

$$C\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} + \frac{1}{\lambda_1^3}\right) < \frac{\mu}{8} \qquad \forall u \in B_1 \cap J_{c_j - \mu/2}, \tag{236}$$

where

$$\frac{C}{2}\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\pi\right)^{\Theta/2}\right) \geqslant \left|O\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2}\right)\right| \tag{2.37}$$

$$\frac{C}{2}\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} + \frac{1}{\lambda_1^3}\right) \geqslant \left|O\left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} + \frac{1}{\lambda_1^3}\right)\right|, \tag{2.38}$$

the quantities $O(\cdot)$ being the ones provided by the expansion of Lemma 10. For $t \in [0, 1]$, we have

$$|\alpha_1\delta_1 + (1-t)v|^2_{-L} = |\alpha_1\delta_1|^2 + (1-t)^2\,|v|^2_{-L} = 1 + ((1-t)^2 - 1)|\,|v|^2_{-L} \tag{239}$$

since v satisfies (Vo) and $|\alpha_1\delta_1 + v|_{-L} = 1$. Thus

$$\frac{\alpha_1\delta_1 + (1-t)v}{|\alpha_1\delta_1 + (1-t)v|_{-L}} = \alpha'_1\delta_1 + v_1; \qquad \begin{cases} \alpha'_1 = \alpha_1(1 + O(|v|^2_{-L})) \\ v_1 = (1-t)\,v(1 + O(|v|^2_{-L})). \end{cases} \tag{240}$$

Using the expansion of Lemma 10, we derive

$$J\left(\frac{|\alpha_1\delta_1 + (1-t)v|}{|\alpha_1\delta_1 + (1-t)v|_{-L}}\right) = J(\alpha'_1\delta_1 + v_1)$$
$$\leqslant J(\alpha_1\delta_1 + v) + C\left(|v|^3 + \frac{1}{\lambda_1^3} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2}\right). \tag{241}$$

If $u \in A \cap B_1$, we have

$$J(u) = J(\alpha_1\delta_1 + v) \leqslant c_j + \mu, \tag{242}$$

hence, using (235) and the fact that $1/\lambda_1 < \varepsilon$

$$\forall t \in [0, 1], \qquad J(U(t, \alpha_1\delta_1 + v)) \leqslant c_j + \mu + \frac{\mu}{8} + C\varepsilon^3 < b \tag{243}$$

if ε and M are small enough. Moreover if $u = \alpha_1\delta_1 + v \in J_{c_j - \mu/2} \cap B_1$, then by (236)

$$\forall t \in [0, 1], \qquad J(U(t, \alpha_1\delta_1 + v)) \leqslant c_j - \frac{\mu}{2} + \frac{\mu}{8} < c_j - \mu/4. \tag{244}$$

Therefore, for $t \in [0, 1]$, by (243) and (244), $U(t, \cdot)$ is a homotopy of i_μ as a map of pairs $(A \cap B_1, J_{c_j - \mu/2} \cap B_1)$ into $(J_b, J_{c_j - \mu/4})$. Let $t \in [1, 2]$.

Observe that by the choice of $\bar{\lambda}_1$ in (209) and by the fact that $1/\lambda_1 < \varepsilon$ we have

$$\forall u = \alpha_1\delta_1(x_1, \lambda_1) + v,\ u \in B_1,\ O\left(\frac{1}{\lambda_1^3}(t)\right) < c\varepsilon^3 + o(\mu) \tag{245}$$

and by (233)

$$\forall u = \alpha_1\delta_1(x_1, \lambda_1) + v,\ u \in J_{c_j - \mu/2} \cap B_1\ O\left(\frac{1}{\lambda_1^3(t)}\right) \leqslant \frac{\mu}{16} + o(\mu). \tag{246}$$

Using the expansions of Lemma 10 and the fact that $-\int_{S^3}(K(x)-K(x_j))\,\delta_1^6$ decreases with λ_1

$$J\left(\frac{\delta_1(x_1,\lambda_1(t))}{|\delta_1(x_1,\lambda_1(t))|_{-L}}\right)=J(U(t,\alpha_1\delta_1(x_1,\lambda_1)+v))$$
$$\leqslant J\left(\frac{\delta_1(x_1,\lambda_1)}{|\delta_1(x_1,\lambda_1)|_{-L}}\right)+\frac{\mu}{16}+O\left(\frac{1}{\lambda_1^3}\right)+O\left(\frac{1}{\lambda_1^3(t)}\right). \tag{247}$$

Therefore, using (241) with $t=1$, (235), (245), (247), and the fact that $1/\lambda_1<\varepsilon$, we now derive

$$J\left(\frac{\delta_1(x_1,\lambda_1(t))}{|\delta_1(x_1,\lambda_1(t))|_{-L}}\right)$$
$$\leqslant J(\alpha_1\delta_1+v)+C\left(|v|^3+\frac{1}{\lambda_1^3}+\left(\sum_{i=1}^{3}|h_i|^2\right)^{3/2}\right)+\frac{\mu}{16}+O\left(\frac{1}{\lambda_1^3(t)}\right)$$
$$\leqslant c_j+\mu+\frac{\mu}{4}+2c\varepsilon^3<b \qquad \forall u=\alpha_1\delta_1+v\in B_1. \tag{248}$$

If μ belongs to $B_1\cap J_{c_j-\mu/2}$, then, from (236), (246), and (247), we derive

$$J\left(\frac{\delta_1(x_1,\lambda_1(t))}{|\delta_1(x_1,\lambda_1(t))|_{-L}}\right)\leqslant c_j-\frac{\mu}{2}+\frac{3\mu}{16}+o(\mu)<c_j-\mu/4 \tag{249}$$

for μ small enough. Relations (248) and (249) imply that $U(t,\cdot)$ is again valued in $(J_b, J_{c_j-\mu/4})$ for $t\in[1,2]$. The proof of Corollary 1 is thereby complete.

Proof of Lemma 10. Rather than assuming that $|\alpha_1\delta_1+v|_{-L}=1$, we will relax this last constraint and provide an expansion of the ratio

$$\frac{|\alpha_1\delta_1+v|_{-L}^6}{\int K(x)(\alpha_1\delta_1+v)^6}=\frac{N}{D};\qquad \frac{1}{3\,|\delta_1|_{-L}}\leqslant\alpha_1\leqslant\frac{3}{|\delta_1|_{-L}};$$
$$|v|_{-L}<\varepsilon_0;\qquad \frac{1}{\lambda_1}<\varepsilon_0. \tag{250}$$

v satisfies (Vo). Therefore

$$N=(|\alpha_1\delta_1+v|_{-L}^2)^3=(\alpha_1^2\,|\delta_1|_{-L}^2+|v|_{-L}^2)^3$$
$$=\alpha_1^6\,|\delta_1|_{-L}^6\left(1+\frac{|v|_{-L}^2}{\alpha_1^2\,|\delta_1|_{-L}^2}\right)^3. \tag{251}$$

Since $|\delta_1|^2_{-L} = \int \delta_1^6 = \int \delta^6$, N is also equal to

$$N = \delta_1^6 = \int \delta^6 \left(1 + \frac{|v|^2_{-L}}{\alpha_1^2 \, |\delta_1|^2_{-L}}\right)^3$$

$$= \alpha_1^6 \left(\int \delta^6\right)^3 \left(1 + \frac{3\,|v|^2_{-L}}{\alpha_1^2\,|\delta_1|^2_{-L}} + O\,|v|^3_{-L}\right). \tag{252}$$

We expand D:

$$\begin{aligned} D &= \int (K(x))(\alpha_1 \delta_1 + v)^6 \\ &= \alpha_1^6 \int K(x)\,\delta_1^6 + 6 \int K(x)(\alpha_1\delta_1)^5\, v + 15 \int K(x)\,\alpha_1^4 \delta_1^4 + v^2 + O(|v|^3_{-L}). \end{aligned} \tag{253}$$

We have, by Lemma A8,

$$\int K(x)\,\delta_1^6 = K(x_1) \int \delta^6 + \int (K(x) - K(x_1))\,\delta_1^6$$

$$= K(x_1) \int \delta^6 + c_9 \, \frac{\Delta K(x_1)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right)$$

$$= K(x_1) \int \delta^6 + c_9 \, \frac{\Delta K(y_j)}{\lambda_1^2} + o\left(\frac{1}{\lambda_1^2}\right). \tag{254}$$

Since $x_1 \in N_\rho$, we also have, by Lemma A6 and the fact that $DK(y_j) = 0$,

$$\left|\int K(x)\,\delta_1^5 v\right| = \left|\int (K(x) - K(x_1))\,\delta_1^5 v\right| \leqslant C\left(\frac{|DK(x_1)|}{\lambda_1} + \frac{1}{\lambda_1^2}\right) |v|_{-L}$$

$$\leqslant C' \left(|v|^3_{-L} + \left(\sum_{i=1}^{3} |h_i|^2\right)^{3/2} + \frac{1}{\lambda_1^3}\right) \tag{255}$$

$$\int K(x)\,\delta_1^4 v^2 = K(x_1) \int \delta_1^4 v^2 + \int (K(x) - K(x_1))\,\delta_1^4 v^2$$

$$= K(x_1) \int \delta_1^4 v^2 + \left(\frac{|v|^2_{-L}}{\lambda_1}\right)$$

$$= K(x_1) \int \delta_1^4 v + O\left(|v|^3_{-L} + \frac{1}{\lambda_1^3}\right). \tag{256}$$

Thus,

$$\frac{N}{D}=\frac{(\int\delta^6)^2\,((1+3\,|v|^2_{-L})/\alpha_1^2\int\delta^6+O(|v|^3_{-L}))}{\begin{bmatrix}K(x_1)[(1+c_9)/K(x_1)\times(\Delta K(y_j)/\lambda_1^2\int\delta^6)(15\int\delta_1^4v^2/\alpha_1^2\int\delta^6)\\ +O(|v|^3_{-L}+(\sum_{i=1}^3|h_i|^2))^{3/2}+o(1/\lambda_1^2)]\end{bmatrix}}. \tag{257}$$

We may complete a Morse Lemma reduction for $1/K$ around y_j in (h_1, h_2, h_3) local coordinates.

