View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 314 (2007) 267–302

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Morita equivalences induced by bimodules over Hopf–Galois extensions [☆]

Stefaan Caenepeel [a,*], Septimiu Crivei [b], Andrei Marcus [b], Mitsuhiro Takeuchi [c]

[a] *Faculty of Engineering, Vrije Universiteit Brussel, VUB, Pleinlaan 2, B-1050 Brussels, Belgium*

[b] *Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Str. Mihail Kogălniceanu 1, RO-400084 Cluj-Napoca, Romania*

[c] *Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan*

Received 23 August 2006

Available online 12 March 2007

Communicated by Susan Montgomery

Abstract

Let H be a Hopf algebra, and A, B be H-Galois extensions. We investigate the category ${}_A\mathcal{M}_B^H$ of relative Hopf bimodules, and the Morita equivalences between A and B induced by them.

© 2007 Published by Elsevier Inc.

Keywords: Hopf–Galois extension; Morita equivalence

[☆] This research was supported by the bilateral project BWS04/04 “New techniques in Hopf algebras and graded ring theory” of the Flemish and Romanian governments and by the research project G.0622.06 “Deformation quantization methods for algebras and categories with applications to quantum mechanics” from FWO-Vlaanderen. The second author acknowledges the support of a grant CEEX-ET 47/2006. The third author acknowledges the support of a Bolyai fellowship of the Hungarian Academy of Science.

[*] Corresponding author.

E-mail addresses: scaenepe@vub.ac.be (S. Caenepeel), crivei@math.ubbcluj.ro (S. Crivei), marcus@math.ubbcluj.ro (A. Marcus), takeu@math.tsukuba.ac.jp (M. Takeuchi).

URL: http://homepages.vub.ac.be/~scaenepe/ (S. Caenepeel).

0021-8693/$ – see front matter © 2007 Published by Elsevier Inc.
doi:10.1016/j.jalgebra.2007.02.033

Introduction

This paper is a contribution to the representation theory of Hopf–Galois extensions, as originated by Schneider in [16]. More specifically, we consider the following questions. Let H be a Hopf algebra, and A, B right H-comodule algebras. Moreover, assume that A and B are right faithfully flat H-Galois extensions.

(1) If A and B are Morita equivalent, does it follow that $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$ are also Morita equivalent?
(2) Conversely, if $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$ are Morita equivalent, when does it follow that A and B are Morita equivalent?

These questions have been considered in [12] in the context of strongly group graded algebras, the motivation coming from problems raised in the modular representation theory of finite groups. The results of the present paper generalize the results of [12, Sections 2 and 3].

Given a right H-comodule algebra A, and a left H-comodule algebra B, we consider $(A \otimes B, H)$-Hopf modules. These are at the same time left $A \otimes B$-modules and right H-comodules, with a suitable compatibility condition. There are various ways to look at these Hopf modules: they are Doi–Hopf modules (see [9]) over a certain Doi–Hopf datum (with two possible descriptions of the underlying module coalgebra), and they can also be viewed as comodules over a coring (see Section 3). The main result of Section 2, and also the main tool used during the rest of the paper, is a Structure Theorem for $(A \otimes B, H)$-Hopf modules, stating that the category of $(A \otimes B, H)$-Hopf modules is equivalent to the category of left modules over the cotensor product $A \,\square_H\, B$, under the condition that A is a faithfully flat H-Galois extension.

The results from Section 2 can be applied to relative Hopf bimodules: let A and B be right H-comodule algebras, and consider (A, B)-bimodules with a right H-coaction, satisfying a certain compatibility condition. The category of relative Hopf bimodules is then isomorphic to the category of $(A \otimes B^{\mathrm{op}}, H)$-Hopf modules. In Section 4, we state the Structure Theorem for relative Hopf bimodules, and we investigate the compatibility of the category equivalence with the Hom and tensor functors.

In Section 5, we apply our results to discuss the two problems stated above. We introduce the notion of H-Morita contexts, and we show that if two right faithfully flat H-Galois extensions are connected by a (strict) H-Morita context, then the algebras of coinvariants are also connected by a (strict) Morita context. Our main result is the following converse result: if the algebras of coinvariants are Morita equivalent, in such a way that the bimodule structure on one of the connecting modules can be extended to a left-action by the cotensor product $A \,\square_H\, B^{\mathrm{op}}$, then A and B are H-Morita equivalent.

In Section 6, we show that the Morita equivalence coming from a strict H-Morita context between two faithfully flat H-Galois extensions respects the Miyashita–Ulbrich action. In Section 7, we investigate the behavior of H-Morita equivalences with respect to Hopf subalgebras.

The category of relative Hopf modules and A-linear (not necessarily H-colinear) modules is an H-colinear category. If two right H-comodule algebras are H-Morita equivalent, then the induced equivalence between their categories of relative Hopf modules is H-colinear. In Section 8, we study the converse property: when does every H-colinear equivalence between two categories of relative Hopf modules come from a strict H-Morita context. This leads to a generalization of the Eilenberg–Watts Theorem (Proposition 8.3). The main result is Corollary 8.5, stating that every H-colinear equivalence comes from a strict H-Morita context if the Hopf algebra H is projective, and the H-comodule algebras A and B are H-Galois extensions of their subalgebras of coinvariants.

For basic results on Hopf algebras, we refer the reader to [8] or [13]. For a concise treatment of corings and their applications, we refer to [4].

1. Preliminary results

Throughout this paper H is a Hopf algebra over a commutative ring k, with bijective antipode S. We use the Sweedler notation for the comultiplication on H: $\Delta(h) = h_{(1)} \otimes h_{(2)}$. $\mathcal{M}^H$ (respectively ${}^H\mathcal{M}$) is the category of right (respectively left) H-comodules. For a right H-coaction ρ (respectively a left H-coaction λ) on a k-module M, we denote

$$\rho(m) = m_{[0]} \otimes m_{[1]} \quad \text{and} \quad \lambda(m) = m_{[-1]} \otimes m_{[0]}.$$

The submodule of coinvariants $M^{\mathrm{co}\,H}$ of a right (respectively left) H-comodule M consists of the elements $m \in M$ satisfying $\rho(m) = m \otimes 1$ (respectively $\lambda(m) = 1 \otimes m$).

Let A be a right H-comodule algebra. ${}_A\mathcal{M}^H$ and $\mathcal{M}_A^H$ are the categories of left and right relative Hopf modules. We have two pairs of adjoint functors $(F_1 = A \otimes_{A^{\mathrm{co}\,H}} -,\ G_1 = (-)^{\mathrm{co}\,H})$ and $(F_2 = - \otimes_{A^{\mathrm{co}\,H}} A,\ G_2 = (-)^{\mathrm{co}\,H})$ between the categories ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ and ${}_A\mathcal{M}^H$, and between $\mathcal{M}_{A^{\mathrm{co}\,H}}$ and $\mathcal{M}_A^H$. The unit and counit of the adjunction (F_1, G_1) are given by the formulas

$$\eta_{1,N} : N \to (A \otimes_{A^{\mathrm{co}\,H}} N)^{\mathrm{co}\,H}, \quad \eta_{1,N}(n) = 1 \otimes n;$$
$$\varepsilon_{1,M} : A \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} \to M, \quad \varepsilon_{1,M}(a \otimes m) = am.$$

The formulas for the unit and counit of (F_2, G_2) are similar. Consider the canonical maps

$$\mathrm{can} : A \otimes_{A^{\mathrm{co}\,H}} A \to A \otimes H, \quad \mathrm{can}(a \otimes b) = ab_{[0]} \otimes b_{[1]};$$
$$\mathrm{can}' : A \otimes_{A^{\mathrm{co}\,H}} A \to A \otimes H, \quad \mathrm{can}(a \otimes b) = a_{[0]}b \otimes a_{[1]}.$$

It is well known (see for example [11]) that can is an isomorphism if and only if can′ is an isomorphism.

Theorem 1.1. *Let A be a right H-comodule algebra. Consider the following statements*:

(1) (F_2, G_2) *is a pair of inverse equivalences*;
(2) (F_2, G_2) *is a pair of inverse equivalences and $A \in {}_{A^{\mathrm{co}\,H}}\mathcal{M}$ is flat*;
(3) can *is an isomorphism and $A \in {}_{A^{\mathrm{co}\,H}}\mathcal{M}$ is faithfully flat*;
(4) (F_1, G_1) *is a pair of inverse equivalences*;
(5) (F_1, G_1) *is a pair of inverse equivalences and $A \in \mathcal{M}_{A^{\mathrm{co}\,H}}$ is flat*;
(6) can′ *is an isomorphism and $A \in \mathcal{M}_{A^{\mathrm{co}\,H}}$ is faithfully flat.*

We have the following implications:

$$(3) \iff (2) \implies (1); \qquad (6) \iff (5) \implies (4).$$

If H is flat as a k-module, then (1) ⇔ (2) *and* (4) ⇔ (5). *If k is field then the six conditions are equivalent.*

If the first three conditions of Theorem 1.1 hold, then we call A a left faithfully flat H-Galois extension; if the three other conditions hold, then we call A a right faithfully flat H-Galois extension.

Proof. The equivalence of (2) and (3) is well known. It is essentially [15, Theorem 3.7], which is an improvement of [10, Theorem 2.11]. For the equivalence of (5) and (6), we observe that A is a left H^{cop}-comodule, so, by the left-handed version of the equivalence (3) $\Leftrightarrow$ (2), (6) is equivalent to flatness of $A \in \mathcal{M}_{A^{\mathrm{co}\,H}}$ and equivalence between the categories ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ and ${}^{H^{\mathrm{cop}}}{}_A\mathcal{M} \cong {}_A\mathcal{M}^H$.

The implications (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) are trivial.

If H is flat as a k-module, then $\mathcal{M}_A^H$ is an abelian category and the forgetful functor $\mathcal{M}_A^H \to \mathcal{M}_A$ is exact. If F_2 is an equivalence, then the functor $-\otimes_{A^{\mathrm{co}\,H}} : \mathcal{M}_{A^{\mathrm{co}}} \to \mathcal{M}_A$ is exact since it is the composition of the forgetful functor and the equivalence F_2. This shows that A is flat as a left $A^{\mathrm{co}\,H}$-module, and the implication (1) $\Rightarrow$ (2) follows. (4) $\Rightarrow$ (5) can be proved in a similar way.

If k is a field, then the equivalence of the six statements in the theorem follows from [15, Theorem I]. $\square$

Let M be a right H-comodule, and N a left H-comodule. The cotensor product $M \,\square_H\, N$ is the k-module

$$M \,\square_H\, N = \left\{ \sum_i m_i \otimes n_i \in M \otimes N \;\middle|\; \sum_i \rho(m_i) \otimes n_i = \sum_i m_i \otimes \lambda(n_i) \right\}.$$

If H is cocommutative, then $M \,\square_H\, N$ is also a right (or left) H-comodule.

