JOURNAL OF FUNCTIONAL ANALYSIS **66**, 105–122 (1986)

On Operators Preserving Commutativity

MATJAŽ OMLADIČ*

Department of Mathematics, E. K. University of Ljubljana, Jadranska 19, 61000 Ljubljana, Yugoslavia

Communicated by C. Foias

Received April 22, 1985; revised October 5, 1985

Let $L(X)$ be the algebra of all bounded operators on a non-trivial complex Banach space X and $F: L(X) \to L(X)$ a bijective linear operator such that F and F^{-1} both send commuting pairs of operators into commuting pairs. Then, either $F(A) = \sigma UAU^{-1} + p(A)\, I$, or $F(A) = \sigma UA'U^{-1} + p(A)\, I$, where p is a linear functional on $L(X)$, U is a bounded linear bijective operator between the appropriate two spaces, σ is a complex constant, and A' is the adjoint of A. The form of an operator F for which F and F^{-1} both send projections of rank one into projections of rank one is also determined.

1. INTRODUCTION

In this paper we study linear operators $F: L(X) \to L(X)$, where X is a non-trivial complex Banach space and $L(X)$ is the algebra of all bounded linear operators on X. We do not assume in general that F is bounded. What we assume is that F preserves some properties of bounded operators. We show that it must then be of a very special form.

We shall say that F preserves commutativity in both directions if for any two $A, B \in L(X)$ the operators $F(A)$ and $F(B)$ commute if and only if A and B do. The main result of this paper is theorem 1.1. The dual of X will be denoted by X' and the adjoint of $A \in L(X)$ by A' throughout.

THEOREM 1.1. *If the dimension of X is greater than 2 and $F: L(X) \to L(X)$ is bijective linear and preserves commutativity in both directions, then there is a linear functional p on $L(X)$, a non-zero complex number σ and either*

(a) *a bounded bijective linear $U: X \to X$ such that*

$$F(A) = \sigma UAU^{-1} + p(A)\, I$$

for every $A \in L(X)$; or

* This work was supported in part by the Research Council of Slovenia.

0022-1236/86 $3.00

(b) *a bounded bijective linear* $U: X' \to X$ *such that*

$$F(A) = \sigma U A' U^{-1} + p(A) I$$

for every $A \in L(X)$. *In this case,* X *is reflexive.*

Note that we do not assume continuity of the operator F, and as a consequence we get a functional p which need not be continuous either. However, U and U^{-1} must be bounded and F can get some discontinuity only through the functional p. Besides, note that we did not assume either multiplicativity, antimultiplicativity or any other property of this kind on F. Nevertheless, we get as a result that our F differs from an algebraic homomorphism or an algebraic antihomomorphism in a multiplicative constant and an additive operator of rank at most 1. The reason for this seems to be in the fact that preserving commutativity implies preserving commutants which are algebras. This fact will be used essentially in the proof. The commutant of a set $\mathcal{S} \subset L(X)$ which is by definition the set of all operators from $L(X)$ commuting with every operator from $\mathcal{S}$ will be denoted by $\mathcal{S}^{\sim}$ throughout. Finally, note that the converse of the theorem is almost trivial.

PROPOSITION 1.2. *Let* p *be a linear functional on* $L(X)$, *let* σ *be a nonzero constant with* $p(I) + \sigma \neq 0$, *let either*

(a) $U: X \to X$ *be a bijective bounded linear operator and define* $F(A) = \sigma U A U^{-1} + p(A) I$; *or*

(b) X *be reflexive,* $U: X' \to X$ *be a bijective bounded linear operator and define* $F(A) = \sigma U A' U^{-1} + p(A) I$.

Then, $F: L(X) \to L(X)$ *is linear bijective and preserves commutativity in both directions.*

Proof. (a) From $F(A) = 0$ we get $p(A)(\sigma + p(I)) = 0$ which forces $A = 0$. For any $B \in L(X)$ set

$$A = \frac{1}{\sigma}\left(U^{-1} B U - \frac{p(U^{-1} B U)}{\sigma + p(I)} I\right)$$

to get $F(A) = B$. The proof of (b) goes similarly.

The proof of the theorem will be given in Section 4, while in Section 3 some auxiliary results will be presented. It seems that the first result in the direction of our theorem was given by Watkins [9] for X of finite dimension greater than 3; in the same paper a counterexample was constructed in 2 dimensions. The case of dimension 3 was settled almost simultaneously by Pierce and Watkins [7] and by Beasley [1]. Extensions of these results to the case of symmetric matrices were given by Chan and Lim [3] and by Radjavi [8].

It seems that the first attack on the infinite dimensional case was made by Choi, Jafarian, and Radjavi in their nice paper [4], where X is assumed a Hilbert space and F an adjoint-preserving bijective linear operator on $L(X)$ which also preserves commutativity; the results obtained under these assumptions are analogous to ours. Also, some ideas used in the proof of our theorem are similar to those presented in [4]. However, note that in a Hilbert space a bounded operator can be represented as a matrix which makes the proof and even the formulation of this result closer to the finite dimensional case. In this paper we give the result in a general Banach space, where the transpose of a matrix is formally replaced by the adjoint operator. We have to make greater use of the commutativity relations to obtain this result. We also need a stronger assumption on F: it must preserve commutativity in both directions.

In Section 2 we give a result (Theorem 2.1) which is probably of some independent interest. But, a side result of this section (Proposition 2.6) will be needed in the proof of the main theorem. What we shall actually show in Section 2 is that every operator $F: L(X) \to L(X)$ which is continuous in the weak operator topology, linear bijective, and preserves the property of being a projection of rank 1 in both directions, is either of the form UAU^{-1} or of the form $UA'U^{-1}$, where U is a bijective bounded linear operator acting between the appropriate Banach spaces. It seems that the study of operators preserving the rank or some other characteristics of matrices was started in the papers [5 and 6]. A rather fresh reference [2] could help an interested reader fo find some further references on the problem. It seems that the problems of this kind are still almost untouched in infinite dimensions.