We then have

$$\frac{1}{K(x_1)}=\frac{1}{K(y_j)}\left(1+\sum_1^{k_j}|h_i|^2-\sum_{k_j+1}^3|h_i|^2\right). \tag{258}$$

Thus

$$\frac{N}{D}=c_j\left(1+\sum_1^{k_j}|h_i|^2-\sum_{k_j+1}^3|h_i|^2-c\,\frac{\Delta K(y_j)}{K(y_j)\,\lambda_1^2}+\frac{3}{\alpha_1^2\int\delta^6}\left(|v|^2_{-L}-5\int\delta_1^4v^2\right)\right.$$
$$\left.+O\left(\left(\sum_{i=1}^3|h_i|^2\right)^{3/2}\right)+o\left(\frac{1}{\lambda_1^2}\right)\right). \tag{259}$$

Observe that $-c(\Delta K(y_j)/\lambda_1^2)+o(1/\lambda_1^2)=-c'\int_{S^3}(K(x)-K(x_1))\,\delta_1^6+O(1/\lambda_1^3)$ since the term $o(1/\lambda_1^2)$ comes from $\int(K(x)-K(x_1))\,\delta_1^6$ in (254), the other estimates (255) and (256) providing $O(1/\lambda_1^3)$. The behaviour of $-\int_{S^3}(K(x)-K(x_1))\,\delta_1^6$, $x_1\in N_\rho(y_j)$, is studied in Lemma A8. This completes the proof of Lemma 10.

Proof of Theorem 1. Let

$$a_1<\operatorname*{Min}_{u\in\Sigma^+}J(u)=c_1<a_2<c_2<\cdots<c_j<a_{j+1}<c_{j+1}<\cdots$$
$$<a_s<c_s<a_{s-1}. \tag{260}$$

If X is a topological set, then $\chi(X)$ is its Euler–Poincaré characteristic with rational coefficients.

We know, by Lemmas 5 and 7, that Σ^+ retracts by deformation on $J_{a_{s-1}}$. Therefore $\chi(J_{s_{s-1}})=1$.

Since (1) has no solution, we have by Corollary 1

$$\begin{aligned}&H_q(J_{a_{r+1}},J_{a_r})=0 \qquad \text{if}\quad q\neq 3-k_r\\ &H_{3-k_r}(J_{a_{r+1}},J_{a_r})=\mathbf{Q}.\end{aligned} \tag{261}$$

Thus

$$\chi(J_{a_{r+1}})=\chi(J_{a_r})+(-1)^{3-k_r}. \tag{262}$$

Of course $\chi(J_{a_1}) = \chi(\phi) = 0$. Therefore

$$1 = \sum_{r=1}^{s} (-1)^{3-k_r}. \tag{263}$$

If (263) is violated, (1) has a solution. Q.E.D.

Appendix A

LEMMA A1. *Let*

$$\left.\begin{aligned} &\frac{\partial u}{\partial s} = -\partial J(u) \\ &u(0) = u_0 \in \Sigma^+ \end{aligned}\right. \quad s \geqslant 0 \textit{ be a decreasing flow-line.} \tag{A1}$$

Then $\int_0^{+\infty} |\partial J(u)|^2 \, dt < +\infty$ *and* $\lim_{s \to +\infty} |\partial J(u)| = 0$.

Proof of Lemma A1. Since J is bounded below on Σ^+, $\int_0^{+\infty} |\partial J(u)|^2 \, dt$ has to be bounded on any flow-line. Therefore, we can find a sequence (s'_k), s'_k tending to $+\infty$, such that

$$\lim_{k \to +\infty} |\partial J(u)| \, (s'_k) = 0. \tag{A2}$$

Let

$$b = J(u_0). \tag{A3}$$

Using the expansion of $\partial J(u)$ provided in (10) and Lemma 1, we derive, since $J(u(s)) \leqslant b$ $\forall s \geqslant 0$,

$$\begin{aligned} &\exists C_1 \text{ such that } \forall s \geqslant 0, \\ &|\partial J(u(s)) - \partial J(u(\tau))| \leqslant C_1 \, |u(s) - u(\tau)|_{-L} \qquad \forall \tau \geqslant 0. \end{aligned} \tag{A4}$$

Let us assume, arguing by contradiction, that there exist a sequence $(\tau_k) \to +\infty$ and a number $\varepsilon_1 > 0$ such that

$$|\partial J(u(\tau_k))| \geqslant \varepsilon_1. \tag{A5}$$

We may assume

$$s'_k < \tau_k < s'_{k+1} < \tau_{k+1} \tag{A6}$$

and, by continuity of $|\partial J(u(s))|$, we may find a sequence (s_k) such that

$$\begin{aligned} &|\partial J(u(s_k))| = \frac{\varepsilon_1}{2}; \cdots < s_k < \tau_k < s_{k+1} < \tau_{k+1} < \cdots; \\ &|\partial J(u(\tau))| \geqslant \frac{\varepsilon_1}{2} \qquad \forall \tau \in [s_k, \tau_k] \; \forall k. \end{aligned} \tag{A7}$$

We then have

$$\int_{s_k}^{\tau_k} |\partial J(u(\tau))|^2 \, dt \geqslant \frac{\varepsilon_1}{2} \int_{s_k}^{\tau_k} |\partial J(u(\tau))| \, dt. \tag{A8}$$

Since $u(\cdot)$ satisfies (A2), we also have

$$|u(\tau_k) - u(s_k)| \leqslant \int_{s_k}^{\tau_k} |\partial J(u(\tau))| \, d\tau. \tag{A9}$$

Finally, by (A5), we have

$$|u(\tau_k) - u(s_k)| \geqslant \frac{1}{C_1} \cdot \frac{\varepsilon_1}{2}. \tag{A10}$$

Combining (A8), (A9), and (A10), we derive

$$\int_{s_k}^{\tau_k} |\partial J(u)(\tau)|^2 \, d\tau \geqslant \frac{1}{C_1} \left(\frac{\varepsilon_1}{2}\right)^2. \tag{A11}$$

This contradicts the finiteness of $\int_0^{+\infty} |\partial J(u)|^2 \, d\tau$. The proof of Lemma A1 is thereby complete.

LEMMA A2. *Let $p \in \mathbb{N}^*$ be given. There exists $\varepsilon_0(p) > 0$ and $\alpha_0 > 0$ such that for any $\delta_1 = \delta(a_1, \lambda_1), ..., \delta_p = \delta(a_p, \lambda_p)$, $a_i \in S^3$, $\lambda_i > 0$, satisfying*

$$\varepsilon_{iju} < \varepsilon_0(p) \qquad \forall i \neq j \tag{A12}$$

the following estimate holds:

$$|v|^2_{-L} - 5\left(\sum_{i=1}^{p} \int_{S^3} \delta_1^4 v^2 \geqslant \alpha_0 \, |v|^2_{-L}\right)$$

for any v satisfying (Vo).

Proof of Lemma A2. For the sake of simplicity in the presentation, we will work on $\mathbb{R}^3$ rather than on S^3, having completed a stereographic projection. We thus have functions

$$\hat{\delta}(a, \lambda) = c_0 \frac{\sqrt{k}}{(1 + \lambda^2 |x-a|^2)^{1/2}} \tag{A13}$$

and we denote

$$\hat{\varepsilon}_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j \mid x_i - x_j|^2\right)^{-1/2}. \tag{A14}$$

The correspondence between $\hat{\delta}_i$, $\hat{\varepsilon}_{ij}$ and δ_i, ε_{ij} is given through stereographic projection. Details are available in [20].

The proof of Lemma A2 requires the following construction: For $\hat{\delta}_i$ given, $i = 1, ..., p$, we introduce

$$\Omega_i = \left\{ x \in \mathbb{R}^3 \text{ s.t. } |x - x_i| < \frac{1}{8\lambda_i} \operatorname*{Min}_{i \neq j} \hat{\varepsilon}_{ij}^{-1}; \ |x - x_j| < \frac{1}{8\lambda_i} \varepsilon_{ij}^{-1} \right.$$

$$\left. \text{for those } j \text{ s.t. } \lambda_j \geqslant \lambda_i \right\}. \tag{A15}$$

By construction, $\Omega_i \cap \Omega_j = \phi$ for $i \neq j$. Let

$$Q_i \text{ be the orthogonal projection on } H_0^1(\Omega_i) \tag{A16}$$

and for φ in $H = \{\varphi \in L^6; \nabla\varphi \in L^2\}$, let

$$\varphi_i = Q_i \varphi. \tag{A17}$$

We then have the estimate

$$\int \hat{\delta}_j^5 |\varphi - \varphi_j| \leqslant C \sqrt{\sum_{i \neq j} \hat{\varepsilon}_{ij}} \left(\int |\nabla\varphi|^2\right)^{1/2}. \tag{A18}$$

Assuming that (A18) holds, we give the proof of Lemma A2. Let

$$v_i = Q_i v \tag{A19}$$

$$E_i^- = \operatorname{span}\left\{\hat{\delta}_i, \frac{\partial \hat{\delta}_i}{\partial x_i}, \frac{\partial \hat{\delta}_i}{\partial \lambda_i}\right\}; \qquad E_i^+ = (E_i^-)^\perp; \tag{A20}$$

the orthogonal being taken in the sense of the scalar product $\int \nabla\varphi \nabla\psi$, for ψ and φ in H. α_1 is in the sequel a strictly positive fixed constant. On E_i^+, we have

$$\int |\nabla\varphi|^2 - 5 \int \hat{\delta}_i^4 \varphi^2 \geqslant \alpha_1 \int |\nabla\varphi|^2 \qquad \forall \varphi \in E_i^+. \tag{A21}$$

On E_i^-, we have

$$\left|\int |\nabla\varphi|^2 - 5\int \hat{\delta}_i^4\varphi^2\right| \leqslant C\,|\nabla\varphi|^2 \qquad \forall\varphi\in E_i^-. \tag{A22}$$