Proposition 1.2. *Let R be a k-algebra, and assume that $P \in \mathcal{M}_R$ is flat. Take $M \in {}_R\mathcal{M}^H$ and $N \in {}^H\mathcal{M}$, and assume that we have a right H-coaction on M that is left R-linear. Then the map*

$$P \otimes_R (M \,\square_H\, N) \to (P \otimes_R M) \,\square_H\, N, \qquad p \otimes \left(\sum_i m_i \otimes n_i \right) \mapsto \sum_i (p \otimes m_i) \otimes n_i$$

is bijective.

2. A Structure Theorem for $(A \otimes B, H)$-Hopf modules

Under our assumption on H, $H \otimes H^{\mathrm{cop}}$ is also a Hopf algebra, and H is a left $H \otimes H^{\mathrm{cop}}$-module coalgebra; the left $H \otimes H^{\mathrm{cop}}$-action is given by

$$(k \otimes l) \cdot h = khS(l),$$

for all $h, k, l \in H$.

We present an alternative description of H as a left $H \otimes H^{\mathrm{cop}}$-module coalgebra. $H \otimes H^{\mathrm{cop}} \in {}_{H \otimes H^{\mathrm{cop}}}\mathcal{M}_H$, with right H-action induced by the comultiplication on H, and $k \in {}_H\mathcal{M}$ via ε, so we have the left $H \otimes H^{\mathrm{cop}}$-module $(H \otimes H^{\mathrm{cop}}) \otimes_H k$. $(H \otimes H^{\mathrm{cop}}) \otimes_H k$ is a coalgebra with comultiplication and counit given by

$$\Delta\big((h\otimes h')\otimes_H 1\big)=\big(h_{(1)}\otimes h'_{(2)}\big)\otimes_H 1\otimes\big(h_{(2)}\otimes h'_{(1)}\big)\otimes_H 1;$$
$$\varepsilon\big((h\otimes h')\otimes_H 1\big)=\varepsilon(hh').$$

It is easy to show that $(H\otimes H^{\mathrm{cop}})\otimes_H k$ is an $H\otimes H^{\mathrm{cop}}$-module coalgebra.

Proposition 2.1. *$(H\otimes H^{\mathrm{cop}})\otimes_H k$ and H are isomorphic as $H\otimes H^{\mathrm{cop}}$-module coalgebras.*

Proof. Define

$$f:\big(H\otimes H^{\mathrm{cop}}\big)\otimes_H k\to H,\quad f\big((h\otimes h')\otimes_H 1\big)=hS(h');$$
$$g:H\to\big(H\otimes H^{\mathrm{cop}}\big)\otimes_H k,\quad g(h)=(h\otimes 1)\otimes_H 1.$$

f is well defined since for all $h,h',l\in H$

$$f\big((h\otimes h')l\otimes_H 1\big)=hl_{(1)}S(h'l_{(2)})=hS(h')\varepsilon(l)=f\big((h\otimes h')\otimes_H\varepsilon(l)\big).$$

f is $H\otimes H^{\mathrm{cop}}$-linear since for all $h,h',k,k'\in H$

$$f\big((kh\otimes k'h')\otimes_H 1\big)=khS(k'h')=(k\otimes k')\cdot\big(hS(k')\big)=(k\otimes k')f\big((h\otimes h')\otimes_H 1\big).$$

f is a coalgebra map since for all $h,h'\in H$

$$\big((f\otimes f)\circ\Delta\big)\big((h\otimes h')\otimes_H 1\big)=h_{(1)}S\big(h'_{(2)}\big)\otimes h_{(2)}S\big(h'_{(1)}\big)=\Delta\big(hS(h')\big),$$

and

$$(\varepsilon\circ f)\big((h\otimes h')\otimes_H 1\big)=\varepsilon\big(hS(h')\big)=\varepsilon(hh').$$

It is obvious that $f\circ g=H$. Finally for all $h,k\in H$

$$\begin{aligned}(g\circ f)\big((h\otimes k)\otimes_H 1\big)&=g\big(hS(k)\big)\\&=\big(hS(k)\otimes 1\big)\otimes_H 1=\big(hS\big(k_{(1)}\big)\otimes 1\big)\otimes_H\varepsilon(k_{(2)})\\&=\big(hS(k_{(1)})k_{(2)}\otimes k_{(3)}\big)\otimes_H 1=(h\otimes k)\otimes_H 1.\qquad\square\end{aligned}$$

Let A be a right H-comodule algebra, and B a left H-comodule algebra. Then $A\otimes B$ is a right $H\otimes H^{\mathrm{cop}}$-comodule algebra, with coaction

$$\rho(a\otimes b)=a_{[0]}\otimes b_{[0]}\otimes a_{[1]}\otimes b_{[-1]}.$$

Then $(H\otimes H^{\mathrm{cop}},A\otimes B,H)$ is a left–right Doi–Hopf datum (see [6] or [9] for details), and we can consider the category ${}_{A\otimes B}\mathcal{M}(H\otimes H^{\mathrm{cop}})^H$ of Doi–Hopf modules. The objects of this category are k-modules M with a left $A\otimes B$-action and a right H-coaction such that

$$\rho\big((a\otimes b)m\big)=(a_{[0]}\otimes b_{[0]})m_{[0]}\otimes a_{[1]}m_{[1]}S(b_{[-1]}),$$

for all $a \in A$, $b \in B$ and $m \in M$. The objects of ${}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$ are called $(A \otimes B, H)$-Hopf modules. It is well known and easily verified that $A \otimes B \in {}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$, with coaction defined by

$$\rho(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1]}S(b_{[-1]}).$$

Lemma 2.2. *With notation as above, we have that* $(A \otimes B)^{\mathrm{co}\,H} = A \,\square_H\, B$.

Proof. Take $x = \sum_i a_i \otimes b_i \in (A \otimes B)^{\mathrm{co}\,H}$. Then

$$\sum_i a_i \otimes b_i \otimes 1 = \sum_i a_{i[0]} \otimes b_{i[0]} \otimes a_{i[1]}S(b_{i[-1]}).$$

Apply λ to the second tensor factor. Then switch the second and fourth tensor factor, and multiply the third and fourth tensor factor. It follows that

$$\sum_i a_i \otimes b_{i[0]} \otimes b_{i[-1]} = \sum_i a_{i[0]} \otimes b_{i[0]} \otimes a_{i[1]}S(b_{i[-2]})b_{i[-1]} = \sum_i a_{i[0]} \otimes b_i \otimes a_{i[1]},$$

and then $x \in A \,\square_H\, B$. The converse inclusion is proved in a similar way. $\square$

Recall (see for example [9]) that we have a pair of adjoint functors (F, G):

$$F : {}_{A\square_H B}\mathcal{M} \to {}_{A\otimes B}\mathcal{M}\big(H \otimes H^{\mathrm{cop}}\big)^H, \quad F(N) = (A \otimes B) \otimes_{A\square_H B} N;$$
$$G : {}_{A\otimes B}\mathcal{M}\big(H \otimes H^{\mathrm{cop}}\big)^H \to {}_{A\square_H B}\mathcal{M}, \quad G(M) = M^{\mathrm{co}\,H}.$$

The unit and counit of the adjunction are the following:

$$\eta_N : N \to \big((A \otimes B) \otimes_{A\square_H B} N\big)^{\mathrm{co}\,H}, \quad \eta_N(n) = 1_A \otimes 1_B \otimes n;$$
$$\varepsilon_M : (A \otimes B) \otimes_{A\square_H B} M^{\mathrm{co}\,H} \to M, \quad \varepsilon_M(a \otimes b \otimes m) = (a \otimes b)m.$$

Proposition 2.3. *Assume that H is flat as a k-algebra. Let A be a right H-comodule algebra, and B a left H-comodule algebra. We have a right H-colinear map*

$$f : A \otimes_{A^{\mathrm{co}\,H}} (A \,\square_H\, B) = F_1(A \,\square_H\, B) \to A \otimes B, \quad f\left(a \otimes \left(\sum_i a_i \otimes b_i\right)\right) = \sum_i aa_i \otimes b_i.$$

If A is a right faithfully flat H-Galois extension, then f is an isomorphism.

Proof. f is right H-colinear since

$$\begin{aligned}\rho\left(f\left(a \otimes \left(\sum_i a_i \otimes b_i\right)\right)\right) &= \sum_i a_{[0]}a_{i[0]} \otimes b_{i[0]} \otimes a_{[1]}a_{i[1]}S(b_{i[-1]}) \\ &= \sum_i a_{[0]}a_i \otimes b_{i[0]} \otimes a_{[1]}b_{i[-2]}S(b_{i[-1]})\end{aligned}$$

$$= \sum_i a_{[0]} a_i \otimes b_i \otimes a_{[1]}$$
$$= (f \otimes H)\biggl(\rho\biggl(a \otimes \biggl(\sum_i a_i \otimes b_i\biggr)\biggr)\biggr).$$

On $A \otimes_{A^{\mathrm{co}\,H}} A$ and $A \otimes H$, we consider the following right H-coactions:

$$\rho(a \otimes b) = a \otimes b_{[0]} \otimes b_{[1]}; \qquad \rho(a \otimes h) = a \otimes h_{(1)} \otimes h_{(2)}.$$

Then $\mathrm{can}: A \otimes_{A^{\mathrm{co}\,H}} A \to A \otimes H$ is right H-colinear, so we can consider the map $\mathrm{can} \,\square_H\, B : (A \otimes_{A^{\mathrm{co}\,H}} A) \,\square_H\, B \to (A \otimes H) \,\square_H\, B$. If A is a right faithfully flat H-Galois extension, then $\mathrm{can} \,\square_H\, B$ is bijective, and applying Proposition 1.2, we see that f is the composition of the following isomorphisms:

$$\begin{aligned} A \otimes_{A^{\mathrm{co}\,H}} (A \,\square_H\, B) &\cong (A \otimes_{A^{\mathrm{co}\,H}} A) \,\square_H\, B \\ &\cong (A \otimes H) \,\square_H\, B \cong A \otimes (H \,\square_H\, B) \cong A \otimes B. \qquad \square \end{aligned}$$

The following Structure Theorem is the main result of this section.

