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Non-abelian Hopf cohomology II—The general case

Philippe Nuss*, Marc Wambst

*Institut de Recherche Mathématique Avancée, Université Louis-Pasteur et CNRS, 7, rue René-Descartes,
67084 Strasbourg Cedex, France*

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Abstract

We introduce and study non-abelian cohomology sets of Hopf algebras with coefficients in Hopf comodule algebras. We prove that these sets generalize as well Serre's non-abelian group cohomology theory as the cohomological theory constructed by the authors in a previous article. We establish their functoriality and compute explicit examples. Further we classify Hopf torsors.

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Introduction

The present article, conceived as the continuation of [9], is devoted to the study of non-abelian cohomology theory in the Hopf algebra setting. We define a general cohomology theory analogous to that for groups adapted to Hopf algebras and suitable coefficient objects [10,11]. In order to clarify the purpose of our work, we recall first some basic facts about the classical constructions in the framework of groups. Let G be a group acting on a group A . The non-abelian cohomology theory $H^*(G, A)$ for groups may be organized in three different stages depending on the properties of the coefficient group A .

Stage 1. *The group of coefficients A is abelian.* The classical Eilenberg–MacLane cohomology theory $H^*(G, A) = \text{Ext}_{\mathbf{Z}[G]}^*(\mathbf{Z}, A)$ produces a sequence of commutative groups.

* Corresponding author.

E-mail addresses: nuss@math.u-strasbg.fr (P. Nuss), wambst@math.u-strasbg.fr (M. Wambst).

It provides useful invariants in homological algebra, algebraic topology and algebraic number theory.

- Stage 2. *The group of coefficients A is not abelian.* The previous construction fails in this case. However it is still possible to define a group $H^0(G, A)$ and a pointed set $H^1(G, A)$. This theory, called the non-abelian cohomology theory of groups, was introduced by Lang and Tate [4] for Galois groups with coefficients in an algebraic group, and was studied in full generality by Serre [10,11]. It is for instance well known that the non-abelian cohomology set $H^1(G, A)$ classifies the G -torsors (or principal homogeneous spaces) on A (see [11]).
- Stage 3. *The group of coefficients A is the group of automorphisms of a G -Galois extension.* Suppose that the group G is finite and acts as a Galois group on a Galois extension S/R of non-commutative rings (for this generalization of Galois extensions of fields, see [5]). Let M be an S -module endowed with a compatible G -action. The latter induces a G -group structure on the group $\text{Aut}_S(M)$ of S -linear automorphisms of M . One of the authors [8] showed that in this context non-abelian cohomology theory comes into play. In particular he proved that the set $H^1(G, \text{Aut}_S(M))$ classifies objects which arise in descent theory along S/R , for example descent cocycles on M or twisted forms of M .

Hopf algebras naturally generalize groups. Kreimer and Takeuchi [3] widened Galois extensions to Hopf-Galois extensions of rings in the following spirit. As a group acting on rings plays the rôle of the symmetry object for Galois extensions, a Hopf algebra coacting on the rings does for Hopf-Galois extensions. In [9], we answered the natural question of extending Stage 3 to this setting. For a Hopf algebra H , an H -Hopf comodule algebra S , and an (H, S) -Hopf module M , we introduced a group $H^0(H, M)$ and a pointed set $H^1(H, M)$. This construction, here called *restricted non-abelian Hopf cohomology theory*, replaces $H^*(G, \text{Aut}_S(M))$. It offers a generalization of Stage 3 in the following two senses (see [9]):

- If S/R is an H -Hopf-Galois extension, then $H^1(H, M)$ classifies the analogue of descent cocycles on M along S/R and the twisted forms of M .
- Given a group G , a G -Galois extension of rings S/R is nothing but a \mathbf{Z}^G -Hopf-Galois extension, where \mathbf{Z}^G stands for the Hopf algebra of functions on G . Then $H^*(\mathbf{Z}^G, M)$ is isomorphic to $H^*(G, \text{Aut}_S(M))$.

The aim of this article is to define a non-abelian cohomology theory in the Hopf context corresponding to Stage 2. More precisely, let H be a Hopf algebra over a commutative ring k . For any H -comodule k -algebra E we introduce the *general non-abelian Hopf cohomology theory of H with coefficients in E* . We define a group $\mathcal{H}^0(H, E)$ and a pointed set $\mathcal{H}^1(H, E)$. These constructions are based on the non-abelian cohomology theory associated to a pre-cosimplicial group. We prove three main results (the precise wording and definitions will be found in the core of the article):

- (a) We show (Theorem 1.5) that the cohomology theory $\mathcal{H}^*(k^G, E)$ is isomorphic to $H^*(G, E^\times)$, where k^G denotes the Hopf algebra of functions on G and E^\times is the group of invertible elements of E .
- (b) Let S be an H -Hopf comodule algebra and M be an (H, S) -Hopf module. We establish (Theorem 2.6) that under lax technical conditions, $\mathcal{H}^*(H, \text{End}_S(M))$ and $H^*(H, M)$ are isomorphic.

(c) Finally, if E is an H -comodule algebra, we classify (H, E) -Hopf torsors via the pointed set $\mathcal{H}^1(H, E)$ (Theorem 3.4).

The article is built in the following way. The first section is devoted to the definition and the properties of general non-abelian Hopf cohomology theory. There we prove Result (a), give some examples, explicit computations (Sections 1.2 and 1.4), and show that the Hopf module structures may be deformed with the help of 1-cocycles (Proposition 1.7). In Section 1.6 we study the functoriality of the general non-abelian Hopf cohomology sets and write down an exact sequence associated to a sub-comodule algebra. In the second section, we clarify the links between general and restricted non-abelian Hopf cohomology theory. To this end, we state a technical condition (Condition (\mathcal{F}_n) in Section 2.2) which allows to endow the endomorphism algebra of a Hopf module with a comodule structure (Lemma 2.4). We then deduce Result (b). The third and last section deals with Hopf torsors. We define them as a generalization of usual torsors (Definition 3.2, Proposition 3.7, and Corollary 3.8) and prove Result (c).

We mention here that an attempt of generalizing the non-abelian group cohomology theory to the Hopf context was done by Blanco Ferro [1]. This author adapted Sweedler’s theory [12], which can be viewed as a generalization of Stage 1. Blanco Ferro defined a 1-cohomology set $H^1(H, A)$, where H is a cocommutative Hopf algebra and A is an algebra not necessarily commutative. His construction is in some sense dual to ours. But if one tries to apply it to the Hopf-Galois extensions, one has to restrict oneself to a very particular case: not only does H have to be a commutative finitely generated k -projective Hopf algebra, but the Hopf-Galois extension S/k is over the ground field and moreover has to be commutative.

0. Conventions, notations, and terminology

Let k be a fixed commutative and unital ring. The unadorned symbol \otimes between a right k -module and a left k -module stands for \otimes_k . By *(co-)algebra* we mean a (co-)unital (co-)associative k -(co-)algebra. By *(co-)module* over a (co-)algebra D , we always understand a right D -(co-)module unless otherwise stated. Let M be a k -module. We identify in a systematic way $M \otimes k$ with M .

For any algebra D , we denote by D^\times the group of invertible elements in D . If M is a D -module, $\text{End}_D(M)$ (respectively $\text{Aut}_D(M)$) is the algebra (respectively the group) of D -linear endomorphisms (respectively automorphisms) of M .

Let H be a Hopf algebra with multiplication μ_H , unity map η_H , comultiplication Δ_H , counity map ε_H , and antipode σ_H . Recall that an H -comodule algebra E is a k -module which is both an algebra and an H -comodule such that the coaction map is a morphism of algebras. A morphism of H -comodule algebras is simultaneously a morphism of algebras and of H -comodules. Suppose that E is an H -comodule algebra. Let M be both an E -module and an H -comodule. If the coaction map $\Delta_M : M \rightarrow M \otimes H$ verifies the equality

$$\Delta_M(ms) = \Delta_M(m)\Delta_E(s)$$

for any $m \in M$ and $s \in E$, we say that M is an (H, E) -Hopf module (also called a *relative Hopf module* in the literature) and that Δ_M is (H, E) -linear. A *morphism of (H, E) -Hopf modules* is an E -linear map $f : M \rightarrow M'$ such that $(f \otimes \text{id}_M) \circ \Delta_M = \Delta_{M'} \circ f$. Observe that E itself is naturally an (H, E) -Hopf module.

To denote the coactions on elements, we use the Sweedler–Heyneman convention, that is, for $m \in M$, we write $\Delta_M(m) = m_0 \otimes m_1$, with summation implicitly understood. More generally, when we write down a tensor we usually omit the summation sign \sum .

Let G be a finite group with neutral element e . Denote by k^G the k -free Hopf algebra over the k -basis $\{\delta_g\}_{g \in G}$, with the following structure maps: the multiplication is given by $\delta_g \cdot \delta_{g'} = \delta_{g, g'} \delta_g$, where $\delta_{g, g'}$ stands for the Kronecker symbol of g and g' ; the comultiplication Δ_{k^G} is defined by $\Delta_{k^G}(\delta_g) = \sum_{ab=g} \delta_a \otimes \delta_b$; the unit in k^G is the element $1 = \sum_{g \in G} \delta_g$; the counit ε_{k^G} is defined by $\varepsilon_{k^G}(\delta_g) = \delta_{g, e} 1$; the antipode σ_{k^G} sends δ_g on $\delta_{g^{-1}}$. When k is a field, then k^G is the dual of the usual group algebra $k[G]$.

1. General non-abelian Hopf cohomology theory

The first section is devoted to the definition, the properties and examples of general non-abelian Hopf cohomology theory. The constructions are provided in simplicial terms (a résumé about the simplicial language may be found in [6]).

