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Jordan higher all-derivable points in triangular algebras ☆

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ABSTRACT

Let $\mathcal{T}$ be a triangular algebra. We say that $D = \{D_n : n \in N\} \subseteq L(\mathcal{T})$ is a Jordan higher derivable mapping at G if $D_n(ST + TS) = \sum_{i+j=n}(D_i(S)D_j(T) + D_i(T)D_j(S))$ for any $S, T \in \mathcal{T}$ with $ST = G$. An element $G \in \mathcal{T}$ is called a Jordan higher all-derivable point of $\mathcal{T}$ if every Jordan higher derivable linear mapping $D = \{D_n\}_{n \in N}$ at G is a higher derivation. In this paper, under some mild conditions on $\mathcal{T}$, we prove that some elements of $\mathcal{T}$ are Jordan higher all-derivable points. This extends some results in [6] to the case of Jordan higher derivations.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a ring (or algebra) with the unit I. An additive or linear mapping δ from $\mathcal{A}$ into itself is called a derivation if $\delta(ST) = \delta(S)T + S\delta(T)$ for any $S, T \in \mathcal{A}$ and is said to be a Jordan derivation if $\delta(ST + TS) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S)$ for any $S, T \in \mathcal{A}$. We say that a mapping δ is Jordan derivable at a given point $G \in \mathcal{A}$ if $\delta(ST + TS) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S)$ for any $S, T \in \mathcal{A}$ with $ST = G$, and G is called a Jordan all-derivable point of $\mathcal{A}$ if every Jordan derivable mapping at G is a derivation. We say that $D = \{D_n\} \subseteq L(\mathcal{A})$ is a Jordan higher derivable mapping at G if $D_n(ST + TS) = \sum_{i+j=n}(D_i(S)D_j(T) + D_i(T)D_j(S))$ for any $S, T \in \mathcal{A}$ with $ST = G$. An element $G \in \mathcal{A}$ is called a Jordan higher all-derivable point of $\mathcal{A}$ if every Jordan higher derivable linear mapping $D = \{D_n\}$ at G is a higher derivation. There have been a number of papers on the study of conditions under which derivations of operator algebras can be completely determined by the action on some sets of operators. In [3], Jing showed that I is a Jordan all-derivable point of $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ is a Hilbert space. In [7], Zhu proved that every invertible operator in nest algebra is an all-derivable point in the

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strong operator topology. Also it was showed that every element in the algebra of all upper triangular matrices is a Jordan all-derivable point by Sha and Zhu [6].

With the development of derivation, higher derivation has attracted much attention of mathematicians as an active subject of research in algebras. In [4] Xiao and Wei showed that any Jordan higher derivation on a triangular algebra is a higher derivation. In this paper we will extend the conclusion of [6] to the case of Jordan higher derivations. For other relative reference, see [1,2,5].

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital rings (or algebras) with the unit I_1, I_2, and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithfull as a left $\mathcal{A}$-module and as a right $\mathcal{B}$-module. The ring (or algebra)

$$\mathcal{T} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\},$$

under the usual matrix operations is said to be a triangular algebra. We mainly proved that 0 and $\begin{bmatrix} I_1 & X_0 \\ 0 & I_2 \end{bmatrix}$ are Jordan higher all-derivable points for any given point $X_0 \in \mathcal{M}$.

2. Jordan higher all-derivable points in ring algebras

In this section, we always assume that the characteristics of $\mathcal{A}$ and $\mathcal{B}$ are not 2 and 3, and for any $X \in \mathcal{A}$, $Y \in \mathcal{B}$, there are some integers n_1, n_2 such that $n_1 I_1 - X$ and $n_2 I_2 - Y$ are invertible. The following two theorems are the main results in this paper.

Theorem 2.1. *Let $D = (D_n)_{n \in N}$ be a family of additive or linear mappings on $\mathcal{T}$ with $D_0 = id_{\mathcal{T}}$ (identical mapping on $\mathcal{T}$). If D is Jordan higher derivable at* 0*, then D is a higher derivation.*

Proof. For any $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}$, we can write

$$D_n\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} \delta_n^{11}(X) + \varphi_n^{11}(Y) + \tau_n^{11}(Z) & \delta_n^{12}(X) + \varphi_n^{12}(Y) + \tau_n^{12}(Z) \\ 0 & \delta_n^{22}(X) + \varphi_n^{22}(Y) + \tau_n^{22}(Z) \end{bmatrix},$$

where $\delta_n^{ij} : \mathcal{A} \to \mathcal{A}_{ij}$, $\varphi_n^{ij} : \mathcal{M} \to \mathcal{A}_{ij}$, $\tau_n^{ij} : \mathcal{B} \to \mathcal{A}_{ij}$, $1 \leqslant i \leqslant j \leqslant 2$ are additive maps with $\mathcal{A}_{11} = \mathcal{A}$, $\mathcal{A}_{12} = \mathcal{M}$, $\mathcal{A}_{22} = \mathcal{B}$. It follows from the fact $D_0 = id_{\mathcal{T}}$ that when $i = j = 1$, $\delta_0^{ij} = id_{\mathcal{A}}$, else $\delta_0^{ij} = 0$; when $i = 1, j = 2$, $\varphi_0^{ij} = id_{\mathcal{M}}$, else $\varphi_0^{ij} = 0$; when $i = j = 2$, $\tau_0^{ij} = id_{\mathcal{B}}$, else $\tau_0^{ij} = 0$.

We set $S = \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ for every $X \in \mathcal{A}, W \in \mathcal{M}$. Then $ST = 0$ and $TS = \begin{bmatrix} 0 & XW \\ 0 & 0 \end{bmatrix}$. So

$$\begin{aligned} \begin{bmatrix} \varphi_n^{11}(XW) & \varphi_n^{12}(XW) \\ 0 & \varphi_n^{22}(XW) \end{bmatrix} &= D_n(ST + TS) = \sum_{i+j=n} (D_i(S)D_j(T) + D_i(T)D_j(S)) \\ &= \sum_{i+j=n} \left(\begin{bmatrix} \varphi_i^{11}(W) & \varphi_i^{12}(W) \\ 0 & \varphi_i^{22}(W) \end{bmatrix} \begin{bmatrix} \delta_j^{11}(X) & \delta_j^{12}(X) \\ 0 & \delta_j^{22}(X) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \delta_i^{11}(X) & \delta_i^{12}(X) \\ 0 & \delta_i^{22}(X) \end{bmatrix} \begin{bmatrix} \varphi_j^{11}(W) & \varphi_j^{12}(W) \\ 0 & \varphi_j^{22}(W) \end{bmatrix} \right) \end{aligned}$$

$$= \sum_{i+j=n} \begin{bmatrix} \varphi_i^{11}(W)\delta_j^{11}(X) + \delta_i^{11}(X)\varphi_j^{11}(W) & \varphi_i^{11}(W)\delta_j^{12}(X) + \delta_i^{11}(X)\varphi_j^{12}(W) \\ & +\varphi_i^{12}(W)\delta_j^{22}(X) + \delta_i^{12}(X)\varphi_j^{22}(W) \\ 0 & \varphi_i^{22}(W)\delta_j^{22}(X) + \delta_i^{22}(X)\varphi_j^{22}(W) \end{bmatrix}.$$