We split v_i:

$$v_i = v_i^- + v_i^+; \qquad v_i^-\in E_i^-;\ v_i^+\in E_i^+. \tag{A23}$$

We have

$$\begin{aligned} v_i^- &= \left(\int \nabla v_i\nabla\hat{\delta}_i\right)\frac{\hat{\delta}_i}{\int |\nabla\hat{\delta}_i|^2} + \left(\int \nabla v_i\nabla\frac{\partial\hat{\delta}_i}{\partial x_i}\right)\frac{\partial\hat{\delta}_i/\partial x_i}{\int |\nabla(\partial\hat{\delta}_i/\partial x_i)|^2} \\ &\quad + \left(\int \nabla v_i\nabla\frac{\partial\hat{\delta}_i}{\partial\lambda_i}\right)\frac{\partial\hat{\delta}_i/\partial\lambda_i}{\int |\nabla(\partial\hat{\delta}_i/\partial\lambda_i)|^2}. \end{aligned} \tag{A24}$$

The notation $(\int \nabla v_i\nabla(\partial\hat{\delta}_i/\partial x_i))(\partial\hat{\delta}_i/\partial x_i)$ should be understood as a summation on each component of x_i in $\mathbb{R}^3$. We estimate $\int |\nabla v_i^-|^2$. We have

$$\int \nabla v_i\nabla\hat{\delta}_i = -\int \hat{\delta}_i^5(v_i - v) \tag{A25}$$

since $\int \hat{\delta}_i^5 v = 0$ (v satisfies (Vo)).

Thus, by (A18)

$$\left|\int \nabla v_i\nabla \sqrt{\sum_{i\neq j}\hat{\varepsilon}_{ij}}\left(\int |\nabla v|^2\right)^{1/2}.\right. \tag{A26}$$

By (Vo), we also have

$$\begin{aligned} \left|\int \nabla v_i\nabla\frac{\partial\hat{\delta}_i}{\partial x_i}\right| &= \left|\int \nabla(v_i - v)\nabla\frac{\partial\hat{\delta}_i}{\partial x_i}\right| = \left|-5\int \hat{\delta}_i^4\nabla\frac{\partial\hat{\delta}_i}{\partial x_i}(v_i - v)\right| \\ &\leqslant C\lambda_i\int \hat{\delta}_i^5|v_i - v| \leqslant C'\lambda_i\sqrt{\sum_{i\neq j}\hat{\varepsilon}_{ij}}\left(\int |\nabla v|^2\right)^{1/2}. \end{aligned} \tag{A27}$$

Similarly, we have

$$\left|\int \nabla v_i\nabla\frac{\partial\hat{\delta}_i}{\partial\lambda_i}\right| \leqslant \frac{C''}{\lambda_i}\sqrt{\sum_{i\neq j}\hat{\varepsilon}_{ij}}\left(\int |\nabla u|^2\right)^{1/2}. \tag{A28}$$

Thus, combining (A26), (A27), we derive from (A24)

$$\left(\int |\nabla v_i^-|^2\right) \leqslant C\sqrt{\sum_{i\neq j}\hat{\varepsilon}_{ij}}\left(\int |\nabla v|^2\right)^{1/2}. \tag{A29}$$

Using then (A21) and (A22), we derive

$$\begin{aligned}\int |\nabla v_i|^2 - 5\int \delta_i^4 v_i^2 \geqslant \int |\nabla v_i^+|^2 - 5\int \delta_i^4 v_i^{+2} - C\int |\nabla v_i^-|^2 \\ - C\left(\int |\nabla v_i^+|^2\right)^{1/2}\left(\int |\nabla v_i^-|^2\right)^{1/2} \\ \geqslant \alpha_1 \int |\nabla v_i|^2 - C\left(\sum_{i\neq j} \hat{\varepsilon}_{ij}\right)\int |\nabla v|^2 \\ - C\left(\int |\nabla v_i^+|^2\right)^{1/2}\left(\int |\nabla v_i^-|^2\right)^{1/2}. \qquad \text{(A30)}\end{aligned}$$

With a suitable constant $M = M(\alpha_1, C)$, we have

$$-C\left(\int |\nabla v_i^+|^2\right)^{1/2}\left(\int |\nabla v_i^-|^2\right)^{1/2} \leqslant \alpha_{1/2}\int |\nabla v_i^+|^2 + M\int |\nabla v_i^-|^2. \qquad \text{(A31)}$$

Since the Ω_i's are disjoint, we have

$$\sum_i \int |\nabla v_i|^2 \leqslant \sum_i \int_{\Omega_i} |\nabla v|^2. \qquad \text{(A32)}$$

On the other hand

$$\begin{aligned}\int \delta_i^4\, |v^2 - v_i^2| \leqslant \left(\int |v + v_i|^6\right)^{1/6}\left(\int \delta_i^{24/5}\, |v - v_i|^{6/5}\right)^{5/6} \\ \leqslant C\left(\int |\nabla v|^2\right)^{1/2}\left(\int \delta_i^5 |v - v_i|\right)^{24/25}\left(\int |v - v_i|^6\right)^{1/25}. \qquad \text{(A33)}\end{aligned}$$

Thus, by (A18)

$$\int \delta_i^4\, |v^2 - v_i^2| \leqslant C\int |\nabla v_i|^2 \left(\sum_{i\neq j} \hat{\varepsilon}_{ij}\right)^{12/25}. \qquad \text{(A34)}$$

Relations (A30), (A31), (A32), and (A34) imply

$$\begin{aligned}\int |\nabla v|^2 - 5\sum_i \int \delta_i^4 v^2 \geqslant \int_{(U\Omega_i)^c} |\nabla v|^2 + \sum_i \left[\int_{\Omega_i} (|\nabla v|^2 - |\nabla v_i|^2)\right] \\ + \frac{\alpha_1}{2}\int |\nabla v_i|^2 - o(1)\int |\nabla v|^2. \qquad \text{(A35)}\end{aligned}$$

By (A32), we have

$$\sum_i \int_{\Omega_i} (|\nabla v|^2 - |\nabla v_i|^2) + \frac{\alpha_1}{2} \sum_i \int |\nabla v_i|^2 \geqslant \inf\left(\frac{\alpha_1}{2}, 1\right) \sum_i \int_{\Omega_i} |\nabla v|^2$$

$$= \bar{\alpha}_0 \sum_i \int_{\Omega_i} |\nabla v_i|^2. \qquad \text{(A36)}$$

Relations (A35) and (A36) imply

$$\int |\nabla v|^2 - 5 \sum_i \int \delta_i^4 v^2 \geqslant \inf(\bar{\alpha}_0, 1) \int |\nabla v|^2 - o(1) \int |\nabla v|^2. \qquad \text{(A37)}$$

The proof of Lemma A2 will thus be complete once (A18) is established. Since this proof is rather lengthy, we have delayed it until Appendix B.

LEMMA A3. *Using the notations $\delta_i = \delta(a_i, \lambda_i)$; $\delta_j = \delta(a_j, \lambda_j)$, the following estimates hold if ε_{ij} is small enough*:

$$(\delta_i, \delta_i)_{-L} = C; \qquad \left(\frac{\partial \delta_i}{\partial x_i}, \frac{\partial \delta_i}{\partial x_i}\right)_{-L} = C_1 \lambda_i^2$$

$$\left(\frac{\partial \delta_i}{\partial \lambda_i}, \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L} = C_2/\lambda_i^2; \qquad |(\delta_i, \delta_j)_{-L}| \leqslant C\varepsilon_{ij};$$

$$\left|\left(\frac{\partial \delta_i}{\partial x_i}, \delta_j\right)_{-L}\right| \leqslant C\lambda_i \varepsilon_{ij}; \qquad \left|\left(\frac{\partial \delta_i}{\partial \lambda_i}, \delta_j\right)_{-L}\right| \leqslant \frac{C\varepsilon_{ij}}{\lambda_i};$$

$$\left|\left(\frac{\partial \delta_i}{\partial x_i}, \frac{\partial \delta_j}{\partial x_j}\right)_{-L}\right| \leqslant C\lambda_i \lambda_j \varepsilon_{ij}; \qquad \left|\left(\frac{\partial \delta_i}{\partial x_i}, \frac{\partial \delta_j}{\partial \lambda_j}\right)_{-L}\right| \leqslant C \frac{\lambda_i}{\lambda_j} \varepsilon_{ij}$$

Proof of Lemma A3. The three first equalities are straightforward computations, using the inequalities

$$\left|\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i}\right| \leqslant C\delta_i \qquad \text{(A38)}$$

$$\left|\lambda_i \frac{\partial \delta_i}{\partial \lambda_i} v\right| \leqslant C\delta_i \qquad \text{(A39)}$$

and the equations

$$-L\delta_i = \delta_i^5 \qquad \text{(A40)}$$

$$-L \frac{\partial \delta_i}{\partial x_i} = 5\delta_i^4 \frac{\partial \delta_i}{\partial x_i} \qquad \text{(A41)}$$

$$-L \frac{\partial \delta_i}{\partial \lambda_i} = 5\delta_i^4 \frac{\partial \delta_i}{\partial \lambda_i}. \qquad \text{(A42)}$$

The four last inequalities of Lemma A3 are easily derived from

$$(\delta_i, \delta_j)_{-L} \leqslant C\varepsilon_{ij}. \tag{A43}$$

In the next lemma, we give a much more precise expansion of $(\delta_i, \delta_j)_{-L} = \int_{S^3} \delta_i^5 \delta_j$ in function of ε_{ij} which implies (A43) in particular.