1.1. Definitions

Let

$$\mathcal{A}^* = A^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{\quad} \end{array} A^1 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \xrightarrow{\quad} \end{array} A^2$$

be a pre-cosimplicial group. The *non-abelian 0-cohomology group* $\mathbb{H}^0(\mathcal{A}^*)$ is the equalizer of the pair (d^0, d^1) :

$$\mathbb{H}^0(\mathcal{A}^*) = \{x \in A^0 \mid d^1(x) = d^0(x)\}.$$

The *non-abelian 1-cohomology pointed set* $\mathbb{H}^1(\mathcal{A}^*)$ is the right quotient

$$\mathbb{H}^1(\mathcal{A}^*) = A^0 \backslash \mathbb{Z}^1(\mathcal{A}^*).$$

Here the set $\mathbb{Z}^1(\mathcal{A}^*)$ of 1-cocycles is the subset of A^1 defined by

$$\mathbb{Z}^1(\mathcal{A}^*) = \{X \in A^1 \mid d^2(X)d^0(X) = d^1(X)\}.$$

The group A^0 acts on the right on A^1 by

$$X \longleftarrow x = (d^1 x^{-1})X(d^0 x),$$

where $X \in A^1$ and $x \in A^0$. Using the pre-cosimplicial relations, one easily checks that this action restricts to $\mathbb{Z}^1(\mathcal{A}^*)$. Two 1-cocycles X and X' are said to be *cohomologous* if they belong to the same orbit under this action. The quotient set $\mathbb{H}^1(\mathcal{A}^*) = A^0 \backslash \mathbb{Z}^1(\mathcal{A}^*)$ is pointed with distinguished point the class of the neutral element of A^1 .

Let H be a Hopf algebra, let E be an H -comodule algebra with multiplication μ_E and coaction Δ_E . We define two maps $d^i : E \rightarrow E \otimes H$ ($i = 0, 1$) and three maps $d^i : E \otimes H \rightarrow E \otimes H \otimes H$ ($i = 0, 1, 2$) by the formulae

$$d^0(x) = \Delta_E(x), \quad d^1(x) = x \otimes 1,$$

$$d^0(X) = (\Delta_E \otimes \text{id}_H)(X), \quad d^1(X) = (\text{id}_E \otimes \Delta_H)(X), \quad d^2(X) = X \otimes 1,$$

where $x \in E$ and $X \in E \otimes H$.

Lemma 1.1. *The diagram $\mathcal{C}_{\leq 2}(H, E)$ given by*

$$E \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \longrightarrow \end{array} E \otimes H \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \longrightarrow \end{array} E \otimes H \otimes H$$

is a pre-cosimplicial object in the category of algebras.

Proof. The maps d^i are easily seen to be morphisms of algebras. The pre-cosimplicial relations $d^i d^j = d^j d^{i-1}$ for $i > j$ follow from the Hopf axioms for H and E . \square

Lemma 1.1 allows us to deduce a pre-cosimplicial diagram $\mathcal{C}_{\leq 2}^\times(H, E)$ in the category of groups by setting:

$$E^\times \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \longrightarrow \end{array} (E \otimes H)^\times \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \longrightarrow \end{array} (E \otimes H \otimes H)^\times$$

(we still denote by d^i the restrictions of the maps $d^i : E \otimes H^{\otimes j} \rightarrow E \otimes H^{\otimes(j+1)}$ to the corresponding multiplicative groups).

Remark. Both $\mathcal{C}_{\leq 2}(H, E)$ and $\mathcal{C}_{\leq 2}^\times(H, E)$ are in fact cosimplicial objects. The codegeneracy maps on $\mathcal{C}_{\leq 2}(H, E)$ are given by

$$s^0 = \text{id}_E \otimes \varepsilon_H : E \otimes H \rightarrow E,$$

$$s^0 = \text{id}_E \otimes \varepsilon_H \otimes \text{id}_H : E \otimes H \otimes H \rightarrow E \otimes H \quad \text{and}$$

$$s^1 = \text{id}_E \otimes \text{id}_H \otimes \varepsilon_H : E \otimes H \otimes H \rightarrow E \otimes H.$$

The codegeneracy maps on $\mathcal{C}_{\leq 2}^\times(H, E)$ are again obtained by restriction.

Definition 1.2. The general non-abelian Hopf cohomology objects $\mathcal{H}^*(H, E)$ of a Hopf algebra H with coefficients in an H -comodule algebra E is the non-abelian cohomology theory associated to the pre-cosimplicial diagram $\mathcal{C}_{\leq 2}^\times(H, E)$.

In other words

$$\mathcal{H}^0(H, E) = \mathbb{H}^0(\mathcal{C}_{\leq 2}^\times(H, E)) = \{x \in E^\times \mid d^1(x) = d^0(x)\} \quad \text{and}$$

$$\mathcal{H}^1(H, E) = \mathbb{H}^1(\mathcal{C}_{\leq 2}^\times(H, E)) = E^\times \setminus \mathcal{Z}^1(H, E).$$

Observe that $\mathcal{H}^0(H, E)$ is the group $(E^{\text{co}H})^\times$ of invertible coinvariant elements of E . The set $\mathcal{Z}^1(H, E)$ of Hopf 1-cocycles of H with coefficients in E is the subset of $(E \otimes H)^\times$ given by

$$\mathcal{Z}^1(H, E) = \{X \in (E \otimes H)^\times \mid d^2(X)d^0(X) = d^1(X)\}.$$

We refer to $d^2(X)d^0(X) = d^1(X)$ as the Hopf 1-cocycle relation.

Remark 1.3.

(a) For any Hopf algebra H and any H -comodule algebra E , one proves the inclusion

$$\mathcal{Z}^1(H, E) \subseteq \text{Ker}(\text{id}_E \otimes \varepsilon_H : (E \otimes H)^\times \longrightarrow E^\times)$$

by applying the map $\text{id}_E \otimes \varepsilon_H \otimes \text{id}_H$ to the Hopf 1-cocycle relation.

(b) If the algebras E and H are both commutative, the sets $\mathcal{Z}^1(H, E)$ and $\mathcal{H}^1(H, E)$ become groups with product induced by the multiplication of $E \otimes H$.

1.2. First examples

(1) *The Hopf algebra is trivial.* Any algebra E is naturally a k -comodule algebra with the coaction Δ_E equal to id_E . One then has:

$$\mathcal{H}^0(k, E) = E^\times \quad \text{and} \quad \mathcal{H}^1(k, E) = \{1\}.$$

Indeed, the first equality is obvious. One checks that $\mathcal{Z}^1(k, E)$ is the pointed set of invertible idempotent elements of E , that is nothing else than $\{1\}$.

(2) *The coefficients are trivial.* Let H be a Hopf algebra. The ground ring k is an H -comodule algebra through the coaction Δ_k given by the unity map η_H . Denote by $\text{Gr}(H)$ the group of grouplike elements in H . One then has:

$$\mathcal{H}^0(H, k) = k^\times \quad \text{and} \quad \mathcal{H}^1(H, k) \cong \text{Gr}(H),$$

the latter relation being an isomorphism of groups. The calculation of $\mathcal{H}^0(H, k)$ is straightforward. We compute now $\mathcal{Z}^1(H, k)$. A 1-cocycle is in particular an element $h \in H$ verifying the 1-cocycle relation, here $h \otimes h = \Delta_H(h)$. So the element h is grouplike, hence incidentally also invertible in H . The action of k^\times on $\mathcal{Z}^1(H, k)$ is trivial; therefore $\mathcal{H}^1(H, k)$ is the whole group of grouplike elements of H .

(3) *The coefficients are the Hopf algebra itself.* A Hopf algebra H is a comodule algebra over itself. One has:

$$\mathcal{H}^0(H, H) = k^\times \quad \text{and} \quad \mathcal{H}^1(H, H) = \{1\}.$$

The first equality follows from the very definition: $\mathcal{H}^0(H, H) = (H^{\text{co}H})^\times$. To prove the second equality, pick $X \in \mathcal{Z}^1(H, H)$ and apply the map $\varepsilon_H \otimes \text{id}_H \otimes \text{id}_H$ to the cocycle relation $d^2(X)d^0(X) = d^1(X)$. One gets $(x \otimes 1)X = \Delta_H(x)$, with $x = (\varepsilon_H \otimes \text{id}_H)(X)$. So $\mathcal{Z}^1(H, H)$ is contained in the set $\{(x^{-1} \otimes 1)\Delta_H(x) \mid x \in H^\times\}$, which is equal to $\{d^1(x^{-1})d^0(x) \mid x \in H^\times\}$. Conversely, if $X = (x^{-1} \otimes 1)\Delta_H(x)$ for $x \in H^\times$, then X fulfills the cocycle relation. So $\mathcal{Z}^1(H, H)$ equals $\{d^1(x^{-1})d^0(x) \mid x \in H^\times\}$, and therefore the 1-cohomology set is trivial.

1.3. Link with non-abelian group cohomology

We first recall the definitions given by Serre [10,11] of the non-abelian cohomology theory $H^i(G, A)$ (with $i = 0, 1$) of a group G with coefficients in a (left) G -group A . The 0-cohomology object $H^0(G, A)$ is the group A^G of invariant elements of A under the action of G . The set $Z^1(G, A)$ of 1-cocycles is given by

$$Z^1(G, A) = \{\alpha : G \longrightarrow A \mid \alpha(gg') = \alpha(g)^g \alpha(g'), \forall g, g' \in G\}.$$

It is pointed with distinguished point the constant map $1 : G \longrightarrow A$. The group A acts on the right on $Z^1(G, A)$ by

$$(\alpha \longleftarrow a)(g) = a^{-1} \alpha(g)^g a,$$

where $a \in A$, $\alpha \in Z^1(G, A)$, and $g \in G$. Two 1-cocycles α and α' are *cohomologous* if they belong to the same orbit under this action. The non-abelian 1-cohomology set $H^1(G, A)$ is the left quotient $A \backslash Z^1(G, A)$. It is pointed with distinguished point the class of the constant map $1 : G \longrightarrow A$.