Theorem 2.4. *Let A be a right H-comodule algebra, and B a left H-comodule algebra. If A is a right faithfully flat H-Galois extension, then (F, G) is a pair of inverse equivalences between the categories ${}_{A \square_H B}\mathcal{M}$ and ${}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$.*

Proof. Take $N \in {}_{A\square_H B}\mathcal{M}$. We have a well-defined algebra map $A^{\mathrm{co}\,H} \to A \,\square_H\, B$, sending a to $a \otimes 1_B$, and N is a left $A^{\mathrm{co}\,H}$-module, by restriction of scalars. Consider the isomorphism

$$\begin{aligned} &\alpha_N = f \otimes_{A\square_H B} N : F_1(N) = A \otimes_{A^{\mathrm{co}\,H}} N \cong A \otimes_{A^{\mathrm{co}\,H}} (A \,\square_H\, B) \otimes_{A\square_H B} N \\ &\quad \to F(N) = (A \otimes B) \otimes_{A\square_H B} N. \end{aligned}$$

It is easy to see that $\alpha_N(a \otimes n) = (a \otimes 1) \otimes_{A\square_H B} n$, and α_N is right H-colinear since

$$(\alpha_N \otimes H)\bigl((a_{[0]} \otimes n) \otimes a_{[1]}\bigr) = \bigl((a_{[0]} \otimes 1) \otimes_{A\square_H B} n\bigr) \otimes a_{[1]} = \rho\bigl((a \otimes 1) \otimes n\bigr).$$

It follows that α_N restricts to an isomorphism

$$\alpha_N^{\mathrm{co}\,H} : (A \otimes_{A^{\mathrm{co}\,H}} N)^{\mathrm{co}\,H} \to \bigl((A \otimes B) \otimes_{A\square_H B} N\bigr)^{\mathrm{co}\,H}.$$

It is then easily seen that

$$\eta_N = \alpha_N^{\mathrm{co}\,H} \circ \eta_{1,N}.$$

$\eta_{1,N}$ is an isomorphism by Theorem 1.1, and it follows that η_N is an isomorphism.

Take $M \in {}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$. Then M is a left A-module, by restriction of scalars, and a relative Hopf module since

$$\rho(am) = \rho\bigl((a \otimes 1)m\bigr) = (a_{[0]} \otimes 1)m_{[0]} \otimes a_{[1]}m_{[1]}S(1) = a_{[0]}m_{[0]} \otimes a_{[1]}m_{[1]}.$$

It is then easy to see that

$$\varepsilon_M \circ \alpha_{M^{\mathrm{co}H}} = \varepsilon_{1,M}.$$

It follows from Theorem 1.1 that $\varepsilon_{1,M}$ is an isomorphism, and this implies that ε_M is an isomorphism. $\square$

3. Connection to comodules over corings

Let A be a ring. Recall that an A-coring $\mathcal{C}$ is a comonoid in the monoidal category ${}_A\mathcal{M}_A$. For a detailed discussion of the theory of corings and comodules, we refer to [4]. One of the results is that we can associate a coring to a Doi–Hopf datum, and that the category of Doi–Hopf modules is isomorphic to the category of comodules over this coring.

Let us describe the $A \otimes B$-coring $\mathcal{C}$ associated to the left–right Doi–Hopf datum $(H \otimes H^{\mathrm{cop}}, A \otimes B, H)$ that we have discussed in the previous section. We have that $\mathcal{C} = H \otimes A \otimes B$, with left and right $A \otimes B$-action given by

$$(a' \otimes b')(h \otimes a \otimes b)(a'' \otimes b'') = a'_{[1]} h S\big(b'_{[-1]}\big) \otimes a'_{[0]} a a'' \otimes b'_{[0]} b b''.$$

The comultiplication and counit are given by the formulas

$$\Delta(h \otimes a \otimes b) = (h_{(2)} \otimes 1_A \otimes 1_B) \otimes_{A\otimes B} (h_{(1)} \otimes a \otimes b);$$
$$\varepsilon(h \otimes a \otimes b) = \varepsilon(h) a \otimes b.$$

The category ${}^{\mathcal{C}}\mathcal{M}$ of left $\mathcal{C}$-comodules is isomorphic to ${}_{A\otimes B}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$.

A Galois theory for corings can be developed (see [3,5]). Let x be a group-like element of a coring $\mathcal{C}$, and let

$$A^{\mathrm{co}\mathcal{C}} = \{a \in A \mid ax = xa\}.$$

Then we have an adjoint pair of functors between ${}_{A^{\mathrm{co}\mathcal{C}}}\mathcal{M}$ and ${}^{\mathcal{C}}\mathcal{M}$. If this adjoint pair is a pair of inverse equivalences, then the map

$$\mathrm{can}: A \otimes_{A^{\mathrm{co}\mathcal{C}}} A \to \mathcal{C}, \quad \mathrm{can}(a \otimes b) = axb$$

is an isomorphism of corings (see [5, Proposition 3.1]). If, in addition, A is flat as a right $A^{\mathrm{co}\mathcal{C}}$-module, then it also follows that A is faithfully flat as a right $A^{\mathrm{co}\mathcal{C}}$-module (see [5, Proposition 3.8, (2) $\Rightarrow$ (1)]). We will apply this to the coring $\mathcal{C} = H \otimes A \otimes B$. $1_H \otimes 1_A \otimes 1_B$ is a group-like element of $H \otimes A \otimes B$, and the associated pair of adjoint functors is precisely (F, G). It can be easily verified that the corresponding canonical map is precisely the map

$$\mathrm{can}: (A \otimes B) \otimes_{A \square_H B} (A \otimes B) \to H \otimes A \otimes B,$$
$$\mathrm{can}\big((a \otimes b) \otimes (a' \otimes b')\big) = a_{[1]} S(b_{[-1]}) \otimes a_{[0]} a' \otimes b_{[0]} b'. \tag{1}$$

Proposition 3.1. *Let A be a right H-comodule algebra, and B a left H-comodule algebra. Assume that A is a right faithfully flat H-Galois extension. Then* can *is an isomorphism. Furthermore, $A \otimes B$ is faithfully flat as a right $A \square_H B$-module.*

Proof. It follows from Theorem 2.4 that (F, G) is a pair of inverse equivalences, hence can is an isomorphism.

We will now show that $A \otimes B$ is flat as a right $A \square_H B$-module. Assume that $N \to N'$ is a monomorphism of left $A \square_H B$-modules. Using Proposition 2.3 and the fact that A is flat as a right $A^{\mathrm{co}\,H}$-module, we find that

$$(A \otimes B)_{A\square_H B} N \cong A \otimes_{A^{\mathrm{co}\,H}} N \to A \otimes_{A^{\mathrm{co}\,H}} N' \cong (A \otimes B)_{A\square_H B} N'$$

is injective. As explained above, it then follows from [5, Proposition 3.8] and Lemma 2.2 that $A \otimes B$ is faithfully flat as a right $A \square_H B$-module. □

4. Application to Hopf bimodules

Now let A and B be right H-comodule algebras. A two-sided relative Hopf module is a k-module with a left A-action, a right B-action, and a right H-coaction, such that

$$\rho(amb) = a_{[0]} m_{[0]} b_{[0]} \otimes a_{[1]} m_{[1]} b_{[1]},$$

for all $a \in A$, $b \in B$ and $m \in M$. ${}_A\mathcal{M}_B^H$ is the category of two-sided relative Hopf modules with k-module maps that are A-linear, B-linear and H-colinear.

B^{op} is a left H-comodule algebra, with left coaction λ given by $\lambda(b) = S^{-1}(b_{[1]}) \otimes b_{[0]}$. We can then apply the above results to A and B^{op}. In particular, $A \otimes B^{\mathrm{op}}$ is a right $H \otimes H^{\mathrm{cop}}$-comodule algebra.

Lemma 4.1. *Let A and B be right H-comodule algebras. Then the Doi–Hopf modules category ${}_{A\otimes B^{\mathrm{op}}}\mathcal{M}(H \otimes H^{\mathrm{cop}})^H$ is isomorphic to the category of two-sided relative Hopf modules ${}_A\mathcal{M}_B^H$.*

Proof. It is well known that ${}_{A\otimes B^{\mathrm{op}}}\mathcal{M}$ is isomorphic to the category of bimodules ${}_A\mathcal{M}_B$. The isomorphism respects the compatibility of the action and coaction. □

$A \otimes B^{\mathrm{op}}$ is a two-sided Hopf module, with coaction $\rho(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1]} b_{[1]}$. Furthermore $(A \otimes B^{\mathrm{op}})^{\mathrm{co}\,H} = A \square_H B^{\mathrm{op}}$. Applying Theorem 2.4, we obtain the following Structure Theorem for two-sided Hopf modules.

Theorem 4.2. *Let H be a Hopf algebra over the commutative ring k, with bijective antipode, and consider two right H-comodule algebras A and B. We have a pair of adjoint functors $(F = A \otimes B^{\mathrm{op}} \otimes_{A\square_H B^{\mathrm{op}}} -,\ G = (-)^{\mathrm{co}\,H})$ between the categories ${}_{A\square_H B^{\mathrm{op}}}\mathcal{M}$ and ${}_A\mathcal{M}_B^H$. If A is a right faithfully flat H-Galois extension, then (F, G) is a pair of inverse equivalences.*

Remark 4.3. Assume that A (respectively B) is a right (respectively left) faithfully flat H-Galois extension. The proof of Theorem 2.4 shows that via appropriate transport of structure, the functors

$$\left(A \otimes B^{\mathrm{op}}\right) \otimes_{A\square B^{\mathrm{op}}} -, \quad A \otimes_{A^{\mathrm{co}\,H}} -, \quad - \otimes_{B^{\mathrm{co}\,H}} B : {}_{A\square B^{\mathrm{op}}}\mathcal{M} \to {}_A\mathcal{M}_B^H$$

are naturally isomorphic equivalences of categories. It follows immediately that we may define the functors

$$-\otimes_{A^{\mathrm{co}H}} - : {}_{B\Box A^{\mathrm{op}}}\mathcal{M} \times {}_A\mathcal{M}_C \to {}_B\mathcal{M}_C,$$
$$-\otimes_{A^{\mathrm{co}H}} - : {}_B\mathcal{M}_A \times {}_{A\Box C^{\mathrm{op}}}\mathcal{M} \to {}_B\mathcal{M}_C.$$

Proposition 4.4. *Let A, B, C be right H-comodule algebras. If $M \in {}_A\mathcal{M}_B^H$ and $N \in {}_B\mathcal{M}_C^H$, then $M \otimes_B N \in {}_A\mathcal{M}_C^H$. If A and B are right faithfully flat H-Galois extensions, then the map*

$$f : M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \to (M \otimes_B N)^{\mathrm{co}\,H}, \quad f(m \otimes n) = m \otimes n,$$

is an isomorphism. Consequently $M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H}$ is a left $A \Box_H C^{\mathrm{op}}$-module.

Proof. It is clear that $M \otimes_B N$ is an (A, C)-bimodule. A right H-coaction on $M \otimes_B N$ is defined as follows:

$$\rho(m \otimes_B n) = m_{[0]} \otimes_B n_{[0]} \otimes m_{[1]}n_{[1]}.$$

It is easy to show that ρ is well defined, and that this coaction makes $M \otimes_B N \in {}_A\mathcal{M}_C^H$.