This implies that

$$\varphi_n^{11}(XW) = \sum_{i+j=n} (\varphi_i^{11}(W)\delta_j^{11}(X) + \delta_i^{11}(X)\varphi_j^{11}(W)), \tag{1}$$

$$\varphi_n^{12}(XW) = \sum_{i+j=n} (\varphi_i^{11}(W)\delta_j^{12}(X) + \delta_i^{11}(X)\varphi_j^{12}(W) + \varphi_i^{12}(W)\delta_j^{22}(X) + \delta_i^{12}(X)\varphi_j^{22}(W)), \tag{2}$$

and

$$\varphi_n^{22}(XW) = \sum_{i+j=n} (\varphi_i^{22}(W)\delta_j^{22}(X) + \delta_i^{22}(X)\varphi_j^{22}(W)) \tag{3}$$

for any $X \in \mathcal{A}$, $W \in \mathcal{M}$. One obtains that

$$\varphi_n^{11}(W) = \sum_{i+j=n} (\varphi_i^{11}(W)\delta_j^{11}(I_1) + \delta_i^{11}(I_1)\varphi_j^{11}(W)), \tag{4}$$

$$\varphi_n^{22}(W) = \sum_{i+j=n} (\varphi_i^{22}(W)\delta_j^{22}(I_1) + \delta_i^{22}(I_1)\varphi_j^{22}(W)) \tag{5}$$

by taking $X = I_1$ in Eqs. (1) and (3). Now we prove the fact that $\varphi_n^{11}(W) = 0$ and $\varphi_n^{22}(W) = 0$ by induction on n. When $n = 0$, it is easily verified that $\varphi_0^{11}(W) = 0$ and $\varphi_0^{22}(W) = 0$ from the characterizations of φ_0^{11} and φ_0^{22}. When $n = 1$, $\varphi_1^{11}(W) = 0$ and $\varphi_1^{22}(W) = 0$ can be obtained by the proof in [6, Theorem 2.1]. We assume that $\varphi_m^{11}(W) = 0$ and $\varphi_m^{22}(W) = 0$ for all $1 \leqslant m < n$. In fact, by the Eq. (4) and $\delta_0^{11} = id_{\mathcal{A}}$, we have $\varphi_n^{11}(W) = \varphi_n^{11}(W) + \varphi_n^{11}(W) = 2\varphi_n^{11}(W)$. Thus $\varphi_n^{11}(W) = 0$. Similarly combining Eq. (5) with the fact that $\delta_0^{22}(I_1) = 0$, we can get $\varphi_n^{22}(W) = 0$ for any $W \in M$ and $n \in N$. For any $X \in \mathcal{A}$, $W \in \mathcal{M}$ and $Y \in \mathcal{B}$, setting $S = \begin{bmatrix} 0 & W \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$, then $ST = 0$, $TS = \begin{bmatrix} 0 & XW \\ 0 & 0 \end{bmatrix}$. One gets

$$\begin{aligned} \begin{bmatrix} 0 & \varphi_n^{12}(XW) \\ 0 & 0 \end{bmatrix} &= D_n(ST + TS) = \sum_{i+j=n} (D_i(S)D_j(T) + D_i(T)D_j(S)) \\ &= \sum_{i+j=n} \left(\begin{bmatrix} \tau_i^{11}(Y) & \varphi_i^{12}(W) + \tau_i^{12}(Y) \\ 0 & \tau_i^{22}(Y) \end{bmatrix} \begin{bmatrix} \delta_j^{11}(X) & \delta_j^{12}(X) \\ 0 & \delta_j^{22}(X) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \delta_i^{11}(X) & \delta_i^{12}(X) \\ 0 & \delta_i^{22}(X) \end{bmatrix} \begin{bmatrix} \tau_j^{11}(Y) & \varphi_j^{12}(W) + \tau_j^{12}(Y) \\ 0 & \tau_j^{22}(Y) \end{bmatrix} \right). \end{aligned}$$

Hence the following three equations hold

$$\sum_{i+j=n} (\tau_i^{11}(Y)\delta_j^{11}(X) + \delta_i^{11}(X)\tau_j^{11}(Y)) = 0, \tag{6}$$

$$\sum_{i+j=n}(\tau_i^{22}(Y)\delta_j^{22}(X)+\delta_i^{22}(X)\tau_j^{22}(Y))=0, \tag{7}$$

$$\begin{aligned}\varphi_n^{12}(XW) &= \sum_{i+j=n}(\tau_i^{11}(Y)\delta_j^{12}(X)+\varphi_i^{12}(W)\delta_j^{22}(X)+\tau_i^{12}(Y)\delta_j^{22}(X)\\ &\quad +\delta_i^{11}(X)\varphi_j^{12}(W)+\delta_i^{11}(X)\tau_j^{12}(Y)+\delta_i^{12}(X)\tau_j^{22}(Y))\end{aligned} \tag{8}$$

for any $X\in\mathcal{A}, W\in\mathcal{M}$. One can see that

$$\sum_{i+j=n}(\tau_i^{11}(Y)\delta_j^{11}(I_1)+\delta_i^{11}(I_1)\tau_j^{11}(Y))=0 \tag{9}$$

by taking $X=I_1$ in Eq. (6). Using Eq. (9) and induction, one has $\tau_n^{11}(Y)=0$ for every $n\in N$. Similarly taking $Y=I_2$ in Eq. (7), by inducting and using the fact that $\tau_0^{22}(Y)=Y$, we get $\delta_n^{22}(X)=0$ for every $n\in N$ and $X\in\mathcal{A}$.

We can obtain that

$$\sum_{i+j=n}(\delta_i^{11}(X)\tau_j^{12}(Y)+\delta_i^{12}(X)\tau_j^{22}(Y))=0 \tag{10}$$

by $\delta_i^{22}(X)=0$, $\tau_i^{11}(Y)=0$ and taking $W=0$ in Eq. (8).

By Eq. (2) and the fact that $\delta_n^{22}(X)=0$, $\varphi_n^{11}(W)=0$, $\varphi_n^{22}(W)=0$ and $\varphi_0^{12}=id_{\mathcal{M}}$, we have

$$\varphi_n^{12}(XW)=\sum_{i+j=n}\delta_i^{11}(X)\varphi_j^{12}(W). \tag{11}$$

We claim that $\delta=\{\delta_n^{11}:n\in N\}$ is a higher derivation on $\mathcal{A}$. In fact, we know that δ_1^{11} is a derivation by [6, Theorem 2.1]. It follows that $\delta_1^{11}(X_1X_2)=\delta_1^{11}(X_1)X_2+X_1\delta_1^{11}(X_2)$ for any X_1, X_2 in $\mathcal{A}$. Now we assume that $\delta_m^{11}(X_1X_2)=\sum_{i+j=m}\delta_i^{11}(X_1)\delta_j^{11}(X_2)$ for any $1\leqslant m<n$ with $m\in N$. Summing up Eq. (11) and $\varphi_0^{12}=id_M$, we get

$$\begin{aligned}\varphi_n^{12}(X_1(X_2W)) &= \sum_{i+j=n}\delta_i^{11}(X_1)\varphi_j^{12}(X_2W)\\ &= \sum_{i+e=n}\delta_i^{11}(X_1)\delta_e^{11}(X_2)W+\sum_{i+e+k=n,k>0}\delta_i^{11}(X_1)\delta_e^{11}(X_2)\varphi_k^{12}(W)\end{aligned} \tag{12}$$

for any $X_1, X_2\in\mathcal{A}$ and $W\in\mathcal{M}$. On the other hand

$$\begin{aligned}\varphi_n^{12}((X_1X_2)W) &= \sum_{i+j=n,j>0}\delta_i^{11}(X_1X_2)\varphi_j^{12}(W)+\delta_n^{11}(X_1X_2)W\\ &= \sum_{e+k+j=n,j>0}\delta_e^{11}(X_1)\delta_k^{11}(X_2)\varphi_j^{12}(W)+\delta_n^{11}(X_1X_2)W\end{aligned} \tag{13}$$

for any $X_1, X_2\in\mathcal{A}$ and $W\in\mathcal{M}$. Combining Eq. (12) with Eq. (13), we get $\left[\delta_n^{11}(X_1X_2)-\sum_{e+i=n}\delta_i^{11}(X_1)\delta_e^{11}(X_2)\right]W=0$. Since M is faithful, we get $\delta_n^{11}(X_1X_2)=\sum_{i+j=n}\delta_i^{11}(X_1)\delta_j^{11}(X_2)$, i.e. $\delta=\{\delta_n^{11}:n\in N\}$ is a higher derivation.