Throughout the paper, we will denote for any $x \in X$ and $f \in X'$ by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)\, y = f(y)\, x$ for $y \in X$. Note that every operator of rank 1 can be written in this form. The operator $x \otimes f$ is a projection if and only if $f(x) = 1$. Note that $(x \otimes f)' = f \otimes (\mathscr{K} x)$, where $\mathscr{K}$ is the natural embedding of X into X''. Recall also that every operator of finite rank is a linear combination of operators of rank 1.

2. Operators Preserving Projections of Rank 1

We shall say that an operator $F: L(X) \to L(X)$ preserves projections of rank 1 in both directions, if for every $A \in L(X)$ the operator $F(A)$ is a projection of rank 1 if and only if A is a projection of rank 1.

Theorem 2.1. *Let $F: L(X) \to L(X)$ be linear, bijective, and continuous in* *the weak operator topology. Besides, let F preserve projections of rank* 1 *in both directions. Then either*

(a) *there is a bounded bijective linear operator $U: X \to X$ such that $F(A) = UAU^{-1}$; or*

(b) *there is a bounded bijective linear operator $U: X' \to X$ such that $F(A) = UA'U^{-1}$. In this case the space X must be reflexive.*

The proof of Theorem 2.1 will be given through a series of lemmas. Throughout this section we shall suppose that all the assumptions of the theorem, except for the continuity of F in the weak operator topology, are in effect.

LEMMA 2.2. *If for some linearly independent vectors $x, y \in X$ and a non-zero functional $f \in X'$ it holds that $f(x) = f(y) = 1$, then there are $u, v \in X$, $\varphi, \psi \in X'$ with $\varphi(u) = \psi(v) = 1$, $\varphi(v)\,\psi(u) = 1$, $F(x \otimes f) = u \otimes \varphi$, and $F(y \otimes f) = v \otimes \psi$, such that either*

(a) *the vectors u and v are linearly independent and $\psi = \psi(u)\,\varphi$; or*

(b) *the functionals φ, ψ are linearly independent and $v = \varphi(v)\,u$.*

Proof. Since $f(x) = f(y) = 1$, the operators $P = x \otimes f$ and $Q = y \otimes f$ are projections of rank 1. Hence, by the assumption on F both $F(P)$ and $F(Q)$ are projections of rank 1, thus $F(P) = u \otimes \varphi$, $F(Q) = v \otimes \psi$ for some $u, v \in X$, $\varphi, \psi \in X'$ such that $\varphi(u) = \psi(v) = 1$. From the definition of P and Q we get for any λ that $\lambda P + (1-\lambda)\,Q$ is also a projection of rank 1. Hence $\lambda F(P) + (1-\lambda)\ F(Q) = (\lambda F(P) + (1-\lambda)\ F(Q))^2$ which gives $F(P)\ F(Q) + F(Q)\ F(P) = F(P) + F(Q)$ and from this it follows that

$$u \otimes (\varphi - \varphi(v)\,\psi) + v \otimes (\psi - \psi(u)\,\varphi) = 0. \tag{1}$$

Suppose u and v are linearly independent, then (1) gives $\varphi = \varphi(v)\,\psi$, $\psi = \psi(u)\,\varphi$, and since φ and ψ are non-trivial, we also get $\varphi(v)\,\psi(u) = 1$ and (a) holds. In the case φ and ψ are linearly independent, we can rewrite (1) in the form

$$(u - \psi(u)\,v) \otimes \varphi + (v - \varphi(v)\,u) \otimes \psi = 0$$

which implies $u = \psi(u)\ v$, $v = \varphi(v)\ u$. But, u and v are non-trivial, hence $\varphi(v)\ \psi(u) = 1$ and (b) holds. It remains to consider the case when neither (a) nor (b) are true. Then both pairs u, v and φ, ψ are linearly dependent which forces $F(P)$ and $F(Q)$ to be linearly dependent, contrary to the fact that P and Q are linearly independent and that F is injective.

LEMMA 2.3. *If for a non-zero vector $x \in X$ and some linearly independent* *functionals* $f, g \in X'$ *it holds that* $f(x) = g(x) = 1$, *then there are* $u, v \in X$, $\varphi, \psi \in X'$ *with* $\varphi(u) = \psi(v) = 1$, $\varphi(v)\ \ \psi(u) = 1$, $F(x \otimes f) = u \otimes \varphi$, *and* $F(x \otimes g) = v \otimes \psi$, *such that either*

(a) *the vectors* u *and* v *are linearly independent and* $\psi = \psi(u)\, \varphi$; *or*

(b) *the functionals* φ, ψ *are linearly independent and* $v = \varphi(v)\, u$.

Proof. Define $P = x \otimes f$ and $Q = x \otimes g$ to reach the same situation as in the proof of the Lemma 2.2. Naturally, this must yield the same conclusion.

LEMMA 2.4. *For every* $f_0 \in X'$ *there exists either*

(a) *a bijective linear operator* $U: X \to X$ *and a functional* $\varphi_0 \in X'$ *such that for every* $x \in X$,

$$F(x \otimes f_0) = (Ux) \otimes \varphi_0;$$

or

(b) *a bijective linear operator* $V: X \to X'$ *and a vector* $u_0 \in X$ *such that for every* $x \in X$,