By restriction of scalars, $M \in {}_A\mathcal{M}^H$ and $N \in {}_B\mathcal{M}^H$. It follows from Theorem 1.1 that

$$\varepsilon_{1,M} : A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}\,H} \to M \quad \text{and} \quad \varepsilon_{1,N} : B \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \to N$$

are isomorphisms. Let g be the composition of the maps

$$\varepsilon_{1,M} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} : A \otimes_{A^{\mathrm{co}}} M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \to M \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H}$$

and

$$M \otimes_B \varepsilon_{1,N} : M \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \cong M \otimes_B B \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \to M \otimes_B N.$$

g is bijective, and is given by the formula

$$g(a \otimes m \otimes n) = am \otimes_B n,$$

for $a \in A$, $m \in M^{\mathrm{co}\,H}$ and $n \in N^{\mathrm{co}\,H}$. It is clear that g is left H-linear. g is also right H-colinear, since $g(a_{[0]} \otimes m \otimes n) \otimes a_{[1]} = a_{[0]}m \otimes n \otimes a_{[1]} = \rho(am \otimes_B n)$, and it follows that g is an isomorphism in ${}_A\mathcal{M}^H$, and, by Theorem 1.1 that

$$g^{\mathrm{co}\,H} : \left(A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H}\right)^{\mathrm{co}\,H} \to (M \otimes_B N)^{\mathrm{co}\,H}$$

is an isomorphism. The map f is an isomorphism since it is the composition of $g^{\mathrm{co}\,H}$ and the isomorphism

$$\eta_{1, M^{\mathrm{co}H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}H}} : M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H} \to \left(A \otimes_{A^{\mathrm{co}H}} M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H}\right)^{\mathrm{co}\,H}.$$

Finally, the left $A \Box_H C^{\mathrm{op}}$-action on $(M \otimes_B N)^{\mathrm{co}\,H}$ can be transported using f to $M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}H}} N^{\mathrm{co}\,H}$. $\Box$

In the sequel, we will use the adjoint pair of functors (F,G), with unit η and counit ε introduced after Lemma 2.2, in the cases where the algebras involved are respectively A and B^{op}, A and C^{op} and B and C^{op}. If A and B are right faithfully flat H-Galois extensions, then these three adjunctions are pairs of inverse equivalences, by Theorem 4.2. We will use the same notation (F,G) and (η,ε) for the three adjunctions, no confusion will arise from this.

Take $M_1 \in {}_{A\Box_H B^{\mathrm{op}}}\mathcal{M}$ and $N_1 \in {}_{B\Box_H C^{\mathrm{op}}}\mathcal{M}$, and denote

$$M = \left(A \otimes B^{\mathrm{op}}\right) \otimes_{A\Box_H B^{\mathrm{op}}} M_1 \in {}_A\mathcal{M}_B^H;$$
$$N = \left(B \otimes C^{\mathrm{op}}\right) \otimes_{B\Box_H C^{\mathrm{op}}} N_1 \in {}_B\mathcal{M}_C^H.$$

Using Theorem 4.2 and Proposition 4.4, we find isomorphisms

$$M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \cong M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} \cong (M \otimes_B N)^{\mathrm{co}\,H} \in {}_{A\Box_H C^{\mathrm{op}}}\mathcal{M}.$$

Transporting structure, we find that $M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \in {}_{A\Box_H C^{\mathrm{op}}}\mathcal{M}$, and we have a functor

$$- \otimes_{B^{\mathrm{co}\,H}} - : {}_{A\Box_H B^{\mathrm{op}}}\mathcal{M} \times {}_{B\Box_H C^{\mathrm{op}}}\mathcal{M} \to {}_{A\Box_H C^{\mathrm{op}}}\mathcal{M}.$$

Corollary 4.5. *Let A, B, C be right H-comodule algebras, and assume that A and B are right faithfully flat H-Galois extensions. Take $M_1 \in {}_{A\Box_H B^{\mathrm{op}}}\mathcal{M}$ and $N_1 \in {}_{B\Box_H C^{\mathrm{op}}}\mathcal{M}$. With notation as above, we have that $M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \in {}_{A\Box_H C^{\mathrm{op}}}\mathcal{M}$, and we have an isomorphism*

$$h : \left(A \otimes C^{\mathrm{op}}\right) \otimes_{A\Box_H C^{\mathrm{op}}} (M_1 \otimes_{B^{\mathrm{co}\,H}} N_1) \to M \otimes_B N$$

in ${}_A\mathcal{M}_C^H$. This isomorphism is natural in M_1 and N_1.

For later use, we observe that the naturality of h means the following. Let $\mu_1 : M_1 \to M_1'$ and $\nu_1 : N_1 \to N_1'$ be morphisms in respectively ${}_{A\Box_H B^{\mathrm{op}}}\mathcal{M}$ and ${}_{B\Box_H C^{\mathrm{op}}}\mathcal{M}$, and let $\mu = F(\mu_1)$, $\nu = F(\nu_1)$. Then $\mu_1 \otimes_{B^{\mathrm{co}\,H}} \nu_1$ is a morphism in ${}_{A\Box_H C^{\mathrm{op}}}\mathcal{M}$, and the following diagram commutes

$$\begin{array}{ccc} F(M_1 \otimes_{B^{\mathrm{co}\,H}} N_1) & \xrightarrow{F(\mu_1 \otimes \nu_1)} & F(M_1' \otimes_{B^{\mathrm{co}\,H}} N_1') \\ \downarrow h & & \downarrow h \\ M \otimes_B N & \xrightarrow{\mu \otimes \nu} & M' \otimes_B N'. \end{array} \tag{2}$$

From now on, let H be a projective Hopf algebra (this condition is always fulfilled if k is a field); let A be a right H-comodule algebra, and $M, N \in {}_A\mathcal{M}^H$. Then the map

$$\nu : {}_A\mathrm{Hom}(M,N) \otimes H \to {}_A\mathrm{Hom}(M, N \otimes H), \quad \nu(f \otimes h)(m) = f(m) \otimes h$$

is injective (see for example [2, Proposition II.4.2, p. AII.75]). A direct computation shows that the map $\tilde{\rho} : {}_A\mathrm{Hom}(M,N) \to {}_A\mathrm{Hom}(M, N \otimes H)$ defined by

$$\tilde{\rho}(f)(m) = f(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]}) f(m_{[0]})_{[1]}$$

is left A-linear. Let ${}_A\mathrm{HOM}(M,N)$ be the k-submodule of ${}_A\mathrm{Hom}(M,N)$ consisting of the maps f for which $\tilde\rho(f)$ factorizes through ${}_A\mathrm{Hom}(M,N)$, or, equivalently, for which there exists $f_{[0]}\otimes f_{[1]}\in {}_A\mathrm{Hom}(M,N)\otimes H$ such that

$$f_{[0]}(m)\otimes f_{[1]}=f(m_{[0]})_{[0]}\otimes S^{-1}(m_{[1]})f(m_{[0]})_{[1]}, \tag{3}$$

for all $m\in M$. It follows from the injectivity of ν that $f_{[0]}\otimes f_{[1]}$ is unique if it exists. ${}_A\mathrm{HOM}(M,N)$ is called the rational part of ${}_A\mathrm{Hom}(M,N)$. If H is finitely generated and projective, then ν is bijective, and ${}_A\mathrm{HOM}(M,N)={}_A\mathrm{Hom}(M,N)$. We have a map

$$\rho=\nu^{-1}\circ\tilde\rho : {}_A\mathrm{HOM}(M,N)\to {}_A\mathrm{Hom}(M,N)\otimes H,\quad \rho(f)=f_{[0]}\otimes f_{[1]}.$$

Proposition 4.6. *Let H be a projective Hopf algebra, A a right H-comodule algebra, and $M,N\in {}_A\mathcal{M}^H$. Then $({}_A\mathrm{HOM}(M,N),\rho)$ is a right H-comodule.*

Proof. $N\otimes H\in {}_A\mathcal{M}^H$ under the diagonal coaction. We know that five of the six faces of the following diagram, namely all faces except the top one, commute.

$$\begin{tikzcd}
{}_A\mathrm{HOM}(M,N) \arrow[rr, "\rho"] \arrow[dr, "\rho"] \arrow[dd, "\subset"'] & & {}_A\mathrm{Hom}(M,N)\otimes H \arrow[dr, "\tilde\rho\otimes H"] \arrow[dd, "\nu"] & \\
& {}_A\mathrm{Hom}(M,N)\otimes H \arrow[rr, "(\nu\circ H)\circ(id\otimes\Delta)"] \arrow[dd, "\nu"] & & {}_A\mathrm{Hom}(M,N\otimes H)\otimes H \arrow[dd, "\nu"] \\
{}_A\mathrm{Hom}(M,N) \arrow[rr, "\tilde\rho"] \arrow[dr, "\tilde\rho"] & & {}_A\mathrm{Hom}(M,N\otimes H) \arrow[dr, "\tilde\rho"] & \\
& {}_A\mathrm{Hom}(M,N)\otimes H \arrow[rr, "{}_A\mathrm{Hom}(M,N\otimes\Delta)"] & & {}_A\mathrm{Hom}(M,N\otimes H\otimes H)
\end{tikzcd}$$

This implies that the top face also commutes; this means that, for all $f\in {}_A\mathrm{HOM}(M,N)$,

$$(\nu\otimes H)\big(f_{[0]}\otimes\Delta(f_{[1]})\big)=\tilde\rho(f_{[0]})\otimes f_{[1]},$$

and therefore $f_{[0]}\otimes f_{[1]}\in {}_A\mathrm{HOM}(M,N)\otimes H$. We then also have that

$$(\nu\otimes H)\big(f_{[0]}\otimes\Delta(f_{[1]})\big)=(\nu\otimes H)\big(\rho(f_{[0]})\otimes f_{[1]}\big),$$

and, since $\nu\otimes H$ is injective,

$$f_{[0]}\otimes\Delta(f_{[1]})=\rho(f_{[0]})\otimes f_{[1]}.$$

We therefore have shown that $\rho : {}_A\mathrm{HOM}(M,N)\to {}_A\mathrm{HOM}(M,N)\otimes H$ is a coassociative map. Finally, it follows immediately from (3) that $\varepsilon(f_{[1]})f_{[0]}=f$, for all $f\in {}_A\mathrm{HOM}(M,N)$. $\square$

An alternative description of ${}_A\mathrm{HOM}(M,N)$ is the following: ${}_A\mathrm{Hom}(M,N)$ is a left H^*-module, with action (see [8, 6.5] in the case where k is a field):

$$(h^*\cdot f)(m)=\big\langle h^*,S^{-1}(m_{[1]})f(m_{[0]})_{[1]}\big\rangle f(m_{[0]})_{[0]}.$$

${}_A\mathrm{HOM}(M,N)$ is then the subspace of ${}_A\mathrm{Hom}(M,N)$ consisting of left A-linear $f:M\to N$ for which there exists a (unique) $f_{[0]}\otimes f_{[1]}\in {}_A\mathrm{Hom}(M,N)\otimes H$ such that

$$(h^*\cdot f)(m)=\langle h^*,f_{[1]}\rangle f_{[0]}(m).$$

Proposition 4.7. *Let A be a right H-comodule algebra, with H a projective Hopf algebra, and $M,N\in {}_A\mathcal{M}^H$. If M is finitely generated projective as a left A-module, then ${}_A\mathrm{HOM}(M,N)$ coincides with ${}_A\mathrm{Hom}(M,N)$. For $f\in {}_A\mathrm{HOM}(M,N)$, we have*

$$\rho(f)=\sum_i m_i^*\cdot f(m_{i[0]})_{[0]}\otimes S^{-1}(m_{i[1]})f(m_{i[0]})_{[1]},\tag{4}$$

where $\sum_i m_i^*\otimes_A m$ is a finite dual basis of $M\in {}_A\mathcal{M}$.