LEMMA A4. *The following estimates hold if ε_{ij} is small enough: There exists a constant $\bar{c}$ such that*

$$\int_{S^3} \delta_i^5 \delta_j = \bar{c}\varepsilon_{ij} + O(\varepsilon_{ij}^3); \qquad \int_{S^3} \delta_i^3 \delta_j^3 = O\left(\varepsilon_{ij}^3 \log \frac{1}{\varepsilon_{ij}}\right)$$

$$5 \int_{S^3} \delta_i^4 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} \delta_j = \int_{S^3} \delta_j^5 \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = \left(\delta_j, \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}\right)_{-L}$$

$$= \bar{c}\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij}) = O(\varepsilon_{ij}).$$

Proof of Lemma A4. The proof of the three estimates is roughly the same. We refer the reader to [10] for the detailed computations. In [10], the aim is different and the computations are therefore carried out in much more detail. We give here the proof of the second estimate, as an example illustrating how these estimates are established in the three cases. For the sake of simplicity in the presentation, we will work here also on $\mathbb{R}^3$ with the functions

$$\hat{\delta}(a, \lambda) = c_0 \frac{\sqrt{\lambda}}{(1 + \lambda^2 |x - a|^2)^{1/2}} \tag{A44}$$

$$\hat{\varepsilon}_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2\right)^{-1/2}. \tag{A45}$$

We consider

$$S = \int_{\mathbb{R}^3} \hat{\delta}_i \hat{\delta}_j^3 = c_0^6 \int_{\mathbb{R}^3} \frac{\sqrt{\lambda_i \lambda_j}}{(1 + \lambda_i^2 |x - x_i|^2)^{3/2} (1 + \lambda_j^2 |x - x_j|^2)^{3/2}} \, dx. \tag{A46}$$

Let

$$a_{ij} = \frac{x_j - x}{2}; \qquad z = x - \frac{x_i + x_j}{2}. \tag{A47}$$

We have

$$S = c_0^6 \frac{1}{(\lambda_i \lambda_j)^{3/2}} \int_{\mathbb{R}^3} \frac{dz}{(1/\lambda_i^2 + |z + a_{ij}|^2)^{3/2} \, (1/\lambda_j^2 + |z - a_{ij}|^2)^{3/2}}. \tag{A48}$$

By symmetry, we may assume

$$\lambda_j \leqslant \lambda_i. \tag{A49}$$

Consider first the case

$$\lambda_i \lambda_j \, |a_{ij}|^2 \geqslant C \lambda_i / \lambda_j; \qquad C \text{ being a large constant.} \tag{A50}$$

Thus

$$\lambda_j^2 \, |a_{ij}|^2 \geqslant C; \qquad \lambda_i^2 \, |a_{ij}|^2 \geqslant C. \tag{A51}$$

We upper bound then S as

$$\begin{aligned} S \leqslant \frac{1}{(\lambda_i \lambda_j)^{3/2}} & \left\{ \int_{|z + a_{ij}| \leqslant 1/\lambda_i} \frac{C_1 \lambda_i^3}{(1/\lambda_i^2 + |a_{ij}|^2)^{3/2}} \, dz \right. \\ & + \int_{|z + a_{ij}| \leqslant 1/\lambda_i} \frac{C_1 \lambda_j^3}{(1/\lambda_i^2 + |a_{ij}|^2)^{3/2}} \, dz \\ & \left. + \int_{\substack{|z + a_{ij}| \geqslant 1/\lambda_i \\ |z + a_{ij}| \geqslant 1/\lambda_j}} \frac{C_1 \, dz}{(|z + a_{ij}|^3 \, |z - a_{ij}|^2)^3} \right\}, \end{aligned} \tag{A52}$$

where C_1 is a suitable constant. Indeed, if $|z + a_{ij}| \leqslant 1/\lambda_i$, then by (A51), $|z + a_{ij}| \leqslant |a_{ij}|$ and thus $|z - a_{ij}| \geqslant |a_{ij}|$. If $|z - a_{ij}| \leqslant 1/\lambda_j$, then by (A51), $|z - a_{ij}| \leqslant |a_{ij}|$ and thus $|z + a_{ij}| \geqslant |a_{ij}|$. Let us estimate

$$\int_{\substack{|z + a_{ij}| \geqslant 1/\lambda_i \\ |z + a_{ij}| \geqslant 1/\lambda_j}} \frac{dz}{|z + a_{ij}|^3 \, |z - a_{ij}|^3}. \tag{A53}$$

We have, using the fact that $|z \pm a_{ij}| \leqslant |a_{ij}|$ implies $|z \mp a_{ij}| \geqslant |a_{ij}|$,

$$\begin{aligned} & \int_{\substack{|z + a_{ij}| \geqslant 1/\lambda_i \\ |z + a_{ij}| \geqslant 1/\lambda_j}} \frac{dz}{|z + a_{ij}|^3 \, |z - a_{ij}|^3} \\ & \quad \leqslant \int_{1/\lambda_i \leqslant |z + a_{ij}| \leqslant |a_{ij}|} \frac{dz}{|z + a_{ij}|^3 \, |a_{ij}|^3} \int_{1/\lambda_i \leqslant |z - a_{ij}| \leqslant |a_{ij}|} \frac{dz}{|z - a_{ij}|^3 \, |a_{ij}|^3} \\ & \quad \quad + \int_{\substack{|z + a_{ij}| \geqslant |a_{ij}| \\ |z - a_{ij}| \geqslant |a_{ij}|}} \frac{dz}{|z + a_{ij}|^3 \, |z - a_{ij}|^3}. \end{aligned} \tag{A54}$$

Observe that

$$|z+a_{ij}| \geqslant \tfrac{1}{4}\,|z-a_{ij}| \qquad \text{if} \quad |z+a_{ij}| \geqslant |a_{ij}|. \tag{A55}$$

Thus

$$\int_{\substack{|z+a_{ij}| \geqslant |a_{ij}| \\ |z-a_{ij}| \geqslant |a_{ij}|}} \frac{dz}{|z+a_{ij}|^3\,|z-a_{ij}|^3} \leqslant 4^3 \int_{|z+a_{ij}| \geqslant |a_{ij}|} \frac{dz}{|z+a_{ij}|^6} = c_3 \times \frac{1}{|a_{ij}|^3}. \tag{A56}$$

On the other hand

$$\begin{aligned} \int_{1/\lambda_i \leqslant |z+a_{ij}| \leqslant |a_{ij}|} \frac{dz}{|z+a_{ij}|^3} &= \log \lambda_i\,|a_{ij}| \\ \int_{1/\lambda_i \leqslant |z-a_{ij}| \leqslant |a_{ij}|} \frac{dz}{|z-a_{ij}|^3} &= \log \lambda_j\,|a_{ij}|. \end{aligned} \tag{A57}$$

Relations (A52), (A54), (A56), and (A57) imply

$$\begin{aligned} S \leqslant C_3 \Bigg\{ & \frac{1}{(\lambda_i/\lambda_j + \lambda_i\lambda_j\,|a_{ij}|^2)^{3/2}} + \frac{1}{(\lambda_j/\lambda_i + \lambda_i\lambda_j\,|a_{ij}|^2)^{3/2}} \\ & + \frac{1}{(\lambda_i\lambda_j\,|a_{ij}|^2)^{3/2}} \log(\lambda_i\lambda_j\,|a_{ij}|^2+1)\Bigg\}. \end{aligned} \tag{A58}$$

Relation (A58) provides the desired estimate under (A50).
We assume now

$$\lambda_i\lambda_j\,|a_{ij}|^2 \leqslant C\lambda_i/\lambda_j, \tag{A59}$$

i.e.,

$$\lambda_j^2\,|a_{ij}|^2 \leqslant C. \tag{A60}$$

Let

$$z = \lambda_i(x-x_i). \tag{A61}$$

Then

$$S = \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \int_{\mathbb{R}^3} \frac{dz}{(1+|z|^2)^{3/2}\,(1+|(\lambda_j/\lambda_i)z - 2a_{ij}\lambda_j|^2)^{3/2}}. \tag{A62}$$

We split S into two parts,

$$S = S_1 + S_2, \tag{A63}$$

where

$$S_1 = \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \int_{|z| \leqslant 4C(\lambda_i/\lambda_j)} \frac{dz}{(1+|z|^2)^{3/2}\,(1+|(\lambda_j/\lambda_i)z - 2a_{ij}\lambda_i|^2)^{3/2}}$$
$$S_2 = \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \int_{|z| \geqslant 4C(\lambda_i/\lambda_j)} \frac{dz}{(1+|z|^2)^{3/2}\,(1+z - 2a_{ij}\lambda_i|^2)^{3/\lambda_i|^2)^{3/2}}. \qquad \text{(A64)}$$

We have, by (A59),

$$S_1 \leqslant C_1 \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \int_{|z| \leqslant 4C(\lambda_i/\lambda_j)} \frac{dz}{(1+|z|^2)^{3/2}} \leqslant C_2 \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \log(\lambda_i/\lambda_j) \qquad \text{(A65)}$$
$$= O\left(\hat{\varepsilon}_{ij}^3 \log \frac{1}{\hat{\varepsilon}_{ij}}\right).$$

If $|z| \geqslant 4C\lambda_i/\lambda_j$, then by (A60),

$$\left|\frac{\lambda_j}{\lambda_i} z - 2a_{ij}\lambda_j\right| \geqslant \frac{\lambda_j}{\lambda_i}\frac{|z|}{2}. \qquad \text{(A66)}$$

Thus

$$S_2 \leqslant C_1' \left(\frac{\lambda_j}{\lambda_i}\right)^{3/2} \left(\frac{\lambda_i}{\lambda_j}\right)^{3/2} \int_{|z| \geqslant 4C(\lambda_i/\lambda_j)} \frac{dz}{|z|^6} = C_2' \left(\frac{\lambda_j}{\lambda_i}\right)^3 = O(\hat{\varepsilon}_{ij}^6). \qquad \text{(A67)}$$

Relations (A65) and (A67) imply the desired estimate under (A60). Q.E.D.

