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On critical normal sections for two-dimensional immersions in $\mathbb{R}^4$ and a Riemann–Hilbert problem

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ABSTRACT

For orthonormal normal sections of two-dimensional immersions in $\mathbb{R}^4$ we define torsion coefficients and a functional for the total torsion. We discuss normal sections which are critical for this functional. In particular, a global estimate for the torsion coefficients of a critical normal section in terms of the curvature of the normal bundle is provided.

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1. Introduction

Consider a two-dimensional, conformally parametrized immersion

$$X = X(u,v) = \big(x^1(u,v), x^2(u,v), x^3(u,v), x^4(u,v)\big) \in C^4\big(B,\mathbb{R}^4\big) \tag{1.1}$$

on the closed unit disc $B = \{(u,v)\in\mathbb{R}^2\colon\ u^2+v^2 \leqslant 1\} \subset \mathbb{R}^2$, together with an orthogonal moving 4-frame

$$\{X_u, X_v, N_1, N_2\}, \tag{1.2}$$

which consists of the orthogonal tangent vectors X_u, X_v, and orthogonal unit normal vectors $N_1, N_2 \in C^3(B,\mathbb{R}^4)$:

$$\begin{aligned}
&X_u\cdot X_u^t =: g_{11} = W = g_{22} := X_v\cdot X_v^t, \qquad g_{12} := X_u\cdot X_v^t = 0,\\
&X_u\cdot N_\sigma^t = 0 = X_v\cdot N_\sigma^t \quad \text{for } \sigma = 1,2,\\
&|N_1| = 1 = |N_2|, \qquad N_1\cdot N_2^t = 0.
\end{aligned} \tag{1.3}$$

Here W denotes the area element of X, and X^t means the transposed vector of X.

Finally, we set $\mathring{B} = \{(u,v)\in\mathbb{R}^2\colon\ u^2+v^2<1\}$ for the open unit disc and $\partial B = \{(u,v)\in\mathbb{R}^2\colon\ u^2+v^2=1\}$ for its boundary.

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Remarks.

1. Note the relation $W > 0$ in B.
2. For the introduction of conformal parameters into a Riemannian metric we refer to [4].

With the present notes we want to draw the reader's attention to a definition of torsion coefficients $T^{\vartheta}_{\sigma,i}$ for an orthonormal normal section $\{N_1, N_2\}$, which is deduced from the theory of space curves.

In addition, we introduce the associated functional of total torsion $\mathcal{T}_X(N_1, N_2)$ in Section 3, and we study its critical points. This functional appears e.g. in the second variation formula of the area functional, which in turn can be applied in certain area estimates of geodesic discs.

The paper is organized as follows:

- In Section 2, we define torsion coefficients of orthonormal normal sections.
- In Section 3, we introduce the concept of the total torsion of an orthonormal normal section. We provide conditions for a normal section to be critical and optimal for the total torsion.
- Section 4 contains some aspects about generalized analytic functions and Riemann-Hilbert problems. These results are used to prove a global pointwise estimate for the torsion coefficients of critical normal sections.
- Finally, an example of a critical normal section for holomorphic graphs $(w, \Phi(w))$ will be discussed in Section 5.

2. Torsion coefficients and curvature of the normal bundle

According to the classical theory of curves in $\mathbb{R}^3$, we introduce *torsion coefficients* as follows:

Definition 1. For an orthonormal normal section $\{N_1, N_2\}$ we define

$$T^{\vartheta}_{\sigma,i} := N_{\sigma,u^i} \cdot N^t_{\vartheta}, \quad i = 1,2, \ \sigma, \vartheta = 1,2, \tag{2.1}$$

setting $u^1 \equiv u$ and $u^2 \equiv v$.

Remarks.

