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Advances in Mathematics 218 (2008) 87–104

www.elsevier.com/locate/aim

Simple homotopy types and finite spaces

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Received 29 November 2006; accepted 28 November 2007

Available online 3 January 2008

Communicated by Mark Hovey

Abstract

We present a new approach to simple homotopy theory of polyhedra using finite topological spaces. We define the concept of *collapse* of a finite space and prove that this new notion corresponds exactly to the concept of a simplicial collapse. More precisely, we show that a collapse $X \searrow Y$ of finite spaces induces a simplicial collapse $\mathcal{K}(X) \searrow \mathcal{K}(Y)$ of their associated simplicial complexes. Moreover, a simplicial collapse $K \searrow L$ induces a collapse $\mathcal{X}(K) \searrow \mathcal{X}(L)$ of the associated finite spaces. This establishes a one-to-one correspondence between simple homotopy types of finite simplicial complexes and simple equivalence classes of finite spaces. We also prove a similar result for maps: We give a complete characterization of the class of maps between finite spaces which induce simple homotopy equivalences between the associated polyhedra. This class describes all maps coming from simple homotopy equivalences at the level of complexes. The advantage of this theory is that the elementary move of finite spaces is much simpler than the elementary move of simplicial complexes: It consists of removing (or adding) just a single point of the space.

MSC: 57Q10; 55U05; 57N65; 55P15; 06A06

Keywords: Finite spaces; Simplicial complexes; Simple homotopy types; Posets; Weak homotopy equivalences; Simple homotopy equivalences

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.11.019

1. Introduction

J.H.C. Whitehead's theory of simple homotopy types is inspired by Tietze's theorem in combinatorial group theory, which states that any finite presentation of a group could be deformed into any other by a finite sequence of elementary moves, which are now called Tietze transformations. Whitehead translated these algebraic moves into the well-known geometric moves of elementary collapses and expansions of finite simplicial complexes. His beautiful theory turned out to be fundamental for the development of piecewise-linear topology: The s-cobordism theorem, Zeeman's conjecture [18], the applications of the theory in surgery, Milnor's classical paper on Whitehead Torsion [10] and the topological invariance of torsion are some of its major uses and advances.

In this paper we show how to use finite topological spaces to study simple homotopy types. There is a strong relationship between finite spaces and finite simplicial complexes, which was discovered by McCord [9]. Explicitly, given a finite simplicial complex K, one can associate to K a finite T_0-space $\mathcal{X}(K)$ which corresponds to the poset of simplices of K ordered by inclusion. Moreover, a simplicial map $\varphi : K \to L$ gives rise to a continuous map $\mathcal{X}(\varphi)$ between the associated finite spaces. Conversely, one can associate to a finite T_0-space X a simplicial complex $\mathcal{K}(X)$, whose simplices are the non-empty chains of X, and a weak homotopy equivalence $\mathcal{K}(X) \to X$. This construction is also functorial.

In [2] we showed that finite spaces are very useful for studying homotopy invariants of (general) spaces. In fact, in that article we were looking *for minimal finite models* of some spaces, i.e. the smallest finite spaces which are weak (homotopy) equivalent to a given space. Finite spaces are closely related to finite posets, which have become an important tool in algebraic and geometric topology (see for example Quillen's paper [12], Björner's paper [3] and Björner, Wachs and Welker's [4]). The finite space point of view adds a new dimension to finite posets and allows the development of new and more appropriate techniques based on the combinatorics and the topology of these objects.

It is easy to prove that if two finite T_0-spaces X, Y are homotopy equivalent, their associated simplicial complexes $\mathcal{K}(X), \mathcal{K}(Y)$ are also homotopy equivalent. Furthermore, Osaki [11] showed that in this case, the latter have the same simple homotopy type. Nevertheless, we noticed that the converse of this result is not true in general: There are finite spaces with different homotopy types whose associated simplicial complexes have the same simple homotopy type. Starting from this point, we were looking for the relation that X and Y should satisfy for their associated complexes to be simple homotopy equivalent. More specifically, we wanted to find an elementary move in the setting of finite spaces (if it existed) which corresponds exactly to a simplicial collapse of the associated polyhedra.

We discovered this elementary move when we were looking for a homotopically trivial finite space (i.e. weak equivalent to a point) which was non-contractible. In order to construct such a space, we developed a method of reduction, i.e. a method that allows us to reduce a finite space to a smaller weak equivalent space. This method of reduction together with the homotopically trivial and non-contractible space (of 11 points) that we found are exhibited in Section 3. Surprisingly, this method, which consists of removing a *weak point* of the space (see Definition 3.2), turned out to be the key to solve the problem of translating simplicial collapses into this setting.

We will say that two finite spaces are *simply equivalent* if we can obtain one of them from the other by adding and removing weak points. If Y is obtained from X by only removing weak points, we say that X *collapses* to Y and write $X \searrow Y$. The first main result of this article is the following

Theorem 3.10.

(a) *Let X and Y be finite T_0-spaces. Then, X and Y are simply equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*
(b) *Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simply equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

In particular, the functors $\mathcal{K}$ and $\mathcal{X}$ induce a one-to-one correspondence between simple equivalence classes of finite spaces and simple homotopy types:

$$\{\text{Finite } T_0\text{-Spaces}\}\Big/\diagup\!\!\!\searrow \quad \underset{\mathcal{X}}{\overset{\mathcal{K}}{\rightleftarrows}} \quad \{\text{Finite Simplicial Complexes}\}\Big/\diagup\!\!\!\searrow$$

We are now able to study finite spaces using all the machinery of Whitehead's simple homotopy theory for CW-complexes. But also, what is more important, we can use finite spaces to strengthen the classical theory. The elementary move in this setting is much simpler to handle and describe because it consists of adding or removing just one single point.

As an example or application of this theorem, we study *collapsible* finite spaces and their relationship with collapsible complexes. We also relate simple types of finite spaces with the notion of minimal finite model introduced in [2].

In the last section of this article we investigate the class of maps between finite spaces which induce simple homotopy equivalences between their associated simplicial complexes. To this end, we introduce the notion of a *distinguished* map. Similarly to the classical case, the class of simple equivalences between finite spaces can be generated, in a certain way, by expansions and a kind of formal homotopy inverses of expansions. Remarkably this class, denoted by $\mathcal{S}$, is also generated by the distinguished maps. The second main result of the article is the following

Theorem 4.13.

(a) *Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a simple equivalence if and only if $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a simple homotopy equivalence.*
(b) *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. Then φ is a simple homotopy equivalence if and only if $\mathcal{X}(\varphi)$ is a simple equivalence.*

2. Preliminaries

In this section we recall various results on finite spaces which are needed in Sections 3 and 4. For more details on finite spaces we refer the reader to [9,14] and P. May's notes [7,8].