Proof. We used the following notation: for $m^*\in {}_A\mathrm{Hom}(M,A)$, and $n\in N$, $m^*\cdot n\in {}_A\mathrm{Hom}(M,N)$ is defined by

$$(m^*\cdot n)(m)=m^*(m)n.$$

For every $m\in M$, we have that $m=\sum_i m_i^*(m)m_i$, hence

$$\rho(m)=\sum_i m_i^*(m)_{[0]}m_{i[0]}\otimes m_i^*(m)_{[1]}m_{i[1]}.\tag{5}$$

We then compute that

$$\begin{aligned}
&f(m_{[0]})_{[0]}\otimes S^{-1}(m_{[1]})f(m_{[0]})_{[1]}\\
&\overset{(5)}{=}\sum_i f\big(m_i^*(m)_{[0]}m_{i[0]}\big)_{[0]}\otimes S^{-1}\big(m_i^*(m)_{[1]}m_{i[1]}\big)f\big(m_i^*(m)_{[0]}m_{i[0]}\big)_{[1]}\\
&=\sum_i m_i^*(m)_{[0]}f(m_{i[0]})_{[0]}\otimes S^{-1}(m_{i[1]})S^{-1}\big(m_i^*(m)_{[2]}\big)m_i^*(m)_{[1]}f(m_{i[0]})_{[1]}\\
&=\sum_i m_i^*(m)f(m_{i[0]})_{[0]}\otimes S^{-1}(m_{i[1]})(m_{i[0]})_{[1]}\\
&=\sum_i m_i^*\cdot f(m_{i[0]})_{[0]}(m)\otimes S^{-1}(m_{i[1]})f(m_{i[0]})_{[1]},
\end{aligned}$$

and (4) follows from (3). □

Proposition 4.8. *Let H be a projective Hopf algebra, and A, B, C right H-comodule algebras. If $M \in {}_A\mathcal{M}_B^H$ and $N \in {}_A\mathcal{M}_C^H$, then*

$$ {}_A\mathrm{HOM}(M,N) \in {}_B\mathcal{M}_C^H. $$

We have a map

$$ \beta : {}_A\mathrm{HOM}(M,N)^{\mathrm{co}\,H} \to {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}\big(M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}\big). $$

If A is a right faithfully flat H-Galois extension, then β is an isomorphism of left $B \,\square\, C^{\mathrm{op}}$-modules.

Proof. We consider the following (B,C)-bimodule structure on ${}_A\mathrm{Hom}(M,N)$:

$$ (b \cdot f \cdot c)(m) = f(mb)c. $$

It is clear that $b \cdot f \cdot c$ is then left A-linear. Take $f \in {}_A\mathrm{HOM}(M,N)$; in order to show that $b \cdot f \cdot c \in {}_A\mathrm{HOM}(M,N)$, it suffices to show that $b_{[0]} \cdot f_{[0]} \cdot c_{[0]} \otimes b_{[1]} f_{[1]} c_{[1]}$ satisfies (3). This can be seen as follows: for all $m \in M$, we have

$$ \begin{aligned} &(b_{[0]} \cdot f_{[0]} \cdot c_{[0]})(m) \otimes b_{[1]} f_{[1]} c_{[1]} \\ &\quad = f_{[0]}(mb_{[0]})c_{[0]} \otimes b_{[1]} f_{[1]} c_{[1]} \\ &\quad = f\big((mb_{[0]})_{[0]}\big)_{[0]} c_{[0]} \otimes b_{[1]} S^{-1}\big((mb_{[0]})_{[1]}\big) f\big((mb_{[0]})_{[0]}\big)_{[1]} c_{[1]} \\ &\quad = f(m_{[0]}b_{[0]})_{[0]} c_{[0]} \otimes b_{[2]} S^{-1}(b_{[1]}) S^{-1}(m_{[1]}) f(m_{[0]}b_{[0]})_{[1]} c_{[1]} \\ &\quad = f(m_{[0]}b)_{[0]} c_{[0]} \otimes S^{-1}(m_{[1]}) f(m_{[0]}b)_{[1]} c_{[1]} \\ &\quad = \big(f(m_{[0]}b)c\big)_{[0]} \otimes S^{-1}(m_{[1]}) \big(f(m_{[0]}b)c\big)_{[1]} \\ &\quad = (b \cdot f \cdot c)(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]}) (b \cdot f \cdot c)(m_{[0]})_{[1]}, \end{aligned} $$

as needed. This shows also that $\rho(b \cdot f \cdot c) = b_{[0]} \cdot f_{[0]} \cdot c_{[0]} \otimes b_{[1]} f_{[1]} c_{[1]}$, hence that ${}_A\mathrm{HOM}(M,N) \in {}_B\mathcal{M}_C^H$.

Now take $f \in {}_A\mathrm{HOM}(M,N)^{\mathrm{co}\,H}$. Then $\rho(f) = f_{[0]} \otimes f_{[1]}$, so

$$ f(m) \otimes 1 = f(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]}) f(m_{[0]})_{[1]}, $$

for all $m \in M$. If $m \in M^{\mathrm{co}\,H}$, then it follows that $f(m) \otimes 1 = \rho(f(m))$, so $f(m) \in N^{\mathrm{co}\,H}$. Thus f restricts to a map $\beta(f) = f^{\mathrm{co}\,H} : M^{\mathrm{co}\,H} \to N^{\mathrm{co}\,H}$. Using the fact that f is left A-linear, we see that the diagram

$$ \begin{array}{ccc} A \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} & \xrightarrow{A \otimes f^{\mathrm{co}\,H}} & A \otimes_{A^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} \\ \Big\downarrow{\scriptstyle \varepsilon_{1,M}} & & \Big\downarrow{\scriptstyle \varepsilon_{1,N}} \\ M & \xrightarrow{\quad f \quad} & N \end{array} $$

commutes. If A is right faithfully flat H-Galois, then we define the inverse of β as follows:

$$\beta^{-1}(g) = \varepsilon_{1,N} \circ (A \otimes g) \circ \varepsilon_{1,M}^{-1}. \qquad \square$$

Combining Proposition 4.8 with Theorem 4.2, we obtain the following result.

Corollary 4.9. *Let A, B, C be right H-comodule algebras, and assume that A and B are right faithfully flat H-Galois extensions. Let $M_1 \in {}_{A\Box B^{\rm op}}\mathcal{M}$ and $N_1 \in {}_{A\Box C^{\rm op}}\mathcal{M}$, and consider*

$$M = \left(A \otimes B^{\rm op}\right) \otimes_{A\Box B^{\rm op}} M_1 \in {}_A\mathcal{M}_B^H,$$
$$N = \left(A \otimes C^{\rm op}\right) \otimes_{A\Box C^{\rm op}} N_1 \in {}_A\mathcal{M}_C^H.$$

Then

$${}_{A^{{\rm co}\,H}}\mathrm{Hom}(M_1, N_1) \cong {}_A\mathrm{HOM}(M, N)^{{\rm co}\,H} \in {}_{B\Box C^{\rm op}}\mathcal{M}$$

and

$${}_A\mathrm{HOM}(M, N) \cong \left(B \otimes C^{\rm op}\right) \otimes_{B\Box C^{\rm op}} {}_{A^{{\rm co}\,H}}\mathrm{Hom}(M_1, N_1).$$

Proposition 4.10. *Let A, B, C be right H-comodule algebras, and consider $M \in {}_A\mathcal{M}_B^H$, $N \in {}_A\mathcal{M}_C^H$. Then the evaluation map*

$$\varphi : M \otimes_B {}_A\mathrm{HOM}(M, N) \to N, \quad \varphi(m \otimes_B f) = f(m)$$

is a morphism in ${}_A\mathcal{M}_C^H$.

If A and B are right faithfully flat H-Galois extensions, then the evaluation map

$$M^{{\rm co}\,H} \otimes_{B^{{\rm co}\,H}} {}_{A^{{\rm co}\,H}}\mathrm{Hom}\left(M^{{\rm co}\,H}, N^{{\rm co}\,H}\right) \to N^{{\rm co}\,H}$$

is left $A \,\Box_H\, C^{\rm op}$-linear.

Proof. We first show that φ is right H-colinear.

$$\begin{aligned}(\varphi \otimes H)\big(\rho(m \otimes f)\big) &= (\varphi \otimes H)(m_{[0]} \otimes_B f_{[0]} \otimes m_{[1]} f_{[1]}) \\ &= f_{[0]}(m_{[0]}) \otimes m_{[1]} f_{[1]} \\ &\overset{(3)}{=} f(m_{[0]})_{[0]} \otimes m_{[2]} S^{-1}(m_{[1]}) f(m_{[0]})_{[1]} \\ &= \rho\big(f(m)\big) = \rho\big(\varphi(m \otimes_B f)\big).\end{aligned}$$

φ is left A-linear and right C-linear since

$$\varphi(am \otimes f \cdot c) = (f \cdot c)(am) = f(am)c = af(m)c = a\varphi(m \otimes_B f)c.$$

The composition

$$
\begin{aligned}
&M^{\operatorname{co}H} \otimes_{B^{\operatorname{co}H}} {}_{A^{\operatorname{co}H}}\operatorname{Hom}\big(M^{\operatorname{co}H}, N^{\operatorname{co}H}\big) \xrightarrow{M^{\operatorname{co}H}\otimes\beta} M^{\operatorname{co}H} \otimes_{B^{\operatorname{co}H}} {}_A\operatorname{HOM}(M,N)^{\operatorname{co}H}\\
&\quad \xrightarrow{f} \big(M \otimes_B {}_A\operatorname{HOM}(M,N)\big)^{\operatorname{co}H} \xrightarrow{\varphi^{\operatorname{co}H}} N^{\operatorname{co}H}
\end{aligned}
$$

is the required evaluation map. If A is right faithfully flat H-Galois, then β is an isomorphism of $B \,\square_H\, C^{\mathrm{op}}$-modules, by Proposition 4.8, and then $M^{\operatorname{co}H} \otimes \beta$ is an isomorphism of $A \,\square_H\, C^{\mathrm{op}}$-modules, by Corollary 4.5. If B is right faithfully flat H-Galois, then f is an isomorphism of $A \,\square_H\, C^{\mathrm{op}}$-modules, by Proposition 4.4. φ is a morphism in ${}_A\mathcal{M}_C^H$, hence $\varphi^{\operatorname{co}H}$ is left $A \,\square_H\, C^{\mathrm{op}}$-linear, since $(-)^{\operatorname{co}H}$ is a functor from ${}_A\mathcal{M}_C^H$ to ${}_{A\square_H C^{\mathrm{op}}}\mathcal{M}$. $\square$