The non-abelian cohomology theory of groups may be interpreted as the non-abelian cohomology theory associated to the pre-cosimplicial diagram of groups

$$\mathcal{G}_{\leq 2}(G, A) = \left(A = \text{Map}(G^0, A) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} \text{Map}(G, A) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \text{Map}(G^2, A) \right).$$

Here $\text{Map}(G^i, A)$ stands for the set of the maps from G^i to A , which is endowed with the group structure induced by pointwise multiplication. The coboundaries are given by

$$\begin{aligned} d^0(x) : g &\longmapsto {}^g x, & d^1(x) : g &\longmapsto x, \\ d^0(\alpha) : (g, g') &\longmapsto {}^g \alpha(g'), & d^1(\alpha) : (g, g') &\longmapsto \alpha(gg'), & d^2(\alpha) : (g, g') &\longmapsto \alpha(g), \end{aligned}$$

where $x \in A$, $g, g' \in G$ and $\alpha \in \text{Map}(G, A)$. The reader may easily check that the pre-cosimplicial relations are satisfied and that one has the equality

$$\mathbb{H}^*(\mathcal{G}_{\leq 2}(G, A)) = H^*(G, A).$$

We now connect the general non-abelian Hopf cohomology theory with the non-abelian cohomology theory of groups. Let G be a finite group and E be a k^G -comodule algebra. For any $x \in E$, write

$$\Delta_E(x) = \sum_{g \in G} {}^g x \otimes \delta_g.$$

This formula defines an action of the group G on the algebra E , hence on the group E^\times . One has the following result:

Proposition 1.4. *Let G be a finite group and E be a k^G -comodule algebra. The pre-cosimplicial groups $\mathcal{G}_{\leq 2}(G, E^\times)$ and $\mathcal{C}_{\leq 2}(k^G, E)$ are isomorphic.*

Before we give the proof, we state the following immediate consequence:

Theorem 1.5. *Let G be a finite group and E be a k^G -comodule algebra. There is the equality of groups*

$$\mathcal{H}^0(k^G, E) = H^0(G, E^\times)$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(k^G, E) \cong H^1(G, E^\times).$$

Proof of Proposition 1.4. First remark that any element in $(E \otimes k^G)^\times$ is of the form $\sum_{g \in G} x_g \otimes \delta_g$, where for all $g \in G$, the element x_g belongs to E^\times . In the same way any element in $(E \otimes k^G \otimes k^G)^\times$ is of the form $\sum_{g, g' \in G} x_{g, g'} \otimes \delta_g \otimes \delta_{g'}$, where for all $g, g' \in G$, the element $x_{g, g'}$ belongs to E^\times . We consider the map $\gamma_* : \mathcal{C}_{\leq 2}(k^G, E) \rightarrow \mathcal{G}_{\leq 2}(G, E^\times)$, given by

$$\begin{aligned} \gamma_0 &= \text{id}_{E^\times}, \\ \gamma_1 \left(\sum_{g \in G} x_g \otimes \delta_g \right) &: u \mapsto x_u, \\ \gamma_2 \left(\sum_{g, g' \in G} x_{g, g'} \otimes \delta_g \otimes \delta_{g'} \right) &: (u, v) \mapsto x_{u, v}, \end{aligned}$$

for any $u, v \in G$. On each level, γ_* is an isomorphism of groups since $(E \otimes k^G)^\times$ (respectively $(E \otimes k^G \otimes k^G)^\times$) is isomorphic to $(E^\times)^{|G|}$ (respectively to $(E^\times)^{|G|^2}$).

It remains to check that γ_* is a morphism of pre-cosimplicial objects, in other words γ_* verifies $\gamma_j d^i = d^i \gamma_{j-1}$ for any $1 \leq j \leq 2$ and $0 \leq i \leq j$. This is done by direct computations. For example, set

$$v = \gamma_2 d^0 \left(\sum_{g \in G} x_g \otimes \delta_g \right) = \gamma_2 \left(\sum_{g, g' \in G} {}^{g'} x_g \otimes \delta_{g'} \otimes \delta_g \right).$$

So $v(u, v) = {}^u x_v$, for any $u, v \in g$. Hence $v = d^0 \gamma_1(\sum_{g \in G} x_g \otimes \delta_g)$. As an other example, set

$$v' = \gamma_2 d^1 \left(\sum_{g \in G} x_g \otimes \delta_g \right) = \gamma_2 \left(\sum_{h, h' \in G} x_{hh'} \otimes \delta_h \otimes \delta_{h'} \right).$$

So $v'(u, v) = x_{uv}$, for any $u, v \in g$. Hence $v' = d^1 \gamma_1(\sum_{g \in G} x_g \otimes \delta_g)$. We leave to the reader the three remaining computations. \square

Two direct applications of Theorem 1.5

(1) Let G be a finite group. One may recover the isomorphism between the group $\text{Gr}(k^G)$ of grouplike elements of k^G and the Pontryagin dual $\hat{G} = \text{Hom}(G, k^\times)$ of G . Indeed, by Example 2 of Section 1.2, the group $\text{Gr}(k^G)$ is isomorphic to $\mathcal{H}^1(k^G, k)$. In this situation, the identification $\mathcal{H}^1(k^G, k) \cong H^1(G, k^\times)$ given by Theorem 1.5 is in fact an isomorphism of groups, and one sees that $H^1(G, k^\times)$ is isomorphic to \hat{G} .

(2) For any finite subgroup G of a group L , the group ring $k[L]$ is canonically equipped with a k^G -comodule algebra structure $\Delta_{k[L]}$ given by $\Delta_{k[L]}(h) = \sum_{g \in G} ghg^{-1} \otimes \delta_g$, for any $h \in L$, and extended by linearity. Theorem 1.5 claims the isomorphism $\mathcal{H}^*(k^G, k[L]) \cong H^*(G, k[L]^\times)$. However the computation of the group of units in $k[L]$ is in general a very difficult problem: the group $k[L]^\times$ is known only for some particular groups L .

1.4. An explicit example where the Hopf algebra is not an algebra of functions on a group

Let here k be a field and H_4 be the Sweedler four-dimensional Hopf algebra over k . Recall that H_4 is generated by two elements g and h submitted to the relations:

$$g^2 = 1, \quad h^2 = 0, \quad gh + hg = 0.$$

On the generators, the comultiplication, the antipode, and the counit of H_4 are given by

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(h) &= h \otimes g + 1 \otimes h, \\ \sigma(g) &= g, & \sigma(h) &= gh, \\ \varepsilon(g) &= 1, & \varepsilon(h) &= 0. \end{aligned}$$

Denote by E_2 the algebra of dual numbers, viewed as the subalgebra of H_4 generated by h . Via Δ , the algebra E_2 is naturally endowed with a structure of H_4 -comodule algebra.

Proposition 1.6. *There is an equality of groups*

$$\mathcal{H}^0(H_4, E_2) = k^\times$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(H_4, E_2) \cong \{1 \otimes 1, 1 \otimes g\}.$$

Proof. The proof consists in calculating explicitly the invariants on the 0-level (we leave this point to the reader) and in writing down the cocycle relations on generic elements on the 1-level. The computation of $\mathcal{Z}^1(H_4, E_2)$ is lightened by remarking that $E_2 \otimes H_4 = \{1 \otimes U + h \otimes V \mid U, V \in H_4\}$ and that $H_4 = F \oplus Fh$, where F is the sub-Hopf algebra of H_4 generated by g . The cocycle relation is then equivalent to the following system of two conditions on $\Delta(U)$ and $\Delta(V)$:

$$\Delta(U) = U \otimes U + Uh \otimes V, \tag{1}$$

$$\Delta(V) = (Ug) \otimes V + V \otimes U + (Vh) \otimes V. \tag{2}$$

In Eq. (1), if one replaces U by $x + yh$ and V by $z + th$, with $x, y, z, t \in F$, one gets a system of four equations in x, y, z, t . Solving them, one deduces U and V , which automatically satisfy Eq. (2).

Finally one obtains

$$\mathcal{Z}^1(H_4, E_2) = \{X_u, Y_u \mid u \in k\},$$

where the elements X_u and Y_u of $\mathcal{Z}^1(H_4, E_2)$ are given by

$$X_u = 1 \otimes 1 + u(1 \otimes h) - u(h \otimes 1) + u(h \otimes g) - u^2(h \otimes h),$$

$$Y_u = 1 \otimes g + u(1 \otimes gh) - u(h \otimes g) + u(h \otimes 1) - u^2(h \otimes gh).$$

The distinguished point of $\mathcal{Z}^1(H_4, E_2)$ is $X_0 = 1 \otimes 1$. One may observe that $\mathcal{Z}^1(H_4, E_2)$ contains a group, the set $\{X_u \mid u \in k\}$, which acts on the right on $\mathcal{Z}^1(H_4, E_2)$ by way of the multiplication in $(E_2 \otimes H_4)^\times$. Indeed, for any $u, v \in k$ one has the formulae:

$$X_u X_v = X_{u+v}, \quad Y_u X_v = Y_{u+v}.$$

It remains to describe the action of E_2^\times on $\mathcal{Z}^1(H_4, E_2)$. A generic element in E_2^\times is of the form $\alpha + \beta h$, with $\alpha \in k^\times$ and $\beta \in k$. A direct computation gives the identities

$$X_u \longleftarrow (\alpha + \beta h) = X_{u+\beta/\alpha} \quad \text{and} \quad Y_u \longleftarrow (\alpha + \beta h) = Y_{u+\beta/\alpha},$$

from which we deduce the isomorphism $\mathcal{H}^1(H_4, E_2) \cong \{X_0, Y_0\} = \{1 \otimes 1, 1 \otimes g\}$. \square

Remark. The Sweedler four-dimensional Hopf algebra H_4 is in fact the simplest example of the Taft algebras $H_{2^n} = k\langle g, h \rangle / \langle g^n - 1, h^n, hg - \zeta gh \rangle$ generated by g and h , with ζ a primitive n th root of unity in the field k [13].

As underlined by the referee, the computation of the non-abelian cohomology sets of H_{2^n} in various coefficient objects E should gather information about the Taft algebras. The main non-trivial examples of H_{2^n} -comodule algebras E one may choose are:

- (1) the cleft right H_{2^n} -Galois object $A_{2^n, \alpha, \beta} = k\langle x, y \rangle / \langle x^n - \alpha, y^n - \beta, yx - \zeta xy \rangle$ generated by x and y , with $\alpha \in k^\times$ and $\beta \in k$ (see [7]);
- (2) the subalgebra E_n of H_{2^n} generated by h .

It is easy to see that in the two above cases $\mathcal{H}^0(H_{2^n}, E)$ is isomorphic to k^\times . However, the computations of the 1-cohomology sets $\mathcal{H}^1(H_{2^n}, E)$ are very complex even in the second, apparently simpler, case. Nevertheless, our first partial results allows us to conjecture that $\mathcal{H}^1(H_{2^n}, E_n)$ is isomorphic to $\{1 \otimes 1, 1 \otimes g, \dots, 1 \otimes g^{n-1}\}$.