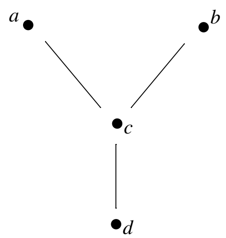
2.1. The correspondence between finite spaces and finite posets

There is a natural relationship between topologies and preorders defined on a finite set X. This correspondence, which was studied in first place by Alexandroff [1], can be described as follows. Given a topology τ on X, consider for each point x in X, the intersection U_x of all open sets containing x. This is clearly an open set for each x and the family $B = \{U_x, x \in X\}$ is a basis for the topology τ. This basis is called the *minimal basis* of X for obvious reasons. Associated to τ, there is a preorder structure on X (i.e. a reflexive and transitive relation), defined by $x \leqslant y$ if $x \in U_y$. Conversely, if a preorder $\leqslant$ on the finite set X is given, we define for each $x \in X$ the subset $U_x = \{y \in X \mid y \leqslant x\}$. It is not hard to see that these subsets form a basis for a topology on X, which is the topology associated to the preorder $\leqslant$.

The applications described above define a one-to-one correspondence between topological structures and preorders on X. Moreover, the T_0 separation axiom is equivalent to the antisymmetry of the associated preorder and therefore, T_0-topologies on X correspond to order relations. Having this equivalence in mind, we will regard finite T_0-spaces as finite posets and vice versa. We will use both structures according to convenience.

It is very useful to represent finite spaces using *Hasse diagrams*. The Hasse diagram of a finite T_0-space X is a digraph whose vertex set is X and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in X$ with $x < z < y$.

Example 2.1. Consider the space $X = \{a, b, c, d\}$ whose proper open sets are $\{a, c, d\}$, $\{b, c, d\}$, $\{c, d\}$ and $\{d\}$. Its Hasse diagram is



Instead of representing an edge (x, y) with an arrow, one simply writes y over x.

Sometimes it is convenient to consider the opposite preorder of a finite space X. The space associated to this preorder will be denoted by X^{op}. Concretely, the open sets of X^{op} are the closed sets of X.

Note that a map $f : X \to Y$ between finite spaces is continuous if and only if it is order preserving. There is also a nice way to describe homotopies. Given two functions $f, g : X \to Y$, we will say that $f \leqslant g$ if $f(x) \leqslant g(x)$ for every $x \in X$. It is not difficult to prove that if f and g are continuous and $f \leqslant g$, then f is homotopic to g (see [7,14] for more details). In particular, any finite space with maximum or minimum is contractible.

2.2. Homotopy types

In 1966 R.E. Stong [14] found a combinatorial way to describe conclusively the homotopy types of finite spaces. He introduced the notions of *linear* and *colinear* points and proved that these two kinds of points generate all homotopy equivalences between finite spaces. Essentially, two finite T_0-spaces X and Y have the same homotopy type if and only if there exists a sequence $X = X_0, X_1, \dots, X_n = Y$ such that each space is obtained from the previous one by adding or removing a linear or colinear point. Afterwards, Peter May called these points *beat points* [7].

Following Peter May's language, we will say that a point x of a finite T_0-space X is an *up beat point* if the set of points which are greater than x has a minimum. On the other hand, $x \in X$ is said to be a *down beat point* if the set of points below it has a maximum. This is equivalent to say that x is an up beat point of X^{op}. When there is no need to precise if x is an up or a down beat point, we simply say that x is a beat point.

The next obvious remark plays an important role in Theorem 3.10.

Remark 2.2. If $x \in X$ is a beat point, there exists $y \in X$, $y \neq x$, such that any point which is comparable with x is also comparable with y.

It is not difficult to see that if $x \in X$ is a beat point, the inclusion of $X \smallsetminus \{x\}$ in X is a strong deformation retract. Therefore, given a finite T_0-space X, one can remove beat points, one at the time, to obtain a strong deformation retract of X with no beat points. Such a subspace is called a *core* of X. A finite T_0-space with no beat points is called a *minimal finite space*.

In [14] Stong proves that every homotopy equivalence between minimal finite spaces is a homeomorphism and therefore, the core of any finite space X is unique up to homeomorphism. It can be described as the smallest space which is homotopy equivalent to X. Note that a finite T_0-space X is contractible if and only if there exists a sequence $X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_n = *$, where X_{i+1} is obtained from X_i by removing a beat point. Note also that a point $x \in X$ is a beat point if and only if x is a beat point of X^{op}. Therefore, X is contractible if and only if X^{op} is contractible.

2.3. *Finite spaces and simplicial complexes*

In contrast to Stong's combinatorial approach to homotopy theory of finite spaces, M.C. McCord [9] investigated their relationship with polyhedra. Finite spaces are not in general subspaces of Euclidean spaces. Moreover, they do not have in general the homotopy type of any T_1 topological space [2]. Nevertheless, their weak homotopy types describe all weak homotopy types of compact polyhedra.

Following McCord [9] (cf. also [8]) one can associate to any finite T_0-space X a simplicial complex $\mathcal{K}(X)$, whose simplices are the non-empty chains of X (see Fig. 1).

There exists a weak homotopy equivalence from the geometric realization $|\mathcal{K}(X)|$ to X, i.e. a continuous map $|\mathcal{K}(X)| \to X$ which induces isomorphisms in all homotopy groups. The application $\mathcal{K}$ is in fact functorial. A continuous map $f : X \to Y$ between finite T_0-spaces induces a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ which coincides with f on vertices. Besides, it is easy to see that this construction makes the following diagram commutative

$$\begin{tikzcd} |\mathcal{K}(X)| \arrow[r, "|\mathcal{K}(f)|"] \arrow[d] & |\mathcal{K}(Y)| \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

If two maps $f, g : X \to Y$ between finite T_0-spaces are homotopic, it can be proved that the simplicial maps $\mathcal{K}(f), \mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$ lie in the same contiguity class. In particular $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$.

The functor $\mathcal{K}$ can be used to find a simplicial complex with the same weak homotopy type of a given finite space. Recall that two spaces X and Y (non-necessarily finite) are said to be weak (homotopy) equivalent (or to have the same weak homotopy type) if there exists a sequence

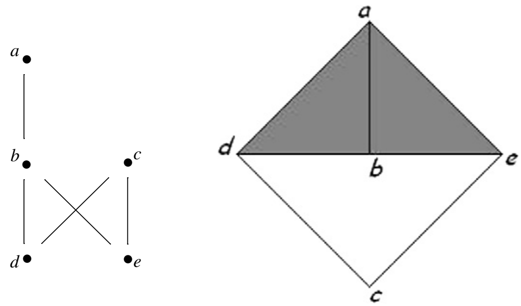


Fig. 1. A finite space and its associated simplicial complex.

of spaces $X = X_1, X_2, \ldots, X_n = Y$ such that for each $1 \leqslant i < n$ there is a weak homotopy equivalence $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$. We will denote this by $X \overset{\text{we}}{\approx} Y$.