Letting $S=\begin{bmatrix}0 & -X^{-1}WY\\ 0 & Y\end{bmatrix}$ and $T=\begin{bmatrix}X & W\\ 0 & 0\end{bmatrix}$ for any $Y\in\mathcal{B}, W\in\mathcal{M}$, and invertible $X\in\mathcal{A}$. Then $ST=TS=0$. So we get

$$\begin{aligned}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= D_n(ST+TS) = \sum_{i+j=n}(D_i(S)D_j(T)+D_i(T)D_j(S)) \\ &= \sum_{i+j=n}\left(\begin{bmatrix} 0 & -\varphi_i^{12}(X^{-1}WY)+\tau_i^{12}(Y) \\ 0 & \tau_i^{22}(Y) \end{bmatrix}\begin{bmatrix} \delta_j^{11}(X) & \delta_j^{12}(X)+\varphi_j^{12}(W) \\ 0 & 0 \end{bmatrix}\right. \\ &\quad \left. + \begin{bmatrix} \delta_i^{11}(X) & \delta_i^{12}(X)+\varphi_i^{12}(W) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\varphi_j^{12}(X^{-1}WY)+\tau_j^{12}(Y) \\ 0 & \tau_j^{22}(Y) \end{bmatrix}\right).\end{aligned}$$

The above equation implies that

$$0 = \sum_{i+j=n}[\delta_i^{11}(X)(-\varphi_j^{12}(X^{-1}WY)+\tau_j^{12}(Y))+(\delta_i^{12}(X)+\varphi_i^{12}(W))\tau_j^{22}(Y)]. \tag{14}$$

By replacing W by λW in the above equation, dividing the equation by λ and letting $\lambda \to +\infty$, we obtain that

$$0 = \sum_{i+j=n}[-\delta_i^{11}(X)\varphi_j^{12}(X^{-1}WY)+\varphi_i^{12}(W)\tau_j^{22}(Y)]. \tag{15}$$

So we can get

$$0 = \sum_{i+j=n}[-\delta_i^{11}(I_1)\varphi_j^{12}(WY)+\varphi_i^{12}(W)\tau_j^{22}(Y)] \tag{16}$$

by setting $X = I_1$ in the above equation. Since $\delta = \{\delta_n^{11} : n \in N\}$ is a higher derivation, $\delta_n^{11}(I_1) = 0$ when $n \geqslant 1$. It follows from Eq. (16) that

$$\varphi_n^{12}(WY) = \sum_{i+j=n}\varphi_i^{12}(W)\tau_j^{22}(Y). \tag{17}$$

We claim that $\tau = \{\tau_n^{22} : n \in N\}$ is a higher derivation on $\mathcal{B}$. In fact, by the proof of [6, Theorem 2.1] we know that τ_1^{22} is a derivation. This implies that $\tau_1^{22}(Y_1Y_2) = \tau_1^{22}(Y_1)Y_2 + Y_1\tau_1^{22}(Y_2)$ for any $Y_1, Y_2 \in \mathcal{B}$. We now assume that $\tau_m^{22}(Y_1Y_2) = \sum_{i+j=m}\tau_i^{22}(Y_1)\tau_j^{22}(Y_2)$ for all $1 \leqslant m < n$ with $m \in N$. It follows from Eq. (17) that

$$\begin{aligned}\varphi_n^{12}(WY_1Y_2) &= \varphi_n^{12}(W(Y_1Y_2)) \\ &= W\tau_n^{22}(Y_1Y_2) + \sum_{i+j=n, j<n}\varphi_i^{12}(W)\tau_j^{22}(Y_1Y_2) \\ &= W\tau_n^{22}(Y_1Y_2) + \sum_{i+e+k=n, i>0}\varphi_i^{12}(W)\tau_e^{22}(Y_1)\tau_k^{22}(Y_2)\end{aligned} \tag{18}$$

for any $Y_1, Y_2 \in \mathcal{B}$ and $W \in \mathcal{M}$. On the other hand by Eq. (17) and the fact that $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{B})$-bimodule, we have

$$\begin{aligned}\varphi_n^{12}(WY_1Y_2) &= \varphi_n^{12}((WY_1)Y_2) \\ &= \sum_{i+j=n}\varphi_i^{12}(WY_1)\tau_j^{22}(Y_2) = \sum_{e+k+j=n}\varphi_e^{12}(W)\tau_k^{22}(Y_1)\tau_j^{22}(Y_2) \\ &= W\sum_{k+j=n}\tau_k^{22}(Y_1)\tau_j^{22}(Y_2) + \sum_{e+k+j=n, e>0}\varphi_e^{12}(W)\tau_k^{22}(Y_1)\tau_j^{22}(Y_2).\end{aligned} \tag{19}$$

Combining Eq. (18) with Eq. (19), we get $W[\tau_n^{22}(Y_1Y_2) - \sum_{k+j=n}\tau_k^{22}(Y_1)\tau_j^{22}(Y_2)]W = 0$. Since M is faithful, we get $\tau_n^{22}(Y_1Y_2) = \sum_{i+j=n}\tau_i^{22}(Y_1)\tau_j^{22}(Y_2)$.

Now we prove that $(D_n)_{n\in N}$ is a higher derivation. For any $S = \begin{bmatrix} X_1 & W_1 \\ 0 & Y_1 \end{bmatrix}, T = \begin{bmatrix} X_2 & W_2 \\ 0 & Y_2 \end{bmatrix} \in \mathcal{T}$, where $X_1, X_2 \in \mathcal{A}$, $W_1, W_2 \in \mathcal{M}$ and $Y_1, Y_2 \in \mathcal{B}$. Summing up the above results and using the definition of D_n, we obtain that

$$\begin{aligned} D_n(ST) &= D_n\left(\begin{bmatrix} X_1X_2 & X_1W_2 + W_1Y_2 \\ 0 & Y_1Y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \delta_n^{11}(X_1X_2) & \delta_n^{12}(X_1X_2) + \varphi_n^{12}(X_1W_2 + W_1Y_2) + \tau_n^{12}(Y_1Y_2) \\ 0 & \tau_n^{22}(Y_1Y_2) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{i+j=n} D_i(S)D_j(T) \\ &= \sum_{i+j=n}\left(\begin{bmatrix} \delta_i^{11}(X_1) & \delta_i^{12}(X_1) + \varphi_i^{12}(W_1) + \tau_i^{12}(Y_1) \\ 0 & \tau_i^{22}(Y_1) \end{bmatrix}\right. \\ &\quad \left.\times \begin{bmatrix} \delta_j^{11}(X_2) & \delta_j^{12}(X_2) + \varphi_j^{12}(W_2) + \tau_j^{12}(Y_2) \\ 0 & \tau_j^{22}(Y_2) \end{bmatrix}\right) \\ &= \begin{bmatrix} \delta_n^{11}(X_1X_2) & \begin{array}{c} \sum_{i+j=n} (\delta_i^{11}(X_1)\delta_j^{12}(X_2) + \delta_i^{11}(X_1)\tau_j^{12}(Y_2) + \delta_i^{12}(X_1)\tau_j^{22}(Y_2) \\ +\tau_i^{12}(Y_1)\tau_j^{22}(Y_2)) + \varphi_n^{12}(X_1W_2 + W_1Y_2) \end{array} \\ 0 & \tau_n^{22}(Y_1Y_2) \end{bmatrix} \end{aligned}$$

by Eq. (17) and the fact that both δ and τ are higher derivations. So D is a higher derivations if and only if the equation

$$\begin{aligned} &\delta_n^{12}(X_1X_2) + \varphi_n^{12}(X_1W_2 + W_1Y_2) + \tau_n^{12}(Y_1Y_2) \\ &\quad = \sum_{i+j=n} (\delta_i^{11}(X_1)\delta_j^{12}(X_2) + \delta_i^{11}(X_1)\tau_j^{12}(Y_2) + \delta_i^{12}(X_1)\tau_j^{22}(Y_2) + \tau_i^{12}(Y_1)\tau_j^{22}(Y_2)) \\ &\quad\quad + \varphi_n^{12}(X_1W_2 + W_1Y_2) \end{aligned}$$

holds.

We get that $\tau_n^{22}(I_2) = 0 (n \geqslant 1)$ from [4, Lemma 2.2]. So we can write

$$\delta_n^{12}(X) = -\sum_{i+j=n} \delta_i^{11}(X)\tau_j^{12}(I_2)$$

by setting $Y = I_2$ in Eq. (10). Letting $X = I_1$ in the above equation, one gets $\delta_n^{12}(I_1) = -\tau_n^{12}(I_2)$. So

$$\delta_n^{12}(X) = \sum_{i+j=n} \delta_i^{11}(X)\delta_j^{12}(I_1). \tag{20}$$

Similarly by taking $X = I_1$ in Eq. (10) and noting the fact $\delta_n^{11}(I_1) = 0 (n \geqslant 1)$, we have

$$\tau_n^{12}(Y) = -\sum_{i+j=n} \delta_i^{12}(I_1)\tau_j^{22}(Y). \tag{21}$$

Thus it follows from Eq. (20) and Eq. (21) that

$$\begin{aligned}\delta_n^{12}(X_1X_2)+\tau_n^{12}(Y_1Y_2) &= \sum_{i+j=n}\delta_i^{11}(X_1X_2)\delta_j^{12}(I_1)-\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y_1Y_2)\\ &= \sum_{k+l+j=n}\delta_k^{11}(X_1)\delta_l^{11}(X_2)\delta_j^{12}(I_1)-\sum_{i+k+l=n}\delta_i^{12}(I_1)\tau_k^{22}(Y_1)\tau_l^{22}(Y_2).\end{aligned} \tag{22}$$