1.5. Deforming the Hopf module structure with a cocycle

Let H be a Hopf algebra and E be an H -comodule algebra. We show how the natural structure of (H, E) -Hopf module on E may be deformed with the help of a Hopf cocycle. To this end, for any element X of $E \otimes H$, denote by Δ_E^X the map from E to $E \otimes H$ given on $x \in E$ by

$$\Delta_E^X(x) = X \Delta_E(x).$$

One has then the following result:

Proposition 1.7. *Let H be a Hopf algebra, E be an H -comodule algebra, and X be an element of $(E \otimes H)^\times$. Then*

- (1) *the element X is a Hopf 1-cocycle if and only if (E, Δ_E^X) is an (H, E) -Hopf module;*
- (2) *two Hopf 1-cocycles X and X' are cohomologous if and only if the (H, E) -Hopf modules (E, Δ_E^X) and $(E, \Delta_E^{X'})$ are isomorphic.*

Proof. (1) Let us prove that Δ_E^X defines a coaction on E if and only if X belongs to $\mathcal{Z}^1(H, E)$. Suppose that X is a Hopf 1-cocycle. We have to show the two identities $(\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X = (\text{id}_E \otimes \Delta_H) \circ \Delta_E^X$ and $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E^X = \text{id}_E$. Pick an element x in E . On the one hand, since Δ_E is a morphism of algebras, one has the equalities

$$\begin{aligned} ((\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X)(x) &= (\Delta_E^X \otimes \text{id}_H)(X \Delta_E(x)) \\ &= (X \otimes 1)((\Delta_E \otimes \text{id}_H)(X \Delta_E(x))) \\ &= ((X \otimes 1)((\Delta_E \otimes \text{id}_H)(X)))(((\Delta_E \otimes \text{id}_H) \circ \Delta_E)(x)). \end{aligned}$$

On the other hand, the following equalities hold:

$$\begin{aligned} ((\text{id}_E \otimes \Delta_H) \circ \Delta_E^X)(x) &= (\text{id}_E \otimes \Delta_H)(X \Delta_E(x)) \\ &= ((\text{id}_E \otimes \Delta_H)(X))(((\text{id}_E \otimes \Delta_H) \circ \Delta_E)(x)). \end{aligned}$$

Since $((\Delta_E \otimes \text{id}_H) \circ \Delta_E)(x)$ is equal to $((\text{id}_E \otimes \Delta_H) \circ \Delta_E)(x)$, it remains to remark that the identity $(X \otimes 1)((\Delta_E \otimes \text{id}_H)(X)) = (\text{id}_E \otimes \Delta_H)(X)$ is exactly the cocycle relation $d^2(X)d^0(X) = d^1(X)$. In a similar way, using Remark 1.3(a) and the identity $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E = \text{id}_E$, one proves the equality $(\text{id}_E \otimes \varepsilon_H) \circ \Delta_E^X = \text{id}_E$.

The map Δ_E is a morphism of algebras, whence for any x and x' in E , one has the equality $\Delta_E(xx') = \Delta_E(x)\Delta_E(x')$. So one gets $X \Delta_E(xx') = X \Delta_E(x)\Delta_E(x')$, or $\Delta_E^X(xx') = \Delta_E^X(x)\Delta_E^X(x')$. This proves that (E, Δ_E^X) is an (H, E) -Hopf module, where the E -module structure of E is still given by the multiplication.

Conversely, assume that Δ_E^X endows E with a structure of (H, E) -Hopf module. Applying the identity $(\Delta_E^X \otimes \text{id}_H) \circ \Delta_E^X = (\text{id}_E \otimes \Delta_H) \circ \Delta_E^X$ to the element $x = 1$, one obtains the cocycle relation for X .

(2) Suppose now given two cohomologous Hopf 1-cocycles X and X' . Let x be an element of E^\times such that $X' = (d^1 x^{-1})X(d^0 x)$. One easily checks that $\tau_x : E \rightarrow E$, the left multiplication by x , realizes an isomorphism of (H, E) -Hopf module from $(E, \Delta_E^{X'})$ to (E, Δ_E^X) .

Conversely, assume that for two Hopf 1-cocycles X and X' , there exists an isomorphism of (H, E) -Hopf modules $\varphi : (E, \Delta_E^X) \rightarrow (E, \Delta_E^{X'})$. By E -linearity, φ is entirely determined by $\varphi(1)$, more precisely $\varphi = \tau_{\varphi(1)}$. Since φ is surjective, the element $\varphi(1)$ is invertible in E . The comodule compatibility relation $\Delta_E^{X'} \circ \varphi = (\varphi \otimes \text{id}_H) \circ \Delta_E^X$ then implies $d^1(\varphi(1))X = X'd^0(\varphi(1))$. \square

1.6. The cohomology exact sequence associated to a sub-comodule algebra

Let H be a Hopf algebra. By the very definition, any morphism $\varphi : D \rightarrow E$ of H -comodule algebras gives rise to a group homomorphism $\mathcal{H}^0(\varphi) : \mathcal{H}^0(H, D) \rightarrow \mathcal{H}^0(H, E)$ and to a morphism of pointed sets $\mathcal{H}^1(\varphi) : \mathcal{H}^1(H, D) \rightarrow \mathcal{H}^1(H, E)$. Our aim is to produce an exact sequence in cohomology associated to any inclusion $\varphi : D \hookrightarrow E$ of H -comodule algebras. To this purpose, we state the following lemma, which is a slight generalization to the cosimplicial case of Serre’s exact sequence announced in the framework of non-abelian cohomology theory of groups.

Lemma 1.8. *Let $\varphi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ be an injective morphism of two pre-cosimplicial groups*

$$\mathcal{A}^* = A^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} A^1 \begin{array}{c} \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \xrightarrow{d^3} \end{array} A^2 \quad \text{and} \quad \mathcal{B}^* = B^0 \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} B^1 \begin{array}{c} \xrightarrow{d^1} \\ \xrightarrow{d^2} \\ \xrightarrow{d^3} \end{array} B^2.$$

Let $\mathcal{C}^* = \mathcal{B}^*/\varphi(\mathcal{A}^*)$ be the pre-cosimplicial left quotient object in the category of pointed sets and let $\pi : \mathcal{B}^* \rightarrow \mathcal{C}^*$ be the quotient map. Then there is an exact sequence of pointed sets

$$1 \rightarrow \mathbb{H}^0(\mathcal{A}^*) \xrightarrow{\mathbb{H}^0(\varphi)} \mathbb{H}^0(\mathcal{B}^*) \xrightarrow{\mathbb{H}^0(\pi)} \mathbb{H}^0(\mathcal{C}^*) \xrightarrow{\partial} \mathbb{H}^1(\mathcal{A}^*) \xrightarrow{\mathbb{H}^1(\varphi)} \mathbb{H}^1(\mathcal{B}^*).$$

Moreover, if $\varphi(A^i)$ is for $i = 0, 1, 2$ a normal subgroup of B^i , then the above exact sequence extends to the right in the following way:

$$1 \rightarrow \mathbb{H}^0(\mathcal{A}^*) \xrightarrow{\mathbb{H}^0(\varphi)} \mathbb{H}^0(\mathcal{B}^*) \xrightarrow{\mathbb{H}^0(\pi)} \mathbb{H}^0(\mathcal{C}^*) \xrightarrow{\partial} \mathbb{H}^1(\mathcal{A}^*) \xrightarrow{\mathbb{H}^1(\varphi)} \mathbb{H}^1(\mathcal{B}^*) \xrightarrow{\mathbb{H}^1(\pi)} \mathbb{H}^1(\mathcal{C}^*).$$

Proof. The connecting morphism ∂ is obtained by usual diagram-chasing. We leave the reader check the functoriality of \mathbb{H}^* as well as the exactness of the two sequences. \square

We mention here that the definition of the non-abelian 0-cohomology object $\mathbb{H}^0(\mathcal{A}^*)$ as an equalizer does in fact not require any algebraic structure on the set A^0 . This observation leads to the following definition. For any inclusion of H -comodule algebras $\varphi : D \hookrightarrow E$, we introduce the relative non-abelian 0-cohomology set

$$\mathcal{H}^0(H, D \hookrightarrow E) = \mathbb{H}^0(\mathcal{C}_{\leq 2}^\times(H, E)/\mathcal{C}_{\leq 2}^\times(H, D)),$$

where $\mathcal{C}_{\leq 2}^{\times}(H, E)/\mathcal{C}_{\leq 2}^{\times}(H, D)$ is the pre-cosimplicial diagram of pointed sets

$$E^{\times}/D^{\times} \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} (E \otimes H)^{\times}/(D \otimes H)^{\times} \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} (E \otimes H \otimes H)^{\times}/(D \otimes H \otimes H)^{\times}.$$

In the particular case where D^{\times} , $(D \otimes H)^{\times}$, and $(D \otimes H \otimes H)^{\times}$, are normal subgroups respectively of E^{\times} , $(E \otimes H)^{\times}$, and $(E \otimes H \otimes H)^{\times}$, then $\mathcal{C}_{\leq 2}^{\times}(H, E)/\mathcal{C}_{\leq 2}^{\times}(H, D)$ is a pre-cosimplicial group, and the definition

$$\mathcal{H}^1(H, D \hookrightarrow E) = \mathbb{H}^1(\mathcal{C}_{\leq 2}^{\times}(H, E)/\mathcal{C}_{\leq 2}^{\times}(H, D))$$

makes sense. Next result is a corollary of Lemma 1.8.