Proposition 4.11. *Let A be a right H-comodule algebra, and $M \in {}_A\mathcal{M}^H$. Then ${}_A\operatorname{END}(M)^{\mathrm{op}}$ is a right H-comodule algebra.*

Proof. Applying Proposition 4.8 (with $M = N$, $B = C = k$), we see that ${}_A\operatorname{END}(M)$ is a right H-comodule. We have to show the compatibility relation

$$\rho(g \circ f) = g_{[0]} \circ f_{[0]} \otimes f_{[1]}g_{[1]}, \tag{6}$$

for all $f, g \in {}_A\operatorname{END}(M)$. To this end, it suffices to show that the right-hand side of (6) satisfies (3). Indeed, for all $m \in M$, we have

$$
\begin{aligned}
&(g_{[0]} \circ f_{[0]})(m) \otimes f_{[1]}g_{[1]}\\
&\overset{(3)}{=} g_{[0]}\big(f(m_{[0]})_{[0]}\big) \otimes S^{-1}(m_{[1]})f(m_{[0]})_{[1]}g_{[1]}\\
&\overset{(3)}{=} g\big(f(m_{[0]})_{[0]}\big)_{[0]} \otimes S^{-1}(m_{[1]})f(m_{[0]})_{[2]}S^{-1}\big(f(m_{[0]})_{[1]}\big)g\big(f(m_{[0]})_{[0]}\big)_{[1]}\\
&= g\big(f(m_{[0]})\big)_{[0]} \otimes S^{-1}(m_{[1]})g\big(f(m_{[0]})\big)_{[1]}\\
&= (g \circ f)(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]})(g \circ f)(m_{[0]})_{[1]}. \qquad \square
\end{aligned}
$$

Proposition 4.12. *Let A, B be right H-comodule algebras, and consider $M \in {}_A\mathcal{M}_B^H$. Then the map*

$$\psi : B \to {}_A\operatorname{END}(M), \quad \psi(b)(m) = mb$$

is a morphism in ${}_B\mathcal{M}_B^H$.

If A is a right faithfully flat H-Galois extension, then the map

$$\psi^{\operatorname{co}H} : B^{\operatorname{co}H} \to {}_A\operatorname{END}(M)^{\operatorname{co}H} \cong {}_{A^{\operatorname{co}H}}\operatorname{End}\big(M^{\operatorname{co}H}\big)$$

is left $B \,\square_H\, B^{\mathrm{op}}$-linear.

Proof. We first show that ψ is right H-colinear and well defined. Indeed,

$$\psi(b)_{[0]} \otimes \psi(b)_{[1]} = \psi(b_{[0]}) \otimes b_{[1]},$$

since

$$\begin{aligned}
&\psi(b)(m_{[0]})_{[0]} \otimes S^{-1}(m_{[1]})\psi(b)(m_{[0]})_{[1]} \\
&\quad = (m_{[0]}b)_{[0]} \otimes S^{-1}(m_{[1]})(m_{[0]}b)_{[1]} \\
&\quad = m_{[0]}b_{[0]} \otimes S^{-1}(m_{[2]})m_{[1]}b_{[1]} \\
&\quad = mb_{[0]} \otimes b_{[1]} = \psi(b_{[0]})(m) \otimes b_{[1]}.
\end{aligned}$$

ψ is left and right B-linear since

$$\psi(b'bb'')(m) = mb'bb'' = \big((b' \cdot \psi \cdot b'')(b)\big)(m),$$

for all $b, b', b'' \in B$ and $m \in M$. The second statement then follows immediately from Corollary 4.9. $\square$

Remark 4.13. The map ψ in Proposition 4.12 is also a morphism of right H-comodule algebras between B and ${}_A\mathrm{END}(M)^{\mathrm{op}}$.

Proposition 4.14. *Let A, B, C be right H-comodule algebras, and consider $M \in {}_A\mathcal{M}_B^H$, $N \in {}_A\mathcal{M}_C^H$. Then the map*

$$\mu : {}_A\mathrm{HOM}(M, A) \otimes_A N \to {}_A\mathrm{HOM}(M, N), \quad \mu(f \otimes n)(m) = f(m)n$$

is a morphism in ${}_B\mathcal{M}_C^H$. If A is a right faithfully flat H-Galois extension, then the map

$$\begin{aligned}
\mu^{\mathrm{co}\,H} : {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}\big(M^{\mathrm{co}\,H}, A^{\mathrm{co}\,H}\big) \otimes_{A^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} &\cong \big({}_A\mathrm{HOM}(M, A) \otimes_A N\big)^{\mathrm{co}\,H} \\
\to {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}\big(M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}\big) &\cong {}_A\mathrm{HOM}(M, N)^{\mathrm{co}\,H}
\end{aligned}$$

is left $B \,\square_H\, C^{\mathrm{op}}$-linear.

Proof. In order to prove that μ is right H-colinear, we have to show that

$$\rho\big(\mu(f \otimes n)\big) = \mu(f_{[0]} \otimes n_{[0]}) \otimes f_{[1]}n_{[1]}.$$

It suffices to compute that

$$\begin{aligned}
\mu(f_{[0]} \otimes n_{[0]})(m) \otimes f_{[1]}n_{[1]} &\overset{(3)}{=} f(m_{[0]})_{[0]}n_{[0]} \otimes S^{-1}(m_{[1]})f(m_{[0]})_{[1]}n_{[1]} \\
&= \big(f(m_{[0]})n\big)_{[0]} \otimes S^{-1}(m_{[1]})\big(f(m_{[0]})n\big)_{[1]} \\
&= \big(\mu(f \otimes n)(m_{[0]})\big)_{[0]} \otimes S^{-1}(m_{[1]})\big(\mu(f \otimes n)(m_{[0]})\big)_{[1]}.
\end{aligned}$$

Finally, μ is left B-linear and right C-linear, since

$$\big(\mu(b \cdot f \otimes nc)\big)(m) = f(mb)nc = \mu(f \otimes n)(mb)c = \big(b \cdot \mu(f \otimes n) \cdot c\big)(m). \qquad \square$$

5. Morita equivalences

In this section, we study Morita equivalences induced by two-sided relative Hopf modules.

Definition 5.1. Let A and B be right H-comodule algebras. An H-Morita context connecting A and B is a Morita context (A,B,M,N,α,β) such that $M\in {}_A\mathcal{M}_B^H$, $N\in {}_B\mathcal{M}_A^H$, $\alpha: M\otimes_B N\to A$ is a morphism in ${}_A\mathcal{M}_A^H$ and $\beta: N\otimes_A M\to B$ is a morphism in ${}_B\mathcal{M}_B^H$.

A morphism between two H-Morita contexts (A,B,M,N,α,β) and $(A',B',M',N',\alpha',\beta')$ is defined in the obvious way: it consists of a fourtuple (κ,λ,μ,ν), where $\kappa: A\to A'$ and $\lambda: B\to B'$ are H-comodule algebra maps, $\mu: M\to M'$ is a morphism in ${}_A\mathcal{M}_B^H$ and $\nu: N\to N'$ is a morphism in ${}_B\mathcal{M}_A^H$ such that $\kappa\circ\alpha=\alpha'\circ(\mu\otimes\nu)$ and $\lambda\circ\beta=\beta'\circ(\nu\otimes\mu)$. $\underline{\underline{\mathrm{Morita}}}^H(A,B)$ will be the subcategory of the category of H-Morita contexts, consisting of H-Morita contexts connecting A and B, and morphisms with the identity of A and B as the underlying algebra maps.

Proposition 5.2. *Let (A,B,M,N,α,β) be a strict H-Morita context. Then we have a pair of inverse equivalences $(M\otimes_B -, N\otimes_A -)$ between the categories ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}^H$.*

Proof. Let $P\in {}_B\mathcal{M}^H$. Then $M\otimes_B P\in {}_A\mathcal{M}^H$, with right H-action

$$\rho(m\otimes_B p)=m_{[0]}\otimes_B p_{[0]}\otimes m_{[1]}p_{[1]}.$$

The rest of the proof is straightforward. □

We will now give an H-comodule version of [1, Proposition 4.2.1].

Example 5.3. Let A be a right H-comodule algebra, and $M\in {}_A\mathcal{M}^H$. Then $B={}_A\mathrm{END}(M)^{\mathrm{op}}$ is also a right H-module algebra, by Proposition 4.11. Then $M\in {}_A\mathcal{M}_B^H$, with right B-action given by $m\cdot f=f(m)$, for all $f\in B$ and $m\in M$. Indeed, $(m\cdot f)\cdot g=m\cdot(g\circ f)$, and

$$\begin{aligned} m_{[0]}\cdot f_{[0]}\otimes m_{[1]}f_{[1]} &= f_{[0]}(m_{[0]})\otimes m_{[1]}f_{[1]}\\ &\overset{(3)}{=} f(m_{[0]})_{[0]}\otimes m_{[2]}S^{-1}(m_{[1]})f(m_{[0]})_{[1]}\\ &= \rho\big(f(m)\big)=\rho(m\cdot f). \end{aligned}$$

It follows from Proposition 4.8 that $N={}_A\mathrm{HOM}(M,A)\in {}_B\mathcal{M}_A^H$, and from Proposition 4.10 that the map

$$\alpha: M\otimes_B N\to A,\quad \alpha(m\otimes n)=n(a)$$

is a morphism in ${}_A\mathcal{M}_A^H$. It follows from Proposition 4.14 that the map

$$\beta: N\otimes_A M\to {}_A\mathrm{END}(M),\quad \beta(n\otimes m)(x)=n(x)m$$

is a morphism in ${}_B\mathcal{M}_B^H$. Straightforward computations then show that (A,B,M,N,α,β) is an H-Morita context. We call it the H-Morita context associated to $M\in {}_A\mathcal{M}^H$.