$$F(x \otimes f_0) = u_0 \otimes (Vx).$$

Proof. Fix any $x_0 \in X$ with $f_0(x_0) = 1$ and write $F(x_0 \otimes f_0) = u_0 \otimes \varphi_0$, then $\varphi_0(u_0) = 1$. Now, choose a vector $x \in X$, linearly independent of x_0 and such that $f_0(x) = 1$. Then, use Lemma 2.2 for x_0, x, and f. In the first place, assume that the case (a) of that lemma holds. Then, we can take $F(x \otimes f_0) = v \otimes \varphi_0$, where v and u_0 are linearly independent and $\varphi_0(v) = 1$. If an additional vector $y \in X$ can be found which does not depend linearly on either of the vectors x_0 or x, then we can use Lemma 2.2 for x_0, y and f_0. If the case (b) occurred this time, we would get $F(y \otimes f_0) = u_0 \otimes \varphi$, φ being linearly independent of φ_0 which whould lead to a contradiction with Lemma 2.2 used for x, y, and f_0. Thus, the case (a) of the Lemma 2.2 holds. We have proved in this way that for every $x \in X$ with $f_0(x) = 1$ there is a unique $v \in X$ with $\varphi_0(v) = 1$ and $F(x \otimes f_0) = v \otimes \varphi_0$. For every $x \in X$ of this kind define $Ux = v$. For $x \in X$ with $f_0(x) \neq 0$ put $Ux = f_0(x)\, U(f_0(x)^{-1} x)$ to get $F(x \otimes f_0) = (Ux) \otimes \varphi_0$ again. Finally, if $f_0(x) = 0$, define $Ux = U(x + x_0) - Ux_0$ to get $F(x \otimes f_0) = (Ux) \otimes \varphi_0$ for every $x \in X$. It is then easy to see that U is linear and injective. To complete the proof of case (a) of this lemma we only need to verify surjectivity of the operator U. If the image of U has dimension 1, then there is nothing to prove. If the rank of U is greater than 1, choose any two linearly independent vectors y_1 and y_2 in the image of U. If there is a vector $y \in X$ which is not in the image of U, apply Lemma 2.2 to the vectors y_1, y, φ_0 and to the vectors y_2, y, φ_0

respectively, using the operator F^{-1} instead of F in both cases, to reach a contradiction.

Let us now return to the point of this proof when Lemma 2.2 was used for the first time. If the case (b) of that lemma occurred at that point, we could use similar arguments as above to get the case (b) of this lemma.

LEMMA 2.5. *Let a functional $f_0 \in X'$ be given. Then there exists either*

(a) *a bijective, linear operator $U: X \to X$ and for every $x \in X$ with $f_0(x) = 1$ a bijective, linear operator $V: X' \to X'$ such that for every $f \in X'$ it holds that $(Vf)(Ux) = f(x)$ and $F(x \otimes f) = (Ux) \otimes (Vf)$; or*

(b) *a bijective, linear operator $V: X \to X'$ and for every $x \in X$ with $f_0(x) = 1$ a bijective, linear operator $U: X' \to X$ such that for every $f \in X'$ it holds that $(Vx)(Uf) = f(x)$ and $F(x \otimes f) = (Uf) \otimes (Vx)$.*

Proof. In the first place assume that for the functional $f_0 \in X'$ we are in the case (a) of Lemma 2.4. Fix a vector $x_0 \in X$ with $f_0(x_0) = 1$, choose any further $f \in X'$ with $f(x_0) = 1$, linearly independent of f_0 (note that if this is not possible, there is nothing to prove) and apply Lemma 2.3 to the vectors x_0, f_0, and f. We shall see that the case (b) of that lemma must hold. Suppose, on the contrary, that (a) holds. Then, $F(x_0 \otimes f_0) = Ux_0 \otimes \varphi_0$, $F(x_0 \otimes f) = u \otimes \varphi_0$, and the vectors u and Ux_0 are linearly independent. But, for any $x \in X$ such that x and x_0 are linearly independent we have $F(x \otimes f_0) = (Ux) \otimes \varphi_0$. Since $x_0 \otimes f$ and $x \otimes f_0$ are linearly independent operators, so must be $u \otimes \varphi_0$ and $Ux \otimes \varphi_0$, hence u and Ux are linearly independent. Applying Lemma 2.2 to the vectors u, Ux, φ_0 with operator F^{-1} instead of F we get that either x and x_0 are linearly dependent or f and f_0 are which is in contradiction with the above assumptions. Therefore, we must be in the case (b) of Lemma 2.3 and so $F(x_0 \otimes f) = (Ux_0) \otimes \varphi$ for a functional $\varphi \in X'$. Define $Vf = \varphi$ for $f \in X'$ with $f(x_0) = 1$. Choose now $f \in X'$ such that $f(x_0) \neq 0$ and define $Vf = f(x_0)\ V(f(x_0)^{-1} f)$. Then we have

$$F(x_0 \otimes f) = Ux_0 \otimes Vf \tag{2}$$

for every $f \in X'$ with $f(x_0) \neq 0$. For $f \in X'$ with $f(x_0) = 0$ define $Vf = V(f + f_0) - Vf_0$ to get that (2) holds for every $f \in X'$. We can then use similar arguments as in the proof of Lemma 2.4 to see that the operator V is linear and bijective. Moreover, we get $(Vf)(Ux_0) = f(x_0)$, first for every $f \in X'$ with $f(x_0) = 1$, then for $f \in X'$ with $f(x_0) \neq 0$ and finally for every $f \in X'$.

In the second place assume that for the functional $f_0 \in X'$ we are in the case (b) of Lemma 2.4. Choose $x_0 \in X$ and then $f \in X'$ such that $f(x_0) =$

$f_0(x_0)=1$ and f, f_0 are linearly independent (if possible). Using similar arguments as above, we can see that the case (a) of Lemma 2.3, applied to x_0, f_0, and f, must hold. Define $Uf=u$ and extend this operator U by linearity to the whole space X' to get a bijective, linear operator U from X' to X for which $F(x_0 \otimes f)=(Uf)\otimes(Vx_0)$ holds for every $f \in X'$ and moreover, $(Vx_0)(Uf)=f(x_0)$.

PROPOSITION 2.6. *There exists either*

(a) *a bijective, bounded, linear operator $U: X \to X$ such that*

$$F(A)=UAU^{-1}$$

for every $A \in L(X)$ of finite rank; or

(b) *a bijective, bounded, linear operator $U: X' \to X$ such that*

$$F(A)=UA'U^{-1}$$

for every $A \in L(X)$ of finite rank. In this case the space X must be reflexive.