LEMMA A5. *The following estimates hold, if* ε_{ij}, ε_{jk}, *and* $1/\lambda_i$ *are small enough*:

$$\left(\int_{S^3} \delta_i^{24/5} \delta_j^{6/5}\right)^{5/6} = O\left(\left(\varepsilon_{ij}^3 \left(\log \frac{1}{\varepsilon_{ij}}\right)\right)^{1/3}\right)$$

$$\int_{S^3} \delta_j^4 \delta_k \delta_i = o(\varepsilon_{ij} + \varepsilon_{jk})$$

$$\int |K(x) - K(x_i)|\, \delta_i^5 \delta_j = o(\varepsilon_{ij}).$$

Proof of Lemma A5. By Hölder's inequality, we have

$$\int_{S^3} \delta_i^{24/5} \delta_j^{6/5} \leqslant \left(\int_{S^3} \delta_i^3 \delta_j^3\right)^{2/5} \left(\int_{S^3} \delta_i^6\right)^{3/5}. \qquad \text{(A68)}$$

Using Lemma A4, we thus have

$$\int_{S^3} \delta_i^{24/5} \delta_j^{6/5} = O\left(\varepsilon_{ij}^{6/5}\left(\log \frac{1}{\varepsilon_{ij}}\right)^{2/5}\right) \tag{A69}$$

hence the first estimate in Lemma A5. For the second estimate, let us consider a very large real $M > 0$.

We have

$$\int_{S^3} \delta_j^4 \delta_k \delta_i \leqslant M \int_{S^3} \delta_j^3(\delta_k^3 + \delta_i^3) + \frac{1}{M} \int_{S^3} (\delta_j^5 \delta_k + \delta_j^5 \delta_i). \tag{A70}$$

Thus, by Lemma A4, with C a fixed constant,

$$\int_{S^3} \delta_j^4 \delta_k \delta_i \leqslant CM\left(\varepsilon_{ij}^3 \log \frac{1}{\varepsilon_{ij}} + \varepsilon_{jk}^3 \log \frac{1}{\varepsilon_{jk}}\right) + \frac{1}{M} C(\varepsilon_{ij} + \varepsilon_{jk}) = o(\varepsilon_{ij}) \tag{A71}$$

hence the second estimate in Lemma A5.

Lastly, we have, for $\varepsilon > 0$ given, with a suitable constant C independent of ε,

$$\begin{aligned} \int |K(x) - K(x_i)|\, \delta_i^5 \delta_j &\leqslant \int_{d(x_1 x_i) \leqslant \varepsilon} |K(x) - K(x_i)|\, \delta_i^5 \delta_j + C \int_{d(x_1 x_i) \geqslant \varepsilon} \delta_i^5 \delta_j \\ &\leqslant C\varepsilon\varepsilon_{ij} + \frac{C}{\lambda_i^{5/2} \varepsilon^5 \sqrt{\lambda_j}} \\ &\leqslant C\varepsilon_{ij}\left(\varepsilon + \frac{1}{\lambda_i^2}\right) = o(\varepsilon_{ij}). \end{aligned} \tag{A72}$$

The proof of Lemma A5 is complete.

LEMMA A6. *Let δ_1 be $\delta(x_1, \lambda_1)$ and let v satisfy* (Vo). *We then have*

$$\left(\int |K(x) - K(x_1)|^{6/5}\, \delta_1^6\right)^{5/6} \leqslant C\left(\frac{|DK(x_1)|}{\lambda_1} + \frac{1}{\lambda_1^2}\right)$$

$$\int |(K(x) - K(x_1))|\, \delta_1^6 \leqslant C\left(\frac{|DK(x_1)|}{\lambda_1} + \frac{1}{\lambda_1^2}\right)$$

$$\left|\int (K(x) - K(x_1))\, \delta_1^5 \lambda_1 \frac{\partial \delta_1}{\partial \lambda_1}\right| \leqslant \frac{C}{\lambda_1^2}$$

$$\left|\int (K(x) - K(x_1))\, \delta_1^5 v\right| \leqslant C\left(\frac{|DK(x_1)|}{\lambda_1} + \frac{1}{\lambda_1^2}\right) |v|_{-L}$$

$$\int K(x)\, \delta_1^4 v^2 = K(x_1) \int \delta_1^4 v^2 + O\left(\frac{|v|_{-L}^2}{\lambda_1}\right).$$

Proof of Lemma A6. We can upper bound $|(K(x)-K(x_1))|$ by $C\,[|DK(x_1)|\,d(x,x_1)+d(x,x_1)^2]$. Therefore

$$\begin{aligned}&\int |(K(x)-K(x_1))|^{6/5}\,\delta_1^6\\ &\leqslant C\,|DK(x_1)|^{6/5}\int d(x,x_1)^{6/5}\,\delta_1^6+C\int d(x,x_1)^{12/5}\,\delta_1^6\\ &\leqslant C'\left(\frac{|DK(x_1)|^{6/5}}{\lambda_1^{6/5}}+\frac{1}{\lambda_1^{12/5}}\right).\end{aligned}\tag{A73}$$

The last inequality in (A73) is a direct estimate.
Relation (A73) implies the first inequality in Lemma A6. The proof of the second inequality is very similar.

Next, we have

$$\begin{aligned}\left|\int (K(x)-K(x_1))\,\delta_1^5 v\right| &\leqslant C\left(\int v^6\right)^{1/6}\left(\int |K(x)-K(x_1)|^{6/5}\,\delta_1^6\right)^{5/6}\\ &\leqslant C'\,|v|_{-L}\left(\frac{DK(x_1)|}{\lambda_1}+\frac{1}{\lambda_1^2}\right)\end{aligned}\tag{A74}$$

$$\begin{aligned}\left|\int (K(x)-K(x_1))\,\delta_1^4 v^2\right| &\leqslant C\left(\int v^6\right)^{1/3}\left(\int |K(x)-K(x_1)|^{3/2}\,\delta_1^6\right)^{2/3}\\ &\leqslant C'\,|v|_{-L}\left(\int d(x,x_1)^{3/2}\,\delta_1^6\right)^{2/3}\leqslant\frac{C''\,|v|_{-L}^2}{\lambda_1}.\end{aligned}\tag{A75}$$

This establishes the two last inequalities in Lemma A6. We are thus left with the third inequality. Let $\varepsilon>0$ be given. We have, using the symmetry of $\partial\delta_1/\partial\lambda_1$ around x_1,

$$\begin{aligned}&\left|\int_{d(x,x_1)\leqslant\varepsilon} (K(x)-K(x_1))\,\delta_1^5\lambda_1\,\frac{\partial\delta_1}{\partial\lambda_1}\right|\\ &=\underbrace{\left|\int_{d(x,x_1)\leqslant\varepsilon} DK(x_1)\cdot(x-x_1)\,\delta_1^5\lambda_1\,\frac{\partial\delta_1}{\partial\lambda_1}\right|}_{=0}\\ &\quad+O\left(\int d(x,x_1)^2\,\delta_1^6\right)=O\left(\frac{1}{\lambda_1^2}\right)\end{aligned}\tag{A76}$$

$$\left|\int_{d(x,x_1)\leqslant\varepsilon}(K(x)-K(x_1))\,\delta_1^5\lambda_1\frac{\partial\delta_1}{\partial\lambda_1}\right|$$

$$\leqslant C\int_{d(x,x_1)\geqslant\varepsilon}\delta_1^6\leqslant\frac{C'}{\varepsilon^6\lambda_1^3}=O\left(\frac{1}{\lambda_1^2}\right). \tag{A77}$$

Relations (A76) and (A77) imply the third estimate in Lemma A6. The proof of this lemma is thereby complete.

LEMMA A7. *Let* $\delta_1=\delta(x_1,\lambda_1)$. *We then have*

$$\int K(x)\,\delta_1^5\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial\lambda_1}=-c\,\frac{DK(x_1)}{\lambda_1}+O\left(\frac{1}{\lambda_1^2}\right)$$

$$\int K(x)\,\delta_i^5\lambda_1\frac{\partial\delta_1}{\partial\lambda_1}=-c'\,\frac{DK(x_1)}{\lambda_1^2}+o\left(\frac{1}{\lambda_1^2}\right).$$

Proof of Lemma A7. For the second expansion, observe that

$$\int K(x)\,\delta_1^5\lambda_1\frac{\partial\delta_1}{\partial\lambda_1}=\frac{\lambda_1}{6}\frac{\partial}{\partial\lambda_1}\int K(x)\,\delta_1^6$$

$$=\frac{\lambda_1}{6}\frac{\partial}{\partial\lambda_1}\left(\int(K(x)-K(x_1))\,\delta_1^6\right). \tag{A78}$$

In the course of the proof of the following lemma, Lemma A8, we will give an expansion of $-(\partial/\partial\lambda_1)\int(K(x)-K(x_1))\,\delta_1^6$; see (A82) to (A92). This expansion, in particular in (A92), is the same as the one proposed here.

We consider now

$$\int K(x)\,\delta_1^5\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}=\int(K(x)-K(x_1))\,\delta_1^5\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}. \tag{A79}$$

Expanding K around x_1, we derive

$$\int K(x)\,\delta_1^5\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}=\int(DK(x_1)\cdot x-x_1)\,\delta_1^5\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}$$

$$+O\left(\int d(x,x_1)^2\,\delta_1^5\frac{1}{\lambda_1}\left|\frac{\partial\delta_1}{\partial x_1}\right|\right). \tag{A80}$$

Using the symmetry of δ_1 around x_1 and (A38), we derive

$$\int K(x)\,\delta_1^5\,\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1} = -\frac{DK(x_1)}{3}\int \delta_1^5\cdot\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}\cdot(x-x_1)+O\left(\int d(x,x_1)^2\,\delta_1^6\right)$$

$$= -\frac{DK(x_1)}{3\lambda_1}\int \delta_1^5\cdot\frac{1}{\lambda_1}\frac{\partial\delta_1}{\partial x_1}\cdot(x-x_1)+O\left(\frac{1}{\lambda_1^2}\right)$$

$$= -c\,\frac{DK(x_1)}{\lambda_1}+O\left(\frac{1}{\lambda_1^2}\right). \tag{A81}$$

The proof of Lemma A7 is thereby complete.