On the other hand

$$\begin{aligned}&\sum_{i+j=n}(\delta_i^{11}(X_1)\delta_j^{12}(X_2)+\delta_i^{11}(X_1)\tau_j^{12}(Y_2)+\delta_i^{12}(X_1)\tau_j^{22}(Y_2)+\tau_i^{12}(Y_1)\tau_j^{22}(Y_2))\\ &\quad=\sum_{i+j=n}\sum_{k+l=j}\delta_i^{11}(X_1)\delta_k^{11}(X_2)\delta_l^{12}(I_1)-\sum_{i+j=n}\sum_{k+l=j}\delta_i^{11}(X_1)\delta_k^{12}(I_1)\tau_l^{22}(Y_2)\\ &\quad\quad+\sum_{i+j=n}\sum_{k+l=i}\delta_k^{11}(X_1)\delta_l^{12}(I_1)\tau_j^{22}(Y_2)-\sum_{i+j=n}\sum_{k+l=i}\delta_k^{11}(I_1)\tau_l^{22}(Y_1)\tau_j^{22}(Y_2)\\ &\quad=\sum_{i+k+l=n}\delta_i^{11}(X_k)\delta_k^{11}(X_2)\delta_l^{12}(I_1)-\sum_{j+k+l=n}\delta_k^{12}(I_1)\tau_l^{22}(Y_1)\tau_j^{22}(Y_2).\end{aligned} \tag{23}$$

Thus combining Eq. (22) with Eq. (23), we arrive at

$$\begin{aligned}&\delta_n^{12}(X_1X_2)+\varphi_n^{12}(X_1W_2+W_1Y_2)+\tau_n^{12}(Y_1Y_2)\\ &\quad=\sum_{i+j=n}(\delta_i^{11}(X_1)\delta_j^{12}(X_2)+\delta_i^{11}(X_1)\tau_j^{12}(Y_2)+\delta_i^{12}(X_1)\tau_j^{22}(Y_2)\\ &\quad\quad+\tau_i^{12}(Y_1)\tau_j^{22}(Y_2))+\varphi_n^{12}(X_1W_2+W_1Y_2).\end{aligned}$$

Finally we obtain the desired result. □

Theorem 2.2. *Let $D=\{D_n\}$ be a family of additive or linear mappings on $\mathcal{T}$ with $D_0=id_{\mathcal{T}}$. If D is Jordan higher derivable at $G=\begin{bmatrix} I_1 & X_0\\ 0 & I_2\end{bmatrix}$, then D is a higher derivation.*

Proof. We set $S=\begin{bmatrix} X & 0\\ 0 & Y\end{bmatrix}$ and $T=\begin{bmatrix} X^{-1} & X^{-1}X_0\\ 0 & Y^{-1}\end{bmatrix}$ for every invertible element $X\in\mathcal{A}$ and $Y\in\mathcal{B}$. Then $ST=G$ and $TS=\begin{bmatrix} I_1 & X^{-1}X_0Y\\ 0 & I_2\end{bmatrix}$, so we obtain

$$\begin{aligned}&\begin{bmatrix} 2\delta_n^{11}(I_1)+2\tau_n^{11}(I_2)+\varphi_n^{11}(X_0+X^{-1}X_0Y) & 2\delta_n^{12}(I_1)+2\tau_n^{12}(I_2)+\varphi_n^{12}(X_0+X^{-1}X_0Y)\\ 0 & 2\delta_n^{22}(I_1)+\varphi_n^{22}(X_0+X^{-1}X_0Y)+2\tau_n^{22}(I_2)\end{bmatrix}\\ &\quad=D_n(ST+TS)=\sum_{i+j=n}(D_i(S)D_j(T)+D_i(T)D_j(S))\\ &=\sum_{i+j=n}\left(\begin{bmatrix} \delta_i^{11}(X)+\tau_i^{11}(Y) & \delta_i^{12}(X)+\tau_i^{12}(Y)\\ 0 & \delta_i^{22}(X)+\tau_i^{22}(Y)\end{bmatrix}\right.\end{aligned}$$

$$\times \begin{bmatrix} \delta_j^{11}(X^{-1}) + \varphi_j^{11}(X^{-1}X_0) & \delta_j^{12}(X^{-1}) + \varphi_j^{12}(X^{-1}X_0) \\ +\tau_j^{11}(Y^{-1}) & +\tau_j^{12}(Y^{-1}) \\ 0 & \delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0) + \tau_j^{22}(Y^{-1}) \end{bmatrix}$$

$$+ \begin{bmatrix} \delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0) & \delta_i^{12}(X^{-1}) + \varphi_i^{12}(X^{-1}X_0) \\ +\tau_i^{11}(Y^{-1}) & +\tau_i^{12}(Y^{-1}) \\ 0 & \delta_i^{22}(X^{-1}) + \varphi_i^{22}(X^{-1}X_0) + \tau_i^{22}(Y^{-1}) \end{bmatrix}$$

$$\times \left.\begin{bmatrix} \delta_j^{11}(X) + \tau_j^{11}(Y) & \delta_j^{12}(X) + \tau_j^{12}(Y) \\ 0 & \delta_j^{22}(X) + \tau_j^{22}(Y) \end{bmatrix}\right).$$

So according to the above matrix equation, we get

$$\begin{aligned} &2\delta_n^{11}(I_1) + 2\tau_n^{11}(I_2) + \varphi_n^{11}(X_0 + X^{-1}X_0Y) \\ &= \sum_{i+j=n} [(\delta_i^{11}(X) + \tau_i^{11}(Y))(\delta_j^{11}(X^{-1}) + \varphi_j^{11}(X^{-1}X_0) + \tau_j^{11}(Y^{-1})) \\ &\quad + (\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0) + \tau_i^{11}(Y^{-1}))(\delta_j^{11}(X) + \tau_j^{11}(Y))], \end{aligned} \tag{24}$$

$$\begin{aligned} &2\delta_n^{12}(I_1) + 2\tau_n^{12}(I_2) + \varphi_n^{12}(X_0 + X^{-1}X_0Y) \\ &= \sum_{i+j=n} [(\delta_i^{11}(X) + \tau_i^{11}(Y))(\delta_j^{12}(X^{-1}) + \varphi_j^{12}(X^{-1}X_0) + \tau_j^{12}(Y^{-1})) \\ &\quad + (\delta_i^{12}(X) + \tau_i^{12}(Y))(\delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0) + \tau_j^{22}(Y^{-1})) \\ &\quad + (\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0) + \tau_i^{11}(Y^{-1}))(\delta_j^{12}(X) + \tau_j^{12}(Y)) \\ &\quad + (\delta_i^{12}(X^{-1}) + \varphi_i^{12}(X^{-1}X_0) + \tau_i^{12}(Y^{-1}))(\delta_j^{22}(X) + \tau_j^{22}(Y))], \end{aligned} \tag{25}$$

$$\begin{aligned} &2\delta_n^{22}(I_1) + 2\tau_n^{22}(I_2) + \varphi_n^{22}(X_0 + X^{-1}X_0Y) \\ &= \sum_{i+j=n} [(\delta_i^{22}(X) + \tau_i^{22}(Y))(\delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0) + \tau_j^{22}(Y^{-1})) \\ &\quad + (\delta_i^{22}(X^{-1}) + \varphi_i^{22}(X^{-1}X_0) + \tau_i^{22}(Y^{-1}))(\delta_j^{22}(X) + \tau_j^{22}(Y))]. \end{aligned} \tag{26}$$

We claim that $\delta_n^{11}(I_1) = \tau_n^{11}(I_2) = \varphi_n^{11}(X_0) = 0$ when $n \geqslant 1$. In fact, we could obtain

$$\begin{aligned} &2\delta_n^{11}(I_1) + 2\tau_n^{11}(I_2) + \varphi_n^{11}(X_0 + X_0) \\ &= \sum_{i+j=n} [(\delta_i^{11}(I_1) + \tau_i^{11}(I_2))(\delta_j^{11}(I_1) + \varphi_j^{11}(X_0) + \tau_j^{11}(I_2)) \\ &\quad + (\delta_i^{11}(I_1) + \varphi_i^{11}(X_0) + \tau_i^{11}(I_2))(\delta_j^{11}(I_1) + \tau_j^{11}(I_2))] \end{aligned} \tag{27}$$

by setting $X = I_1$ and $Y = I_2$ in Eq. (24). When $n = 1$, the result that $\delta_1^{11}(I_1) = \tau_1^{11}(I_2) = \varphi_1^{11}(X_0) = 0$ holds according to the [6, Theorem 2.2]. So we assume that $\delta_m^{11}(I_1) = \tau_m^{11}(I_2) = \varphi_m^{11}(X_0) = 0$ for all $1 \leqslant m < n, m \in N$. Combining Eq. (27) with the fact $\delta_0^{11}(I_1) = I_1, \tau_0^{11}(I_2) = 0$ and using the induction hypothesis, we have

$$\begin{aligned} 2\delta_n^{11}(I_1) + 2\tau_n^{11}(I_2) + 2\varphi_n^{11}(X_0) &= \delta_n^{11}(I_1) + \tau_n^{11}(I_2) + \delta_n^{11}(I_1) + \tau_n^{11}(I_2) \\ &\quad + 2\delta_n^{11}(I_1) + 2\tau_n^{11}(I_2) + 2\varphi_n^{11}(X_0). \end{aligned}$$