Proposition 5.4. *The H-Morita context associated to $M \in {}_A\mathcal{M}^H$ is strict if and only if M is a progenerator as a left A-module.*

Proof. If the Morita context is strict, then M is a left A-progenerator by [1, Theorem III.3.5]. Conversely, if M is a left A-progenerator, then $M \in {}_A\mathcal{M}$ is finitely generated and projective, hence ${}_A\mathrm{Hom}(M, X) = {}_A\mathrm{HOM}(M, X)$, for all $X \in {}_A\mathcal{M}^H$. If we forget the H-comodule structure in the H-Morita context, then we obtain the Morita context associated to $M \in {}_A\mathcal{M}$, as in [1, Proposition II.4.1]. By [1, Proposition II.4.4], this Morita context is strict. □

Proposition 5.5. *Let $(A, B, M, N, \alpha, \beta)$ be a strict H-Morita context. Then M is a left A-progenerator, and the H-Morita context is isomorphic to the H-Morita context associated to $M \in {}_A\mathcal{M}^H$.*

Proof. M is a left A-progenerator by [1, Theorem III.3.5]. Then ${}_A\mathrm{End}(M) = {}_A\mathrm{END}(M)$, and by [1, Theorem II.3.4], $\psi : B \to {}_A\mathrm{END}(M)^{\mathrm{op}}$, $\psi(b)(m) = mb$ is an isomorphism of k-algebras. It is an isomorphism of H-comodule algebras, by Remark 4.13. It follows from [1, Theorem 3.4] that

$$\varphi : N \to {}_A\mathrm{HOM}(M, A) = {}_A\mathrm{Hom}(M, A), \quad \varphi(n)(m) = \alpha(m \otimes n)$$

is an isomorphism of (B, A)-bimodules. We verify that φ is H-colinear. For every $n \in N$, we have to show that

$$\varphi(n_{[0]}) \otimes n_{[1]} = \varphi(n)_{[0]} \otimes \varphi(n)_{[1]}. \tag{7}$$

Using the right H-colinearity of α, we find

$$\begin{aligned}&\alpha(m_{[0]} \otimes_B n)_{[0]} \otimes S^{-1}(m_{[1]})\alpha(m_{[0]} \otimes_B n)_{[1]}\\ &\quad = \alpha(m_{[0]} \otimes_B n_{[0]}) \otimes S^{-1}(m_{[2]})m_{[1]}n_{[1]} = \alpha(m \otimes_B n_{[0]}) \otimes n_{[1]},\end{aligned}$$

and (7) follows from (3). From classical Morita theory (see [1]), we know that (A, ψ, M, φ) is an isomorphism of Morita contexts; since ψ and φ are H-colinear, it follows that is an isomorphism of H-Morita contexts. □

Definition 5.6. Assume that A and B are right faithfully flat H-Galois extensions of $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$. A $\square_H$-Morita context between $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$ is a Morita context $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ such that M_1 (respectively N_1) is a left $A \,\square_H\, B^{\mathrm{op}}$-module (respectively $B \,\square_H\, A^{\mathrm{op}}$-module) and

- $\alpha_1 : M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \to A^{\mathrm{co}\,H}$ is left $A \,\square_H\, A^{\mathrm{op}}$-linear,
- $\beta_1 : N_1 \otimes_{A^{\mathrm{co}\,H}} M_1 \to B^{\mathrm{co}\,H}$ is left $B \,\square_H\, B^{\mathrm{op}}$-linear.

A morphism between two $\square_H$-Morita contexts connecting $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$, is a morphism between Morita contexts of the form $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, \mu_1, \nu_1)$, where μ_1 is left $A \,\square_H\, B^{\mathrm{op}}$-linear and ν_1 is left $B \,\square_H\, A^{\mathrm{op}}$-linear. The category of $\square_H$-Morita contexts connecting $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$ will be denoted by $\underline{\underline{\mathrm{Morita}}}^{\square_H}(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H})$.

Theorem 5.7. *Let A and B be right faithfully flat H-Galois extensions of $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$. Then the categories $\underline{\underline{\mathrm{Morita}}}^H(A,B)$ and $\underline{\underline{\mathrm{Morita}}}^{\square_H}(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H})$ are equivalent. The equivalence functors send strict contexts to strict contexts.*

Proof. Let (A,B,M,N,α,β) be an H-Morita context. It follows from Theorem 4.2 that $M^{\mathrm{co}\,H} \in {}_{A\square_H B^{\mathrm{op}}}\mathcal{M}$, and $N^{\mathrm{co}\,H} \in {}_{B\square_H A^{\mathrm{op}}}\mathcal{M}$. It follows from Proposition 4.4 that we have a left $A\,\square_H\, A^{\mathrm{op}}$-linear map

$$\alpha_1 = \alpha^{\mathrm{co}\,H} \circ f : M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} \to (M \otimes_B N)^{\mathrm{co}\,H} \to A^{\mathrm{co}\,H},$$

and a left $B\,\square_H\, B^{\mathrm{op}}$-linear isomorphism

$$\beta_1 = \beta^{\mathrm{co}\,H} \circ f : N^{\mathrm{co}\,H} \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} \to (N \otimes_A M)^{\mathrm{co}\,H} \to B^{\mathrm{co}\,H}.$$

From the description of f in Proposition 4.4, it follows that we have a commutative diagram of isomorphisms

$$\begin{tikzcd}
M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} \arrow[r] \arrow[d] & (M \otimes_B N)^{\mathrm{co}\,H} \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} \arrow[d] \\
M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}\,H}} (N \otimes_A M)^{\mathrm{co}\,H} \arrow[r] & (M \otimes_B N \otimes_A M)^{\mathrm{co}\,H}.
\end{tikzcd}$$

Now $\alpha \otimes_A M = M \otimes_B \beta$ implies $(\alpha \otimes_A M)^{\mathrm{co}\,H} = (M \otimes_B \beta)^{\mathrm{co}\,H}$, and it follows that

$$\alpha_1 \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} = M^{\mathrm{co}\,H} \otimes_{B^{\mathrm{co}\,H}} \beta.$$

In a similar way, we have that

$$\beta_1 \otimes_{B^{\mathrm{co}\,H}} N^{\mathrm{co}\,H} = N^{\mathrm{co}\,H} \otimes_{A^{\mathrm{co}\,H}} \alpha$$

and it follows that $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}, \alpha_1, \beta_1)$ is a Morita context. If (A,B,M,N,α,β) is strict, then $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}, \alpha_1, \beta_1)$ is also strict.

Conversely, let $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ be a $\square_H$-Morita context. Then $M = F(M_1) = (A \otimes B^{\mathrm{op}}) \otimes_{A\square_H B^{\mathrm{op}}} M_1 \in {}_A\mathcal{M}_B^H$ and $N = F(N_1) = (B \otimes A^{\mathrm{op}}) \otimes_{B\square_H A^{\mathrm{op}}} N_1 \in {}_B\mathcal{M}_A^H$. Also observe that $A \cong F(A^{\mathrm{co}\,H}) = (A \otimes A^{\mathrm{op}}) \otimes_{A\square_H A^{\mathrm{op}}} A^{\mathrm{co}\,H}$ and $B \cong F(B^{\mathrm{co}\,H}) = (B \otimes B^{\mathrm{op}}) \otimes_{B\square_H B^{\mathrm{op}}} B^{\mathrm{co}\,H}$. We define $\alpha : M \otimes_B N \to A$ and $\beta : N \otimes_A M \to N$ by the commutativity of the following two diagrams, where the isomorphisms h are defined as in Corollary 4.5:

$$\begin{tikzcd}
F(M_1 \otimes_{B^{\mathrm{co}\,H}} N_1) \arrow[r, "F(\alpha_1)"] \arrow[d, "h"] & F(A^{\mathrm{co}\,H}) \arrow[d, "\cong"] \\
M \otimes_B N \arrow[r, "\alpha"] & A,
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
F(N_1 \otimes_{B^{\mathrm{co}\,H}} M_1) \arrow[r, "F(\beta_1)"] \arrow[d, "h"] & F(B^{\mathrm{co}\,H}) \arrow[d, "\cong"] \\
N \otimes_A M \arrow[r, "\beta"] & B.
\end{tikzcd} \tag{9}$$

It is clear that $\alpha \in {}_A\mathcal{M}_B^H$ and $\beta \in {}_B\mathcal{M}_A^H$. We claim that $(A, B, M, N, \alpha, \beta)$ is an H-Morita context. To this end, consider the following diagram

$$\begin{tikzcd}
M \otimes_B N \otimes_A M \arrow[r, "M\otimes\beta"] \arrow[d, "h^{-1}"] & M \otimes_B B \arrow[d, "h^{-1}"] \\
F(M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \otimes_{A^{\mathrm{co}\,H}} M_1) \arrow[r, "F(M_1\otimes\beta_1)"] \arrow[d, "="] & F(M_1 \otimes_{B^{\mathrm{co}\,H}} B^{\mathrm{co}\,H}) \arrow[d, "\cong"] \\
F(M_1 \otimes_{B^{\mathrm{co}\,H}} N_1 \otimes_{A^{\mathrm{co}\,H}} M_1) \arrow[r, "F(\alpha_1\otimes N_1)"] \arrow[d, "h"] & F(A^{\mathrm{co}\,H} \otimes_A^{\mathrm{co}\,H} N_1) \arrow[d, "h"] \\
M \otimes_B N \otimes_A M \arrow[r, "\alpha\otimes_A M"] & A \otimes_A M.
\end{tikzcd}$$

The top square and the bottom square commute by the definition of α and β, and because of the naturality of h (see (2)). The square in the middle commutes because $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ is a Morita context. So the whole diagram commutes. The composition of the left vertical morphisms is the identity of $M \otimes_B N \otimes_A M$, and the composition of the right vertical morphisms is the natural isomorphism $M \otimes_B B \cong A \otimes_A M$. So it follows that the diagram

$$\begin{tikzcd}
M \otimes_B N \otimes_A M \arrow[r, "M\otimes\beta"] \arrow[d, "\alpha\otimes M"] & M \otimes_B B \arrow[d, "\cong"] \\
A \otimes_A M \arrow[r, "\cong"] & M
\end{tikzcd}$$

commutes. The commutativity of the second diagram in the definition of a Morita context is proved in a similar way. $\square$

Recall that $M \in {}_A\mathcal{M}$ is a progenerator if and only if A and M are mutually direct summands of finite direct sums of copies of the other. Now let $M \in {}_A\mathcal{M}^H$. If this property holds in the category ${}_A\mathcal{M}^H$, then we call M an H-progenerator.

Corollary 5.8. *Assume that A and B are right faithfully flat H-Galois extensions. If $(A, B, M, N, \alpha, \beta)$ is a strict H-Morita context, then M is an H-progenerator.*

Proof. Let $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ be the corresponding strict $\square_H$-Morita context, as in Theorem 5.7. It follows from classical Morita theory that M_1 is a left $A^{\mathrm{co}\,H}$-progenerator.