Conversely, given a finite simplicial complex K, one would like to find a *finite model* of $|K|$, i.e. a finite space which is weak equivalent to $|K|$. With this aim, McCord defined another functor, denoted by $\mathcal{X}$, that associates to each finite simplicial complex K a finite T_0-space $\mathcal{X}(K)$, which is the poset of simplices of K ordered by inclusion. Note that $\mathcal{K}(\mathcal{X}(K)) = K'$ is the barycentric subdivision of K, which implies that there exists a weak homotopy equivalence $|K| \to \mathcal{X}(K)$. The functor $\mathcal{X}$ on maps is defined as follows. Given a simplicial map $\varphi : K \to L$, we define $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ by $\mathcal{X}(\varphi)(S) = \varphi(S)$ for every simplex S of K. In this case one does not have a commutative diagram as before, but a diagram that commutes up to homotopy

$$\begin{tikzcd} {|K|} \arrow[r, "|\varphi|"] \arrow[d] & {|L|} \arrow[d] \\ \mathcal{X}(K) \arrow[r, "\mathcal{X}(\varphi)"] & \mathcal{X}(L). \end{tikzcd}$$

By Whitehead theorem, if X, Y are finite T_0-spaces, $X \overset{\text{we}}{\approx} Y$ if and only if $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ have the same homotopy type. On the other hand, if K and L are finite simplicial complexes, $|K|$ and $|L|$ are homotopy equivalent if and only if $\mathcal{X}(K) \overset{\text{we}}{\approx} \mathcal{X}(L)$.

2.4. Simplicial collapses and expansions

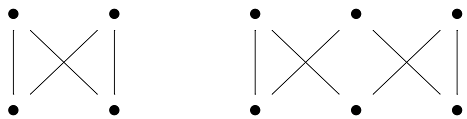
We finish this introductory section by recalling the basic notions on simple homotopy theory for simplicial complexes. Mainly, we want to fix the notations that we will use in Sections 3 and 4. The standard references for this are Whitehead's papers [15–17], Milnor's article [10] and M.M. Cohen's book [5].

Let L be a subcomplex of a finite simplicial complex K. There is an *elementary simplicial collapse* from K to L if there is a simplex S of K and a vertex a of K not in S such that $K = L \cup aS$ and $L \cap aS = a\dot{S}$. Here aS denotes the join of a and S and $\dot{S}$ denotes the boundary of S. This is equivalent to say that there are only two simplices S, S' of K which are not in L and such that S is a free face of S'. Elementary collapses will be denoted, as usual, $K \overset{e}{\searrow} L$.

We say that K *(simplicially) collapses* to L (or that L *expands* to K) if there exists a sequence $K = K_1, K_2, \ldots, K_n = L$ of finite simplicial complexes such that $K_i \overset{e}{\searrow} K_{i+1}$ for all i. This is denoted by $K \searrow L$ or $L \nearrow K$. Two complexes K and L have the same simple homotopy type if there is a sequence $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all i. Following M.M. Cohen's notation, we denote this by $K \diagup\!\!\searrow L$. It is well known that $K \diagup\!\!\searrow L$ if and only if $|K|$ and $|L|$ are simple homotopy equivalent [17].

3. Simple homotopy types: The first main theorem

The first mathematician who investigated the relationship between finite spaces and simple homotopy types of polyhedra was T. Osaki [11]. He showed that if $x \in X$ is a beat point, $\mathcal{K}(X)$ collapses to $\mathcal{K}(X \smallsetminus \{x\})$. In particular, if two finite T_0-spaces, X and Y are homotopy equivalent, their associated simplicial complexes, $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, have the same simple homotopy type. However, there exist finite spaces which are not homotopy equivalent but whose associated complexes have the same simple homotopy type. Consider, for instance, the spaces with the following Hasse diagrams:



They are not homotopy equivalent because they are non-homeomorphic minimal finite spaces. However their associated complexes are triangulations of S^1 and therefore, have the same simple homotopy type.

A more interesting example is the following.

Example 3.1 *(The Wallet).* Let W be a finite T_0-space, whose Hasse diagram is

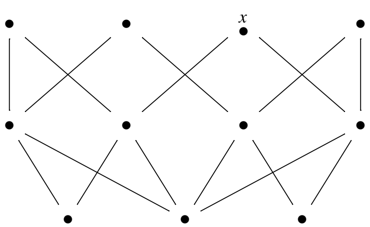


Fig. 2. W.

This finite space is not contractible since it does not have beat points, but it is not hard to see that $|\mathcal{K}(W)|$ is contractible and therefore, it has the same simple homotopy type as a point. In fact we will deduce from Proposition 3.3 that W is a homotopically trivial space, i.e. all its homotopy groups are trivial. This example also shows that Whitehead theorem does not hold in the context of finite spaces, not even for homotopically trivial spaces.

We introduce now the notion of a *weak beat point* which generalizes Stong's definition of beat points. The following notations will be used in the rest of the paper. Given a point $x \in X$, we denote by F_x the closure of x in X, i.e. the set of points which are greater than or equal to x.

We let $\hat{U}_x = U_x \smallsetminus \{x\}$ and $\hat{F}_x = F_x \smallsetminus \{x\}$. In case we need to specify the ambient space X, we will write F_x^X, $\hat{U}_x^X$ and $\hat{F}_x^X$ respectively.

Definition 3.2. Let X be a finite T_0-space. We will say that $x \in X$ is a *weak beat point of* X (or a *weak point*, for short) if either $\hat{U}_x$ is contractible or $\hat{F}_x$ is contractible. In the first case we say that x is a *down weak point* and in the second, that x is an *up weak point*.

Note that beat points are in particular weak points, for if $x \in X$ is a down beat point, $\hat{U}_x$ has a maximum and if x is an up beat point, $\hat{F}_x$ has a minimum. When x is a beat point of X, we have seen in the previous section that the inclusion $i : X \smallsetminus \{x\} \hookrightarrow X$ is a homotopy equivalence. This is not the case if x is just a weak point. However, a slightly weaker result holds.

Proposition 3.3. *Let x be a weak point of a finite T_0-space X. Then the inclusion map $i : X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

Proof. We may suppose that x is a down weak point since the other case follows immediately from this one, considering X^{op} instead of X. Note that $\mathcal{K}(X^{\mathrm{op}}) = \mathcal{K}(X)$.

Given $y \in X$, the set $i^{-1}(U_y) = U_y \smallsetminus \{x\}$ has a maximum if $y \neq x$ and is contractible if $y = x$. Therefore $i|_{i^{-1}(U_y)} : i^{-1}(U_y) \to U_y$ is a weak homotopy equivalence for every $y \in X$. Now the result follows from Theorem 6 of [9] applied to the basis-like cover given by the minimal basis of X. $\square$

As an application of the last proposition, we verify that the space W defined above, is a non-contractible homotopically trivial space. As we pointed out in Example 3.1, W is not contractible since it is a minimal finite space with more than one point. However, it contains a weak point x (see Fig. 2), since $\hat{U}_x$ is contractible (see Fig. 3).

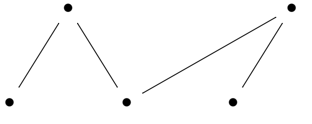


Fig. 3. $\hat{U}_x$.

Therefore W is weak homotopy equivalent to $W \smallsetminus \{x\}$.

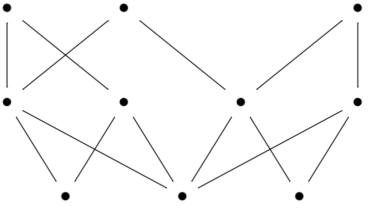


Fig. 4. $W \smallsetminus \{x\}$.