Hence $\delta_n^{11}(I_1) + \tau_n^{11}(I_2) = 0(n \geqslant 1)$. Similarly we also can set that $X = I_1$ and $Y = -I_2$ in Eq. (24). Using the induction hypothesis, we get $\delta_n^{11}(I_1) - \tau_n^{11}(I_2) = -\varphi_n^{11}(X_0)$. Summing up the above equations we get $2\delta_n^{11}(I_1) = -2\tau_n^{11}(I_2) = \varphi_n^{11}(X_0)$. □

Setting $X = \frac{1}{2}I_1$ and $Y = I_2$ in Eq. (24) and using $\delta_n^{11}(I_1) + \tau_n^{11}(I_2) = 0$, we have

$$\begin{aligned} 3\varphi_n^{11}(X_0) = \sum_{i+j=n} \bigg[&\left(\frac{1}{2}\delta_i^{11}(I_1) + \tau_i^{11}(I_2)\right)(2\delta_j^{11}(I_1) + \tau_j^{11}(I_2) + 2\varphi_j^{11}(X_0)) \\ &+ (2\delta_i^{11}(I_1) + \tau_i^{11}(I_2) + 2\varphi_i^{11}(X_0))\left(\frac{1}{2}\delta_j^{11}(I_1) + \tau_j^{11}(I_2)\right)\bigg]. \end{aligned}$$

Thus combining $2\delta_n^{11}(I_1) = -2\tau_n^{11}(I_2) = \varphi_n^{11}(X_0)$ with the assumption and using $\delta_0^{11}(I_1) = I_1$, one obtains

$$\begin{aligned} 3\varphi_n^{11}(X_0) = &\frac{1}{2}(2\delta_n^{11}(I_1) + \tau_n^{11}(I_2) + 2\varphi_n^{11}(X_0)) + 2(\delta_n^{11}(I_1) + \tau_n^{11}(I_2)) + 2(\delta_n^{11}(I_1) + \tau_n^{11}(I_2)) \\ &+ \frac{1}{2}(2\delta_n^{11}(I_1) + \tau_n^{11}(I_2) + 2\varphi_n^{11}(X_0)). \end{aligned}$$

So $\varphi_n^{11}(X_0) = 4\delta_n^{11}(I_1) + 5\tau_n^{11}(I_2)$. We can claim that $\delta_n^{11}(I_1) = \tau_n^{11}(I_2) = \varphi_n^{11}(X_0) = 0$. Hence the Eq. (24) can be rewritten into

$$\begin{aligned} \varphi_n^{11}(X^{-1}X_0Y) = \sum_{i+j=n} [&(\delta_i^{11}(X) + \tau_i^{11}(Y))(\varphi_j^{11}(X^{-1}X_0) + \tau_j^{11}(Y^{-1}) + \delta_j^{11}(X^{-1})) \\ &+ (\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0) + \tau_i^{11}(Y^{-1}))(\delta_j^{11}(X) + \tau_j^{11}(Y))]. \end{aligned} \tag{28}$$

Similarly by setting $X = I_1$ and $Y = I_2$ in Eq. (26) and using the induction, we can get $\delta_n^{22}(I_1) + \tau_n^{22}(I_2) = 0$. We also can obtain $\delta_n^{22}(I_1) = \tau_n^{22}(I_2) = \varphi_n^{22}(X_0) = 0$ if we take $X = I_1$ and $Y = \frac{1}{2}I_2$ in Eq. (26). Thus

$$\begin{aligned} \varphi_n^{22}(X^{-1}X_0Y) = \sum_{i+j=n} [&(\delta_i^{22}(X) + \tau_i^{22}(Y))(\varphi_j^{22}(X^{-1}X_0) + \tau_j^{22}(Y^{-1}) + \delta_j^{22}(X^{-1})) \\ &+ (\delta_i^{22}(X^{-1}) + \varphi_i^{22}(X^{-1}X_0) + \tau_i^{22}(Y^{-1}))(\delta_j^{22}(X) + \tau_j^{22}(Y))]. \end{aligned} \tag{29}$$

We take $X = I_1$ and $Y = I_2$ in Eq. (25), then we can get $\delta_n^{12}(I_1) + \tau_n^{12}(I_2) = 0$. Letting respectively $Y = I_2$ and $Y = \frac{1}{2}I_2$ in Eq. (25) and using the above equation we have

$$\begin{aligned} \varphi_n^{12}(X_0 + X^{-1}X_0) = &\sum_{i+j=n} [\delta_i^{11}(X)(\delta_j^{12}(X^{-1}) + \varphi_j^{12}(X^{-1}X_0) + \tau_j^{12}(I_2)) \\ &+ (\delta_i^{12}(X) + \tau_i^{12}(I_2))(\delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0)) \\ &+ (\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0))(\delta_j^{12}(X) + \tau_j^{12}(I_2)) \\ &+ (\delta_i^{12}(X^{-1}) + \varphi_i^{12}(X^{-1}X_0) + \tau_i^{12}(I_2))\delta_j^{22}(X)] \\ &+ \delta_n^{12}(X) + \tau_n^{12}(I_2) + \delta_n^{12}(X^{-1}) + \varphi_n^{12}(X^{-1}X_0) + \tau_n^{12}(I_2), \end{aligned} \tag{30}$$

$$\begin{aligned} \varphi_n^{12}\left(X_0 + \frac{1}{2}X^{-1}X_0\right) = &\sum_{i+j=n} [\delta_i^{11}(X)(\delta_j^{12}(X^{-1}) + \varphi_j^{12}(X^{-1}X_0) + 2\tau_j^{12}(I_2)) \\ &+ (\delta_i^{12}(X) + \frac{1}{2}\tau_i^{12}(I_2))(\delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0)) \\ &+ (\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0))(\delta_j^{12}(X) + \frac{1}{2}\tau_j^{12}(I_2)) \end{aligned}$$

$$+ (\delta_i^{12}(X^{-1}) + \varphi_i^{12}(X^{-1}X_0) + 2\tau_i^{12}(I_2))\delta_j^{22}(X)]$$
$$+ 2\delta_n^{12}(X) + \tau_n^{12}(I_2) + \frac{1}{2}\delta_n^{12}(X^{-1}) + \frac{1}{2}\varphi_n^{12}(X^{-1}X_0) + \tau_n^{12}(I_2), \quad (31)$$

which implies that

$$\frac{1}{2}\varphi_n^{12}(X^{-1}X_0) = \sum_{i+j=n}[-\delta_i^{11}(X)\tau_j^{12}(I_2) + \frac{1}{2}\tau_i^{12}(I_2)(\delta_j^{22}(X^{-1}) + \varphi_j^{22}(X^{-1}X_0)) + \frac{1}{2}(\delta_i^{11}(X^{-1}) + \varphi_i^{11}(X^{-1}X_0))\tau_j^{12}(I_2) - \tau_i^{12}(I_2)\delta_j^{22}(X)] - \delta_n^{12}(X) + \frac{1}{2}\delta_n^{12}(X^{-1}) + \frac{1}{2}\varphi_n^{12}(X^{-1}X_0).$$

So

$$\frac{1}{2}\sum_{i+j=n}[\tau_i^{12}(I_2)\delta_j^{22}(X^{-1}) + \delta_i^{11}(X^{-1})\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\varphi_j^{22}(X^{-1}X_0) + \varphi_i^{11}(X^{-1}X_0)\tau_j^{12}(I_2)] + \frac{1}{2}\delta_n^{12}(X^{-1}) = \sum_{i+j=n}[\delta_i^{11}(X)\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\delta_j^{22}(X)] + \delta_n^{12}(X). \quad (32)$$