Proof. Suppose that for a functional $f_0 \in X'$ we are in the case (a) of Lemma 2.5 and choose linearly independent vectors x_1, $x_2 \in X$ with $f_0(x_1)=f_0(x_2)=1$. Let V_1, $V_2: X' \to X'$ be such that for every $f \in X'$ we have

$$F(x_1 \otimes f)=(Ux_1)\otimes(V_1 f),$$
$$F(x_2 \otimes f)=(Ux_2)\otimes(V_2 f).$$

Since the operator U is injective, the vectors Ux_1 and Ux_2 are linearly independent, therefore the functionals $V_1 f$ and $V_2 f$ are linearly dependent by Lemma 2.2. Fix a non-zero $f \in X'$ to get $V_2 f=\lambda V_1 f$ for a non-zero complex number λ. Similarly, for a complex number α, different from zero and from one, we get $F((\alpha x_1+(1-\alpha)x_2)\otimes f)=U(\alpha x_1+(1-\alpha)\ x_2)\otimes V_3 f$ which implies $V_3 f=\mu V_1 f$ for a non-zero complex number μ. Thence,

$$U((\mu\alpha-\alpha)x_1+(\mu-\lambda)(1-\alpha)x_2)\otimes V_1 f=0$$

which yields $\lambda=\mu=1$ and $V_3=V_2=V_1$. Thus, the operator V in case (a) of Lemma 2.5 does not depend on the choice of $x \in X$ for which $f_0(x)=1$. We can now use the linearity of operator F to obtain the equality $F(x \otimes f)=Ux \otimes Vf$ valid for every $x \in X$ and for every $f \in X'$. From Lemma 2.5 we know that $(Vf)(Ux)=f(x)$ for all $x \in X, f \in X'$, which shows that V is the adjoint of U^{-1}. This forces V to be bounded which implies that U^{-1} and finally U is bounded. Besides, for all $x, y \in X$ and $f \in X'$ we have

$$F(x \otimes f)\, y = (Ux \otimes Vf)\, y = Ux((Vf)(y))$$
$$= Uxf(U^{-1}y) = U(x \otimes f)\, U^{-1}y$$

which proves (a) for every A of rank 1. The general case now follows by the linearity of F.

Suppose now that for a functional $f_0 \in X'$ we are in the case (b) of Lemma 2.5 and choose linearly independent vectors x_1, $x_2 \in X$ such that $f_0(x_1) = f_0(x_2) = 1$. Take the operators U_1, U_2: $X' \to X$ such that for every $f \in X'$ we have

$$F(x_1 \otimes f) = (U_1 f) \otimes (Vx_1),$$
$$F(x_2 \otimes f) = (U_2 f) \otimes (Vx_2).$$

Using arguments as above we see that $U_1 = U_2 = U$ does not depend on the choice of $x \in X$. Therefore, for all $x \in X$, $f \in X'$ we have $F(x \otimes f) = (Uf) \otimes (Vx)$ and, moreover, $V(x)(Uf) = f(x)$. Let $\mathscr{K}$ be the natural embedding of X into X''. Then, V' is defined at least on the image of $\mathscr{K}$ and coincides there with $U^{-1}\mathscr{K}^{-1}$. Thus, U^{-1} is closed and therefore bounded. Besides, the operator $(U^{-1})'$: $X'' \to X'$ is bounded and so is $V = (U^{-1})'\, \mathscr{K}$. But, the operators V and $(U^{-1})'$ and therefore also $\mathscr{K}$ are bijections which implies the reflexivity of X. In this way we get for every $x, y \in X$ and $f \in X'$

$$F(x \otimes f)\, y = ((Uf) \otimes (Vx))\, y = (Uf)((Vx)(y))$$
$$= (Uf)((U^{-1}y)(x)) = U(f \otimes x)\, U^{-1}y$$
$$= U(x \otimes f)'\, U^{-1}y$$

which proves (b) for every A of rank 1. The general case then follows by linearity.

Proof of the Theorem 2.1. Suppose that F satisfies the above assumptions and is also continuous in the weak operator topology. Note that operators of finite rank are a dense subset of $L(X)$ in the weak operator topology to get the theorem as an immediate consequence of the Proposition 2.6.

3. Some Auxiliary Results

In this section we give some results needed in Section 4. The most important among them is Proposition 3.3 which represents the first step in the proof of the main theorem 1.1. We start with a result which is probably well known, but we give here a proof for completeness.

LEMMA 3.1. *The second commutant of an operator with more than one point in its spectrum has dimension* 2 *if and only if the operator is of the form* $\alpha P + \beta I$, *where* P *is a non-trivial projection and* α, β *are complex numbers with* $\alpha \neq 0$.

Proof. Let P be a projection with image Y and kernel Z. Then $A \in \{P\}^{\sim}$ if and only if A leaves the subspaces Y and Z invariant. It follows that every operator $A \in \{P\}^{\sim\sim} \subset \{P\}^{\sim}$ must be of the form $A = \alpha P + \beta I$ for some complex numbers α, β. On the other hand, suppose that for an operator $A \in L(X)$, $\{A\}^{\sim\sim}$ has dimension 2. Since A is not a scalar multiple of the identity operator, A and I form a basis of the second commutant of A. This implies that $(A - \beta)^2 = \alpha(A - \beta)$ for some complex numbers α, β. If $\alpha = 0$, A has only one point in the spectrum, contrary to the assumption. Therefore, $\alpha \neq 0$, $P = (A - \beta)/\alpha$ is a non-trivial projection and $A = \alpha P + \beta I$.

The next somewhat technical result will be needed in the proof of the Proposition 3.3.

LEMMA 3.2. *Let* $P \neq Q$ *be two non-trivial, commuting projections.*

(a) *Either* P *or* $I - P$ *is of rank* 1 *and either* Q *or* $I - Q$ *is of rank* 1 *if and only if the subspace*

$$\mathscr{C} = \{P\}^{\sim} + \{Q\}^{\sim}$$

of $L(X)$ *has codimension* 2 *in* $L(X)$.

(b) *If* P *and* Q *are of rank* 1, *then there are nilpotents* U *and* V *of rank* 1 *with* $P = UV$, $Q = VU$ *such that* $L(X)$ *is a direct sum of* $\mathscr{C}$ *and the linear span of the operators* U *and* V.

