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P-filters and hereditary Baire function spaces

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Abstract

We extend the results of Gul'ko and Sokolov proving that a filter F on ω, regarded as a subspace of the Cantor set 2^ω, is a hereditary Baire space if and only if F is a nonmeager (i.e., second category) P-filter. We also prove related results on hereditary Baire spaces of continuous functions $C_p(X)$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

All spaces are completely regular.

Given a filter F on an infinite countable set T, we regard F as a subspace of the topological copy 2^T of the Cantor set. We consider only free filters on T, i.e., filters containing all cofinite subsets of T. By N_F we denote the space $T \cup \{\infty\}$, where $\infty \notin T$, equipped with the following topology: All points of T are isolated and the family $\{A \cup \{\infty\}\colon A \in F\}$ is a neighborhood base at ∞.

A filter F is a P-filter if for every sequence (U_n) of sets from F there exists an $A \in F$ which is almost contained in every U_n, i.e., $A \setminus U_n$ is finite. P-ultrafilters are also called P-points.

Recall that a space X is a Baire space if the Baire Category Theorem holds for X, i.e., every sequence (U_n) of dense open subsets of X has a dense intersection in X. If every closed subset of X is a Baire space then we call X a hereditary Baire space. A well-known result of Hurewicz (see [6, p. 97]) says that for a separable metrizable

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X this property is equivalent to the fact that X does not contain a closed copy of the rationals $\mathbb{Q}$.

For a space X, $C_p(X)$ denotes the space of all continuous real valued functions on X with the pointwise convergence topology.

The space σ is a subspace of $\mathbb{R}^\omega$ consisting of all eventually zero sequences. For the notions from infinite-dimensional topology that we are using, we refer the reader to [11].

Gul'ko and Sokolov proved in [5] the following theorem:

Theorem 1.1 (Gul'ko, Sokolov). *Let F be an ultrafilter on ω. The following are equivalent:*

(i) *F is a P-point,*
(ii) *$C_p(N_F)$ does not contain a closed copy of the rationals $\mathbb{Q}$ (equivalently $C_p(N_F)$ is a hereditary Baire space),*
(iii) *$C_p(N_F)$ does not contain a closed copy of the space σ.*

In this note we generalize this result:

Theorem 1.2. *Let F be a filter on ω. The following are equivalent:*

(i) *F is a nonmeager P-filter,*
(ii) *F is a hereditary Baire space,*
(iii) *$C_p(N_F)$ is a hereditary Baire space,*
(iv) *$C_p(N_F)$ does not contain a closed copy of the space σ^ω (a copy of σ).*

The proof of the result of Gul'ko and Sokolov uses topological games, in particular, Debs' characterization of hereditary Baire spaces in terms of games [3]. Our proof is based on a more direct approach. Let us also note that after this paper has been completed Michalewski [9] and Sokolov showed that characterizations of P-points in terms of games given in [5] can also be extended for the case of nonmeager P-filters.

It is well known (see [10]) that the continuum hypothesis implies the existence of P-points and in some models of set theory P-points do not exist. But the following question seems to be open (see [1, p. 230]):

Question 1.3. Can the existence of nonmeager P-filters on ω be proved in ZFC?

We prove Theorem 1.2 in Section 2. The last section contains some additional results concerning hereditary Baire function spaces $C_p(X)$.

2. Proof of Theorem 1.2

The following lemma is a modification of Lemma 2.1 from [8].

Lemma 2.1. *Let F be a filter on ω which is not a P-filter. Then F contains a closed copy of the rationals $\mathbb{Q}$ and $C_p(N_F)$ contains a closed copy of σ^ω (hence also a closed copy of σ).*

Proof. Since F is not a P-filter we can find a partition of ω into disjoint infinite subsets A_k, $k \in \omega$, with the following properties:

(a) $(\forall i \in \omega)\ [U_i = \bigcup_{k \geqslant i} A_k \in F]$,

(b) $(\forall A \in F)(\exists i \in \omega)\ [A \setminus U_i$ is infinite$]$.

The condition (b) is obviously equivalent to

(b′) $(\forall A \in F)(\exists k \in \omega)\ [A \cap A_k$ is infinite$]$.