Proposition 1.9. *Let H be a Hopf algebra and $\varphi : D \rightarrow E$ be an injective morphism of H -comodule algebras. The sequence of pointed sets*

$$1 \longrightarrow \mathcal{H}^0(H, D) \xrightarrow{\mathcal{H}^0(\varphi)} \mathcal{H}^0(H, E) \xrightarrow{\mathcal{H}^0(\pi)} \mathcal{H}^0(H, D \hookrightarrow E) \xrightarrow{\partial} \mathcal{H}^1(H, D) \\ \xrightarrow{\mathcal{H}^1(\varphi)} \mathcal{H}^1(H, E)$$

is exact. Moreover, if D^{\times} , $(D \otimes H)^{\times}$, and $(D \otimes H \otimes H)^{\times}$ are normal subgroups respectively of E^{\times} , $(E \otimes H)^{\times}$, and $(E \otimes H \otimes H)^{\times}$, then the above exact sequence can be extended to the right in the following way:

$$1 \longrightarrow \mathcal{H}^0(H, D) \xrightarrow{\mathcal{H}^0(\varphi)} \mathcal{H}^0(H, E) \xrightarrow{\mathcal{H}^0(\pi)} \mathcal{H}^0(H, D \hookrightarrow E) \xrightarrow{\partial} \mathcal{H}^1(H, D) \\ \xrightarrow{\mathcal{H}^1(\varphi)} \mathcal{H}^1(H, E) \xrightarrow{\mathcal{H}^1(\pi)} \mathcal{H}^1(H, D \hookrightarrow E).$$

2. Links between general and restricted non-abelian Hopf cohomology theory

In this section, H is a Hopf algebra, S is an H -comodule algebra, and M is an (H, S) -Hopf module. In [9], we introduced a cohomology theory $H^*(H, M)$ that we qualify from now on as *restricted*. We first briefly recall its definition and then compare it to our general cohomology theory under some lax technical conditions.

2.1. Reminder on restricted non-abelian Hopf cohomology theory

As in [9], we endow the set $W_k^n(M) = \text{Hom}_k(M, M \otimes H^{\otimes n})$ with a k -algebra structure thanks to the composition-type product

$$\odot : W_k^n(M) \times W_k^n(M) \longrightarrow W_k^n(M)$$

given by

$$\begin{cases} \varphi \odot \varphi' = \varphi \circ \varphi' & \text{if } n = 0, \\ \varphi \odot \varphi' = (\text{id}_M \otimes \mu_H^{\otimes n}) \circ (\text{id}_M \otimes \chi_n) \circ (\varphi \otimes \text{id}_H^{\otimes n}) \circ \varphi' & \text{if } n > 0, \end{cases}$$

for $\varphi, \varphi' \in W_k^n(M)$; here $\chi_n : H^{\otimes n} \otimes H^{\otimes n} \rightarrow (H \otimes H)^{\otimes n}$ denotes the intertwining operator defined by

$$\chi_n((a_1 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_n)) = (a_1 \otimes b_1) \otimes \cdots \otimes (a_n \otimes b_n).$$

Denote by $W_S^n(M)$ the subalgebra $\text{Hom}_S(M, M \otimes H^{\otimes n})$ of $W_k^n(M)$, where the S -module structure on $M \otimes H^{\otimes n}$ is given by $(m \otimes \underline{h})s = ms \otimes \underline{h}$, for any $m \in M, \underline{h} \in H^{\otimes n}$, and $s \in S$.

Let R be either the ground ring k or the algebra S . The algebras $W_R^0(M), W_R^1(M)$ and $W_R^2(M)$ may be organized in a pre-cosimplicial diagram of monoids [9, Lemma 1.1]:

$$W_{\leq 2}(H, M)_R = \left(\begin{array}{ccccc} & & b^0 & & \\ & & \rightarrow & & \\ W_R^0(M) & \xrightarrow{b^0} & W_R^1(M) & \xrightarrow{b^0} & W_R^2(M) \\ & \xrightarrow{b^1} & & \xrightarrow{b^1} & \\ & & & & b^2 \\ & & & & \rightarrow \\ & & & & \end{array} \right).$$

The two maps $b^i : W_R^0(M) \rightarrow W_R^1(M)$ ($i = 0, 1$) and the three maps $b^i : W_R^1(M) \rightarrow W_R^2(M)$ ($i = 0, 1, 2$) are given for $\varphi \in W_R^0(M)$ and $\Phi \in W_R^1(M)$ by the formulae

$$\begin{aligned} b^0\varphi &= (\text{id}_M \otimes \mu_H) \circ (\Delta_M \otimes \text{id}_H) \circ (\varphi \otimes \sigma_H) \circ \Delta_M, \\ b^1\varphi &= (\text{id}_M \otimes \eta_H) \circ \varphi, \\ b^0\Phi &= (\text{id}_M \otimes \mu_H \otimes \text{id}_H) \circ (\Delta_M \otimes T) \circ (\Phi \otimes \sigma_H) \circ \Delta_M, \\ b^1\Phi &= (\text{id}_M \otimes \Delta_H) \circ \Phi, \\ b^2\Phi &= (\text{id}_M \otimes \text{id}_H \otimes \eta_H) \circ \Phi = \Phi \otimes \eta_H, \end{aligned}$$

where T denotes the flip of $H \otimes H$ (i.e. the automorphism of $H \otimes H$ which sends an indecomposable tensor $h \otimes h'$ to $h' \otimes h$).

Recall the definitions stated in [9]. The restricted 0-cohomology group $H^0(H, M)$ is the equalizer $\{\varphi \in \text{Aut}_S(M) \mid b^1\varphi = b^0\varphi\}$ of the pair (b^0, b^1) . The restricted 1-cohomology set $H^1(H, M)$ is the quotient set $\text{Aut}_S(M) \setminus Z^1(H, M)$ of the set $Z^1(H, M)$ of restricted Hopf 1-cocycles of H with coefficients in M under the right action of the group $\text{Aut}_S(M)$. Recall that $Z^1(H, M)$ is the subgroup

$$Z^1(H, M) = \left\{ \Phi \in W_k^1(M) \left| \begin{array}{l} (\text{ZC}_1) \ \Phi(ms) = \Phi(m)s, \text{ for all } m \in M \text{ and } s \in S, \\ (\text{ZC}_2) \ (\text{id}_M \otimes \varepsilon_H) \circ \Phi = \text{id}_M, \\ (\text{ZC}_3) \ b^2\Phi \circ b^0\Phi = b^1\Phi \end{array} \right. \right\}$$

of $W_S^1(M)$ and an element $f \in \text{Aut}_S(M)$ acts on the right on an element $\Phi \in Z^1(H, M)$ by

$$(\Phi \longleftarrow f) = b^1 f^{-1} \circ \Phi \circ b^0 f.$$

We now give a new alternative description of $Z^1(H, M)$ which we shall need in the sequel.

Proposition 2.1. *The set $Z^1(H, M)$ may be written as*

$$Z^1(H, M) = \{ \Phi \in W_S^1(M)^\times \mid b^2\Phi \circ b^0\Phi = b^1\Phi \}.$$

Proof. Let Φ be an element of $Z^1(H, M)$. First observe that Condition (ZC₁) means exactly that Φ belongs to $W^1_S(M)$. It suffices to prove that Condition (ZC₂) is equivalent to the \odot -invertibility of Φ under Condition (ZC₃). Set $F = \Phi \odot \Delta_M$. In the proof of Theorem 3.1 in [9], we showed that Φ satisfies (ZC₂) if and only if F satisfies Condition (CC₂), that is $(\text{id}_M \otimes \varepsilon_H) \circ F = \text{id}_M$. Similarly Φ fulfills (ZC₃) if and only if F fulfills Condition (CC₃), that is $(F \otimes \text{id}_H) \circ F = (\text{id}_M \otimes \Delta_H) \circ F$.

(1) Suppose that Φ is invertible in $W^1_S(M)$ with inverse Φ' . Since the comultiplication map Δ_M is invertible in $W^1_k(M)$ with inverse $\Delta'_M = (\text{id}_M \otimes \sigma_H) \circ \Delta_M$, the map F is invertible in $W^1_k(M)$ with inverse $F' = \Delta'_M \odot \Phi'$. Compose both terms of the equality (CC₃) on the left with the map $\text{id}_M \otimes \text{id}_H \otimes \varepsilon_H$. One gets $F = F \circ ((\text{id}_M \otimes \varepsilon_H) \circ F)$, which is equivalent to the relation $F = F \odot (((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H)$. One may simplify by F , and one gets $((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H = \text{id}_{W^1_k(M)} = \text{id}_M \otimes \eta_H$. Applying now $\text{id}_M \otimes \varepsilon_H$ on the right, one obtains (CC₂).

(2) Conversely, assume that Condition (CC₂) holds. We shall show that the map F' defined by $F' = (\text{id}_M \otimes \sigma_H) \circ F$ is the inverse of F in $W^1_k(M)$. We apply therefore $\text{id}_M \otimes (\mu_H \circ (\text{id}_H \otimes \sigma_H))$, respectively $\text{id}_M \otimes (\mu_H \circ (\sigma_H \otimes \text{id}_H))$, on the left to the equality (CC₃). We get $((\text{id}_M \otimes \varepsilon_H) \circ F) \otimes \eta_H = F \odot F'$, respectively $((\text{id}_M \otimes \varepsilon_H) \circ F) = F' \odot F$. By Condition (CC₂), this exactly means that F' is the inverse of F . The map Φ is therefore invertible in $W^1_k(M)$ with inverse

$$\Phi' = \Delta_M \odot F' = \Delta_M \odot ((\text{id}_M \otimes \sigma_H) \circ (\Phi \odot \Delta_M)).$$

It remains to show that Φ' is S -linear. For $m \in M$, we denote the tensor $\Phi(m) \in M \otimes H$ by $m_{[0]} \otimes m_{[1]}$. We have

$$\Phi'(m) = ((m_0)_{[0]})_0 \otimes ((m_0)_{[0]})_1 \sigma_H((m_0)_{[1]} m_1).$$

For any $s \in S$, we obtain

$$\begin{aligned} \Phi'(ms) &= ((m_0 s_0)_{[0]})_0 \otimes ((m_0 s_0)_{[0]})_1 \sigma_H((m_0 s_0)_{[1]} m_1 s_1) \\ &= ((m_0)_{[0]} s_0)_0 \otimes ((m_0)_{[0]} s_0)_1 \sigma_H(s_1) \sigma_H((m_0)_{[1]} m_1) \\ &= ((m_0)_{[0]})_0 s_0 \otimes ((m_0)_{[0]})_1 s_1 \sigma_H(s_2) \sigma_H((m_0)_{[1]} m_1) \\ &= ((m_0)_{[0]})_0 s_0 \varepsilon_H(s_1) \otimes ((m_0)_{[0]})_1 \sigma_H((m_0)_{[1]} m_1) \\ &= \Phi'(m) s. \end{aligned}$$

This computation, which proves the S -linearity of Φ' , uses the Hopf algebra yoga. Moreover the first and the third equalities come from $\Delta_M(ms) = \Delta_M(m) \Delta_S(s)$, whereas the second one is a consequence of the S -linearity of Φ . \square

Denote by

$$\mathcal{W}_{\leq 2}^\times(H, M) = \left(W^0_S(M)^\times \begin{array}{c} \xrightarrow{b^0} \\ \xrightarrow{b^1} \\ \xrightarrow{b^2} \end{array} W^1_S(M)^\times \begin{array}{c} \xrightarrow{b^0} \\ \xrightarrow{b^1} \\ \xrightarrow{b^2} \end{array} W^2_S(M)^\times \right),$$

the pre-cosimplicial diagram of groups obtained by taking the \odot -invertible elements of $W_S^*(M)$. Proposition 2.1 leads us to state the following result:

Theorem 2.2. *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module. One has the equality*

$$H^*(H, M) = \mathbb{H}^*(W_{\leq 2}^\times(H, M)).$$

2.2. Technical conditions

In this paragraph, we first point out technical conditions we shall need in the sequel in order to compare the general and the restricted non-abelian Hopf cohomology theories. We show then that these conditions are fulfilled in two natural cases.