1. Obviously, $T^{\vartheta}_{\sigma,i} = -T^{\sigma}_{\vartheta,i}$ holds for any $i = 1,2$, $\sigma, \vartheta = 1,2$, and consequently $T^{\sigma}_{\sigma,i} \equiv 0$.
2. The $T^{\vartheta}_{\sigma,i}$ are exactly the coefficients of the normal connection. In their terms one defines the coefficients of the curvature tensor $\mathfrak{S}$ of the normal bundle (summation convention!)
$$S^{\vartheta}_{\sigma,ij} := T^{\vartheta}_{\sigma,i,u^j} - T^{\vartheta}_{\sigma,j,u^i} + T^{\omega}_{\sigma,i} T^{\vartheta}_{\omega,j} - T^{\omega}_{\sigma,j} T^{\vartheta}_{\omega,i}, \quad i,j = 1,2, \ \sigma, \vartheta = 1,2. \tag{2.2}$$
In contrast to the case of codimension $n \geqslant 3$, the quadratical terms in (2.2) vanish in $\mathbb{R}^4$, and $\mathfrak{S}$ consists essentially of the single term
$$S := S^2_{1,12} = T^2_{1,1,v} - T^2_{1,2,u} = \operatorname{div}\bigl(-T^2_{1,2}, T^2_{1,1}\bigr). \tag{2.3}$$
This is the reason why we concentrate on immersions in $\mathbb{R}^4$.
3. Note that S does not depend on the choice of the orthonormal section $\{N_1, N_2\}$, compare Section 3.1.
4. Distinguish our definition from that of the *normal torsion* of a surface (see e.g. [1]): It can be defined as the torsion of the one-dimensional normal section (as a curve on the surface), which arises from a suitable intersection of X with a three-dimensional hyperplane.

3. Total torsion and optimal normal sections

To an orthonormal section $\{N_1, N_2\}$ of the normal bundle we assign the *total torsion*

$$\mathcal{T}_X(N_1, N_2) := \sum_{\sigma,\vartheta=1}^{2} \iint\limits_B g^{ij} T^{\vartheta}_{\sigma,i} T^{\vartheta}_{\sigma,j} W \, du \, dv = 2 \iint\limits_B \bigl\{\bigl(T^2_{1,1}\bigr)^2 + \bigl(T^2_{1,2}\bigr)^2\bigr\} \, du \, dv, \tag{3.1}$$

where $g_{ij} g^{jk} = \delta^k_i$, and δ^k_i is the Kronecker symbol (see the conformality relations in (1.3)).

3.1. Critical orthonormal normal sections

The total torsion depends on the chosen orthonormal section $\{N_1, N_2\}$, and it can be controlled by means of a rotation angle $\varphi = \varphi(u, v)$, depending smoothly on $(u, v) \in B$. Indeed, starting with the section $\{N_1, N_2\}$, we write

$$\widetilde{N}_1 = \cos\varphi N_1 + \sin\varphi N_2, \qquad \widetilde{N}_2 = -\sin\varphi N_1 + \cos\varphi N_2 \tag{3.2}$$

for the rotated normal section $\{\widetilde{N}_1, \widetilde{N}_2\}$. Then, the new torsion coefficients are given by

$$\widetilde{T}^2_{1,1} = T^2_{1,1} + \varphi_u, \qquad \widetilde{T}^2_{1,2} = T^2_{1,2} + \varphi_v. \tag{3.3}$$

Due to (3.1), the difference between new and old total torsion now computes to

$$\begin{aligned}\mathcal{T}_X(\widetilde{N}_1, \widetilde{N}_2) - \mathcal{T}_X(N_1, N_2) &= 2\iint\limits_B |\nabla\varphi|^2\,du\,dv + 4\iint\limits_B \big(T^2_{1,1}\varphi_u + T^2_{1,2}\varphi_v\big)\,du\,dv \\ &= 2\iint\limits_B |\nabla\varphi|^2\,du\,dv + 4\int\limits_{\partial B} \big(T^2_{1,1}, T^2_{1,2}\big)\cdot\nu^t\varphi\,ds - 4\iint\limits_B \operatorname{div}\big(T^2_{1,1}, T^2_{1,2}\big)\varphi\,du\,dv.\end{aligned} \tag{3.4}$$

In general, the right-hand side does not vanish.

Proposition. *$\{N_1, N_2\}$ is critical for $\mathcal{T}_X$, iff the torsion coefficients satisfy*

$$\operatorname{div}\big(T^2_{1,1}, T^2_{1,2}\big) = 0 \quad \text{in } B, \qquad \big(T^2_{1,1}, T^2_{1,2}\big)\cdot\nu^t = 0 \quad \text{on } \partial B. \tag{3.5}$$

3.2. Construction of critical orthonormal normal sections

How can we construct a critical section $\{N_1, N_2\}$ from a given section $\{\widetilde{N}_1, \widetilde{N}_2\}$?
If $\{N_1, N_2\}$ is critical, then we have

$$\begin{aligned}&0 = \operatorname{div}\big(T^2_{1,1}, T^2_{1,2}\big) = \operatorname{div}\big(\widetilde{T}^2_{1,1} - \varphi_u, \widetilde{T}^2_{1,2} - \varphi_v\big) \quad \text{in } B, \\ &0 = \big(T^2_{1,1}, T^2_{1,2}\big)\cdot\nu^t = \big(\widetilde{T}^2_{1,1} - \varphi_u, \widetilde{T}^2_{1,2} - \varphi_v\big)\cdot\nu^t \quad \text{on } \partial B,\end{aligned} \tag{3.6}$$

by virtue of (3.3), (3.5). This implies our next result:

Proposition. *The given section $\{\widetilde{N}_1, \widetilde{N}_2\}$ transforms into a critical section by means of (3.2), iff*

$$\begin{aligned}&\Delta\varphi = \operatorname{div}\big(\widetilde{T}^2_{1,1}, \widetilde{T}^2_{1,2}\big) \quad \text{in } B, \\ &\frac{\partial\varphi}{\partial\nu} = \big(\widetilde{T}^2_{1,1}, \widetilde{T}^2_{1,2}\big)\cdot\nu^t \quad \text{on } \partial B\end{aligned} \tag{3.7}$$

holds for the rotation angle $\varphi = \varphi(u, v)$.

Remark. It is well known that the solvability of the Neumann problem

$$\Delta\varphi = f \quad \text{in } B, \qquad \frac{\partial\varphi}{\partial\nu} = g \quad \text{on } \partial B \tag{3.8}$$

depends on the integrability condition

$$\iint\limits_B f\,du\,dv = \int\limits_{\partial B} g\,ds, \tag{3.9}$$

which is fulfilled in our proposition.

3.3. Minimality of critical orthonormal normal sections

Let $\{N_1, N_2\}$ be a critical section. Then, we conclude

$$\mathcal{T}_X(\widetilde{N}_1, \widetilde{N}_2) = \mathcal{T}_X(N_1, N_2) + 2\iint\limits_B |\nabla\varphi|^2\,du\,dv, \tag{3.10}$$

taking (3.4) and (3.5) into account. This proves the following

Proposition. *A critical orthonormal normal section $\{N_1, N_2\}$ minimizes the total torsion, i.e. we have*

$$\mathcal{T}_X(N_1, N_2) \leqslant \mathcal{T}_X(\widetilde{N}_1, \widetilde{N}_2) \tag{3.11}$$

for all smooth orthonormal normal sections $\{\widetilde{N}_1, \widetilde{N}_2\}$. The equality occurs iff $\varphi \equiv \text{const}$.

3.4. Flat normal bundles

For a critical normal section, the vector-field $(-T^2_{1,2}, T^2_{1,1})$ is parallel to ν along ∂B. Applying the Gaussian integral theorem to (2.3), we infer

$$\iint_B S\,du\,dv = \int_{\partial B} \left(-T^2_{1,2}, T^2_{1,1}\right) \cdot \nu^t\,ds = \pm \int_{\partial B} \sqrt{\left(T^2_{1,1}\right)^2 + \left(T^2_{1,2}\right)^2}\,ds. \tag{3.12}$$

In particular, if $S \equiv 0$, that is, the normal bundle is flat, then we find

$$T^{\vartheta}_{\sigma,i} \equiv 0 \quad \text{on } \partial B \tag{3.13}$$

for $i = 1, 2$ and $\sigma, \vartheta = 1, 2$.

Differentiating (2.3) and (3.5), we further obtain

$$\triangle T^2_{1,1} = \frac{\partial}{\partial v} S = 0, \quad \triangle T^2_{1,2} = -\frac{\partial}{\partial u} S = 0 \quad \text{in } B \tag{3.14}$$

for flat normal bundles. Therefore,

$$T^{\vartheta}_{\sigma,i} \equiv 0 \quad \text{in } B \quad (i = 1, 2,\ \sigma, \vartheta = 1, 2) \tag{3.15}$$

follows by the maximum principle.

Remark. Immersions of prescribed mean curvature with flat normal bundles are extensively studied in the literature; see e.g. [3,6] for higher-dimensional surfaces. Special results for two-dimensional immersions without curvature conditions on the normal bundle can be found in [2].

In the following, we investigate the inhomogeneous case of non-flat normal bundles to extend the relation (3.15) appropriately.

