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Noncommutative symmetric functions and an amazing matrix

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We present a simple way to derive the results of Diaconis and Fulman [P. Diaconis, J. Fulman, Foulkes characters, Eulerian idempotents, and an amazing matrix, arXiv:1102.5159] in terms of noncommutative symmetric functions.

1. Introduction

In [4,6,5], Diaconis and Fulman investigate a remarkable family of matrices $P(i,j)$ introduced by Holte [11] in his analysis of the process of "carries" in the addition of random integers in base b. It should be noted that these matrices occur in other contexts, for example in the study of sections of generating functions and of the Veronese embedding [2].

The aim of this note is to show that the results of [6] can be derived in a simple and natural way within the formalism of noncommutative symmetric functions [10].

This is possible thanks to the following equivalent characterization of the "amazing matrix" P (Theorem 2.1 of [4]):

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The number of descents in successive b-shuffles of n cards form a Markov chain on $\{0, 1, \ldots, n-1\}$ with transition matrix $P(i, j)$.

Such random processes involving descents of permutations can usually be interpreted in the descent algebra of the symmetric group. Here, it is only the number of descents which is involved, so that one can in fact compute in the Eulerian subalgebra.

We assume that the reader is familiar with the notations of [10].

2. The Eulerian algebra

This is a commutative subalgebra of dimension n of the group algebra of the symmetric group $\mathfrak{S}_n$, and in fact of its descent algebra Σ_n. It was apparently first introduced in [1] under the name *algebra of permutors*.[1] It is spanned by the Eulerian idempotents, or, as well, by the sums of permutations having the same number of descents.

It is easier to work with all symmetric groups at the same time, with the help of generating functions. Recall that the algebra of noncommutative symmetric functions **Sym** is endowed with an internal product $*$, for which each homogeneous component $\mathbf{Sym}_n$ is anti-isomorphic to Σ_n [10, Section 5.1].

Recall also the following definitions from [10]. We denote by σ_t or $\sigma_t(A)$ the generating series of the complete symmetric functions[2] S_n ([10, Section 3.1] and [12, Section 4]):

$$\sigma_t(A) = \sum_{n\geqslant 0} t^n S_n(A). \tag{1}$$

The *Eulerian idempotents* $E_n^{[k]}$ are the homogenous components of degree n in the series $E^{[k]}$ defined by

$$\sigma_t(A)^x = \sum_{k\geqslant 0} x^k E^{[k]}(A) \tag{2}$$

(see [10, Section 5.3]). We have

$$E_n^{[k]} * E_n^{[l]} = \delta_{kl} E_n^{[k]}, \quad \text{and} \quad \sum_{k=1}^{n} E_n^{[k]} = S_n, \tag{3}$$

so that the $E_n^{[k]}$ span a commutative n-dimensional $*$-subalgebra of $\mathbf{Sym}_n$, denoted by $\mathcal{E}_n$ and called the Eulerian subalgebra.

The *noncommutative Eulerian polynomials* are defined by [10, Section 5.4]

$$\mathcal{A}_n(t) = \sum_{k=1}^{n} t^k \left(\sum_{\substack{|I|=n \\ \ell(I)=k}} R_I \right) = \sum_{k=1}^{n} \mathbf{A}(n,k) t^k, \tag{4}$$

where R_I is the ribbon basis [10, Section 3.2] The following facts can be found (up to a few misprints[3]) in [10]. The generating series of the $\mathcal{A}_n(t)$ is

[1] A self-contained and elementary presentation of the main results of [1] can be found in [10].

[2] In the commutative case, these are denoted by h_n in Macdonald's book [14]. The letter S is used to remind of symmetric powers of a vector space. Indeed, bases of symmetric powers can be labeled by nondecreasing words. Similarly, the noncommutative elementary symmetric functions (sum of decreasing words) are denoted by Λ_n and are interpreted as exterior powers.

[3] Eqs. (93) and (97) of [10] should be read as (12) and (13) of the present paper.

$$\mathcal{A}(t) := \sum_{n\geqslant 0} \mathcal{A}_n(t) = (1-t)(1-t\sigma_{1-t})^{-1}, \tag{5}$$

where $\sigma_{1-t} = \sum (1-t)^n S_n$.