Thus we get

$$\frac{1}{2}\sum_{i+j=n}[\tau_i^{12}(I_2)\delta_j^{22}(X) + \delta_i^{11}(X)\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\varphi_j^{22}(XX_0) + \varphi_i^{11}(XX_0)\tau_j^{12}(I_2)] + \frac{1}{2}\delta_n^{12}(X) = \sum_{i+j=n}[\delta_i^{11}(X^{-1})\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\delta_j^{22}(X^{-1})] + \delta_n^{12}(X^{-1}) \quad (33)$$

for any invertible $X \in \mathcal{A}$ by replacing X^{-1} by X in Eq. (32). It follows that

$$\frac{1}{2}\left[\frac{1}{2}\sum_{i+j=n}[\tau_i^{12}(I_2)\delta_j^{22}(X) + \delta_i^{11}(X)\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\varphi_j^{22}(XX_0) + \varphi_i^{11}(XX_0)\tau_j^{12}(I_2)] + \frac{1}{2}\delta_n^{12}(X)\right] + \frac{1}{2}\sum_{i+j=n}[\tau_i^{12}(I_2)\varphi_j^{22}(X^{-1}X_0) + \varphi_i^{11}(X^{-1}X_0)\tau_j^{12}(I_2)] = \sum_{i+j=n}[\delta_i^{11}(X)\tau_j^{12}(I_2) + \tau_i^{12}(I_2)\delta_j^{22}(X)] + \delta_n^{12}(X).$$

So

$$\frac{1}{4}\left[\sum_{i+j=n}[\tau_i^{12}(I_2)\delta_j^{22}(X) + \delta_i^{11}(X)\tau_j^{12}(I_2)] + \delta_n^{12}(X)\right] + \frac{1}{4}\sum_{i+j=n}[\tau_i^{12}(I_2)\varphi_j^{22}(XX_0) + \varphi_i^{11}(XX_0)\tau_j^{12}(I_2)] + \frac{1}{2}\sum_{i+j=n}[\tau_i^{12}(I_2)\varphi_j^{22}(X^{-1}X_0) + \varphi_i^{11}(X^{-1}X_0)\tau_j^{12}(I_2)] = \sum_{i+j=n}[\tau_i^{12}(I_2)\delta_j^{22}(X) + \delta_i^{11}(X)\tau_j^{12}(I_2)] + \delta_n^{12}(X) \quad (34)$$

for any invertible $X \in \mathcal{A}$.

Similarly by letting $X = I_1$ and $X = 2I_1$ in Eq. (25), it is easily checked that

$$\begin{aligned}\varphi_n^{12}(X_0 + X_0Y) = &\sum_{i+j=n} [\tau_i^{11}(Y)(\varphi_j^{12}(X_0) + \tau_j^{12}(Y^{-1}) + \delta_j^{12}(I_1))\\ &+ (\delta_i^{12}(I_1) + \tau_i^{12}(Y))\tau_j^{22}(Y^{-1}) + \tau_i^{11}(Y^{-1})(\delta_j^{12}(I_1) + \tau_j^{12}(Y))\\ &+ (\varphi_i^{12}(X_0) + \tau_i^{12}(Y^{-1}) + \delta_i^{12}(I_1))\tau_j^{22}(Y)]\\ &+ \varphi_n^{12}(X_0) + \tau_n^{12}(Y^{-1}) + 2\delta_n^{12}(I_1) + \tau_n^{12}(Y), \end{aligned} \tag{35}$$

$$\begin{aligned}\varphi_n^{12}\left(X_0 + \frac{1}{2}X_0Y\right) = &\sum_{i+j=n} \left[\tau_i^{11}(Y)\left(\frac{1}{2}\varphi_j^{12}(X_0) + \tau_j^{12}(Y^{-1}) + \frac{1}{2}\delta_j^{12}(I_1)\right)\right.\\ &+ (2\delta_i^{12}(I_1) + \tau_i^{12}(Y))\tau_j^{22}(Y^{-1}) + \tau_i^{11}(Y^{-1})(2\delta_j^{12}(I_1) + \tau_j^{12}(Y))\\ &\left.+ \left(\frac{1}{2}\varphi_i^{12}(X_0) + \tau_i^{12}(Y^{-1}) + \frac{1}{2}\delta_i^{12}(I_1)\right)\tau_j^{22}(Y)\right]\\ &+ \varphi_n^{12}(X_0) + 2\tau_n^{12}(Y^{-1}) + 2\delta_n^{12}(I_1) + \frac{1}{2}\tau_n^{12}(Y), \end{aligned} \tag{36}$$

which implies that

$$\begin{aligned}\frac{1}{2}\varphi_n^{12}(X_0Y) = &\sum_{i+j=n} \left[\frac{1}{2}\tau_i^{11}(Y)(\varphi_j^{12}(X_0) + \delta_j^{12}(I_1)) - \delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})\right.\\ &\left.- \tau_i^{11}(Y^{-1})\delta_j^{12}(I_1) \quad + \frac{1}{2}(\varphi_i^{12}(X_0) + \delta_i^{12}(I_1))\tau_j^{22}(Y)\right]\\ &+ \frac{1}{2}\tau_n^{12}(Y) - \tau_n^{12}(Y^{-1}). \end{aligned} \tag{37}$$

By considering Eq. (28) and $\varphi_n^{11}(X_0) = 0$ and letting $X = I_1$ and $X = 2I_1$ respectively, it is easily verified that

$$\varphi_n^{11}(X_0Y) = \sum_{i+j=n} [\tau_i^{11}(Y)\tau_j^{11}(Y^{-1}) + \tau_i^{11}(Y^{-1})\tau_j^{11}(Y)] + 2\tau_n^{11}(Y^{-1}) + 2\tau_n^{11}(Y), \tag{38}$$

$$\frac{1}{2}\varphi_n^{11}(X_0Y) = \sum_{i+j=n} [\tau_i^{11}(Y)\tau_j^{11}(Y^{-1}) + \tau_i^{11}(Y^{-1})\tau_j^{11}(Y)] + 4\tau_n^{11}(Y^{-1}) + \tau_n^{11}(Y). \tag{39}$$

When $n = 0$, $\tau_0^{11}(Y) = 0$. When $n = 1$, $\tau_1^{11}(Y) = 0$ according to [6, Theorem 2.2]. We assume that $\tau_m^{11}(Y) = 0$ for any $Y \in \mathcal{B}$ and $1 \leqslant m < n$. So combining Eq. (38) with Eq. (39) and using the induction hypothesis, we have

$$\varphi_n^{11}(X_0Y) = 2\tau_n^{11}(Y^{-1}) + 2\tau_n^{11}(Y), \tag{40}$$

$$\frac{1}{2}\varphi_n^{11}(X_0Y) = 4\tau_n^{11}(Y^{-1}) + \tau_n^{11}(Y). \tag{41}$$

By direct computation, one can verify that $\tau_n^{11}(Y^{-1}) = 0$. There exists $n \in N$ such that $nI_2 - Y$ is invertible for any $Y \in \mathcal{B}$ and $\tau_n^{11}(I_2) = 0$, so $\tau_n^{11}(Y) = 0$ for any $Y \in \mathcal{B}$.

When $n = 0$, $\delta_0^{22}(X) = 0$ for any $X \in \mathcal{A}$. we have $\delta_1^{22}(X) = 0$ by [6, Theorem 2.2]. So now we assume that $\delta_m^{22}(X) = 0$ for all $1 \leqslant m < n$ and $X \in \mathcal{A}$. Taking respectively $Y = I_2$ and $Y = 2I_2$ in Eq. (29) and using $\tau_n^{22}(I_2) = 0$, $n \geqslant 1$, $\tau_0^{22} = id_{\mathcal{B}}$ we have

$$\varphi_n^{22}(X^{-1}X_0) = \sum_{i+j=n}[\delta_i^{22}(X)(\delta_j^{22}(X^{-1})+\varphi_j^{22}(X^{-1}X_0))+(\varphi_i^{22}(X^{-1}X_0)+\delta_i^{22}(X^{-1}))\delta_j^{22}(X)] + 2\delta_n^{22}(X)+2\varphi_n^{22}(X^{-1}X_0)+2\delta_n^{22}(X^{-1}), \tag{42}$$

and

$$2\varphi_n^{22}(X^{-1}X_0) = \sum_{i+j=n}[\delta_i^{22}(X)(\delta_j^{22}(X^{-1})+\varphi_j^{22}(X^{-1}X_0))+(\varphi_i^{22}(X^{-1}X_0)+\delta_i^{22}(X^{-1}))\delta_j^{22}(X)] + \delta_n^{22}(X)+4\varphi_n^{22}(X^{-1}X_0)+4\delta_n^{22}(X^{-1}). \tag{43}$$

Combining the assumption and the above equations, we have the following equations:

$$-\varphi_n^{22}(X^{-1}X_0) = 2\delta_n^{22}(X)+2\delta_n^{22}(X^{-1}),$$
$$-2\varphi_n^{22}(X^{-1}X_0) = \delta_n^{22}(X)+4\delta_n^{22}(X^{-1}).$$

By direct computation, one can verify that $\delta_n^{22}(X)=0$ for any invertible $X\in\mathcal{A}$ and $n\in N$. Because there is some integer n such that nI_1-X is invertible for every $X\in\mathcal{A}$, the conclusion of $\delta_n^{22}(X)=0$ holds for every $X\in\mathcal{A}$.