4. Estimates for the torsion coefficients

4.1. A Riemann–Hilbert problem

Once again, let us consider (2.3) and (3.5) for critical sections:

$$\frac{\partial}{\partial u} T^2_{1,1} + \frac{\partial}{\partial v} T^2_{1,2} = 0, \quad \frac{\partial}{\partial v} T^2_{1,1} - \frac{\partial}{\partial u} T^2_{1,2} = S \quad \text{in } B. \tag{4.1}$$

The *complex-valued torsion* $\Psi = T^2_{1,1} - iT^2_{1,2}$ solves the non-homogeneous Cauchy–Riemann equation

$$\frac{\partial}{\partial \overline{w}} \Psi(w) = \Psi_{\overline{w}}(w) := \frac{1}{2}(\Psi_u + i\Psi_v) = \frac{i}{2} S, \quad w = u + iv \in \mathring{B}. \tag{4.2}$$

In addition, we write the boundary condition in (3.5) as

$$\operatorname{Re}\left[w\Psi(w)\right] = 0, \quad w \in \partial B. \tag{4.3}$$

The relations (4.2) and (4.3) form a linear Riemann–Hilbert problem for Ψ.

Proposition. *The problem (4.2)–(4.3) possesses at most one solution $\Psi \in C^1(\mathring{B}) \cap C^0(B)$.*

Proof. Let Ψ_1, Ψ_2 be two such solutions. Then we set $\Phi(w) := w[\Psi_1(w) - \Psi_2(w)]$ and note

$$\Phi_{\overline{w}} = 0 \quad \text{in } \mathring{B}, \qquad \operatorname{Re}\Phi = 0 \quad \text{on } \partial B. \tag{4.4}$$

Consequently, $\Phi \equiv ic$ holds true in B with some constant $c \in \mathbb{R}$, and the continuity of Ψ_1, Ψ_2 implies $c = 0$. □

4.2. Some facts about generalized analytic functions

As general references for this subsection we name [5,7].

For arbitrary $f \in C^1(B, \mathbb{C})$ we define

$$T_B[f](w) := -\frac{1}{\pi} \iint_B \frac{f(\zeta)}{\zeta - w}\,d\xi\,d\eta, \quad w \in \mathbb{C}, \tag{4.5}$$

using the notation $\zeta = \xi + i\eta$. Then, there hold $g := T_B[f] \in C^1(\mathbb{C} \setminus \partial B) \cap C^0(\mathbb{C})$ as well as

$$\frac{\partial}{\partial \overline{w}} T_B[f](w) = \begin{cases} f(w), & w \in \mathring{B}, \\ 0, & w \in \mathbb{C} \setminus B, \end{cases} \tag{4.6}$$

cf. [7, Kapitel I, §5]. Next, we set

$$\begin{aligned} P_B[f](w) &:= -\frac{1}{\pi} \iint_B \left\{ \frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta}\,\overline{f(\zeta)}}{1 - w\overline{\zeta}} \right\} d\xi\, d\eta \\ &= T_B[f](w) + \frac{1}{w} \overline{T_B[wf]\left(\frac{1}{\overline{w}}\right)}. \end{aligned} \tag{4.7}$$

We obtain $h := P_B[f] \in C^1(\mathring{B}) \cap C^0(B)$, and (4.6) yields

$$\frac{\partial}{\partial \overline{w}} P_B[f](w) = f(w), \quad w \in \mathring{B}. \tag{4.8}$$

Finally, we note the relation

$$P_B[f](w) = T_{\mathbb{C}}[f_*](w), \quad w \in B. \tag{4.9}$$

Here, $T_{\mathbb{C}}$ is defined as T_B but with integration over $\mathbb{C}$, and we have abbreviated

$$f_*(w) := \begin{cases} f(w), & w \in B, \\ \frac{1}{|w|^4} \overline{f(\frac{1}{\overline{w}})}, & w \in \mathbb{C} \setminus B. \end{cases} \tag{4.10}$$

Observe that $f_*(w)$ is not continuous in $\mathbb{C}$, but it belongs to the class $L_{p,2}(\mathbb{C})$ for any $p \in [1, +\infty]$, that means, $f_*(w)$ as well as $|w|^{-2} f_*(\frac{1}{w})$ belong to $L_p(B)$; compare [7, p. 12]. Consequently, Satz 1.24 in [7] yields the following

Proposition. *With the definitions above, we have the uniform estimate*

$$\left|P_B[f](w)\right| = \left|T_{\mathbb{C}}[f_*](w)\right| \leqslant c(p) \|f\|_{L_p(B)}, \quad w \in B, \tag{4.11}$$

where $p \in (2, +\infty]$, and $c(p)$ is a positive constant dependent on p.

4.3. A global pointwise estimate for the torsion coefficients

Theorem. *Consider a conformally parametrized immersion $X \in C^4(B, \mathbb{R}^4)$ and write*

$$s_p := \|S\|_{L_p(B)}, \quad p \in (2, +\infty]. \tag{4.12}$$

Then, the complex-valued torsion $\Psi = T_{1,1}^2 - iT_{1,2}^2$ of a critical orthonormal section $\{N_1, N_2\}$ satisfies

$$\left|\Psi(w)\right| \leqslant c(p) s_p \quad \text{for all } w \in B, \tag{4.13}$$

with some positive constant $c(p)$.