To simplify the notation we may assume that F is a filter on $\omega \times \omega$ and $A_k = \{(k,n)\colon n \in \omega\}$. Let Q be the family of all sets $A \in F$ satisfying the following conditions:

(i) $(\forall k, n, i \in \omega)\ [((k,n) \in A$ and $i \leqslant n) \Rightarrow ((k,i) \in A)]$,

(ii) $(\forall k, n, i \in \omega)\ [((k,n) \in A$ and $k \leqslant i) \Rightarrow ((i,n) \in A)]$.

The set Q is a closed subset of F. Conditions (i), (ii) and (b′) imply that every $A \in Q$ contains some U_i. From this and (i) it follows that Q is countable. One can easily verify that Q is dense-in-itself. Therefore Q is a topological copy of the rationals, closed in F.

By Lemma 2.1 from [8], $C_p(N_F)$ contains a closed nonempty subset R which is an absolute retract and a Z_σ-space. From [2, Lemma 4.1] it follows that the product $(C_p(N_F))^\omega$ can be embedded as a closed subset of $C_p(N_F)$. Therefore $C_p(N_F)$ contains a closed copy of R^ω. Lemma 5.3 from [15] shows that R^ω contains a closed copy of σ^ω. □

Lemma 2.2. *Let F be a P-filter on ω. If F contains a closed copy Q of the rationals, then F is meager.*

Proof. Enumerate the subspace $Q \subseteq F$ as $\{A_n\colon n \in \omega\}$. Since F is a P-filter we can find $A \in F$ which is almost contained in every A_n. This means that, for every $n \in \omega$, the set $B_n = A \setminus A_n$ is finite. Let $P = \{B_n\colon n \in \omega\}$ considered as a subspace of the Cantor set 2^A. We will show that the space P is dense-in-itself.

Suppose that for some $n_0 \in \omega$, the point B_{n_0} is isolated in P. Then there is a finite subset S of A such that $\{B_k \in P\colon B_k \cap S = B_{n_0} \cap S\} = \{B_{n_0}\}$. Let $C = (A_{n_0} \cap A) \in F$. The set $U = \{A_k\colon A_k \cap S = A_{n_0} \cap S\}$ is a nonempty clopen subset of Q, therefore it is also a copy of the rationals $\mathbb{Q}$ closed in F. One can easily verify that for every $A_k \in U$ we have $C \subseteq A_k$. Hence every set X in the closure W of U in 2^ω contains C. It follows that $W \subseteq F$. Obviously, W being a closed dense-in-itself subset of 2^ω is uncountable. Therefore we have $W \setminus U \neq \emptyset$ which contradicts the fact that U is closed in F.

The fact that P is dense-in-itself and consists of finite sets implies the following property $(\star)$ of P:

For every $B \in P$ there is a sequence (B_k), $B_k \in P$ such that

(a) $(\forall k)\ [B \subseteq B_k]$,

(b) $(\forall k)\ [B_k \setminus B \neq \emptyset]$,

(c) $(\forall k, l,\ k \neq l)\ [(B_k \setminus B) \cap (B_l \setminus B) = \emptyset]$.

To obtain such (B_k) it is enough to choose an appropriate subsequence from any sequence in $P \setminus \{B\}$ converging to B.

Let ω^n denote the set of all sequences of nonnegative integers of length n. For $s = (i_0, i_1, \ldots, i_{n-1}) \in \omega^n$, $k \leqslant n$ and $j \in \omega$, we denote the restriction $(i_0, i_1, \ldots, i_{k-1}) \in$

ω^k by $s|k$, and $s^\frown j$ is the sequence $(i_0, i_1, \ldots, i_{n-1}, j) \in \omega^{n+1}$. Let $\omega^{<\omega} = \bigcup\{\omega^n\colon n \in \omega\}$ and let $\{s_n\colon n \in \omega\}$ be an enumeration of $\omega^{<\omega}$ with the following property: If s_m is a restriction of s_n then $m \leqslant n$. Using the property $(\star)$ of P one can inductively construct a family of nonempty finite subsets C_s of A, indexed by $s \in \omega^{<\omega}$, and satisfying the following conditions (for the inductive construction one should use the above enumeration of $\omega^{<\omega}$):

(1) $(\forall s, t \in \omega^{<\omega},\ s \neq t)\ [C_s \cap C_t = \emptyset]$,

(2) $(\forall n)(\forall s \in \omega^n)\ [D_s = \bigcup\{C_{s|k}\colon k \leqslant n\} \in P]$.