Now it is easy to see that this subspace is contractible, because it does have beat points, and one can get rid of them one by one.

Definition 3.4. Let X be a finite T_0-space and let $Y \subsetneq X$. We say that X *collapses* to Y by an *elementary collapse* (or that Y *expands* to X by an *elementary expansion*) if Y is obtained from X by removing a weak point. We denote $X \overset{e}{\searrow} Y$ or $Y \overset{e}{\nearrow} X$. In general, given two finite T_0-spaces X and Y, we say that X *collapses* to Y (or Y *expands* to X) if there is a sequence $X = X_1, X_2, \dots, X_n = Y$ of finite T_0-spaces such that for each $1 \leqslant i < n$, $X_i \overset{e}{\searrow} X_{i+1}$. In this case we write $X \searrow Y$ or $Y \nearrow X$. Two finite T_0-spaces X and Y are *simply equivalent* if there is a sequence $X = X_1, X_2, \dots, X_n = Y$ of finite T_0-spaces such that for each $1 \leqslant i < n$, $X_i \searrow X_{i+1}$ or $X_i \nearrow X_{i+1}$. We denote in this case $X \diagup\!\!\searrow Y$, following the same notation that we adopted for simplicial complexes.

In contrast with the classical situation, where a simple homotopy equivalence is a special kind of homotopy equivalence, homotopy equivalent finite spaces are simply equivalent. It follows from Proposition 3.3 that simply equivalent finite spaces are weak equivalent.

In order to prove Theorem 3.10, we need some previous results. The first one concerns the homotopy type of the associated finite space $\mathcal{X}(K)$ of a simplicial cone K. Suppose $K = aL$ is a cone, i.e. K is the join of a simplicial complex L with a vertex $a \notin L$. Since $|K|$ is contractible, it is clear that $\mathcal{X}(K)$ is homotopically trivial. The following lemma shows that $\mathcal{X}(K)$ is in fact contractible (compare with [12]).

Lemma 3.5. *Let $K = aL$ be a finite cone. Then $\mathcal{X}(K)$ is contractible.*

Proof. Define $f : \mathcal{X}(K) \to \mathcal{X}(K)$ by $f(S) = S \cup \{a\}$. This function is order-preserving and therefore continuous.

If we consider the constant map $g : \mathcal{X}(K) \to \mathcal{X}(K)$ that takes all $\mathcal{X}(K)$ into $\{a\}$, we have that $1_{\mathcal{X}(K)} \leqslant f \geqslant g$. This proves that the identity is homotopic to a constant map. □

The following construction is the analogue to the mapping cylinder of general spaces and the simplicial mapping cylinder of simplicial complexes.

Definition 3.6. Let $f : X \to Y$ be a map between finite T_0-spaces. We define the *non-Hausdorff mapping cylinder* $B(f)$ as the following finite T_0-space. The underlying set is the disjoint union $X \sqcup Y$. We keep the given ordering within X and Y and for $x \in X$, $y \in Y$ we set $x \leqslant y$ in $B(f)$ if $f(x) \leqslant y$ in Y.

Lemma 3.7. *Let $f : X \to Y$ be a map between finite T_0-spaces such that $f^{-1}(U_y)$ is contractible for every $y \in Y$. Then $B(f) \searrow i(X)$ and $B(f) \searrow j(Y)$, where $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$ are the canonical inclusions.*

Proof. Label all the elements $x_1, x_2, \dots, x_n$ of X in such a way that $x_r \leqslant x_s$ implies $r \leqslant s$ and define $Y_r = j(Y) \cup \{i(x_1), i(x_2), \dots, i(x_r)\} \subseteq B(f)$ for each $0 \leqslant r \leqslant n$. Then

$$\hat{F}^{Y_r}_{i(x_r)} = \{ j(y) \mid y \geqslant f(x_r) \}$$

is homeomorphic to the contractible space $F^{Y}_{f(x_r)}$. It follows that $Y_r \overset{e}{\searrow} Y_{r-1}$ for $1 \leqslant r \leqslant n$, and then $B(f) = Y_n$ collapses to $j(Y) = Y_0$. Notice that we have not yet used the fact that f is distinguished.

Now order the elements $y_1, y_2, \dots, y_m$ of Y in such a way that $y_r \leqslant y_s$ implies $r \leqslant s$ and define $X_r = i(X) \cup \{j(y_{r+1}), j(y_{r+2}), \dots, j(y_m)\} \subseteq B(f)$ for every $0 \leqslant r \leqslant m$. Then

$$\hat{U}^{X_{r-1}}_{j(y_r)} = \{ i(x) \mid f(x) \leqslant y_r \}$$

is homeomorphic to $f^{-1}(U_{y_r})$, which is contractible by hypothesis. Thus $X_{r-1} \searrow^{e} X_r$ for $1 \leqslant r \leqslant m$ and therefore $B(f) = X_0$ collapses to $i(X) = X_m$. □

It is well known that any finite simplicial complex K has the same simple homotopy type of its barycentric subdivision K'. We prove next an analogous result for finite spaces. Following [6], the barycentric subdivision of a finite T_0-space X is defined by $X' = \mathcal{X}(\mathcal{K}(X))$. Explicitly, X' consists of the non-empty chains of X ordered by inclusion. It is shown in [6] that there is a weak homotopy equivalence $h : X' \to X$ which takes each chain C to its maximum $\max(C)$.

Proposition 3.8. *Let X be a finite T_0-space. Then X and X' are simply equivalent.*

Proof. It suffices to show that the map $h : X' \to X$ satisfies the hypothesis of Lemma 3.7. This is clear since $h^{-1}(U_x) = \{C \mid \max(C) \leqslant x\} = \mathcal{X}(\mathcal{K}(U_x)) = \mathcal{X}(x\mathcal{K}(\hat{U}_x))$ is contractible by Lemma 3.5. □

Lemma 3.9. *Let L be a subcomplex of a finite simplicial complex K. Let T be a set of simplices of K which are not in L, and let a be a vertex of K which is contained in no simplex of T, but such that aS is a simplex of K for every $S \in T$. Finally, suppose that $K = L \cup \bigcup_{S\in T}\{S, aS\}$ (i.e. the simplices of K are those of L together with the simplices S and aS for every S in T). Then $L \nearrow K$.*

Proof. Number the elements $S_1, S_2, \dots, S_n$ of T in such a way that for every i, j with $i \leqslant j$, $\#S_i \leqslant \#S_j$. Here $\#S_k$ denotes the cardinality of S_k. Define $K_i = L \cup \bigcup_{j=1}^{i}\{S_j, aS_j\}$ for $0 \leqslant i \leqslant n$. Let $S \subsetneq S_i$. If $S \in T$, then $S, aS \in K_{i-1}$, since $\#S < \#S_i$. If $S \notin T$, then $S, aS \in L \subseteq K_{i-1}$. This proves that $aS_i \cap K_{i-1} = a\dot{S}_i$.