For any $n \geq 0$, consider the linear map

$$\omega_n : \text{End}_S(M) \otimes H^{\otimes n} \longrightarrow W_S^n(M) = \text{Hom}_S(M, M \otimes H^{\otimes n})$$

given on an undecomposable tensor $f \otimes \underline{h} \in \text{End}_S(M) \otimes H^{\otimes n}$ by

$$\omega_n(f \otimes \underline{h})(m) = f(m) \otimes \underline{h},$$

where $m \in M$. Notice that ω_0 is the identity map of $\text{End}_S(M)$ and that, for any $n \geq 0$, the map ω_n is a morphism of algebras. For $n \geq 0$, we consider the following condition.

Condition (\mathcal{F}_n) : the map ω_n is an isomorphism of algebras.

By the very definitions, Condition (\mathcal{F}_0) always holds. The first natural case where Condition (\mathcal{F}_n) is satisfied for all $n \geq 0$ appears when H is a finitely generated free k -module. We develop now a second case.

Let $M^* = \text{Hom}_k(M, k)$ be the linear dual of the k -module M . Consider the evaluation map $d_M : M^* \otimes M \longrightarrow k$ given by $d_M(v \otimes m) = v(m)$, for any $v \in M^*$ and $m \in M$.

Proposition 2.3. *Condition (\mathcal{F}_n) is satisfied for all $n \geq 0$ if the two following statements both hold:*

- (1) *the Hopf algebra H is free as a k -module or S is equal to the ground ring k ;*
- (2) *there exists a map $b_M : k \longrightarrow M \otimes M^*$, called birth-map, such that*

$$(\text{id}_M \otimes d_M) \circ (b_M \otimes \text{id}_M) = \text{id}_M \quad \text{and} \quad (d_M \otimes \text{id}_{M^*}) \circ (\text{id}_{M^*} \otimes b_M) = \text{id}_{M^*}.$$

By convention, we set $b_M(1) = \sum_i e_i \otimes e^i$. With this notation, the previous two equalities are equivalent to

$$\sum_i e_i e^i(m) = m \quad \text{and} \quad \sum_i v(e_i) e^i = v,$$

for any $m \in M$ and $v \in M^*$.

Example. When M is a finitely generated free k -module with basis $(e_j)_{j=1,\dots,n}$ such a birth-map b_M exists and is given by $b_M(1) = \sum_{j=1}^n e_j \otimes e_j^*$. Here $(e_j^*)_{j=1,\dots,n}$ is the dual basis of $(e_j)_{j=1,\dots,n}$.

The data of a module together with an evaluation map and a birth-map abstracts the notion of duality in tensor categories (see [2]).

Proof of Proposition 2.3. First of all, we endow M^* with the left S -module structure given by

$$(sv)(m) = v(ms)$$

with $v \in M^*$, $m \in M$, and $s \in S$. The module $M \otimes M^* \otimes H^{\otimes n}$ becomes an algebra through the multiplication given on two elements $m \otimes v \otimes \underline{h}$ and $m' \otimes v' \otimes \underline{h}'$ of $M \otimes M^* \otimes H^{\otimes n}$ by the formula $(m \otimes v \otimes \underline{h})(m' \otimes v' \otimes \underline{h}') = v(m')m \otimes v \otimes \underline{hh}'$. We introduce the subalgebra $E_S^n(M)$ of $M \otimes M^* \otimes H^{\otimes n}$ consisting of the elements $m \otimes v \otimes \underline{h}$ such that, for any $s \in S$, one has $ms \otimes v \otimes \underline{h} = m \otimes sv \otimes \underline{h}$. Notice that under the first statement, one has the equality

$$E_S^n(M) = E_S^0(M) \otimes H^{\otimes n}.$$

We show now that, under the second statement, $E_S^n(M)$ is isomorphic to $W_S^n(M)$ as an algebra. First observe that the existence of a birth-map allows to write the action of s on v as $sv = \sum_i v(e_i s)e^i$. Moreover one has $\sum_i e_i s \otimes e^i = \sum_i e_i \otimes se^i$, in other words, $b_M(1)$ belongs to $E_S^0(M)$.

Consider the morphism $\lambda_n : E_S^n(M) \rightarrow W_S^n(M)$ defined by

$$(\lambda_n(m \otimes v \otimes \underline{h}))(m') = v(m')m \otimes \underline{h},$$

with $m, m' \in M$, $v \in M^*$, and $\underline{h} \in H^{\otimes n}$. One checks that λ_n is well defined with respect to the S -invariance and that it is a morphism of algebras. We prove now that under the existence of a birth-map, λ_n is a bijection. Let us explicit the inverse map. Denote by $\lambda'_n : W_S^n(M) \rightarrow M \otimes M^* \otimes H^{\otimes n}$ the map given on an element $\Phi \in W_S^n(M)$ by

$$\lambda'_n(\Phi) = \sum_i \Phi(e_i)_0 \otimes e^i \otimes \Phi(e_i)_1,$$

where, for any $m \in M$, we set $\Phi(m) = \Phi(m)_0 \otimes \Phi(m)_1 \in M \otimes H^{\otimes n}$.

The map λ'_n takes its values in $E_S^n(M)$. Indeed, using the S -linearity of $\Phi \in W_S^n(M)$ and the fact that $b_M(1)$ belongs to $E_S^0(M)$, we have, for any $s \in S$:

$$\sum_i \Phi(e_i)_0 \otimes se^i \otimes \Phi(e_i)_1 = \sum_i \Phi(e_i s)_0 \otimes e^i \otimes \Phi(e_i s)_1 = \sum_i \Phi(e_i)_0 s \otimes e^i \otimes \Phi(e_i)_1.$$

Moreover the map λ'_n is a morphism of algebras: for $\Phi, \Psi \in W_S^n(M)$, one has

$$\lambda'_n(\Phi)\lambda'_n(\Psi) = \sum_{i,j} e^i(\Psi(e_j)_0)\Phi(e_i)_0 \otimes e^j \otimes \Phi(e_i)_1\Psi(e_j)_1$$

$$\begin{aligned}
 &= \sum_{i,j} \Phi(e^i(\Psi(e_j)_0)e_i)_0 \otimes e^j \otimes \Phi(e^i(\Psi(e_j)_0)e_i)_1 \Psi(e_j)_1 \\
 &= \sum_j \Phi(\Psi(e_j)_0)_0 \otimes e^j \otimes \Phi(\Psi(e_j)_0)_1 \Psi(e_j)_1 \\
 &= \sum_j (\Phi \circ \Psi)(e_j)_0 \otimes e^j \otimes (\Phi \circ \Psi)(e_j)_1 \\
 &= \lambda'_n(\Phi \circ \Psi).
 \end{aligned}$$

It remains to compute the two compositions $\lambda_n \circ \lambda'_n$ and $\lambda'_n \circ \lambda_n$. One has, for any $\Phi \in W_S^n(M)$ and $m \in M$:

$$\lambda_n(\lambda'_n(\Phi))(m) = \sum_i e^i(m)\Phi(e_i)_0 \otimes \Phi(e_i)_1 = \Phi\left(\sum_i e^i(m)e_i\right) = \Phi(m).$$

On the other hand, for $m \otimes v \otimes \underline{h} \in M \otimes M^* \otimes H^{\otimes n}$, one obtains

$$\lambda'_n(\lambda_n(m \otimes v \otimes \underline{h})) = \sum_i v(e_i)m \otimes e^i \otimes \underline{h} = m \otimes \left(\sum_i v(e_i)e^i\right) \otimes \underline{h} = m \otimes v \otimes \underline{h}.$$

To end the proof, we write down the following sequence of isomorphisms, the composition of which is ω_n :

$$\text{End}_S(M) \otimes H^{\otimes n} = W_S^0(M) \otimes H^{\otimes n} \xrightarrow{\lambda_0 \otimes \text{id}_H^{\otimes n}} E_S^0(M) \otimes H^{\otimes n} = E_S^n(M) \xrightarrow{\lambda_n^{-1}} W_S^n(M).$$

Hence ω_n is an isomorphism of algebras, i.e. Condition (\mathcal{F}_n) is fulfilled. \square