Let $\mathcal{A}_n^*(t) = (1-t)^{-n}\mathcal{A}_n(t)$. Then,

$$\mathcal{A}^*(t) := \sum_{n\geqslant 0} \mathcal{A}_n^*(t) = \sum_I \left(\frac{t}{1-t}\right)^{\ell(I)} S^I. \tag{6}$$

This last formula can also be written in the form

$$\mathcal{A}^*(t) = \sum_{k\geqslant 0} \left(\frac{t}{1-t}\right)^k (S_1 + S_2 + S_3 + \cdots)^k \tag{7}$$

or

$$\frac{1}{1-t\sigma_1(A)} = \sum_{n\geqslant 0} \frac{\mathcal{A}_n(t)}{(1-t)^{n+1}}. \tag{8}$$

Let $S^{[k]} = \sigma_1(A)^k$ be the coefficient of t^k in this series. In degree n,

$$S_n^{[k]} = \sum_{I\vDash n,\ \ell(I)\leqslant k} \binom{k}{\ell(I)} S^I = \sum_{i=1}^n k^i E_n^{[i]}. \tag{9}$$

This is another basis of $\mathcal{E}_n$. Expanding the factors $(1-t)^{-(n+1)}$ in the right-hand side of (8) by the binomial theorem, and taking the coefficient of t^k in the term of weight n in both sides, we get

$$S_n^{[k]} = \sum_{i=0}^k \binom{n+i}{i} \mathbf{A}(n, k-i). \tag{10}$$

Conversely,

$$\frac{\mathcal{A}_n(t)}{(1-t)^{n+1}} = \sum_{k\geqslant 0} t^k S_n^{[k]}, \tag{11}$$

so that

$$\mathbf{A}(n,p) = \sum_{i=0}^p (-1)^i \binom{n+1}{i} S_n^{[p-i]}. \tag{12}$$

The expansion of the $E_n^{[k]}$ on the basis $\mathbf{A}(n,i)$, which is a noncommutative analog of Worpitzky's identity (see [9] or [13]) is

$$\sum_{k=1}^n x^k E_n^{[k]} = \sum_{i=1}^n \binom{x+n-i}{n} \mathbf{A}(n,i). \tag{13}$$

Indeed, when x is a positive integer N,

$$\sum_{k=1}^{n} N^k E_n^{[k]} = S_n(NA) = \sum_{I \vDash n} F_I(N) R_I(A) \tag{14}$$

where F_I are the fundamental quasi-symmetric functions, and for a composition $I = (i_1, \ldots, i_r)$ of n,

$$F_I(N) = \binom{N+n-r}{n}. \tag{15}$$

3. The *b*-shuffle process

For a positive integer b, the b-shuffle permutations in $\mathfrak{S}_n$ are the inverses of the permutations with at most $b-1$ descents. Thus, the b-shuffle operator can be identified with $S_n^{[b]}$ (*i.e.*, with $*$-multiplication by $S_n^{[b]}$). It belongs to the Eulerian algebra, so that it preserves it, and it makes sense to compute its matrix in the basis $\mathbf{A}(n,k)$. Note that since $\mathcal{E}_n$ is commutative, it does not matter whether we multiply on the right or on the left.

The b-shuffle process is an example of what Stanley has called the QS-distribution [15]. It is the probability distribution on permutations derived by assigning probability b^{-1} to the first b positive integers, see [7] for a simplified exposition.

Summarizing, we want to compute the coefficients $P_{ij}(b)$ defined by

$$S_n^{[b]} * \mathbf{A}(n, j) = \sum_{i=1}^{n} P_{ij}(b) \mathbf{A}(n, i). \tag{16}$$

From (9), it is clear that

$$S_n^{[p]} * S_n^{[q]} = S_n^{[pq]} \tag{17}$$

so that, using (12), we obtain

$$\begin{aligned} S_n^{[b]} * \mathbf{A}(n, j) &= \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} S_n^{[b(j-r)]} \\ &= \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \sum_{k=0}^{b(j-r)} \binom{n+k}{k} \mathbf{A}\big(n, b(j-r)-k\big). \end{aligned} \tag{18}$$

The coefficient of $\mathbf{A}(n,i)$ in this expression is therefore

$$P_{ij}(b) = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \binom{n+b(j-r)-i}{n}. \tag{19}$$

These are the coefficients of the amazing matrix (up to a shift of 1 on the indices i, j, and a global normalization factor b^n so as the probabilities sum up to 1).

Since the $E_n^{[k]}$ form a basis of orthogonal idempotents in $\mathcal{E}_n$, it is reasonable to introduce a scalar product such that

$$\big\langle E_n^{[i]} \big| E_n^{[j]} \big\rangle = \delta_{ij}. \tag{20}$$

Then, the b-shuffle operator is self-adjoint. Its orthonormal basis of eigenvectors is clearly $E_n^{[k]}$ (with eigenvalues b^k).