By induction, K_i is a simplicial complex for every i, and $K_{i-1} \nearrow^{e} K_i$. Therefore $L = K_0 \nearrow K_n = K$. □

Now we are ready to prove the first main result of this article.

Theorem 3.10.

(a) *Let X and Y be finite T_0-spaces. Then, X and Y are simply equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*
(b) *Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simply equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

Proof. Let X be a finite T_0-space and let $x \in X$ be a weak point. We will show first that $\mathcal{K}(X \smallsetminus \{x\}) \nearrow \mathcal{K}(X)$. We may suppose that x is a down weak point since the other case follows immediately from this one replacing X by X^{op}. Since $\hat{U}_x$ is contractible, there exists a sequence of spaces $\hat{U}_x = X_n \supsetneq X_{n-1} \supsetneq \dots \supsetneq X_1 = \{x_1\}$, with $X_i = \{x_1, x_2, \dots, x_i\}$ and such that x_i is a beat point of X_i for each $i \geqslant 2$. By Remark 2.2, it follows that there exists $y_i \in X_{i-1}$ for each $2 \leqslant i \leqslant n$ with the following property: if $z \in X_i$ is comparable with x_i, then it is comparable with y_i.

Let $K_i \subseteq \mathcal{K}(X)$ be the subcomplex whose simplices are the chains of $X \smallsetminus \{x\}$ together with the chains of $F_x \cup X_i \subseteq X$. In other words, $K_i = \mathcal{K}(X \smallsetminus \{x\}) \cup \mathcal{K}(F_x \cup X_i)$. We will prove that $\mathcal{K}(X \smallsetminus \{x\}) \nearrow K_1 \nearrow K_2 \nearrow \cdots \nearrow K_n = \mathcal{K}(X)$.

In order to prove that $\mathcal{K}(X \smallsetminus \{x\}) \nearrow K_1$, we apply Lemma 3.9 with $L = \mathcal{K}(X \smallsetminus \{x\})$, $K = K_1$, $T = \{S \in K_1 \mid x \in S,\ x_1 \notin S\}$ and $a = x_1$. Note that $x_1 S \in K_1$ for every $S \in T$ since any element of S is greater than or equal to x and therefore, comparable with x_1. In order to see that $K_{i-1} \nearrow K_i$ for $i \geqslant 2$, note that the simplices of K_i which are not in K_{i-1} are the chains of $F_x \cup X_i$ that contain both x and x_i. We apply again Lemma 3.9 with $L = K_{i-1}$, $K = K_i$, $T = \{S \in K_i \mid x, x_i \in S,\ y_i \notin S\}$ and $a = y_i$. Note that if $S \in T$ and $y \in S$, then either $y \in X_i$ and it is comparable with x_i or $y \geqslant x$. In any of these cases y is comparable with y_i, and therefore $y_i S \in K_i$. We have then proved that $X \searrow Y$ implies $\mathcal{K}(X) \searrow \mathcal{K}(Y)$. In particular, $X \diagup\!\!\searrow Y$ implies $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$.

Suppose now that K and L are finite simplicial complexes such that $K \searrow^{e} L$. Then, there exist $S \in K$ and a vertex a of K not in S such that $aS \in K$, $K = L \cup \{S, aS\}$ and $aS \cap L = a\dot{S}$. It follows that S is an up beat point of $\mathcal{X}(K)$, and since $\hat{U}_{aS}^{\mathcal{X}(K) \smallsetminus \{S\}} = \mathcal{X}(a\dot{S})$, by Lemma 3.5, aS is a down weak point of $\mathcal{X}(K) \smallsetminus \{S\}$. Therefore $\mathcal{X}(K) \searrow^{e} \mathcal{X}(K) \smallsetminus \{S\} \searrow^{e} \mathcal{X}(K) \smallsetminus \{S, aS\} = \mathcal{X}(L)$. This proves the first part of (b) and the "moreover" part.

Let X, Y be finite T_0-spaces such that $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. Then $X' = \mathcal{X}(\mathcal{K}(X)) \diagup\!\!\searrow \mathcal{X}(\mathcal{K}(Y)) = Y'$ and by Proposition 3.8, $X \diagup\!\!\searrow Y$. Finally, if K, L are finite simplicial complexes such that $\mathcal{X}(K) \diagup\!\!\searrow \mathcal{X}(L)$, $K' = \mathcal{K}(\mathcal{X}(K)) \diagup\!\!\searrow \mathcal{K}(\mathcal{X}(L)) = L'$ and therefore $K \diagup\!\!\searrow L$. This completes the proof. $\square$

Corollary 3.11. *The functors $\mathcal{K}$, $\mathcal{X}$ induce a one-to-one correspondence between simple equivalence classes of finite spaces and simple homotopy types of finite simplicial complexes*

$$\{\text{Finite } T_0\text{-Spaces}\}\Big/\diagup\!\!\searrow \;\; \underset{\mathcal{X}}{\overset{\mathcal{K}}{\rightleftarrows}} \;\; \{\text{Finite Simplicial Complexes}\}\Big/\diagup\!\!\searrow$$

The following diagrams illustrate the whole situation. Here $\overset{\text{he}}{\simeq}$ denotes the homotopy equivalence relation.

$$\begin{tikzcd}
X \overset{\text{he}}{\simeq} Y \arrow[r, Rightarrow] & X \diagup\!\!\searrow Y \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & X \overset{\text{we}}{\approx} Y \arrow[d, Leftrightarrow] & \\
& \mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y) \arrow[r, Rightarrow] & {|\mathcal{K}(X)| \overset{\text{we}}{\approx} |\mathcal{K}(Y)|} \arrow[r, Leftrightarrow] & {|\mathcal{K}(X)| \overset{\text{he}}{\simeq} |\mathcal{K}(Y)|}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{X}(K) \overset{\text{he}}{\simeq} \mathcal{X}(L) \arrow[r, Rightarrow] & \mathcal{X}(K) \mathrel{\diagup\!\!\searrow} \mathcal{X}(L) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathcal{X}(K) \overset{\text{we}}{\approx} \mathcal{X}(L) \arrow[d, Leftrightarrow] & \\
& K \mathrel{\diagup\!\!\searrow} L \arrow[r, Rightarrow] & {|K|} \overset{\text{we}}{\approx} {|L|} \arrow[r, Leftrightarrow] & {|K|} \overset{\text{he}}{\simeq} {|L|}
\end{tikzcd}$$

The Wallet W satisfies $W \searrow *$, however $W \overset{\text{he}}{\not\simeq} *$. Therefore $X \mathrel{\diagup\!\!\searrow} Y \not\Rightarrow X \overset{\text{he}}{\simeq} Y$. Since $|K| \overset{\text{he}}{\simeq} |L| \not\Rightarrow K \mathrel{\diagup\!\!\searrow} L$, $X \overset{\text{we}}{\approx} Y \not\Rightarrow X \mathrel{\diagup\!\!\searrow} Y$. Note that, if $X \overset{\text{we}}{\approx} Y$ and their Whitehead group $Wh(\pi_1(X))$ is trivial, then $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ are simple homotopy equivalent CW-complexes. It follows from Theorem 3.10 that $X \mathrel{\diagup\!\!\searrow} Y$. Thus, we have proved

Corollary 3.12. *Let X, Y be weak equivalent finite T_0-spaces such that $Wh(\pi_1(X)) = 0$. Then $X \mathrel{\diagup\!\!\searrow} Y$.*

Another immediate consequence of the theorem is the following

Corollary 3.13. *Let X, Y be finite T_0-spaces. If $X \searrow Y$, then $X' \searrow Y'$.*

Note that from Theorem 3.10 one also deduces the following well-known fact: If K and L are finite simplicial complexes such that $K \searrow L$, then $K' \searrow L'$.