Fix $x \in \omega^\omega$. Let

$$D_x = \bigcup\{D_{x|n}\colon n \in \omega\} = \bigcup\{C_{x|n}\colon n \in \omega\}.$$

For every $n \in \omega$ we take $k(n) \in \omega$ such that $B_{k(n)} = D_{x|n}$. Let E be an accumulation point of the sequence $(A_{k(n)})$ in 2^ω. Since $A_{k(n)} \cap A = A \setminus D_{x|n}$ we have $E \cap A = A \setminus D_x$. The set D_x is infinite, therefore A cannot be almost contained in E, so $E \notin Q$. Since Q is closed in F we have that $E \notin F$. Hence the set $A \setminus D_x \subseteq E$ also does not belong to F. This means that, for every $X \in F$, there is $n \in \omega$ such that $X \cap C_{x|n} \neq \emptyset$. It follows that, for every $X \in F$, there exists $s \in \omega^{<\omega}$ such that $X \cap C_{s^\frown n} \neq \emptyset$ for every $n \in \omega$. Otherwise, we could easily construct by induction $x \in \omega^\omega$ contradicting the previous statement. Therefore F is a subspace of

$$H = \bigcup_{s \in \omega^{<\omega}} \{X \subseteq \omega\colon X \cap C_{s^\frown n} \neq \emptyset \text{ for every } n \in \omega\}.$$

One can easily check that, for a fixed $s \in \omega^{<\omega}$ the set $\{X \subseteq \omega\colon X \cap C_{s^\frown n} \neq \emptyset$ for every $n \in \omega\}$ is closed and nowhere dense in 2^ω. Hence H and F are meager. □

If (F_n) is a sequence of filters on ω then we can consider the product $\prod_{n\in\omega} F_n$ as a filter on $\omega \times \omega$. To do this, we identify

$$(A_n) \in \prod_{n\in\omega} F_n \quad \text{with} \quad \bigcup\{A_n \times \{n\}\colon n \in \omega\} \subseteq \omega \times \omega.$$

We need the following standard fact (cf. [1, p. 228]):

Lemma 2.3. *Let (F_n) be a sequence of P-filters on ω. Then $\prod_{n\in\omega} F_n$ is a P-filter on $\omega \times \omega$.*

Proof. Let (A^k) be a sequence of elements of $\prod_{n\in\omega} F_n$. For $k \in \omega$, let $A^k = (A^k_n)$, where $A^k_n \in F_n$. Fix $n \in \omega$ and choose $A_n \in F_n$ which is almost contained in A^k_n, for every k. Put $B_n = A_n \cap \bigcap\{A^k_n\colon k \leqslant n\} \in F_n$. One can easily verify that $B = (B_n)$ is almost contained in every A^k. □

Corollary 2.4. *Let (F_n) be a sequence of nonmeager P-filters on ω. Then $\prod_{n\in\omega} F_n$ is a hereditary Baire space.*

Proof. Since F_n is nonmeager it is a Baire space. Then $\prod_{n\in\omega} F_n$ is also a Baire space (see [12]), so it is of the second category. From Lemma 2.3 it follows that $\prod_{n\in\omega} F_n$

is a P-filter. Lemma 2.2 implies that $\prod_{n\in\omega} F_n$ does not contain a closed copy of the rationals, hence it is a hereditary Baire space. □

Corollary 2.5. *Let F be a filter on ω. If F is a nonmeager P-filter, then $C_p(N_F)$ is a hereditary Baire space.*

Proof. This follows easily from the previous corollary and the fact that every closed zero-dimensional subspace of $C_p(N_F)$ can be embedded as a closed subset in F^ω, see [7, Lemma 4.1]. □

Proof of Theorem 1.2. The implication (i) $\Rightarrow$ (ii) follows from Lemma 2.2. Lemma 2.1 implies (ii) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (iii) is given by Corollary 2.5. The implication (iii) $\Rightarrow$ (iv) is trivial. The remaining implication (iv) $\Rightarrow$ (i) follows from Lemma 2.1 and Lemma 5.15 from [4] saying that, for a meager filter F on ω the space $C_p(N_F)$ contains a closed copy of σ^ω. □