Proof. Define F, G: $L(X) \to L(X)$ by $F(A) = PA(I - P) + (I - P)AP$ and $G(A) = QA(I - Q) + (I - Q)AQ$. Note that F and G are commuting projections on $L(X)$ and that the kernel of F equals $\{P\}^{\sim}$, while the kernel of G equals $\{Q\}^{\sim}$ which yields that $\mathscr{C}$ equals the kernel of FG. Assume that $\mathscr{C}$ has codimension 2; then the image of FG has dimension 2. Define

$$R_1 = PQ, \quad R_2 = P(I - Q), \quad R_3 = (I - P)Q, \quad R_4 = (I - P)(I - Q)$$

to get four disjoint projections on X with sum I. A straightforward computation gives that A belongs to the image of FG if and only if

$$A = R_1 A R_4 + R_2 A R_3 + R_3 A R_2 + R_4 A R_1.$$

We shall see that at least one of the four projections R_1, R_2, R_3, and R_4 is zero. Suppose on the contrary that all of them are non-zero and choose non-trivial vectors x_i from the image of R_i and non-trivial functionals f_i from the image of R_i' for $i=1, 2, 3, 4$. The operators $A_i = x_i \otimes f_{5-i}$, defined for $i=1, 2, 3, 4$ are then linearly independent members of the image of FG, contradicting the fact that its dimension is 2. With no loss of generality we suppose that $R_1 = PQ = 0$. Now, assume that P has rank greater than 1 and choose linearly independent vectors x_1, x_2 from the image of P and x_3 from the image of Q. Next, choose linearly independent functionals f_1, f_2 from the image of P' and f_3 from the image of Q'. Define the operators

$$B_1 = x_1 \otimes f_3, \quad B_2 = x_2 \otimes f_3, \quad B_3 = x_3 \otimes f_1, \quad B_4 = x_3 \otimes f_2,$$

to get four linearly independent operators from the image of FG. Thence the rank of P is 1 and similarly the rank of Q is 1.

On the other hand, suppose that $P = x \otimes f$ and $Q = y \otimes g$ are projections of rank 1. Since they commute, we have necessarily $PQ = QP = 0$. Set $U = x \otimes g$, $V = y \otimes f$. For every $A \in L(X)$ write

$$B = PAP + (I-P)\,A(I-P),$$

$$C = (I-P-Q)\,AP + PA(I-P-Q),$$

and

$$D = PAQ + QAP = f(Ay)\,U + g(Ax)\,V,$$

to get $A = B + C + D$, where $B \in \{P\}^{\sim}$ and $C \in \{Q\}^{\sim}$. We have thus seen that $L(X)$ is the sum of $\mathscr{C}$ and the linear span of U and V. To see that this sum is direct, take any $B \in \{P\}^{\sim}$ and $C \in \{Q\}^{\sim}$ such that $B + C = \alpha U + \beta V$ for some complex numbers α, β. Then

$$\alpha U = P(\alpha U + \beta V)\,Q = PBQ + PCQ = 0$$

and similarly $\beta V = Q(\alpha U + \beta V)\,P = 0$ and the lemma follows.

We shall suppose from now on that the dimension of X is greater than 2.

PROPOSITION 3.3. *If F: $L(X) \to L(X)$ is bijective, linear, and preserves commutativity in both directions, then for every two disjoint projections P and Q of rank 1 there are disjoint projections R and S of rank 1 and complex numbers α, δ, σ with $\sigma \neq 0$ such that*

$$F(P) = \sigma R + \alpha I \qquad \text{and} \qquad F(Q) = \sigma S + \delta I.$$

Proof. The proof will be given in a few steps. In the beginning write the two projections in the form $P = x \otimes f$, $Q = y \otimes g$, where $x, y \in X$ and

$f, g \in X'$ are such that $f(x) = g(y) = 1$ and $f(y) = g(x) = 0$. Next, define $U = x \otimes g$, $V = y \otimes f$, $A = F(P)$, $B = F(U)$, $C = F(V)$, and $D = F(Q)$. Note that the second commutant of the set $\{P, Q, U, V\}$ is the algebra spanned by P, Q, U, V, and I. Since F preserves commutativity in both directions, the second commutant $\mathscr{A}$ of the set $\{A, B, C, D\}$ is spanned by A, B, C, D, and I. Since F is injective, these five operators are linearly independent and form a basis of $\mathscr{A}$. By Lemma 3.1 the second commutants of A and of D respectively have dimension 2. Similarly the second commutants of B and of C respectively have dimension 3. Choose now any points α, β, γ, and δ respectively from the spectra of operators A, B, C, and D.

STEP I. *$A - \alpha$ and $D - \delta$ cannot both be nilpotent.*

Proof. Through the proof of this step we shall assume with no loss of generality that all of the numbers $\alpha, \beta, \gamma, \delta$ are equal to zero. Assume to the contrary of Step I that $A^2 = D^2 = 0$. Then $G = A + D$ is a sum of two commuting nilpotents which implies that G is nilpotent. Since $P + Q$ is in the center of the algebra $\{P, Q, U, V\}^{\sim\sim}$, the operator G is in the center of $\mathscr{A}$. But, this center has dimension 2 which implies $G^2 = 0$. Since the commutant of U in the algebra $\{P, Q, U, V\}^{\sim\sim}$ is spanned by I, $P + Q$, and U, the commutant of B in $\mathscr{A}$ must be spanned by I, G, and B. From this we get $B^2 = \varphi B + \psi G + \nu I$, for some complex numbers φ, ψ, ν. If $\nu \neq 0$, the operator $\psi G + \nu I$ is invertible and so is $B(B - \varphi)$ contrary to the assumption that 0 is in the spectrum of B. Therefore $\nu = 0$ and $B^2 = \varphi B + \psi G$. Define $H = BA + AB - \varphi A$ to get $AH = ABA = HA$ and $BH = BAB = HB$. This implies that $H = \mu G + \nu I$. But $HG = 0$ and therefore $\nu = 0$, which gives also $HD = 0$. Thus $ABD = 0$ and similarly, after interchanging B and C we get also $ACD = 0$. Now, take any $W \in L(X)$ to get, by Lemma 3.2,

$$F(W) = Y + Z + \tau B + \omega C,$$

where Y commutes with A, Z commutes with D, and τ, ω are complex numbers. Then

$$AF(W)\,D = YAD + ADZ + \tau ABD + \omega ACD = 0.$$

Recall now that F is surjective and that A, D are not scalar multiples of identity. Hence, we can choose $W \in L(X)$ such that $F(W) = z \otimes h$, where $z \in X$ is an arbitrary vector which does not belong to the kernel of A and $h \in X'$ is an arbitrary functional which does not belong to the kernel of D'. With this choice we get $AF(W)\,D \neq 0$, contrary to the above results which completes the proof of this step.