One of the most important open results concerning collapsible complexes is Zeeman's Conjecture [18], which states that if K is a contractible polyhedron of dimension 2, $K \times I$ is collapsible. This conjecture implies the 3-dimensional Poincaré Conjecture (see [18]). The notion of collapsibility for finite spaces is closely related with the analogous notion for simplicial complexes: We say that a finite T_0-space is *collapsible* if it collapses to a point. Observe that every contractible finite T_0-space is collapsible, however the converse is not true. The Wallet W introduced in Example 3.1 is collapsible and non-contractible. Note that if a finite T_0-space X is collapsible, its associated simplicial complex $\mathcal{K}(X)$ is also collapsible. Moreover, if K is a collapsible complex, then $\mathcal{X}(K)$ is a collapsible finite space. Therefore, if X is a collapsible finite space, its subdivision X' is also collapsible.

Let us consider now a compact contractible polyhedron X with the property that any triangulation of X is non-collapsible, for instance the Dunce Hat [18]. Let K be any triangulation of X. The associated finite space $\mathcal{X}(K)$ is homotopically trivial because X is contractible. However, $\mathcal{X}(K)$ is not collapsible since K' is not collapsible.

We have therefore the following strict implications in the context of finite spaces:

$$\text{contractible} \quad \Rightarrow \quad \text{collapsible} \quad \Rightarrow \quad \text{homotopically trivial}.$$

As we pointed out in the previous section, the beat points defined by Stong provide an effective way of deciding whether two finite spaces are homotopy equivalent. The problem becomes much harder when one deals with weak homotopy types instead. In [2] we have studied the *minimal finite models* of a given space X, which are the smallest spaces weak equivalent to X. In that article we characterized the minimal finite models of spheres and finite graphs (finite CW-complexes of dimension one). We proved that, in general, the minimal finite models of a space are not unique. For example $\bigvee_{i=1}^{3} S^1$ has three minimal finite models up to homeomorphism. It makes sense to formulate the following definition.

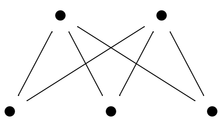
Definition 3.14. A *minimal simple model* of a finite T_0-space X is a finite T_0-space simply equivalent to X of minimum cardinality. We will say that a space is a minimal simple model if it is a minimal simple model of itself.

Clearly, one has the following implications:

$$\text{minimal finite model} \quad \Rightarrow \quad \text{minimal simple model} \quad \Rightarrow \quad \text{minimal finite space.}$$

Note that if the Whitehead group $Wh(\pi_1(X))$ is trivial, the converse of the first implication holds. Therefore, given a finite T_0-space X such that $Wh(\pi_1(X)) = 0$, one could reach any minimal finite model of X just by adding and removing weak points from X. Elementary collapses and expansions provide a tool of reduction when the space has trivial Whitehead group. Unfortunately it is not always possible to obtain a minimal simple model by only removing weak points. For example, take any homotopically trivial non-collapsible finite space.

Of course there is not uniqueness of minimal simple models. Consider for instance the space $\mathbb{S}D_3$



and its opposite, which are minimal simple models because they are minimal finite models. Notice that $\mathbb{S}D_3 \diagup\!\!\searrow (\mathbb{S}D_3)^{\mathrm{op}}$ and they are not homeomorphic.

4. Simple homotopy equivalences: The second main theorem

In this section we prove the second main result of the article, which relates simple homotopy equivalences of complexes with *simple equivalences* between finite spaces. Like in the classical setting, the class of simple equivalences is generated by the elementary expansions. However, in the context of finite spaces this class is also generated by the *distinguished* maps, which play a key role in this theory.

Recall that a homotopy equivalence $f : |K| \to |L|$ between compact polyhedra is a simple homotopy equivalence if it is homotopic to a composition of a finite sequence of maps $|K| \to |K_1| \to \cdots \to |K_n| \to |L|$, each of them an expansion or a homotopy inverse of one [5,13].

We prove first that homotopy equivalences between finite spaces induce simple homotopy equivalences between the associated polyhedra.

Theorem 4.1. *If $f : X \to Y$ is a homotopy equivalence between finite T_0-spaces, then $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a simple homotopy equivalence.*

Proof. Let X_c and Y_c be cores of X and Y. Let $i_X : X_c \to X$ and $i_Y : Y_c \to Y$ be the inclusions and $r_X : X \to X_c$, $r_Y : Y \to Y_c$ retractions of i_X and i_Y such that $i_X r_X \simeq 1_X$ and $i_Y r_Y \simeq 1_Y$.

Since $r_Y f i_X : X_c \to Y_c$ is a homotopy equivalence between minimal finite spaces, it is a homeomorphism. Therefore $\mathcal{K}(r_Y f i_X) : \mathcal{K}(X_c) \to \mathcal{K}(Y_c)$ is an isomorphism and then $|\mathcal{K}(r_Y f i_X)|$ is a simple homotopy equivalence. Since $\mathcal{K}(X) \searrow \mathcal{K}(X_c)$, $|\mathcal{K}(i_X)|$ is a simple homotopy equivalence, and then the homotopy inverse $|\mathcal{K}(r_X)|$ is also a simple homotopy equivalence. Analogously $|\mathcal{K}(i_Y)|$ is a simple homotopy equivalence.

Finally, since $f \simeq i_Y r_Y f i_X r_X$, it follows that $|\mathcal{K}(f)| \simeq |\mathcal{K}(i_Y)||\mathcal{K}(r_Y f i_X)||\mathcal{K}(r_X)|$ is a simple homotopy equivalence. □

In order to describe the class of simple equivalences, we will use a kind of maps that was already studied in Lemma 3.7.

Definition 4.2. A map $f : X \to Y$ between finite T_0-spaces is *distinguished* if $f^{-1}(U_y)$ is contractible for each $y \in Y$. We denote by $\mathcal{D}$ the class of distinguished maps.

Note that by the theorem of McCord [9, Theorem 6], every distinguished map is a weak homotopy equivalence and therefore induces a homotopy equivalence between the associated complexes. We will prove in Theorem 4.4 that in fact the induced map is a simple homotopy equivalence. From the proof of Proposition 3.3, it is clear that if $x \in X$ is a down weak point, the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is distinguished.

Remark 4.3. The map $h : X' \to X$ defined by $h(C) = \max(C)$, is distinguished by the proof of Proposition 3.8.

Clearly, homeomorphisms are distinguished. However it is not difficult to show that homotopy equivalences are not distinguished in general.