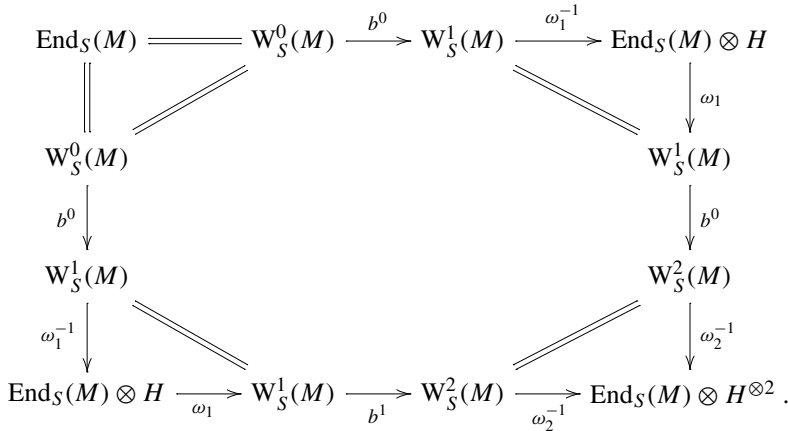
2.3. An H -comodule structure on $\text{End}_S(M)$

Suppose from now on that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. We define the morphism $\Delta_{\text{End}_S(M)} : \text{End}_S(M) \rightarrow \text{End}_S(M) \otimes H$ to be the composition map

$$\text{End}_S(M) = W_S^0(M) \xrightarrow{b^0} W_S^1(M) \xrightarrow{\omega_1^{-1}} \text{End}_S(M) \otimes H.$$

Lemma 2.4. *The map $\Delta_{\text{End}_S(M)}$ endows $\text{End}_S(M)$ with a structure of H -comodule algebra.*

Proof. As a composition of morphisms of algebras, $\Delta_{\text{End}_S(M)}$ is a morphism of algebras. Let us prove that $\Delta_{\text{End}_S(M)}$ is coassociative. To this end, consider the following diagram in which the upper horizontal and the left vertical compositions are $\Delta_{\text{End}_S(M)}$:



The pre-cosimplicial relation $b^0b^0 = b^1b^0$ implies the commutativity of the inner octagon, hence of the whole diagram. One may see that the lower horizontal composition is $\text{id}_{\text{End}_S(M)} \otimes \Delta_H$ and that the right vertical composition is $\Delta_{\text{End}_S(M)} \otimes \text{id}_H$. This shows the coassociativity of $\Delta_{\text{End}_S(M)}$.

The compatibility with the counit $(\text{id}_{\text{End}_S(M)} \otimes \varepsilon_H) \circ \Delta_{\text{End}_S(M)} = \text{id}_{\text{End}_S(M)}$ is a consequence of the relation $(\text{id}_{\text{End}_S(M)} \otimes \varepsilon_H) \circ b^0(\varphi) = \varphi$, which holds for all $\varphi \in \text{End}_S(M)$. \square

This construction allows us to define the cohomology of the Hopf algebra H with values in the H -comodule algebra $\text{End}_S(M)$. So, under the hypothesis that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$, the cohomology sets $\mathcal{H}^i(H, \text{End}_S(M))$ ($i = 0, 1$) make sense.

2.4. The comparison theorem

We are now able to compare restricted and general non-abelian Hopf cohomology theories.

Proposition 2.5. *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. The pre-cosimplicial groups $C_{\leq 2}^{\times}(H, M)$ and $W_{\leq 2}^{\times}(H, \text{End}_S(M))$ are isomorphic.*

Proof. The map ω_n is an isomorphism of algebras since Condition (\mathcal{F}_n) holds. Moreover, one checks the equalities

$$\omega_j d^i = b^i \omega_{j-1}$$

for any $1 \leq j \leq 2$ and $0 \leq i \leq j$. \square

Theorem 2.2 and Proposition 2.5 imply the following result:

Theorem 2.6. *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. Then there is an equality of groups*

$$\mathcal{H}^0(H, \text{End}_S(M)) = H^0(H, M)$$

and an isomorphism of pointed sets

$$\mathcal{H}^1(H, \text{End}_S(M)) \cong H^1(H, M).$$

3. Hopf torsors

In this section, we define Hopf torsors. They generalize the classical torsors used in the framework of groups. We show that Hopf torsors are classified by a general non-abelian Hopf 1-cohomology set.

3.1. Definition of Hopf torsors

Let E be an algebra and T be a right E -module. For any $u \in T$, consider the E -linear map $\vartheta_u : E \rightarrow T$ defined on $x \in E$ by $\vartheta_u(x) = ux$. Denote by T^\times the set

$$T^\times = \{u \in T \mid \vartheta_u \text{ is bijective}\}.$$

From now on, we deal with E -modules T such that T^\times is not empty. For example, if T is E itself the above set coincides with the group E^\times of invertible elements of the algebra E . Moreover, for any E -module T , observe that T^\times inherits the structure of an E^\times -set. In the following lemma, we collect two technical results about T^\times .

Lemma 3.1. *Let E be an algebra and T be a right E -module such that the set T^\times is not empty.*

- (1) *Let u be an element of T^\times . Then $\vartheta_u^{-1}(v)$ is, for any $v \in T$, the unique element of E such that $u\vartheta_u^{-1}(v) = v$.*
- (2) *For any $u \in T^\times$, the map ϑ_u realizes a bijection between E^\times and T^\times .*

Proof. The first point is a direct consequence of the definition of ϑ_u . Let us prove the second point. We have to show that, for any $u \in T^\times$, the bijection $\vartheta_u : E \rightarrow T$ restricts to a bijection between E^\times and T^\times . For any $u \in T^\times$, the set $\vartheta_u(E^\times)$ is contained in T^\times . Indeed if x belongs to E^\times , one has $\vartheta_{ux} = \vartheta_u \circ \tau_x$, where τ_x denotes the left multiplication by x , which is bijective. The induced map remains injective. To prove that it is surjective, it is sufficient to show that $\vartheta_u^{-1}(v)$ belongs to E^\times for any $v \in T^\times$. By definition, one has $u\vartheta_u^{-1}(v)\vartheta_v^{-1}(u) = v\vartheta_v^{-1}(u) = u$, so $\vartheta_u^{-1}(v)\vartheta_v^{-1}(u) = 1$. \square

Let H be a Hopf algebra, E be an H -comodule algebra, and (T, Δ_T) be an (H, E) -Hopf module. In this situation, the tensor product $T \otimes H$ is an $E \otimes H$ -module and $(T \otimes H)^\times$ makes sense. Notice that if u belongs to T^\times , then $u \otimes 1$ lies in $(T \otimes H)^\times$, since $\vartheta_{u \otimes 1} = \vartheta_u \otimes \text{id}_H$. In particular, if T^\times is non-empty, so is $(T \otimes H)^\times$.

We introduce now the set

$$T^\bullet = \{u \in T^\times \mid \Delta_T(u) \in (T \otimes H)^\times\}.$$

Definition 3.2. Let H be a Hopf algebra, E be an H -comodule algebra. An (H, E) -Hopf torsor is an (H, E) -Hopf module (T, Δ_T) such that the set T^\bullet is non-empty.

In particular E is an (H, E) -Hopf torsor. Indeed E is an (H, E) -Hopf module and Δ_E being a morphism of algebras, Δ_E sends any element of E^\times into $(E \otimes H)^\times$.

We denote by $\text{tors}(H, E)$ the set of (H, E) -Hopf torsors. It is pointed with distinguished point (E, Δ_E) . Two (H, E) -torsors (T, Δ_T) and $(T', \Delta_{T'})$ are *equivalent* if T and T' are isomorphic as (H, E) -Hopf modules. We denote by $\text{Tors}(H, E)$ the set of equivalence classes of (H, E) -torsors; it is pointed with distinguished point the class of (E, Δ_E) .

Lemma 3.3. Let T be an (H, E) -Hopf torsor. Then the sets T^\bullet and T^\times coincide.

Proof. Pick v in T^\times and u in T^\bullet . One has $v = u\vartheta_u^{-1}(v)$, thus $\Delta_T(v) = \Delta_T(u)\Delta_E(\vartheta_u^{-1}(v))$. By definition, the term $\Delta_T(u)$ belongs to $(T \otimes H)^\times$, and the factor $\Delta_E(\vartheta_u^{-1}(v))$ is invertible in $E \otimes H$ since Δ_E is a morphism of algebras. In the same way as E^\times acts on T^\times , the group $(E \otimes H)^\times$ acts on $(T \otimes H)^\times$, hence $\Delta_T(v)$ is an element of $(T \otimes H)^\times$, in other words v belongs to T^\bullet . \square

3.2. The non-abelian 1-Hopf cohomology set and Hopf torsors

As in the world of groups, the Hopf torsors are classified by a non-abelian 1-cohomology set. We detail this point now.

Theorem 3.4. Let H be a Hopf algebra and E be an H -comodule algebra. There is an isomorphism of pointed sets

$$\mathcal{H}^1(H, E) \cong \text{Tors}(H, E).$$

Proof. We construct a map $\tilde{\mathcal{T}} : \mathcal{Z}^1(H, E) \rightarrow \text{tors}(H, E)$ in the following way. For any Hopf 1-cocycle X , let $\tilde{\mathcal{T}}(X)$ be the (H, E) -Hopf module (E, Δ_E^X) defined in Section 1.5. It is clearly a torsor (indeed T^\bullet contains for example the unit of E). By Proposition 1.7, the map $\tilde{\mathcal{T}}$ induces a map $\mathcal{T} : \mathcal{H}^1(H, E) \rightarrow \text{Tors}(H, E)$ on the quotients.