STEP II. *$A - \alpha$ and $D - \delta$ are scalar multiples of some projections.*

Proof. To simplify the notations assume again that α, β, γ, δ are all zero. In the first place assume $A^2=0$, then we must have $D^2=\varphi D+\psi I$ for some complex numbers φ, ψ. Since 0 is in the spectrum of D, $\psi=0$. A short computation gives $G^2(G-\varphi)^2=0$. Since G is in the center of $\mathscr{A}$ which is of dimension 2 we must have either $G^2=0$ or $G(G-\varphi)=0$. The first possibility leads directly to $D^2=0$, the second gives $0=(A+D)(A+D-\varphi)D=\varphi AD$ which again forces $D^2=0$. Similarly, the supposition $D^2=0$ yields $A^2=0$. In this way we obtain by Step I that both A^2 and D^2 are non-zero. Consequently, A and D are scalar multiples of non-trivial projections.

Proceed now with the proof of the proposition. Let ρ and σ be non-zero complex numbers such that $R=(A-\alpha)/\rho$ and $S=(D-\delta)/\sigma$ are projections, then $A=\rho R+\alpha I$ and $D=\sigma S+\delta I$. Apply now one direction of Lemma 3.2 to projections P and Q and the other one to projections R and S to get that either R or $I-R$ and either S or $I-S$ are of rank 1. We shall assume with no loss of generality that R and S are of rank 1 and therefore necessarily $RS=SR=0$. It remains to show tha $\rho=\sigma$. But, the second commutant of $P+Q$ has dimension 2 and so has the second commutant of the operator $A+D=\rho R+\sigma S+(\alpha+\delta)I$. Since this operator has more than one point in its spectrum, it must be a linear combination of a non-trivial projection and the identity operator by Lemma 3.1. And for that reason $\rho=\sigma$.

COROLLARY 3.4. *Under the assumptions of Proposition* 3.3., *there is a constant $\sigma\in\mathbb{C}$, $\sigma\neq 0$, such that for every projection R of rank* 1 *there is a projection T of rank* 1 *and a constant $\alpha\in\mathbb{C}$ such that*

$$F(R)=\sigma T+\alpha I.$$

The corollary will follow immediately from the proposition, after proving a simple lemma which will also be needed in the sequel.

LEMMA 3.5. *Let P and R be any projections of rank* 1. *Then there are nilpotents U and V of rank* 1 *such that $P=UV$, further $Q=VU$ is a projection of rank* 1 *disjoint with P, and R is a linear combination of P, Q, U, and V.*

Proof. Write $P=x\otimes f$ and $R=z\otimes h$, where $x, z\in X$ and $f, h\in X'$ are non-zero and $f(x)=h(z)=1$. What we want to find are $y\in X$ and $g\in X'$ with $g(y)=1$, $g(x)=f(y)=0$ and such that h is a linear combination of f, g and that z is a linear combination of x, y. Then, we shall put $U=x\otimes g$, $V=y\otimes f$, and $Q=y\otimes g$ to get the desired operators. If the pairs x, z and f, h are both linearly dependent, then choose any $y\in X$ with $f(y)=0$ and any $g\in X'$ with $g(x)=0$, $g(y)=1$, to solve the problem. If the functionals

g, h are linearly dependent and the vectors x, z are not, then choose any $g \in X'$ with $g(x)=0$, $g(x)=1$, and set $y=z-f(z)\,x$ to meet our requirements. Similarly, in the case when the vectors x, z are linearly dependent and the functionals f, h are not, take arbitrary $y \in X$ with $f(y)=0$, $h(y)=1$ and set $g=h-h(x)\,f$ to get what is needed. Finally, assume that both pairs x, z and f, h are linearly independent and put $y = z-f(z)\,x \neq 0$. Assume, for the moment that $h(y)=0$. Since the intersection of the kernel of h with the subspace spanned by x and z has dimension 1, we must have $h(x)\,z-x=h(x)(z-f(z)\,x)$ which forces $h(x)\,f(z)=1$. The functional $h-h(x)\,f$ has therefore both vectors x and z in its kernel, so it is trivial, contradicting the fact that h and f are linearly independent. Consequently, $h(y) \neq 0$ and the functional $g=(h-h(x)\,f)/h(y)$ solves the problem.

Proof of Corollary 3.4. Apply Proposition 3.3 to see that the only thing to show is that the constant σ does not depend on the choice of the projection R. Choose any two projections of rank 1, say P and R and let U, V, and Q be as in the Lemma 3.5. Choose any further projection S of rank 1, disjoint with both P and Q and note that it is then disjoint also with R. Now, apply the Proposition first to the projections P and S, and then to the projections S and R to get the corollary.

4. Characterization of Operators Preserving Commutativity

Throughout this section it will be assumed that the dimension of the space X is greater than 2. Besides, we shall fix a bijective, linear operator $F: L(X) \to L(X)$ which preserves commutativity in both directions and a projection $P_0 = x \otimes f$, where $x \in X$ and $f \in X'$ with $f(x)=1$. By Proposition 3.3

$$F(P_0) = \sigma(y \otimes g) + \rho I,$$

where $y \in X$ and $g \in X'$ with $g(y)=1$, while ρ, σ are complex numbers and σ non-zero. For every $A \in L(X)$ define

$$p(A) = g(F(A)\,y) - \sigma f(Ax) \tag{1}$$

and

$$G(A) = \frac{1}{\sigma}(F(A) - p(A)\,I). \tag{2}$$

Note that p is a linear functional (not necessarily bounded) on $L(X)$ and that G is an operator on $L(X)$.