Theorem 4.4. *Every distinguished map induces a simple homotopy equivalence.*

Proof. Suppose $f : X \to Y$ is distinguished. Consider the non-Hausdorff mapping cylinder $B(f)$ and the canonical inclusions $i : X \hookrightarrow B(f)$, $j : Y \hookrightarrow B(f)$.

The following diagram

$$\begin{tikzcd} & B(f) & \\ X \arrow[ur, hook, "i"] \arrow[rr, "f"] & & Y \arrow[ul, hook', "j"'] \end{tikzcd}$$

does not commute, but $i \leqslant jf$ and then $i \simeq jf$. Therefore $|\mathcal{K}(i)| \simeq |\mathcal{K}(j)||\mathcal{K}(f)|$. By Lemma 3.7 and Theorem 3.10, $|\mathcal{K}(i)|$ and $|\mathcal{K}(j)|$ are expansions (composed with isomorphisms) and then, $|\mathcal{K}(f)|$ is a simple homotopy equivalence. □

We have already shown that expansions, homotopy equivalences and distinguished maps induce simple homotopy equivalences at the level of complexes. Note that if f, g, h are three maps between finite T_0-spaces such that $fg \simeq h$ and two of them induce simple homotopy equivalences, then so does the third.

Definition 4.5. Let $\mathcal{C}$ be a class of continuous maps between topological spaces. We say that $\mathcal{C}$ is *closed* if it satisfies the following homotopy 2-out-of-3 property: For any f, g, h with $fg \simeq h$, if two of the three maps are in $\mathcal{C}$, then so is the third.

Definition 4.6. Let $\mathcal{C}$ be a class of continuous maps. The class $\bar{\mathcal{C}}$ *generated* by $\mathcal{C}$ is the smallest closed class containing $\mathcal{C}$.

It is clear that $\bar{\mathcal{C}}$ is always closed under composition and homotopy. The class of simple homotopy equivalences between CW-complexes is closed and it is generated by the elementary expansions. Note that every map in the class $\mathcal{E}$ of elementary expansions between finite spaces induces a simple homotopy equivalence at the level of complexes and therefore the same holds for the maps of $\bar{\mathcal{E}}$. Contrary to the case of CW-complexes, a map between finite spaces which induces a simple homotopy equivalence, need not have a homotopy inverse. This is the reason why the definition of $\bar{\mathcal{E}}$ is not as simple as in the setting of complexes. We will prove that $\bar{\mathcal{E}} = \bar{\mathcal{D}}$, the class generated by the distinguished maps.

A map $f : X \to Y$ such that $f^{-1}(F_y)$ is contractible for every y, need not be distinguished. However we will show that $f \in \bar{\mathcal{D}}$. We denote by $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ the map that coincides with f in the underlying sets, and let $\mathcal{D}^{\mathrm{op}} = \{f \mid f^{\mathrm{op}} \in \mathcal{D}\}$.

Lemma 4.7. $\overline{\mathcal{D}^{\mathrm{op}}} = \bar{\mathcal{D}}$.

Proof. Suppose that $f : X \to Y$ lies in $\mathcal{D}^{\mathrm{op}}$. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X' = (X^{\mathrm{op}})' \arrow[l, "h_X"'] \arrow[r, "h_{X^{\mathrm{op}}}"] \arrow[d, "f'"] & X^{\mathrm{op}} \arrow[d, "f^{\mathrm{op}}"] \\
Y & Y' = (Y^{\mathrm{op}})' \arrow[l, "h_Y"'] \arrow[r, "h_{Y^{\mathrm{op}}}"] & Y^{\mathrm{op}}.
\end{tikzcd}$$

Here, f' denotes the map $\mathcal{X}(\mathcal{K}(f))$. Since $\bar{\mathcal{D}}$ satisfies the 2-out-of-3 property and $h_{X^{\mathrm{op}}}$, $h_{Y^{\mathrm{op}}}$, f^{op} are distinguished by Remark 4.3, $f' \in \bar{\mathcal{D}}$. And since h_X, h_Y are distinguished, $f \in \bar{\mathcal{D}}$. This proves that $\overline{\mathcal{D}^{\mathrm{op}}} \subseteq \bar{\mathcal{D}}$. The other inclusion follows analogously from the opposite diagram. $\square$

Proposition 4.8. $\bar{\mathcal{E}} = \bar{\mathcal{D}}$, *and this class contains all homotopy equivalences between finite T_0-spaces.*

Proof. Every expansion of finite spaces is in $\bar{\mathcal{E}}$ because it is a composition of maps in $\mathcal{E}$.

Let $f : X \to Y$ be distinguished. By the proof of Theorem 4.4 there exist expansions (eventually composed with homeomorphisms) i, j, such that $i \simeq jf$. Therefore $f \in \bar{\mathcal{E}}$.

If $x \in X$ is a down weak point, the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is distinguished. If x is an up weak point, $X \smallsetminus \{x\} \hookrightarrow X$ lies in $\bar{\mathcal{D}}$ by the previous lemma and therefore $\bar{\mathcal{E}} \subseteq \bar{\mathcal{D}}$.

Suppose now that $f : X \to Y$ is a homotopy equivalence. From the proof of Theorem 4.1, $fi_X \simeq i_Y r_Y f i_X$ where i_X, i_Y are expansions and $r_Y f i_X$ is a homeomorphism. This implies that $f \in \bar{\mathcal{E}} = \bar{\mathcal{D}}$. $\square$

We denote by $\mathcal{S} = \bar{\mathcal{E}} = \overline{\mathcal{D}}$ the class of *simple equivalences* between finite spaces. In the rest of the paper we study the relationship between simple equivalences of finite spaces and simple homotopy equivalences of polyhedra.

Lemma 4.9. *Let $\varphi, \psi : K \to L$ be simplicial maps which lie in the same contiguity class. Then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$.*

Proof. Assume that φ and ψ are contiguous. Then the map $f : \mathcal{X}(K) \to \mathcal{X}(L)$, defined by $f(S) = \varphi(S) \cup \psi(S)$ is well-defined and continuous. Moreover $\mathcal{X}(\varphi) \leqslant f \geqslant \mathcal{X}(\psi)$, and then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$. $\square$

Given $n \in \mathbb{N}$ we denote by K^n the nth barycentric subdivision of K.