The injectivity of \mathcal{T} is a direct consequence of Proposition 1.7. Let us prove that \mathcal{T} is surjective. Take a torsor (T, Δ_T) and $u \in T^\bullet$. By definition, $\Delta_T(u)$ belongs to $(T \otimes H)^\times$. Applying the map $\vartheta_{u \otimes 1}^{-1} = \vartheta_u^{-1} \otimes \text{id}_H$, we define the element

$$X_T = (\vartheta_{u \otimes 1}^{-1} \circ \Delta_T)(u) = ((\vartheta_u^{-1} \otimes \text{id}_H) \circ \Delta_T)(u),$$

which belongs to $(E \otimes H)^\times$. Writing $\Delta_T(u) = u_0 \otimes u_1$, one gets $X_T = \vartheta_u^{-1}(u_0) \otimes u_1$. Let us compute the product $(u \otimes 1 \otimes 1)(d^2(X_T)d^0(X_T))$. First remark that we have the equalities

$$(u \otimes 1 \otimes 1)d^2(X_T) = (\vartheta_u \vartheta_u^{-1} \otimes \text{id}_H)(\Delta_T(u)) \otimes 1 = \Delta_T(u) \otimes 1.$$

On the other hand, we write

$$d^0(X_T) = (\Delta_E \circ \vartheta_u^{-1} \otimes \text{id}_H)(\Delta_T(u)) = \Delta_E(\vartheta_u^{-1}(u_0)) \otimes u_1.$$

By multiplying the two expressions, we get

$$\begin{aligned} (u \otimes 1 \otimes 1)d^2(X_T)d^0(X_T) &= \Delta_T(u)\Delta_E(\vartheta_u^{-1}(u_0)) \otimes u_1 \\ &= \Delta_T(u\vartheta_u^{-1}(u_0)) \otimes u_1 \\ &= \Delta_T(u_0) \otimes u_1 \\ &= u_0 \otimes \Delta_H(u_1). \end{aligned}$$

Finally we obtain

$$d^2(X_T)d^0(X_T) = \vartheta_u^{-1}(u_0) \otimes \Delta_H(u_1) = (\text{id}_E \otimes \Delta_H)(X_T) = d^1(X_T).$$

Hence X_T is a Hopf 1-cocycle.

We show now that the torsors (T, Δ_T) and $\tilde{T}(X_T) = (E, \Delta_E^{X_T})$ are equivalent. The wished isomorphism of Hopf modules between $(E, \Delta_E^{X_T})$ and (T, Δ_T) is given by the map ϑ_u . Indeed, for any element $x \in E$, one has the equalities

$$\begin{aligned} ((\vartheta_u \otimes \text{id}_H) \circ \Delta_E^{X_T})(x) &= (\vartheta_u \otimes \text{id}_H)(X_T \Delta_E(x)) \\ &= (\vartheta_u \otimes \text{id}_H)((\vartheta_u^{-1} \otimes \text{id}_H) \circ \Delta_T)(u)\Delta_E(x) \\ &= \Delta_T(u)\Delta_E(x) \\ &= \Delta_T(ux) \\ &= (\Delta_T \circ \vartheta_u)(x). \quad \square \end{aligned}$$

Example 3.5. Let, as in Section 1.4, H_4 be the Sweedler four-dimensional Hopf algebra over a field k and E_2 be the algebra of dual numbers. The image of \tilde{T} in $\text{tors}(H, E)$ consists of the (H_4, E_2) -modules $T_{X_a} = (E_2, \Delta^{X_a})$ and $T_{Y_a} = (E_2, \Delta^{Y_a})$, where a runs through k and where the coactions are explicitly given by

$$\begin{aligned} \Delta^{X_a}(1) &= X_a = 1 \otimes 1 + a(1 \otimes h) - a(h \otimes 1) + a(h \otimes g) - a^2(h \otimes h), \\ \Delta^{X_a}(h) &= X_a \Delta(h) = 1 \otimes h + h \otimes g - a(h \otimes h) \end{aligned}$$

and

$$\begin{aligned} \Delta^{Y_a}(1) &= Y_a = 1 \otimes g + a(1 \otimes gh) - a(h \otimes g) + a(h \otimes 1) - a^2(h \otimes gh), \\ \Delta^{Y_a}(h) &= Y_a \Delta(h) = 1 \otimes gh + h \otimes 1 - a(h \otimes gh). \end{aligned}$$

Up to isomorphism, only two equivalence classes of torsors remain: those consisting in the class of (E_2, Δ) itself and the class of (E_2, Δ') , where

$$\Delta'(1) = 1 \otimes g \quad \text{and} \quad \Delta'(h) = h \otimes 1 + 1 \otimes gh.$$

Remark 3.6. Suppose that the algebras E and H are both commutative. Let T and T' be two (H, E) -Hopf torsors. Endow T' with the symmetric E -bimodule action. One may easily check that the tensor product $T \otimes_E T'$ is also an (H, E) -Hopf torsor with coaction given by $\Delta_{T \otimes_E T'}(t \otimes t') = t_0 \otimes t'_0 \otimes t_1 t'_1$. Indeed the set $(T \otimes_E T')^\bullet$ contains all the elements $u \otimes u'$, where u belongs to T^\bullet and u' to T'^\bullet . Whence $\text{tors}(H, E)$ is a monoid with product \otimes_E . Under these hypothesis of commutativity, we already noticed that $\mathcal{Z}^1(H, E)$ and $\mathcal{H}^1(H, E)$ are groups (Remark 1.3(b)). The map $\tilde{T} : \mathcal{Z}^1(H, E) \longrightarrow \text{tors}(H, E)$ is then a morphism of monoids. Following Theorem 3.4, the product of $\text{tors}(H, E)$ induces a group structure on the quotient $\text{Tors}(H, E)$.

3.3. Comparison with the group case

Let us show how to relate Definition 3.2 to the usual notion of torsors. Given a finite group G and a G -group A , a (G, A) -group torsor is a non-empty left G -set P on which A acts on the right in a compatible way with the G -action and such that P is an affine space over A (see [11]). Denote by $\text{Tors}(G, A)$ the set of isomorphism classes of (G, A) -group torsors, which is known to be isomorphic to $H^1(G, A)$ (Proposition I.33 in [11]). If P is a (G, A) -torsor, its class in $\text{Tors}(G, A)$ is written $[P]$.

Proposition 3.7. *Let G be a finite group, let k^G be the Hopf algebra of the functions on G , and E be a k^G -comodule algebra. For any (k^G, E) -Hopf torsor T , the set T^\times is a (G, E^\times) -group torsor.*

Proof. As previously observed, T^\times is an E^\times -set. By Section 1.3, the group E^\times is equipped with a G -group structure. In the same way, if one writes $\Delta_T(u) = \sum_{g \in G} {}^g u \otimes \delta_g$ for $u \in T$, one deduces an action of the group G on the set T . By Lemma 3.3, if u belongs to T^\times , then the element $\Delta_T(u)$ belongs to $(T \otimes k^G)^\times$, which is easily seen to be isomorphic to $(T^\times)^{|G|}$. So, for any $g \in G$, the element ${}^g u$ belongs to T^\times , hence T^\times is a G -group. The compatibility of the two G -structures on E^\times and T^\times is a consequence of the (k^G, E) -Hopf module structure of T . The fact that T^\times is an affine space over E^\times is precisely the bijectivity of ϑ_u proved in Lemma 3.1 for any $u \in T^\times$. \square

Denote by $c : \text{tors}(k^G, E) \longrightarrow \text{Tors}(G, E^\times)$ the map defined for any (k^G, E) -torsor T by

$$c(T) = [T^\times].$$

Corollary 3.8. *Let G be a finite group and E be a k^G -comodule algebra. The map c induces a bijection of pointed sets*

$$\text{Tors}(k^G, E) \cong \text{Tors}(G, E^\times).$$

Proof. The isomorphism is a direct consequence of Theorem 1.5, Theorem 3.4 of this article, and Proposition I.33 in [11]. It is given by the sequence of isomorphisms

$$\text{Tors}(k^G, E) \cong \mathcal{H}^1(k^G, E) \cong H^1(G, E^\times) \cong \text{Tors}(G, E^\times).$$

Let T be a (k^G, E) -torsor and u an element of T^\times . The sequence of isomorphisms associates to T the class of the (G, E^\times) -torsor E_T^\times defined as follows. As a set E_T^\times is nothing but E^\times . It is endowed with the G -action given for $g \in G$ and $x \in E^\times$ by

$$g \longrightarrow x = \vartheta_u^{-1}({}^g u)g x.$$

One verifies that $[E_T^\times] = [T^\times] = c(T)$ in $\text{Tors}(G, E^\times)$ via the isomorphisms $\vartheta_u : E^\times \longrightarrow T^\times$. \square

3.4. Comparison with the restricted case

Let H be a Hopf algebra, S an H -comodule algebra, and M an (H, S) -Hopf module. Recall that what we called M -torsor in [9] is a triple (X, Δ_X, β) , where $\Delta_X : X \longrightarrow X \otimes H$ is a map conferring X a structure of (H, S) -Hopf module and $\beta : M \longrightarrow X$ is an S -linear isomorphism. Here we rename this datum a *restricted M -torsor*. The set of restricted M -torsors is pointed with distinguished point $(M, \Delta_M, \text{id}_M)$. Two restricted M -torsors (X, Δ_X, β) and $(X', \Delta_{X'}, \beta')$ are *equivalent* if there exists $f \in \text{Aut}_S(M)$ such that the composition $\beta \circ f \circ \beta'^{-1} : X' \longrightarrow X$ is a morphism of (H, S) -Hopf modules. Denote by $\text{Tors}(M)$ the set of equivalence classes of restricted M -torsors; it is pointed with distinguished point the class of $(M, \Delta_M, \text{id}_M)$. By Theorems 3.4 and 2.6 of the present article, by Proposition 2.8 and Theorem 3.1 of [9], one deduces the following statement:

Corollary 3.9. *Let H be a Hopf algebra, S be an H -comodule algebra, and M be an (H, S) -Hopf module such that Condition (\mathcal{F}_n) is satisfied for $0 \leq n \leq 2$. Then there is a bijection of pointed sets*

$$\text{Tors}(M) \cong \text{Tors}(H, \text{End}_S(M)).$$

This result shows that, under weak technical conditions on M , the possible structures of $(H, \text{End}_S(M))$ -Hopf module on $\text{End}_S(M)$ are closely related to the possible (H, S) -Hopf-module structures on M . More precisely, if $\text{End}_S(M)$ is equipped with an $(H, \text{End}_S(M))$ -Hopf module structure Δ , then following the track of Δ along the four isomorphisms

$$\text{Tors}(H, \text{End}_S(M)) \cong \mathcal{H}^1(H, \text{End}_S(M)) \cong \mathcal{H}^1(H, M) \cong \text{Tors}(M),$$

one gets an (H, S) -Hopf-module structure Δ' on M defined on an element $m \in M$ by

$$\Delta'(m) = \varphi_0(m_0) \otimes \varphi_1 m_1.$$

Here we denote by $\varphi_0 \otimes \varphi_1$ the element $\Delta(\text{id}_M) \in \text{End}_S(M) \otimes H$, and as usual, we adopt the convention $\Delta_M(m) = m_0 \otimes m_1$.

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