We set $S=\begin{bmatrix} X & XW \\ 0 & Y\end{bmatrix}$ and $T=\begin{bmatrix} X^{-1} & X^{-1}X_0-WY^{-1} \\ 0 & Y^{-1}\end{bmatrix}$ for any $Y\in\mathcal{B}$, $W\in\mathcal{M}$, and for any invertible $X\in\mathcal{A}$, then $ST=G$ and $TS=\begin{bmatrix} I_1 & X^{-1}X_0Y \\ 0 & I_2\end{bmatrix}$. So combining $\delta_n^{12}(I_1)+\tau_n^{12}(I_2)=0$ with the characterization of D, we obtain the following equation:

$$\begin{aligned}
&\begin{bmatrix} \varphi_n^{11}(X^{-1}X_0Y) & \varphi_n^{12}(X_0+X^{-1}X_0Y) \\ 0 & \varphi_n^{22}(X^{-1}X_0Y)\end{bmatrix} \\
&= D_n(ST+TS) = \sum_{i+j=n}(D_i(S)D_j(T)+D_i(T)D_j(S)) \\
&= \sum_{i+j=n}\left(\begin{bmatrix} \delta_i^{11}(X)+\varphi_i^{11}(XW) & \delta_i^{12}(X)+\varphi_i^{12}(XW)+\tau_i^{12}(Y) \\ 0 & \tau_i^{22}(Y)+\varphi_i^{22}(XW)\end{bmatrix}\right. \\
&\quad \begin{bmatrix} \delta_j^{11}(X^{-1})+\varphi_j^{11}(X^{-1}X_0-WY^{-1}) & \delta_j^{12}(X^{-1})+\varphi_j^{12}(X^{-1}X_0-WY^{-1})+\tau_j^{12}(Y^{-1}) \\ 0 & \tau_j^{22}(Y^{-1})+\varphi_j^{22}(X^{-1}X_0-WY^{-1})\end{bmatrix} \\
&\quad + \begin{bmatrix} \delta_i^{11}(X^{-1})+\varphi_i^{11}(X^{-1}X_0-WY^{-1}) & \delta_i^{12}(X^{-1})+\varphi_i^{12}(X^{-1}X_0-WY^{-1})+\tau_i^{12}(Y^{-1}) \\ 0 & \tau_i^{22}(Y^{-1})+\varphi_i^{22}(X^{-1}X_0-WY^{-1})\end{bmatrix} \\
&\quad \left.\begin{bmatrix} \delta_j^{11}(X)+\varphi_j^{11}(XW) & \delta_j^{12}(X)+\varphi_j^{12}(XW)+\tau_j^{12}(Y) \\ 0 & \tau_j^{22}(Y)+\varphi_j^{22}(XW)\end{bmatrix}\right),
\end{aligned}$$

which implies the following three equations:

$$\begin{aligned}
&\varphi_n^{11}(X^{-1}X_0Y) \\
&\quad = \sum_{i+j=n}[(\delta_i^{11}(X)+\varphi_i^{11}(XW))(\delta_j^{11}(X^{-1})+\varphi_j^{11}(X^{-1}X_0-WY^{-1})) \\
&\qquad + (\delta_i^{11}(X^{-1})+\varphi_i^{11}(X^{-1}X_0-WY^{-1}))(\delta_j^{11}(X)+\varphi_j^{11}(XW))],
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\varphi_n^{12}(X_0+X^{-1}X_0Y)\\
&\quad=\sum_{i+j=n}[(\delta_i^{11}(X)+\varphi_i^{11}(XW))(\delta_j^{12}(X^{-1})+\varphi_j^{12}(X^{-1}X_0-WY^{-1})+\tau_j^{12}(Y^{-1}))\\
&\qquad+(\delta_i^{12}(X)+\varphi_i^{12}(XW)+\tau_i^{12}(Y))(\tau_j^{22}(Y^{-1})+\varphi_j^{22}(X^{-1}X_0-WY^{-1}))\\
&\qquad+(\delta_i^{11}(X^{-1})+\varphi_i^{11}(X^{-1}X_0-WY^{-1}))(\delta_j^{12}(X)+\varphi_j^{12}(XW)+\tau_j^{12}(Y))\\
&\qquad+(\delta_i^{12}(X^{-1})+\varphi_i^{12}(X^{-1}X_0-WY^{-1})+\tau_i^{12}(Y^{-1}))(\tau_j^{22}(Y)+\varphi_j^{22}(XW))],
\end{aligned}\tag{45}$$

$$\begin{aligned}
&\varphi_n^{22}(X^{-1}X_0Y)\\
&\quad=\sum_{i+j=n}[(\tau_i^{22}(Y)+\varphi_i^{22}(XW))(\tau_j^{22}(Y^{-1})+\varphi_j^{22}(X^{-1}X_0-WY^{-1}))\\
&\qquad+(\tau_i^{22}(Y^{-1})+\varphi_i^{22}(X^{-1}X_0-WY^{-1}))(\tau_j^{22}(Y)+\varphi_j^{22}(XW))].
\end{aligned}\tag{46}$$

Now we take $X=2I_1$ and $Y=I_2$ in Eq. (44) and Eq. (46), it is checked that

$$\begin{aligned}
\frac{1}{2}\varphi_n^{11}(X_0)&=\sum_{i+j=n}\Big[(2\delta_i^{11}(I_1)+2\varphi_i^{11}(W))\left(\frac{1}{2}\delta_j^{11}(I_1)+\varphi_j^{11}\left(\frac{1}{2}X_0-W\right)\right)\\
&\qquad\left(\frac{1}{2}\delta_i^{11}(I_1)+\varphi_i^{11}\left(\frac{1}{2}X_0-W\right)\right)(2\delta_j^{11}(I_1)+2\varphi_j^{11}(W))\Big],\\
\frac{1}{2}\varphi_n^{22}(X_0)&=\sum_{i+j=n}\Big[(\tau_i^{22}(I_2)+2\varphi_i^{22}(W))(\tau_j^{22}(I_2)+\varphi_j^{22}\left(\frac{1}{2}X_0-W\right)\Big)\\
&\qquad+(\tau_i^{22}(I_2)+\varphi_i^{22}\left(\frac{1}{2}X_0-W\right)\Big)(\tau_j^{22}(I_2)+2\varphi_j^{22}(W))\Big].
\end{aligned}$$

By the fact that $\delta_n^{11}(I_1)=0(n\geqslant 1)$, $\tau_n^{22}(I_2)=0(n\geqslant 1)$ and $\varphi_n^{11}(X_0)=0$, $\varphi_n^{22}(X_0)=0$ for any $n\geqslant 0$, it follows that

$$\begin{aligned}
&0=2\varphi_n^{11}(W)+4\sum_{i+j=n}\varphi_i^{11}(W)\varphi_j^{11}(W),\\
&0=2\varphi_n^{22}(W)+4\sum_{i+j=n}\varphi_i^{22}(W)\varphi_j^{22}(W).
\end{aligned}$$

When $n=0$, $\varphi_0^{11}(W)=\varphi_0^{22}(W)=0$, When $n=1$, $\varphi_1^{11}(W)=\varphi_1^{22}(W)=0$, So we assume that $\varphi_m^{11}(W)=\varphi_m^{22}(W)=0$ for all $1\leqslant m<n$ and $W\in\mathcal{M}$. Combining the above equation with the assumption, we get that $\varphi_n^{11}(W)=\varphi_n^{22}(W)=0$ for all $1\leqslant m<n$.