3. Remarks on hereditary Baire spaces $C_p(X)$

Baire function spaces $C_p(X)$ can be characterized in terms of topological properties of the space X. Such characterizations were obtain by Pytkeev [13], Tkachuk [14] and van Douwen (unpublished). It is natural to ask if there exits a similar characterization of hereditary Baire spaces $C_p(X)$ (see [5]). Theorem 1.2 and the results of this section give only a partial solution of this problem for the case of a countable X.

Proposition 3.1. *If there exist a hereditary Baire space $C_p(X)$, for a countable nondiscrete space X, then there exists a nonmeager P-filter on ω.*

Proof. Let x be an accumulation point of X. Obviously, the space X is zero-dimensional. Therefore we can find a decreasing sequence (U_n) of clopen neighborhoods of x, such that $\bigcap\{U_n\colon n\in\omega\} = \{x\}$. We may also assume that $U_n \setminus U_{n+1} \neq \emptyset$, for every $n \in \omega$. Pick an $x_n \in U_n \setminus U_{n+1}$, for $n \in \omega$. Let $T = \{x_n\colon n \in \omega\}$. Consider the closed subspace E of $C_p(X)$ consisting of all 0, 1-valued functions which are constant on every set $U_n \setminus U_{n+1}$, and which take the value 1 at x. Then E is a hereditary Baire space. The space E can be identified with the filter $F = \{T \cap f^{-1}(1)\colon f \in E\}$. From Theorem 1.2 it follows that F is a nonmeager P-filter. □

Lemma 3.2. *Let (F_n) be a sequence of nonmeager P-filters on ω. Then the product $\prod_{n\in\omega} C_p(N_{F_n})$ is a hereditary Baire space.*

Proof. First, recall that for every filter F the space $C_p(N_F)$ is homeomorphic with the space $c_F = \{f \in C_p(N_F)\colon f(\infty) = 0\}$, see [7, Lemma 2.1]. Hence the space $\prod_{n\in\omega} C_p(N_{F_n})$ is homeomorphic with the product $\prod_{n\in\omega} c_{F_n}$. Let $F = \prod_{n\in\omega} F_n$. By Corollary 2.4, F is a hereditary Baire space. Therefore $C_p(N_F)$ and c_F are also hereditary Baire spaces. One can easily verify that c_F is homeomorphic with $\prod_{n\in\omega} c_{F_n}$. □

Let X be a countable space and let x be an accumulation point of X. On $Y = X\setminus\{x\}$ we can define the following filter $F_x = \{A \subseteq Y\colon\ x$ is an interior point of $A \cup \{x\}\}$.

Proposition 3.3. *Let X be a countable space such that, for every accumulation point $x \in X$, the filter F_x is a nonmeager P-filter. Then the space $C_p(X)$ is a hereditary Baire space.*

Proof. If X is discrete then $C_p(X) = \mathbb{R}^X$ is a completely metrizable space, hence a hereditary Baire space. Therefore we can assume that X is not discrete. Let x be an accumulation point of X. Consider $Z_x = \{x\} \times X$ with the following topology: for every $y \in X$, $x \neq y$ the point (x, y) is isolated in Z_x and the neighborhoods of the point (x, x) have the form $\{x\} \times U$, where U is a neighborhood of x in X. One can easily verify that Z_x is homeomorphic to N_{F_x}. Let Z be a discrete union of Z_x, for all accumulation points $x \in X$. It is obvious that the map $p\colon Z \to X$ defined by $p(x, y) = y$, for every $(x, y) \in Z$, is quotient. Therefore $C_p(X)$ is homeomorphic to the closed subset of $C_p(Z)$ consisting of functions which are constant on fibers of q. It remains to observe that the space $C_p(Z)$ is a hereditary Baire space. This follows easily from Lemma 3.2 and the fact that $C_p(Z)$ is homeomorphic with the product of the spaces $C_p(Z_x)$. □

We refer the reader to the paper [8, Section 3] for some other results concerning hereditary Baire spaces $C_p(X)$.

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