PROPOSITION 4.1. *The operator* G: $L(X) \to L(X)$ *is linear, bijective, and preserves commutativity in both directions. Moreover,*

(a) $G(I) = I$,

(b) $G(P_0 A_0 P_0) = G(P_0)\, G(A_0)\, G(P_0)$ *for all* $A_0 \in L(X)$,

(c) $G(Q_0)$ *is a projection of rank* 1 *for every projection* $Q_0 \in L(X)$ *of rank* 1.

In the proof of the assertion (c) another auxiliary result will be needed. In the following lemma we will assume that P, Q are projections and U, V nilpotents, all of rank 1 and such that $P = UV$, $Q = VU$. Note that an operator $A \in L(X)$ is a linear combination of P, Q, U, and V if and only if $(P+Q)\, A = A(P+Q) = A$.

LEMMA 4.2. *If* A *is a linear combination of* P, Q, U, *and* V, *then its rank is not greater than* 1 *if and only if there exists an operator* $B \in L(X)$ *such that* B *commutes with* A *and* B *does not commute with the projection* $P+Q$.

Proof. Recall the notations of Lemma 3.5. Suppose that $A = z \otimes h$ for some $z = \alpha x + \beta y$, $h = \gamma f + \nu g$. Choose any non-zero $w \in X$ from the intersection of the kernels of f and g. Define $B = w \otimes (\beta f - \alpha g)$ to get $BA = 0$, $AB = 0$, $(P+Q)\, B = 0$, but $B(P+Q) = B \neq 0$. To get the other direction of the lemma, assume that $A \in L(X)$ is of rank 2. Since its image is a subspace of the image of $P+Q$ and the two subspaces have the same dimension, they must be equal and there exists an operator $C \in L(X)$ with $CA = AC = P+Q$. Since A lies in a complex algebra of finite dimension, there is a complex polynomial p with $p(0) = 1$ such that for an integer k it holds that $A^k p(A) = 0$. Multiply by C^k to get $(P+Q)\ p(A) = 0$ and set $q(\lambda) = (1 - p(\lambda))/\lambda$ to obtain $0 = (P+Q)(I - Aq(A))$. Thence, the operator C in $CA = AC = P+Q$ can be interchanged with $q(A)$. Consequently, if an operator $B \in L(X)$ commutes with A, then it commutes with $q(A)$ and also with $P+Q = Aq(A)$.

Proof of Proposition 4.1. It is clear that G is linear and that it preserves commutativity in both directions. If for an operator $A \in L(X)$ we have $G(A) = 0$, then $F(A) = p(A)\, I$ and $p(A) = p(A)\ g(y) - \sigma f(Ay)$. Since F preserves commutativity in both directions, $A = \lambda I$ for a complex λ and therefore $\sigma \lambda f(y) = 0$ which implies $\lambda = 0$. To see that G is surjective, choose any $B \in L(X)$ and put $A = \sigma F^{-1}(B) + p(F^{-1}(B))\, I$ to get $G(A) = B$. The assertion (a) can be verified directly. Note that for every $A_0 \in L(X)$, $G(P_0) = y \otimes g$, and

$$G(P_0)\, G(A_0)\, G(P_0) = (y \otimes g)\, f(Ax) = G(P_0 A_0 P_0)$$

to obtain the assertion (b). To see (c), choose any projection $R_0 \in L(X)$ of rank 1. Apply Lemma 3.5 to get nilpotents U_0 and V_0 of rank 1 such that $P_0 = U_0 V_0$ and $Q_0 = V_0 U_0$ is a projection of rank 1, disjoint with P_0. Besides, R_0 is a linear combination of P_0, Q_0, U_0, and V_0.

Denote $P = G(P_0)$, $Q = G(Q_0)$, $B = G(U_0)$, and $C = G(V_0)$ and note that by Proposition 3.3, $Q - \alpha I$ is a projection of rank 1 disjoint with P, for a complex numer α. Use the assertion (b) of Proposition 4.1 to get

$$\alpha P = PQP = G(P_0 Q_0 P_0) = 0.$$

Therefore, P and Q are disjoint projections of rank 1.

Since $Q_0 + U_0$ is a projection, we get by Corollary 3.4 a complex number λ such that $Q + B - \lambda = S$ is a projection of rank 1, say $S = z \otimes h$ for some $z \in X$, $h \in X'$ with $h(z) = 1$. Denote $u = (P+Q)\, z$, $v = u - z$, $r = (P+Q)'\, h$, and $s = h - r$. Since Q_0 and U_0 commute with $P_0 + Q_0$, the projection $z \otimes h$ commutes with $P + Q$, hence $u \otimes h = z \otimes r$ and therefore $u \otimes s - v \otimes r = 0$. If v depends linearly on u, we get $v = 0$ which forces $s = 0$. If $v \neq 0$, we get necessarily $u = 0$ and $r = 0$. Thence, either

$$(P+Q)\, S = S(P+Q) = S \quad \text{or} \quad (P+Q)\, S = S(P+Q) = 0.$$

The second case yields $QS = SQ = 0$ and Q commutes with B, contrary to the fact that Q_0 does not commute with U_0. Thence, we must have $(P+Q)S = S(P+Q) = S$. Let us now use the assertion (b) of Proposition 4.1 in the following computation:

$$\begin{aligned} -\lambda P &= P(Q + B - \lambda)\, P = P(Q + B - \lambda)^2\, P \\ &= P(Q^2 + Q(B-\lambda) + (B-\lambda)\, Q + (B-\lambda)^2)\, P \\ &= P(B-\lambda)^2\, P. \end{aligned}$$