By setting respectively $Y=\frac{1}{2}I_2$ and $Y=I_2$ in Eq. (45), the following two equations hold

$$\begin{aligned}
\varphi_n^{12}\left(X_0+\frac{1}{2}X^{-1}X_0\right)&=\sum_{i+j=n}\Big[\delta_i^{11}(X)(\delta_j^{12}(X^{-1})+\varphi_j^{12}(X^{-1}X_0-2W)+2\tau_j^{12}(I_2))\\
&\quad+\delta_i^{11}(X^{-1})\left(\delta_j^{12}(X)+\varphi_j^{12}(XW)+\frac{1}{2}\tau_j^{12}(I_2)\right)\Big]+2\delta_n^{12}(X)\\
&\quad+2\varphi_n^{12}(XW)+\tau_n^{12}(I_2)+\frac{1}{2}\delta_n^{11}(X^{-1})\\
&\quad+\frac{1}{2}\varphi_n^{12}(X^{-1}X_0-2W)+\tau_n^{12}(I_2),
\end{aligned}\tag{47}$$

$$\begin{aligned}\varphi_n^{12}(X_0+X^{-1}X_0) = \sum_{i+j=n}[\delta_i^{11}(X)(\delta_j^{12}(X^{-1})+\varphi_j^{12}(X^{-1}X_0-W)+\tau_j^{12}(I_2)) \\ +\delta_i^{11}(X^{-1})(\delta_j^{12}(X)+\varphi_j^{12}(XW)+\tau_j^{12}(I_2))]+\delta_n^{12}(X) \\ +\varphi_n^{12}(XW)+\tau_n^{12}(I_2)+\delta_n^{11}(X^{-1})+\varphi_n^{12}(X^{-1}X_0-W)+\tau_n^{12}(I_2). \end{aligned} \quad (48)$$

which implies that

$$\begin{aligned}-\frac{1}{2}\varphi_n^{12}(X^{-1}X_0) = \sum_{i+j=n}\left[-\delta_i^{11}(X)\varphi_j^{12}(W)+\delta_i^{11}(X)\tau_j^{12}(I_2)+\frac{1}{2}\delta_i^{11}(X^{-1})\tau_j^{12}(I_2)\right] \\ +\delta_n^{12}(X)+\varphi_n^{12}(XW)-\frac{1}{2}\delta_n^{11}(X^{-1})-\frac{1}{2}\varphi_n^{12}(X^{-1}X_0). \end{aligned} \quad (49)$$

It follows from Eq. (34) and the fact $\delta_n^{22}(X)=\varphi_n^{11}(W)=\varphi_n^{22}(W)=0$, we have

$$\delta_n^{12}(X) = -\sum_{i+j=n}\delta_i^{11}(X)\tau_j^{12}(I_2). \quad (50)$$

Hence combing Eq. (49) with Eq. (50), we can see that

$$\varphi_n^{12}(XW) = \sum_{i+j=n}\delta_i^{11}(X)\varphi_j^{12}(W)$$

for any invertible $X \in \mathcal{A}$. There exists some $n \in N$ such that $nI_1 - X$ is invertible for every $X \in \mathcal{A}$, one can check that

$$\varphi_n^{12}(XW) = \sum_{i+j=n}\delta_i^{11}(X)\varphi_j^{12}(W) \quad (51)$$

for any $X \in \mathcal{A}$.

Now we take respectively $X = I_1$ and $X = 2I_1$ in Eq. (45), one gets

$$\begin{aligned}\varphi_n^{12}(X_0+X_0Y) = \sum_{i+j=n}[(\delta_i^{12}(I_1)+\varphi_i^{12}(W)+\tau_i^{12}(Y))\tau_j^{22}(Y^{-1}) \\ +(\delta_i^{12}(I_1)+\varphi_i^{12}(X_0-WY^{-1})+\tau_i^{12}(Y^{-1}))\tau_j^{22}(Y)]+\delta_n^{12}(I_1) \\ +\varphi_n^{12}(X_0-WY^{-1})+\tau_n^{12}(Y^{-1})+\delta_n^{12}(I_1)+\tau_n^{12}(Y)+\varphi_n^{12}(W), \end{aligned} \quad (52)$$

$$\begin{aligned}\varphi_n^{12}(X_0+\frac{1}{2}X_0Y) = \sum_{i+j=n}[(2\delta_i^{12}(I_1)+2\varphi_i^{12}(W)+\tau_i^{12}(Y))\tau_j^{22}(Y^{-1}) \\ +\left(\frac{1}{2}\delta_i^{12}(I_1)+\varphi_i^{12}\left(\frac{1}{2}X_0-WY^{-1}\right)+\tau_i^{12}(Y^{-1})\right)\tau_j^{22}(Y)]+\delta_n^{12}(I_1) \\ +2\varphi_n^{12}\left(\frac{1}{2}X_0-WY^{-1}\right)+2\tau_n^{12}(Y^{-1})+\delta_n^{12}(I_1)+\frac{1}{2}\tau_n^{12}(Y)+\varphi_n^{12}(W), \end{aligned} \quad (53)$$

which implies that

$$\begin{aligned}\frac{1}{2}\varphi_n^{12}(X_0Y) = \sum_{i+j=n}\left[-(\delta_i^{12}(I_1)+\varphi_i^{12}(W))\tau_j^{22}(Y^{-1})+\frac{1}{2}(\delta_i^{12}(I_1)+\varphi_i^{12}(X_0))\tau_j^{22}(Y)\right] \\ +\varphi_n^{12}(WY^{-1})-\tau_n^{12}(Y^{-1})+\frac{1}{2}\tau_n^{12}(Y). \end{aligned} \quad (54)$$

Combining the above equation with Eq. (37) and the fact $\tau_n^{11}(Y)=0$, we get

$$\sum_{i+j=n}\left[-\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\frac{1}{2}\delta_i^{12}(I_1)\tau_j^{22}(Y)+\frac{1}{2}\varphi_i^{12}(X_0)\tau_j^{22}(Y)\right]+\frac{1}{2}\tau_n^{12}(Y)-\tau_n^{12}(Y^{-1})$$
$$=\sum_{i+j=n}\left[-\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\frac{1}{2}\delta_i^{12}(I_1)\tau_j^{22}(Y)+\frac{1}{2}\varphi_i^{12}(X_0)\tau_j^{22}(Y)\right]$$
$$-\sum_{i+j=n}\varphi_i^{12}(W)\tau_j^{22}(Y^{-1})+\frac{1}{2}\tau_n^{12}(Y)-\tau_n^{12}(Y^{-1})+\varphi_n^{12}(WY^{-1}). \tag{55}$$

So

$$\varphi_n^{12}(WY^{-1})=\sum_{i+j=n}\varphi_i^{12}(W)\tau_j^{22}(Y^{-1}). \tag{56}$$

Replacing Y by Y^{-1} in the above equation, we obtain for any invertible $Y\in\mathcal{B}$

$$\varphi_n^{12}(WY)=\sum_{i+j=n}\varphi_i^{12}(W)\tau_j^{22}(Y). \tag{57}$$

Since there is some integer n such that nI_2-Y is invertible for every $Y\in\mathcal{B}$, it is easy to see that Eq. (57) is true for every $Y\in\mathcal{B}$ and $W\in\mathcal{M}$, Summing up Eq. (54) and Eq. (56), Eq. (57), we obtain that

$$\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\tau_n^{12}(Y^{-1})=\frac{1}{2}\left[\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y)+\tau_n^{12}(Y)\right]. \tag{58}$$

Thus

$$\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y)+\tau_n^{12}(Y)=\frac{1}{2}\left[\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\tau_n^{12}(Y^{-1})\right] \tag{59}$$

by replacing Y^{-1} by Y in the Eq. (58). Combining Eq. (58) with Eq. (59), we can obtain

$$\frac{1}{2}\left[\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\tau_n^{12}(Y^{-1})\right]=2\left[\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y^{-1})+\tau_n^{12}(Y^{-1})\right].$$

So using the direct computation, we can claim that

$$\tau_n^{12}(Y)=-\sum_{i+j=n}\delta_i^{12}(I_1)\tau_j^{22}(Y). \tag{60}$$

Now summing up all the above equations and using similar arguments as that in the proof of Theorem 2.1, it is easily checked that both $\{\delta_n^{11}\}_{n\in N}$ and $\{\tau_n^{22}\}_{n\in N}$ are higher derivations. Therefore it is also an easy computation to see that $\{D_n\}_{n\in N}$ is a higher derivation. □

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