Lemma 4.10. *Let $\lambda : K^n \to K$ be a simplicial approximation to the identity. Then $\mathcal{X}(\lambda) \in \mathcal{S}$.*

Proof. It suffices to prove the case $n = 1$. Suppose $\lambda : K' \to K$ is a simplicial approximation of $1_{|K|}$. Then $\mathcal{X}(\lambda) : \mathcal{X}(K)' \to \mathcal{X}(K)$ is homotopic to $h_{\mathcal{X}(K)}$, for if $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_m$ is a chain of simplices of K, then $\mathcal{X}(\lambda)(\{S_1, S_2, \ldots, S_m\}) = \{\lambda(S_1), \lambda(S_2), \ldots, \lambda(S_m)\} \subseteq S_m = h_{\mathcal{X}(K)}(\{S_1, S_2, \ldots, S_m\})$. By Remark 4.3, it follows that $\mathcal{X}(\lambda) \in \mathcal{S}$. $\square$

Lemma 4.11. *Let $\varphi, \psi : K \to L$ be simplicial maps such that $|\varphi| \simeq |\psi|$. If $\mathcal{X}(\varphi) \in \mathcal{S}$, then $\mathcal{X}(\psi)$ also lies in $\mathcal{S}$.*

Proof. There exists an approximation to the identity $\lambda : K^n \to K$ for some $n \geqslant 1$, such that $\varphi\lambda$ and $\psi\lambda$ lie in the same contiguity class. By Lemma 4.9, $\mathcal{X}(\varphi)\mathcal{X}(\lambda) = \mathcal{X}(\varphi\lambda) \simeq \mathcal{X}(\psi\lambda) = \mathcal{X}(\psi)\mathcal{X}(\lambda)$. By Lemma 4.10, $\mathcal{X}(\lambda) \in \mathcal{S}$ and since $\mathcal{X}(\varphi) \in \mathcal{S}$, it follows that $\mathcal{X}(\psi) \in \mathcal{S}$. $\square$

Theorem 4.12. *Let $K_0, K_1, \ldots, K_n$ be finite simplicial complexes and let*

$$|K_0| \xrightarrow{f_0} |K_1| \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} |K_n|$$

be a sequence of continuous maps such that for each $0 \leqslant i < n$ either

(1) *$f_i = |\varphi_i|$ where $\varphi_i : K_i \to K_{i+1}$ is a simplicial map such that $\mathcal{X}(\varphi_i) \in \mathcal{S}$ or*
(2) *f_i is a homotopy inverse of a map $|\varphi_i|$ with $\varphi_i : K_{i+1} \to K_i$ a simplicial map such that $\mathcal{X}(\varphi_i) \in \mathcal{S}$.*

If $\varphi : K_0 \to K_n$ is a simplicial map such that $|\varphi| \simeq f_{n-1} f_{n-2} \ldots f_0$, then $\mathcal{X}(\varphi) \in \mathcal{S}$.

Proof. We may assume that f_0 satisfies condition (1). Otherwise we define $\widetilde{K_0} = K_0$, $\widetilde{f_0} = |1_{K_0}| : |\widetilde{K_0}| \to |K_0|$ and then $|\varphi| \simeq f_{n-1} f_{n-2} \ldots f_0 \widetilde{f_0}$.

We proceed by induction on n. If $n = 1$, $|\varphi| \simeq |\varphi_0|$ where $\mathcal{X}(\varphi_0) \in \mathcal{S}$ and the result follows from Lemma 4.11. Suppose now that $n \geqslant 1$ and let $K_0, K_1, \ldots, K_n, K_{n+1}$ be finite simplicial complexes and $f_i : |K_i| \to |K_{i+1}|$ maps satisfying conditions (1) or (2), f_0 satisfying condition (1). Let $\varphi : K_0 \to K_{n+1}$ be a simplicial map such that $|\varphi| \simeq f_n f_{n-1} \ldots f_0$. We consider two cases: f_n satisfies condition (1) or f_n satisfies condition (2).

In the first case we define $g : |K_0| \to |K_n|$ by $g = f_{n-1}f_{n-2}\dots f_0$. Let $\tilde{g} : K_0^m \to K_n$ be a simplicial approximation to g and let $\lambda : K_0^m \to K_0$ be a simplicial approximation to the identity. Then $|\tilde{g}| \simeq g|\lambda| = f_{n-1}f_{n-2}\dots f_1(f_0|\lambda|)$ where $f_0|\lambda| = |\varphi_0\lambda|$ and $\mathcal{X}(\varphi_0\lambda) = \mathcal{X}(\varphi_0)\mathcal{X}(\lambda) \in \mathcal{S}$ by Lemma 4.10. By induction, $\mathcal{X}(\tilde{g}) \in \mathcal{S}$, and then $\mathcal{X}(\varphi_n\tilde{g}) \in \mathcal{S}$. Since $|\varphi\lambda| \simeq f_n g|\lambda| \simeq f_n|\tilde{g}| = |\varphi_n\tilde{g}|$, by Lemma 4.11, $\mathcal{X}(\varphi\lambda)$ lies in $\mathcal{S}$. Therefore $\mathcal{X}(\varphi) \in \mathcal{S}$.

In the other case, $|\varphi_n\varphi| \simeq f_{n-1}f_{n-2}\dots f_0$ and by induction, $\mathcal{X}(\varphi_n\varphi) \in \mathcal{S}$. Therefore $\mathcal{X}(\varphi)$ also lies in $\mathcal{S}$. $\square$

Theorem 4.13.

(a) *Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a simple equivalence if and only if $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a simple homotopy equivalence.*
(b) *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. Then $|\varphi|$ is a simple homotopy equivalence if and only if $\mathcal{X}(\varphi)$ is a simple equivalence.*

Proof. By definition, if $f \in \mathcal{S}$, $|\mathcal{K}(f)|$ is a simple homotopy equivalence.

Let $\varphi : K \to L$ be a simplicial map such that $|\varphi|$ is a simple homotopy equivalence. Then there exist finite complexes $K = K_0, K_1, \dots, K_n = L$ and maps $f_i : |K_i| \to |K_{i+1}|$, which are simplicial expansions or homotopy inverses of simplicial expansions, and such that $|\varphi| \simeq f_{n-1}f_{n-2}\dots f_0$. By Theorem 3.10, simplicial expansions between complexes induce expansions between the associated finite spaces and therefore, by Theorem 4.12, $\mathcal{X}(\varphi) \in \mathcal{S}$.

Suppose now that $f : X \to Y$ is a map such that $|\mathcal{K}(f)|$ is a simple homotopy equivalence. Then, $f' = \mathcal{X}(\mathcal{K}(f)) : X' \to Y'$ lies in $\mathcal{S}$. Since $fh_X = h_Y f'$, $f \in \mathcal{S}$.

Finally, if $\varphi : K \to L$ is a simplicial map such that $\mathcal{X}(\varphi) \in \mathcal{S}$, $|\varphi'| : |K'| \to |L'|$ is a simple homotopy equivalence. Here $\varphi' = \mathcal{K}(\mathcal{X}(\varphi))$ is the barycentric subdivision of φ. Let $\lambda_K : K' \to K$ and $\lambda_L : L' \to L$ be simplicial approximations to the identities. Then $\lambda_L\varphi'$ and $\varphi\lambda_K$ are contiguous. In particular $|\lambda_L||\varphi'| \simeq |\varphi||\lambda_K|$ and then $|\varphi|$ is a simple homotopy equivalence. $\square$

In the setting of finite spaces one has the following strict inclusions

$$\{\text{homotopy equivalences}\} \subsetneq \mathcal{S} \subsetneq \{\text{weak equivalences}\}.$$

Clearly, if $f : X \to Y$ is a weak homotopy equivalence between finite T_0-spaces with trivial Whitehead group, $f \in \mathcal{S}$.

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