LEMMA A8. *Let y_j be a critical point of K such that $-\Delta K(y_j)>0$; let ρ be small enough so that $-\Delta K(x_1)\geqslant -\Delta K(y_j)/2$ for any x_1 in $N_\rho(y_j)$. Let $\delta_1=\delta(x_1,\lambda_1)$ be such that x_1 belongs to $N_\rho(y_j)$. We then have that there exists exists $A_1>0$, A_1 uniform on $N_\rho(y_j)$ such that for λ_1 in $[A_1, +\infty[$, $-\int (K(x)-K(x_1))\,\delta_1^6$ decreases when λ_1 increases. Furthermore the following expansion holds*:

$$\int K(x)\,\delta_1^6 = K(x_1)\int \delta_1^6 + c'\,\frac{\Delta K(x_1)}{\lambda_1^2}+o\left(\frac{1}{\lambda_1^2}\right).$$

Proof of Lemma A8. For the sake of simplicity, we will first complete a stereographic projection $\pi: S^3\to\mathbb{R}^3$ of north pole $N=-x_1$, the antipodal point of x_1 on S^3. After having completed this transformation, we are left with a new quantity,

$$-\int (\hat{K}(x)-\hat{K}(0))\,\hat{\delta}^6(0,\lambda_1'), \tag{A82}$$

where $\hat{K}$ corresponds to K and where

$$\hat{\delta}(0,\lambda)(x)=c_0\,\frac{\sqrt{\lambda}}{(1+\lambda^2\,|x|^2)^{1/2}}; \qquad c_0 \text{ such that } -\Delta\hat{\delta}=\hat{\delta}^5. \tag{A83}$$

λ_1' satisfies

$$\sqrt{1+2\lambda_1'^2}=\lambda_1. \tag{A84}$$

We leave aside the precise computations, which can be found in [20]. We wish to prove that if $-\Delta\hat{K}(0)$ is larger than $\beta>0$, there exists $A_1(\beta)$ such that if $\lambda_1'\geqslant A_1(\beta)$, (A82) decreases.

We complete the change of variables:

$$\lambda_1' x = y. \tag{A85}$$

We thus have to study

$$-\int_{\mathbb{R}^3}\left(\hat{K}\left(\frac{y}{\lambda_1'}\right)-\hat{K}(0)\right)\frac{1}{(1+|y|^2)^3}\,dy. \tag{A86}$$

Differentiating with respect to λ_1', we obtain

$$\frac{1}{\lambda_1'^2}\int_{\mathbb{R}^3} D\hat{K}\left(\frac{y}{\lambda_1'}\right)\cdot y\,\frac{1}{(1+|y|^2)^3}\,dy=T. \tag{A87}$$

Introducing a number $M>1$, which we will take large, we have, after cutting the integral in two pieces,

$$T=\frac{1}{\lambda_1'}\int_{|y|/\lambda_1'\leqslant M/\lambda_1'} D\hat{K}\left(\frac{y}{\lambda_1'}\right)\cdot\frac{y}{\lambda_1'}\cdot\frac{1}{(1+|y|^2)^3}$$
$$+\frac{1}{\lambda_1'^2}\int_{|y|\geqslant M}\frac{y}{(1+|y|^2)^3}\,dy. \tag{A88}$$

If λ_1' is large enough, M/λ_1' is small and we can expand

$$D\hat{K}\left(\frac{y}{\lambda_1'}\right)=D\hat{K}(0)+D^2K(0)\cdot\frac{y}{\lambda_1'}+o\left(\frac{y}{\lambda_1'}\right)\qquad\text{for}\quad |y|\leqslant M. \tag{A89}$$

Thus

$$T=\frac{1}{\lambda_1'}\int_{|y|/\lambda_1'\leqslant M/\lambda_1'}\left(D^2\hat{K}(0)\cdot\frac{y}{\lambda_1'}\cdot\frac{y}{\lambda_1'}\right)\frac{1}{(1+|y|^2)^3}$$
$$+\frac{1}{\lambda_1'^3}\,o\left(\int_{|y|/\lambda_1'\leqslant M/\lambda_1'}\frac{|y|}{(1+|y|^2)^3}\right)+O\left(\frac{1}{\lambda_1'^2M^2}\right)$$
$$=c'\,\frac{\Delta\hat{K}(0)}{\lambda_1'^3}+o\left(\frac{1}{\lambda_1^2}\right)+O\left(\frac{1}{\lambda_1'^2M^2}\right). \tag{A90}$$

We now choose

$$M=\lambda_1'^{3/4}. \tag{A91}$$

Then M/λ_1' is small when λ_1' is large and we have

$$T=c'\,\frac{\Delta\hat{K}(0)}{\lambda_1'^3}+o\left(\frac{1}{\lambda_1'^3}\right)\leqslant\frac{-\beta c'}{\lambda_1'^3}+o\left(\frac{1}{\lambda_1'^3}\right). \tag{A92}$$

Relation (A92) implies the existence of Λ_1, independent of x_1, such that if $\lambda_1\geqslant\Lambda_1$, T is negative and $-\int(K(x)-K(0))\,\delta_1^6$ decreases when λ_1 increases.

We establish now the expansion, or rather the corresponding expansion, on

$$\int \hat{K}(x)\, \delta^6. \tag{A93}$$

We have

$$\begin{aligned}
&-\int (\hat{K}(x) - \hat{K}(0))\, \delta^6(0, \lambda_1') \\
&= \int_{|y|/\lambda_1' \leqslant M/\lambda_1'} D\hat{K}(0) \cdot \frac{y}{\lambda_1'} \cdot \frac{1}{(1+|y|^2)^3}\, dy \\
&\quad + \int_{|y|/\lambda_1' \leqslant M/\lambda_1'} \left((D^2\hat{K}(0) + o(1)) \cdot \frac{y}{\lambda_1'} \cdot \frac{y}{\lambda_1'} \right) \frac{dy}{(1+|y|^2)^3} \\
&\quad + \int_{|y| \geqslant M} \left(\hat{K}\left(\frac{y}{\lambda_1'}\right) - \hat{K}(0) \right) \frac{1}{(1+|y|^2)^3}\, dy \\
&= \frac{c'\, \Delta \hat{K}(0)}{\lambda_1'^2} + o\left(\frac{1}{\lambda_1'^2}\right) + O\left(\frac{1}{M\lambda_1'^3} + \frac{1}{M^3}\right).
\end{aligned} \tag{A94}$$

Since $M = \lambda_1'^{3/4}$, we derive from (A94) the desired expansion. The proof of Lemma A8 is thereby complete.

Proof of Proposition 3. Let

$$\varphi = \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial x_i} \qquad \text{or} \qquad \varphi = \lambda_i \frac{\partial \delta_i}{\partial \lambda_i}. \tag{A95}$$

In both cases, we have

$$|\varphi| \leqslant C\delta_i; \qquad |\varphi|_{-L} \leqslant C \tag{A96}$$

with a suitable constant C (see Lemma A3 for the second inequality). Observe also that

$$\int L\delta_i \varphi = 0 \tag{A97}$$

since $\int L\delta_i \delta_i$ is independent of λ_i and x_i.

We compute

$$\partial J(u) \cdot \varphi. \tag{A98}$$

We have, by (A97) and (Vo),

$$\partial J(u)\cdot\varphi = -\lambda(u)^5 \int_{S^3} K(x)\left(\sum_{i=1}^{p} \alpha_i\delta_i + v\right)^5 \varphi - \lambda(u) \sum_{j\neq i} \alpha_j \int \delta_j^5\cdot\varphi$$

$$= -\lambda(u)^5\,\alpha_i^4 \int_{S^3} K(x)\,\delta_i^5\varphi + R. \tag{A99}$$

Using (A96), Lemma (A5), and (131), we have

$$|R| \leqslant C\left(\sum_{j\neq i}\int \delta_j^5\delta_i + \sum_{j\neq i}\int \delta_j^4\delta_i\,|v| + \left|\int K(x)\,\delta_i^4 v\varphi\right| + |v|_{-L}^2\right)$$

$$\leqslant C\left(\sum \varepsilon_{ij} + |v|_{-L}\left[\sum \varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/3} + \frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right] + |v|_{-L}^2\right). \tag{A100}$$

Therefore, using Lemma 1 which provides us with a lower bound on $\lambda(u)$, using (A96), and Lemma 2, we derive, for ε_0 small enough,

$$\left|\int K(x)\,\delta_i^5\varphi\right| \leqslant C\left[|\partial J(u)|_{-L} + \sum_{j\neq i}\varepsilon_{ij} + \frac{|DK(x_i)|^2}{\lambda_i^2} + \frac{1}{\lambda_i^4}\right]. \tag{A101}$$

Using Lemma A7, we derive

$$\left|\int K(x)\,\delta_i^5\,\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}\right| + \left|\int K(x)\,\delta_i^5\lambda_i\,\frac{\partial\delta_i}{\partial\lambda_i}\right|$$

$$= c\left|\frac{DK(x_i)}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right)\right| + c'\left|\frac{\Delta K(x_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i^2}\right)\right|. \tag{A102}$$

By (H1), $\Delta K(x_i)$ is nonzero at the critical points of K. Therefore, with a suitable constant c_1,

$$c\left|\frac{DK(x_i)}{\lambda_i} + O\left(\frac{1}{\lambda_i^2}\right)\right| + c'\left|\frac{\Delta K(x_i)}{\lambda_i^2} + o\left(\frac{1}{\lambda_i^2}\right)\right|$$

$$\geqslant c_1\left(\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right). \tag{A103}$$

Relations (A101), (A102), (A103) imply

$$\frac{|DK(x_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} \leqslant C'\left[|\partial J(u)|_{-L} + \sum_{j\neq i}\varepsilon_{ij} + \frac{|DK(x_i)|^2}{\lambda_i^2} + \frac{1}{\lambda_i^4}\right] \tag{A104}$$

hence, for ε_0 small enough,

$$\frac{|DK(x_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\leqslant C''\left[|\partial J(u)|_{-L}+\sum_{j\neq i}\varepsilon_{ij}\right]. \tag{A105}$$

Lemma 2 and (A105) imply Proposition 3. Q.E.D.