In terms of coordinates, since we are working in the non-orthogonal basis $\mathbf{A}(n,i)$, its right eigenvector of eigenvalue b^j is the column vector whose ith component is the coefficient of $E_n^{[j]}$ on $\mathbf{A}(n,i)$, that is, the coefficient of x^j in $\binom{x+n-i}{n}$, thanks to (13). By duality, its left eigenvector associated with the eigenvalue b^i is the row vector whose jth component is

$$\langle \mathbf{A}(n,j) \,|\, E_n^{[i]} \rangle = \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} (j-r)^i. \tag{21}$$

Comparing [8, Theorem 4.1], we see that this is precisely the Foulkes character table (up to indexation, the Frobenius characteristic of $\chi^{n,k}$ is the commutative image[4] of $\mathbf{A}(n,n-k)$).

4. Other examples

4.1. Determinant of the Foulkes character table

This is the determinant of the matrix F

$$F(i,j) = \langle \mathbf{A}(n,i), E_n^{[j]} \rangle, \quad i,j = 1,\dots,n. \tag{22}$$

Because of the triangularity property

$$\mathbf{A}(n,i) = S_n^{[i]} + \sum_{r=1}^{i} (-1)^r \binom{n+1}{r} S_n^{[i-r]}, \tag{23}$$

we have as well

$$\det F = \det G \quad \text{where } G(i,j) = \langle S_n^{[i]}, E_n^{[j]} \rangle = i^j \tag{24}$$

a Vandermonde determinant which evaluates to $n!(n-1)!\cdots 2!1!$. This gives a different proof of [6, (2.9)].

4.2. Descents of b^r-riffle shuffles

Recall from [7] that **FQSym** is an algebra based on all permutations and that it has two bases

$$\mathbf{G}_\sigma = \sum_{\mathrm{std}(w)=\sigma} w = \mathbf{F}_{\sigma^{-1}} \tag{25}$$

which are mutually adjoint for its natural scalar product

$$\langle \mathbf{F}_\sigma, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau}. \tag{26}$$

[4] These commutative symmetric functions have been studied in [3].

Under the embedding of **Sym** into **FQSym**, the b^r-shuffle operator is

$$\left(S_n^{[b]}\right)^{*r} = S_n^{[b^r]} = \sum_{\sigma\, b^r\text{-shuffle}} \mathbf{F}_\sigma. \tag{27}$$

The generating function of b^r-shuffle by number of descents is therefore its scalar product in **FQSym** with the noncommutative Eulerian polynomial

$$\mathcal{A}_n(t) = \sum_{k=1}^{n} t^k \mathbf{A}(n,k) = \sum_{\tau \in \mathfrak{S}_n} t^{d(\tau)+1} \mathbf{G}_\tau. \tag{28}$$

Recall that

$$\mathcal{A}_n(t) = (1-t)^{n+1} \sum_{k=1}^{n} t^k S_n^{[k]} \tag{29}$$

so that

$$\left\langle S_n^{[b^r]}, \mathcal{A}_n(t)\right\rangle = (1-t)^{n+1} \sum_{k=1}^{n} t^k \left\langle S_n^{[b^r]}, S_n^{[k]}\right\rangle. \tag{30}$$

Now, when one factor P of a scalar product $\langle P, Q\rangle$ in **FQSym** is in **Sym**, one has $\langle P, Q\rangle = \langle p, Q\rangle$ where $p = \underline{P}$ is the commutative image of P in *QSym*, and the bracket is now the duality between **Sym** and *QSym*. Furthermore, when p in *Sym*, then, the scalar product reduces to $\langle p, q\rangle$, where $q = \underline{Q}$ is the commutative image of Q in *Sym*, and the bracket is now the ordinary scalar product of symmetric functions (see [7]). Thus,

$$\left\langle S_n^{[b^r]}, S_n^{[k]}\right\rangle = \left\langle h_n\left(b^r X\right), h_n(kX)\right\rangle = h_n\left(b^r k\right) = \binom{b^r k + n - 1}{n} \tag{31}$$

(λ-ring notation) and we are done

$$\left\langle S_n^{[b^r]}, \mathcal{A}_n(t)\right\rangle = (1-t)^{n+1} \sum_{k=1}^{n} t^k \binom{b^r k + n - 1}{n}. \tag{32}$$

This statement is equivalent to Theorem 4.1 of [6].

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