Remark.

1. For a flat normal bundle, i.e. $s_p = 0$, we recover (3.15).
2. The general estimate (4.13) shall be useful, e.g., for proving curvature estimates for immersions with non-flat normal bundle.

Proof of the theorem. Let us write $f := \frac{i}{2} S \in C^1(B)$. We claim that Ψ possesses the integral representation

$$\Psi(w) = P_B[f](w) = -\frac{1}{\pi} \iint_B \left\{ \frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta}\,\overline{f(\zeta)}}{1 - w\overline{\zeta}} \right\} d\xi\, d\eta, \quad w \in B. \tag{4.14}$$

Then, (4.13) follows at once from the proposition in Section 4.2.

An elementary calculation proves

$$w P_B[f](w) = \frac{1}{\pi} \iint_B f(\zeta)\, d\xi\, d\eta + T_B[wf](w) - \overline{T_B[wf]\left(\frac{1}{\overline{w}}\right)}. \tag{4.15}$$

Taking $f = \frac{i}{2}S$ into account, we infer

$$\mathrm{Re}\left\{wP_B[f](w)\right\} = 0, \quad w \in \partial B. \tag{4.16}$$

Consequently—remember (4.8)—$P_B[f](w)$ solves the Riemann–Hilbert problem (4.2)–(4.3). Now the uniqueness result of the proposition in Section 4.1 yields the identity (4.14). □

Remark. We point out that the representation (4.14) relies crucially on the fact that the right-hand side $f = \frac{i}{2}S$ in (4.2) is purely imaginary. In general, a Riemann–Hilbert problem as in (4.2)–(4.3) is solvable, iff the integral of the right-hand side f over B has vanishing real part, cf. (4.15). For details we refer to [7, Kapitel IV, §7].

5. Example: Holomorphic graphs on *B*

Let us consider graphs $X(w) = (w, \Phi(w))$, $w = u + iv \in B$. If $\Phi(w) = \varphi(w) + i\psi(w)$ is holomorphic on B, then the vectors

$$N_1 = \frac{1}{\sqrt{W}}(-\varphi_u, -\varphi_v, 1, 0), \qquad N_2 = \frac{1}{\sqrt{W}}(-\psi_u, -\psi_v, 0, 1) \tag{5.1}$$

form an orthonormal normal section, where $W = 1 + |\nabla\varphi|^2 = 1 + |\Phi'|^2$ is the area element.

Remark. Due to $\varphi_u = \psi_v$, $\varphi_v = -\psi_u$ and thus $\Delta\varphi = \Delta\psi = 0$, the immersion X represents a conformally parametrized minimal graph.

For the torsion coefficients we compute

$$T_{1,1}^2 = \frac{1}{W}(-\varphi_{uu}\varphi_v + \varphi_{uv}\varphi_u) = \frac{1}{2W}\frac{\partial}{\partial v}\left(|\nabla\varphi|^2\right), \qquad T_{1,2}^2 = -\frac{1}{2W}\frac{\partial}{\partial u}\left(|\nabla\varphi|^2\right). \tag{5.2}$$

Consequently, the relation

$$\mathrm{div}\left(T_{1,1}^2, T_{1,2}^2\right) = 0 \quad \text{in } B \tag{5.3}$$

is satisfied. In order to check the boundary condition in (3.5), we introduce polar coordinates $u = r\cos\alpha$, $v = r\sin\alpha$ and note $\frac{1}{r}\frac{\partial}{\partial\alpha} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$. According to (5.2), we then obtain

$$\left(T_{1,1}^2, T_{1,2}^2\right) \cdot \nu^t = \frac{1}{2W}\left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right)\left(|\nabla\varphi|^2\right) = \frac{1}{2W}\frac{\partial}{\partial\alpha}\left(|\Phi'|^2\right) \quad \text{on } \partial B. \tag{5.4}$$

Proposition. *Consider the graph $(w, \Phi(w))$, $w \in B$, with a holomorphic function $\Phi(w) = \varphi(w) + i\psi(w)$. Then the normal section $\{N_1, N_2\}$ defined in (5.1) is critical, that is, it satisfies (3.5), iff $|\Phi'|$ is constant on ∂B.*

Remark. As an example, we mention the graph $X(w) = (w, w^n)$, $w \in B$, for arbitrary $n \in \mathbb{N}$.

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