APPENDIX B

Proof of (A18) *of Lemma* A2. Let

$$\delta=\frac{1}{(1+|y|^2)^{1/2}};\qquad \Delta\delta=-c_2\delta^5;\ c_2>0 \tag{B1}$$

$$w\in L^6;\qquad \nabla w\in L^2 \tag{B2}$$

$$\lambda>0;\qquad B_\lambda=\{x\in\mathbb{R}^n \text{ such that } |x|<\lambda\}. \tag{B3}$$

Let h be defined by

$$\begin{aligned}\Delta h&=0 \quad \text{in} \quad B_\lambda\\ h&=w|_{\partial B_\lambda}.\end{aligned} \tag{B4}$$

We then have

$$\int_{B_\lambda}\delta^5\,|h|\leqslant\frac{C}{\sqrt{\lambda}}\left(\int|\nabla w|^2\,dx\right)^{1/2}. \tag{B5}$$

Indeed, first notice that we can take w to be positive. Otherwise, replacing w by $|w|=\tilde{w}$ and h by $\tilde{h}$, we have

$$|h|\leqslant\tilde{h} \tag{B6}$$

and

$$\int_{\mathbb{R}^3}|\nabla w|^2\,dx=\int_{\mathbf{R}^3}|\nabla\tilde{w}|^2\,dx. \tag{B7}$$

Thus, if (B5) holds for $w\geqslant 0$, i will hold for all w. Assuming that w is positive, (B5) becomes

$$\int_{B_\lambda}\delta^5 h\leqslant\frac{C}{\lambda^{3/2}}\left(\int|\nabla w|^2\right)^{1/2}. \tag{B8}$$

We prove now (B8). We have

$$\int_{B_\lambda} \hat{\delta}^5 h = \frac{1}{c_2} \int_{B_\lambda} \Delta \hat{\delta} h = -\frac{1}{c_2} \int_{\partial B_\lambda} \frac{\partial \hat{\delta}}{\partial n} h + c_2 \int \nabla \left(\hat{\delta} - \frac{1}{\sqrt{1+\lambda^2}} \right) \nabla h$$
$$= -\frac{1}{c_2} \int_{\partial B_\lambda} \frac{\partial \hat{\delta}}{\partial n} h \leqslant \frac{c_3}{\lambda^2} \int_{\partial B_\lambda} h. \tag{B9}$$

Let

$$\bar{h}(x) = \sqrt{\lambda}\, h(\lambda x); \qquad \bar{w}(x) = \sqrt{\lambda}\, w(\lambda x). \tag{B10}$$

We have

$$\begin{aligned} &\Delta \bar{h} = 0 \text{ in } B_1 = \{x \text{ s.t. } |x| < 1\} \\ &\bar{h} = \bar{w}\,|_{\partial B_1} \end{aligned} \tag{B11}$$

and

$$\int_{\partial B_1} \bar{h} \leqslant C \left\{ \left(\int_{B_1} |\nabla \bar{w}|^2 \, dx \right)^{1/2} + \left(\int_{B_1} \bar{w}^6 \right)^{1/6} \right\} \leqslant C \left(\int_{\mathbb{R}_3} |\nabla \bar{w}|^2 \right)^{1/2}. \tag{B12}$$

Since $\int_{\mathbb{R}^3} |\nabla \bar{w}|^2 = \int_{\mathbb{R}^3} |\nabla w|^2$ and $\int_{\partial B_1} \bar{h} = (1/\lambda^{1/2}) \int_{\partial B_\lambda} h$, (B8) follows from (B12). We prove now (A18). First, we assume that $p = 2$, i.e.; we have only two functions $\hat{\delta}_1$ and $\hat{\delta}_2$ and two sets Ω_1 and Ω_2 as in (A15). Next, we derive the general case. We take $j = 1$ in (A18). We also complete a translation and a dilation, so that

$$\lambda_1 = 1; \qquad x_1 = 0. \tag{B13}$$

Thus

$$\begin{aligned} &\bar{\varphi} = \sqrt{\lambda_1}\, \varphi(\lambda_1 x + x_1); \qquad \bar{\delta}_i = \sqrt{\lambda_i}\, \hat{\delta}_1(x_1 + \lambda_1 x) \\ &\bar{\varepsilon}_{12} = \hat{\varepsilon}_{12}; \qquad \bar{\lambda}_1 = 1; \qquad \bar{\lambda}_2 = \lambda_2/\lambda_1; \qquad \bar{x}_1 = 0; \qquad \bar{x}_2 = \lambda_1(x_1 - x_2). \end{aligned} \tag{B14}$$

Assume first that

$$\lambda_1 = \lambda_2, \qquad \text{i.e., } \bar{\lambda}_2 \leqslant 1. \tag{B15}$$

Then

$$\begin{aligned} &\bar{\Omega}_1 = \left\{ x \mid |x| \leqslant \frac{1}{8\hat{\varepsilon}_{12}} \right\} \\ &\bar{\varphi}_1 = \bar{\varphi} - \bar{h}; \text{ with } \Delta \bar{h} = 0 \text{ in } \bar{\Omega}_1; \ \bar{h} = \bar{\varphi} \text{ on } \partial \bar{\Omega}_1. \end{aligned} \tag{B16}$$

We have

$$\int_{\mathbb{R}^3} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1| \leqslant \int_{\bar{\Omega}_1} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1| + \int_{|x| > 1/8\hat{\varepsilon}_{12}} \frac{1}{(1 + |x|^2)^{5/2}} |\bar{\varphi}|. \tag{B17}$$

Relation (B8) implies

$$\int_{\bar{\Omega}_1} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1| \leqslant C(\hat{\varepsilon}_{12})^{3/2} \left(\int_{\mathbb{R}^3} |\nabla\bar{\varphi}|^2 \, dx\right)^{1/2} = C(\hat{\varepsilon}_{12})^{3/2} \left(\int_{\mathbb{R}^3} |\nabla\varphi|^2 \, dx\right)^{1/2} \tag{B18}$$

On the other hand

$$\int_{|x| > 1/8\varepsilon_{12}} \frac{1}{(1 + |x|^2)^{5/2}} |\bar{\varphi}| \leqslant C\left(\int_{\mathbb{R}^3} |\nabla\bar{\varphi}|^2\right)^{1/2} \hat{\varepsilon}_{12}^{5/2}$$

$$= C\hat{\varepsilon}_{12}^{5/2}\left(\int_{\mathbb{R}^3} |\nabla\varphi|^2\right)^{1/2}. \tag{B.19}$$

Since $\int_{\mathbb{R}^3} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1| = \int_{\mathbb{R}^3} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1|$, (B17), (B18), and (B19) imply (A18) under (B15).

We now consider the other case, namely

$$\lambda_2 \geqslant \lambda_1, \qquad \text{i.e., } \bar{\lambda}_2 \geqslant 1. \tag{B20}$$

Then

$$\bar{\Omega}_1 = \left\{x/|x| < \frac{1}{8\hat{\varepsilon}_{12}} \text{ and } |x - \bar{x}_2| > \frac{1}{8\bar{\lambda}_2\hat{\varepsilon}_{12}}\right\}. \tag{B21}$$

Let then

$$\tilde{\Omega} = \left\{x \text{ such that } |x| < \frac{1}{8\hat{\varepsilon}_{12}}\right\}; \qquad \tilde{W} = \left\{x/|x - \bar{x}_2| > \frac{1}{8\bar{\lambda}_2\hat{\varepsilon}_{12}}\right\}. \tag{B22}$$

Let

$$\begin{aligned} \tilde{\varphi}_1 &= \text{orthogonal projection of } \bar{\varphi} \text{ on } H_0^1(\tilde{\Omega}) \\ \tilde{\psi}_1 &= \text{orthogonal projection of } \bar{\varphi} \text{ on } H_0^1(\tilde{W}). \end{aligned} \tag{B23}$$

We may assume that $\bar{\varphi} \geqslant 0$, as we already pointed out. Since $\partial\bar{\Omega}_1 \subset \partial\tilde{\Omega} \cup \partial\tilde{W}$, we then have

$$|\bar{\varphi} - \bar{\varphi}_1| \leqslant 2\bar{\varphi} - [\tilde{\varphi}_1 + \tilde{\psi}_1] \qquad \text{in} \quad \mathbb{R}^3 \tag{B24}$$

and

$$\int_{\mathbb{R}^3} \delta_1^5 |\bar{\varphi} - \bar{\varphi}_1| \leqslant \int_{\tilde{\Omega}} \delta_1^5 (\bar{\varphi} - \tilde{\varphi}_1) + \int_{\tilde{W}} \delta_1^5 (\bar{\varphi} - \tilde{\psi}_1)$$

$$+ \int_{\mathbb{R}^3 - \tilde{\Omega}} \delta_1^5 \bar{\varphi} + \int_{\mathbb{R}^3 - \tilde{\Omega}} \delta_1^5 \bar{\varphi}. \tag{B25}$$

Relation (B19) provides us with a suitable inequality on $\int_{\mathbb{R}^3 - \tilde{\Omega}} \delta_1^5 \bar{\varphi}$. We estimate now

$$\int_{\mathbb{R}^3 - \tilde{W}} \delta_1^5 \bar{\varphi} = \int_{\bar{\lambda}\, |x - \bar{x}_2| < 1/8\hat{\varepsilon}_{12}} \delta_1^5 \bar{\varphi}. \tag{B26}$$

A direct computation shows that if ε_{12} is small enough, then

$$\delta_1 = \frac{1}{(1 + |x|^2)^{1/2}} \leqslant \frac{\sqrt{2}\,\sqrt{\bar{\lambda}_2}}{(1 + \bar{\lambda}_2^2\, |x - \bar{x}_2|^2)^{1/2}} \qquad \text{if} \quad \bar{\lambda}_2\, |x - \bar{x}_2| < \frac{1}{8\hat{\varepsilon}_{12}}. \tag{B27}$$

We then have

$$\int_{\mathbb{R}^3 - \tilde{W}} \delta_1^5 \bar{\varphi} \leqslant C \left(\int_{\mathbb{R}^2 - \tilde{W}} \delta_1^6 \right)^{5/6} \left(\int |\nabla \bar{\varphi}|^2 \right)^{1/2}$$

$$\leqslant C' \left(\int_{\mathbb{R}^3 - \tilde{W}} \delta_1^3 \delta_2^3 \right)^{5/6} \times \left(\int |\nabla \bar{\varphi}|^2 \right)^{1/2}. \tag{B28}$$

By Lemma A4, we derive

$$\int_{\mathbb{R}^3 - \tilde{W}} \delta_1^5 \bar{\varphi} \leqslant C \varepsilon_{ij}^{5/2} \left(\log \frac{1}{\varepsilon_{ij}} \right)^{5/6} \left(\int |\nabla \bar{\varphi}|^2 \right)^{1/2}. \tag{B29}$$

Relation (B29) is also satisfactory for (A18).