Note that by Lemma 4.2 there is an operator $C_0 \in L(x)$ commuting with U_0 which does not commute with $P_0 + Q_0$. Hence, by the same lemma, $B - \lambda$ is of rank 1. Therefore, $(B-\lambda)^2 = \alpha(B-\lambda)$ for a complex number α. If $\alpha \neq 0$, $(B-\lambda)/\alpha$ is a projection of rank 1, contradicting Proposition 3.3, applied for the operator G^{-1} and the fact that $U_0 = G^{-1}(B)$ is a nilpotent. In this way we see that $(B-\lambda)^2 = 0$ which completes the above computation

$$-\lambda P = P(B-\lambda)^2\, P = 0.$$

Thence $\lambda=0$ and B is a nilpotent of rank 1. Find some nilpotents U and V of rank 1 such that $P=UV$ and $Q=VU$. Then, B is a linear combination of P, Q, U, and V. Since $PBP=0$, we have $B=\alpha Q+\beta U+\gamma V$. From $B^2=0$ we get $\alpha\beta=\alpha\gamma=\beta\gamma=\alpha^2=0$. Therefore, $B=\beta U+\gamma V$, where either β or γ is zero. Using the same arguments for $C=F(V_0)$ instead of B, we get $C=\alpha U+\delta V$, where either α or δ is zero. Since U_0 and V_0 are linearly independent, so are B and C. Hence, if $\beta=0$, then $\delta=0$, but if $\beta\neq 0$, then $\gamma=\alpha=0$. Note that $P_0+Q_0+U_0+V_0$ is also a projection of rank 1. Then, $P+Q+B+C-\mu I$ is a projection of rank 1 for a complex number μ. Note that necessarily $\mu=0$ and after a short computation either $\beta=\delta=0$ and $\alpha\gamma=1$, or $\alpha=\gamma=0$ and $\beta\delta=1$. In the first case we can suppose with no loss of generality $F(U_0)=V$ and $F(V_0)=U$, while in the second we can take $F(U_0)=U$ and $F(V_0)=V$.

Now recall the projection R_0 from the beginning of the proof of the assertion (c). Note that for some complex numbers α, β, γ, δ, with $\alpha\gamma+\beta\delta=1$, we can write

$$R_0=\alpha\gamma P_0+\beta\gamma U_0+\alpha\delta V_0+\beta\delta Q_0.$$

This implies that $R=F(R_0)$ is a projection of rank 1 in either of the two cases $F(U_0)=V$, $F(V_0)=U$ or $F(U_0)=U$, $F(V_0)=V$. This completes the proof of the proposition.

THEOREM 4.3. *There exists either*

(a) *a bijective, bounded, linear operator $U: X\to X$ such that*

$$G(A)=UAU^{-1}$$

for every $A\in L(X)$; or

(b) *a bijective, bounded, linear operator $U: X'\to X$ such that*

$$G(A)=UA'U^{-1}$$

for every $A\in L(X)$. In this case the space X is reflexive.

Proof. By the assertion (c) of Proposition 4.1 and by Proposition 2.6, either (a) or (b) of Theorem 4.3 holds for every $A\in B(X)$ of finite rank. Now, let $R_0\in L(X)$ be a projection of rank 1. Then, we have

$$G(R_0A_0R_0)=G(R_0)\,G(A_0)\,G(R_0) \tag{3}$$

for every $A_0\in L(X)$ of finite rank in either of the two cases. We shall prove that (3) holds for every operator $A_0\in L(X)$. Define P_0, Q_0, U_0, V_0 and P, Q, U, V as in the proof of Proposition 4.1. Recall that $P=G(P_0)$,

$Q = G(Q_0)$ and either $U = G(U_0)$, $V = G(V_0)$ or $U = G(V_0)$, $V = G(U_0)$. Fix $A_0 \in L(X)$, introduce $B_0 = (I - P_0 - Q_0)\ A_0(I - P_0 - Q_0)$ and write $B = G(B_0)$. By the definition B_0 commutes with P_0, Q_0, U_0, and V_0 and for that reason B commutes with P, Q, U, and V. Apply the assertion (b) of Proposition 4.1 in the following computations

$$UB = BU = BPU = PBPU = 0,$$

$$BV = VB = VPB = VPBP = 0,$$

which forces also $PB = 0$ and $QB = 0$. In this way we obtain $RBR = 0$ for $R = G(R_0)$. Use the fact that $A_0 - B_0$ is of finite rank and that (3) holds for all the operators of finite rank to see

$$G(R_0)\ G(A_0)\ G(R_0) = RAR = R(A - B)\ R$$

$$= G(R_0(A_0 - B_0)\ R_0) = G(R_0 A_0 R_0).$$

Consequently, (3) is valid for every $A_0 \in L(X)$ and every projection R_0 of rank 1.

Suppose that for the operators of finite rank the case (a) of the theorem holds and fix any $A_0 \in L(X)$. Besides, take any $z \in X$ and any $h \in X'$ with $h(z) = 1$. Set $R_0 = z \otimes h$ and use (3) to get

$$(UR_0 U^{-1})\ h(A_0 z) = UR_0 U^{-1} G(A_0)\ UR_0 U^{-1}$$

which implies

$$h(A_0 z) = h(U^{-1} G(A_0)\ Uz). \tag{4}$$

Note that (4) holds for every $h \in X'$ with $h(z) = 1$ and so, it holds for every $h \in X'$ by linearity. Thus, $A_0 z = U^{-1} G(A_0)\ Uz$ is valid for every $z \in X$ and the case (a) of the theorem is proved.

Now, assume that the case (b) of the theorem holds for operators of finite rank. Then, for every $z \in X$ and $h \in X'$ with $h(z) = 1$, we get from (3) after introducing $R_0 = z \otimes h$

$$UR'_0 U^{-1} h(A_0 z) = UR'_0 U^{-1} G(A_0)\ UR'_0 U^{-1}$$

and therefore

$$h(A_0 z) = h(U' G(A_0)'\ U'^{-1} z).$$

Using similar arguments we obtain $A_0 = U' G(A_0)'\ U'^{-1}$. Consequently, case (b) of the theorem is proved.

We can now obtain Theorem 1.1 as a simple corollary of Theorem 4.3.

Assume F as in Theorem 1.1 and define the functional p by (1) and the operator G by (2). Then,

$$F(A) = \sigma G(A) + p(A) I$$

and the proof is completed.

ACKNOWLEDGMENT

The author is grateful to B. Magajna for some very helpful conversations on the form of rank-preserving operators.

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