We estimate now the two remaining terms in (B25)

$$\int_{\tilde{\Omega}} \delta_1^5 (\bar{\varphi} - \tilde{\varphi}_1) \tag{B30}$$

$$\int_{\tilde{W}} \delta_1^5 (\bar{\varphi} - \tilde{\psi}_1). \tag{B31}$$

Relation (B30) is estimated using (B5). We are left with $\int_{\tilde{W}} \delta_1^5(\bar{\varphi}-\tilde{\psi}_1)$. We have

$$\int_{\tilde{W}} \delta_1^5(\bar{\varphi}-\tilde{\psi}_1) = -c_3 \int_{\tilde{W}} \Delta\delta_1(\bar{\varphi}-\tilde{\psi}_1) = c_3 \int_{\tilde{W}} \nabla(\delta_1(\bar{\varphi}-\bar{\theta}_1))\,\nabla(\bar{\varphi}-\tilde{\psi}_1)$$

$$-c_3 \int_{\partial\tilde{W}} \frac{\partial}{\partial n}(\delta_1-\bar{\theta}_1)(\bar{\varphi}-\tilde{\psi}_1)$$

$$= -c_3 \int_{\partial\tilde{W}} \frac{\partial}{\partial n}(\delta_1-\bar{\theta}_1)(\bar{\varphi}-\tilde{\psi}_1), \tag{B32}$$

where

$$\Delta\bar{\theta}_1 = 0, \qquad \bar{\theta}_1 = \delta_1/\partial\tilde{W}. \tag{B33}$$

Thus

$$\left|\int_{\tilde{W}} \delta_1^5(\bar{\varphi}-\tilde{\psi}_1\right| \leqslant C \sup_{\partial\tilde{W}} \left|\frac{\partial}{\partial n}(\delta_1-\bar{\theta}_1)\right| \int_{\partial\tilde{W}} \bar{\varphi}$$

$$= C \sup_{\partial\tilde{W}} \left|\frac{\partial}{\partial n}(\delta_1-\bar{\theta}_1)\right| \int_{\bar{\lambda}_2\,|x-\bar{x}_2| = 1/8\bar{\varepsilon}_{12}} \bar{\varphi}. \tag{B34}$$

Relation (B12) then implies

$$\left|\int_{\tilde{W}} \delta_1^5(\bar{\varphi}-\tilde{\psi}_1)\right| \leqslant C \sup_{\partial\tilde{W}} \left|\frac{\partial}{\partial n}(\delta_1-\theta_1)\right| \frac{1}{(\bar{\varepsilon}_{12}\bar{\lambda}_2)^{1/2}} \times \left(\int |\nabla\bar{\varphi}|^2\right)^{1/2}. \tag{B35}$$

$\tilde{W}$ is the exterior of a ball around $\bar{x}_2$ of radius $1/8\varepsilon_{12}\bar{\lambda}_2$. By a homogeneity argument, we can check that

$$\sup_{\partial\tilde{W}} \left|\frac{\partial}{\partial n}(\delta_1-\bar{\theta}_1)\right| \leqslant C \sup_{\partial\tilde{W}} \left|\frac{\partial}{\partial n}\delta_1\right| = C' \sup_{\partial\tilde{W}} \frac{|x\cdot((x-x_2)/|x-x_2|)|}{(1+|x|^2)^{3/2}}. \tag{B36}$$

Let

$$r = \frac{1}{8\bar{\lambda}_2\varepsilon_{12}}. \tag{B37}$$

Then

$$\sup_{\partial\tilde{W}} \frac{|x\cdot((x-x_2)/|x-x_2|)|}{(1+|x|^2)^{3/2}} = \frac{r+|x_2|}{(1+r^2+|x_2|^2-2r\,|x_2|)^{3/2}}. \tag{B38}$$

Thus

$$\left|\int_{\tilde{W}} \delta_1^5(\bar{\varphi}-\tilde{\psi}_1)\right| \leqslant C\rho\left(\int |\nabla\varphi|^2\right)^{1/2}, \tag{B39}$$

where

$$\rho = \frac{r + |x_2|}{(1 + r^2 + |x_2|^2 - 2r\,|x_2|)^{3/2}} \cdot \sqrt{r}. \tag{B40}$$

One can check directly that

$$\rho = O(\varepsilon_{12}^{1/2}) \tag{B41}$$

hence (A18) in case we have two indexes only.

For the general case, one introduces the sets, assuming $\lambda_1 = 1$, $x_1 = 0$, and $\varphi \geqslant 0$,

$$\theta_i = \left\{ x \in \mathbb{R}^n \text{ such that } |x| < \frac{1}{\varepsilon_{1i}} \text{ and } |x - x_i| > \frac{1}{\lambda_i \varepsilon_{1i}} \text{ if } \lambda_i > 1 \right\}. \tag{B42}$$

Then $\partial\Omega_1 \subset U\, \partial\theta_i$. Let

$$h_i\colon \Delta h_i = 0, \qquad h_i = \varphi/\partial\theta_i. \tag{B43}$$

$$h\colon \Delta h = 0, \qquad h = \varphi/\partial\Omega_1. \tag{B44}$$

Then $h = \varphi - \varphi_1$ where φ_1 is the orthogonal projection on $H_0^1(\Omega_1)$ of φ and $h_i = \varphi - \tilde{\varphi}_i$, where $\tilde{\varphi}_i$ is the orthogonal projection on $H_0^1(\theta_i)$ of φ. The above arguments show that

$$\int_{\theta_i} \delta_1^5 h_i \leqslant C\sqrt{\varepsilon_{1i}} \left(\int |\nabla\varphi|^2 \right)^{1/2} \tag{B45}$$

$$\int_{\mathbb{R}^3 - \theta_i} \delta_1^5 h_i \leqslant C\sqrt{\varepsilon_{1i}} \left(\int |\nabla\varphi|^2 \right)^{1/2}. \tag{B46}$$

Relations (B45) and (B46) imply the general case. Q.E.D.

Appendix C

We state here a result similar to Theorem 1 which can be derived on (S^4, c). (S^4, c) is the standard euclidian sphere.

We want to solve

$$\begin{aligned} -Lu = -\Delta u + 2u &= \tfrac{1}{6} K u^3 \\ u > 0 \qquad &\text{on} \quad S^4, \end{aligned} \tag{C1}$$

where Δ is the Laplace operator on $S^4 = \{x \in \mathbf{R}^5 \text{ s.t. } |x| = 1\}$. $x_1, ..., x_m$ are the critical points of K; k_i is the Morse index of K at x_i.

Let

$$\bar{c} = \frac{32768}{27} \pi^4 \tag{C2}$$

and let, for $\tau = (i_1, ..., i_l)$, $1 \leqslant i_1 \leqslant i_l \leqslant m$, $M(\tau)$ be the symmetric $l \times l$ matrix defined by

$$\begin{aligned} M_{qq}(\tau) &= -LK(x_{i_q}), \qquad 1 \leqslant q \leqslant l \\ M_{qr}(\tau) &= -\bar{c} \frac{1}{|x_{i_q} - x_{i_r}|^2} \left(\frac{1}{K(x_{i_q}) K(x_{i_r})} \right)^{1/2}, \end{aligned} \tag{C3}$$

where $|x-y|$ is the euclidean distance between x and y in $\mathbf{R}^5$. Let then $J(u)$ be

$$J(u) = \frac{3}{2} \frac{1}{\int Ku^4}. \tag{C4}$$

We make the following assumptions on K:

(A) for any τ, $M(\tau)$ is nondegenerate

(B) $K > 0$

(C) there is no polynomial Q with integer coefficients such that $(1+t) Q(t) - 1 = \sum_{\tau,\, M(\tau) > 0} \prod_{j=1}^{l(\tau)} t^{5-k_{i_j}}$.

Under (A), (B), (C), we have:

THEOREM 1′. *Equation* (C1) *has a solution* $u > 0$.

In this framework, some phenomena are different.

In particular, Lemma 5 does not hold and we need to make a more detailed study of the dynamical system of (99). The analog of Proposition 3 is used in a crucial way at this step, in particular in order to derive the differential equation satisfied by $\dot{x}_i$. Some of the details are available in [10]. The critical values at infinity are now the values

$$c(\tau) = \frac{1}{4 \int \delta^4} \sum_{j=1}^{l(c)} \frac{1}{K(x_{i_j})}. \tag{C5}$$

For the sake of simplicity, we assume $c(\tau) \neq c(\tau')$ if $\tau \neq \tau'$.

In this new framework, the difference of topology at the crossing of the level $c(\tau)$ is given by the formula

$$H_q(J_{c(\tau)+\varepsilon}, J_{c(\tau)-\varepsilon}) = \begin{cases} O & \text{for } q \neq k(\tau) \\ G & \text{for } q = k(\tau) \end{cases} \tag{C6}$$

with $k(\tau) = 5l(\tau) - \sum_{j=1}^{l(\tau)} k_{i_j}$.

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