The claim then follows from the equivalence $A \otimes_{A^{\mathrm{co}\,H}} -$ between the categories ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ and ${}_A\mathcal{M}^H$. $\square$

Theorem 5.9. *Assume that A and B are right faithfully flat H-Galois extensions, and let $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ be a strict Morita context. If M_1 has a left $A \,\square_H\, B^{\mathrm{op}}$-module structure, then there is a unique left $B \,\square_H\, A^{\mathrm{op}}$-module structure on N_1 such that $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$ is a strict $\square_H$-Morita context.*

Proof. We know that $M = A \otimes_{A^{\mathrm{co}\,H}} M_1 \in {}_A\mathcal{M}_B^H$. We have seen in Proposition 4.12 that we have a morphism $\psi : B \to {}_A\mathrm{END}(M)$ in ${}_B\mathcal{M}_B^H$, and a left $B \,\square_H\, B$-linear map $\psi^{\mathrm{co}\,H} : B^{\mathrm{co}\,H} \to {}_A\mathrm{END}(M)^{\mathrm{co}\,H} \cong {}_{A^{\mathrm{co}\,H}}\mathrm{End}(M^{\mathrm{co}\,H})$, see also Corollary 4.9. $\psi^{\mathrm{co}\,H}$ is an isomorphism, because the Morita context is strict. Since B is right faithfully flat H-Galois, it follows that ψ is an isomorphism in ${}_B\mathcal{M}_B^H$. Since M_1 is a progenerator as a left $A^{\mathrm{co}\,H}$-module, M is a progenerator as a left A-module. Let $N = {}_A\mathrm{HOM}(M, A)$. Then $N^{\mathrm{co}\,H} \cong {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}(M_1, A^{\mathrm{co}\,H})$ as left $B \,\square_H\, A^{\mathrm{op}}$-modules (see Corollary 4.9); ${}_{A^{\mathrm{co}\,H}}\mathrm{Hom}(M_1, A^{\mathrm{co}\,H})$ and N_1 are canonically isomorphic as $(B^{\mathrm{co}\,H}$–$A^{\mathrm{co}\,H})$-bimodules, since the Morita context is strict. Using this isomorphism, the left $B \,\square_H\, A^{\mathrm{op}}$-module structure can be transported to N_1. The H-Morita context (A, B, M, N) associated to M is strict by Proposition 5.4. The corresponding $\square_H$-Morita context from Theorem 5.7 is canonically isomorphic to $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M_1, N_1, \alpha_1, \beta_1)$. This proves the claim. $\square$

We end this section with the following result.

Theorem 5.10. *Let A be a (right) faithfully flat Galois extension of $A^{\mathrm{co}\,H}$. Assume that $M \in {}_A\mathcal{M}^H$ is a progenerator as a left A-module. Then $B = {}_A\mathrm{END}(M)^{\mathrm{op}}$ is a (right) faithfully flat H-Galois extension of $B^{\mathrm{co}\,H}$ if and only if M is an H-progenerator.*

Proof. The H-Morita context $(A, B, M, N = {}_A\mathrm{HOM}(M, A), \alpha, \beta)$ from Example 5.3 is strict by Proposition 5.4.

If B is a faithfully flat H-Galois extension, then M is an H-progenerator by Corollary 5.8.

Conversely, let M be an H-progenerator. $M \in {}_A\mathcal{M}_B^H$ (see Example 5.3), hence $M_1 = M^{\mathrm{co}\,H} \in {}_{A^{\mathrm{co}\,H}}\mathcal{M}_{B^{\mathrm{co}\,H}}$. From the fact that the categories ${}_A\mathcal{M}^H$ and ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ are equivalent, it follows that M_1 is a left $A^{\mathrm{co}\,H}$-progenerator. From Proposition 4.8, we know that $B^{\mathrm{co}\,H} \cong {}_{A^{\mathrm{co}\,H}}\mathrm{End}(M_1)^{\mathrm{op}}$ and that $N^{\mathrm{co}\,H} \cong {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}(M_1, B^{\mathrm{co}\,H})$. The Morita context

$$\left(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H} \cong {}_{A^{\mathrm{co}\,H}}\mathrm{End}(M_1)^{\mathrm{op}}, M_1, {}_{A^{\mathrm{co}\,H}}\mathrm{Hom}\left(M_1, B^{\mathrm{co}\,H}\right), \alpha_1, \beta_1\right)$$

associated to $M_1 \in {}_{A^{\mathrm{co}\,H}}\mathcal{M}$ is strict, so $M_1 \otimes_{B^{\mathrm{co}\,H}} - : {}_{B^{\mathrm{co}\,H}}\mathcal{M} \to {}_{A^{\mathrm{co}\,H}}\mathcal{M}$ is a category equivalence. $A \otimes_{A^{\mathrm{co}\,H}} - : {}_{A^{\mathrm{co}\,H}}\mathcal{M} \to {}_A\mathcal{M}^H$ is an equivalence since A is a right faithfully flat H-Galois extension, and $M \otimes_B - : {}_B\mathcal{M}^H \to {}_A\mathcal{M}^H$ is also an equivalence (see Proposition 5.2). Using the fact that $A \otimes_{A^{\mathrm{co}\,H}} M_1 \cong M$ (A is a right faithfully flat Galois extension), we see easily that the following diagram of functors commutes:

$$\begin{tikzcd}
{}_{B^{\mathrm{co}\,H}}\mathcal{M} \arrow[r, "M_1 \otimes_{B^{\mathrm{co}\,H}} -"] \arrow[d, "B \otimes_{B^{\mathrm{co}\,H}} -"'] & {}_{B^{\mathrm{co}\,H}}\mathcal{M} \arrow[d, "A \otimes_{A^{\mathrm{co}\,H}} -"] \\
{}_B\mathcal{M}^H \arrow[r, "M \otimes_B -"] & {}_B\mathcal{M}^H.
\end{tikzcd}$$

Three of the four functors in the diagram are equivalences, hence the fourth one, $B \otimes_{B^{\mathrm{co}\,H}} -$ is also an equivalence (see the observations following Corollary 7.2). M_1, A and M are right faithfully flat over $B^{\mathrm{co}\,H}$, $A^{\mathrm{co}\,H}$ and B respectively, hence it follows that B is right faithfully flat over $B^{\mathrm{co}\,H}$. Thus condition (5) of Theorem 1.1 is fulfilled, and it follows that B is a right faithfully flat H-Galois extension. □

6. Application to the Miyashita–Ulbrich action

Let A be a right faithfully flat right H-Galois extension, and consider the map

$$\gamma_A = \mathrm{can}^{-1} \circ (\eta_A \otimes H) : H \to A \otimes_{A^{\mathrm{co}\,H}} A.$$

Following [14], we use the notation

$$\gamma_A(h) = \sum_i l_i(h) \otimes_{A^{\mathrm{co}\,H}} r_i(h).$$

$\gamma_A(h)$ is then characterized by the property

$$\sum_i l_i(h) r_i(h)_{[0]} \otimes r_i(h)_{[1]} = 1 \otimes h.$$

The following properties are then easy to prove (see [16, 3.4]): for all $h, h' \in H$ and $a \in A$, we have

$$\gamma_A(h) \in (A \otimes_{A^{\mathrm{co}\,H}} A)^{A^{\mathrm{co}\,H}}; \tag{10}$$

$$\gamma_A(h_{(1)}) \otimes h_{(2)} = \sum_i l_i(h) \otimes_{A^{\mathrm{co}\,H}} r_i(h)_{[0]} \otimes r_i(h)_{[1]}; \tag{11}$$

$$\gamma_A(h_{(2)}) \otimes S(h_{(1)}) = \sum_i l_i(h)_{[0]} \otimes_{A^{\mathrm{co}\,H}} r_i(h) \otimes l_i(h)_{[1]}; \tag{12}$$

$$\sum_i l_i(h) r_i(h) = \varepsilon(h) 1_A; \tag{13}$$

$$\sum_i a_{[0]} l_i(a_{[1]}) \otimes r_i(a_{[1]}) = 1 \otimes a; \tag{14}$$

$$\gamma(hh') = \sum_{i,j} l_i(h') l_j(h) \otimes_{A^{\mathrm{co}\,H}} r_j(h) r_j(h'). \tag{15}$$

Combining (11) and (12), we find

$$\begin{aligned}
&\sum_i l_i(h)_{[0]} \otimes_{A^{\mathrm{co}\,H}} r_i(h)_{[0]} \otimes l_i(h)_{[1]} \otimes r_i(h)_{[1]} \\
&\overset{(12)}{=} \sum_i l_i(h_{(2)}) \otimes_{A^{\mathrm{co}\,H}} r_i(h_{(2)})_{[0]} \otimes S(h_{(1)}) \otimes r_i(h_{(2)})_{[1]}
\end{aligned}$$

$$\overset{(11)}{=}\sum_i l_i(h_{(2)})\otimes_{A^{\mathrm{co}\,H}} r_i(h_{(2)})\otimes S(h_{(1)})\otimes h_{(3)}. \tag{16}$$

Let M be an (A,A)-bimodule. On $M^{A^{\mathrm{co}\,H}}$, we can define a right H-action called the Miyashita–Ulbrich action. This was introduced in [10], and we follow here the description given in [14]. It is given by the formula

$$m \leftharpoonup h=\sum_i l_i(h)mr_i(h).$$

It follows from (10) and (15) that we have a well-defined right H-action. In particular, for $X,Y\in\mathcal{M}_A$, $\mathrm{Hom}(X,Y)\in {}_A\mathcal{M}_A$, with left and right A-action given by

$$(a\cdot f\cdot a')(x)=f(xa)a'.$$

It is easy to see that

$$\mathrm{Hom}(X,Y)^{A^{\mathrm{co}\,H}}=\mathrm{Hom}_{A^{\mathrm{co}\,H}}(X,Y),$$

and the Miyashita–Ulbrich action is then given by (see [16, Corollary 3.5])

$$(f\leftharpoonup h)(x)=\sum_i f\bigl(xl_i(h)\bigr)r_i(h).$$

Lemma 6.1. *Let A and B be right faithfully flat right H-Galois extensions. For all $b\in B$, we have that*

$$x:=\gamma\bigl(S^{-1}(b_{[1]})\bigr)\otimes b_{[0]}\in A\otimes_{A^{\mathrm{co}\,H}}\bigl(A\,\Box_H\,B^{\mathrm{op}}\bigr).$$

Proof. We have

$$\begin{aligned}&\sum_i l_i\bigl(S^{-1}(b_{[1]})\bigr)\otimes_{A^{\mathrm{co}\,H}} r_i\bigl(S^{-1}(b_{[1]})\bigr)_{[0]}\otimes r_i\bigl(S^{-1}(b_{[1]})\bigr)_{[1]}\otimes b_{[0]}\\&\quad\overset{(11)}{=}\gamma\bigl(S^{-1}(b_{[1]})\bigr)\otimes S^{-1}(b_{[1]})\otimes b_{[0]},\end{aligned}$$

hence $x\in(A\otimes_{A^{\mathrm{co}\,H}}A)\,\Box_H\,B^{\mathrm{op}}\cong A\otimes_{A^{\mathrm{co}\,H}}(A\,\Box_H\,B^{\mathrm{op}})$. $\square$

Now we assume that (A,B,M,N,α,β) is a strict H-Morita context connecting the right faithfully flat H-Galois extensions A and B. For $X\in\mathcal{M}_A$, we have the isomorphism

$$\varphi: X\otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}\cong X\otimes_A A\otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}\xrightarrow{X\otimes_A\varepsilon_{1,M}} X\otimes_A M,$$

given by

$$\varphi(x\otimes_{A^{\mathrm{co}\,H}} m)=x\otimes_A m.$$

We have that $X\otimes_A M\in\mathcal{M}_B$, and its right B-action can be transported to $X\otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}$. We compute this action in our next lemma.

Lemma 6.2. *The transported right B-action on $X \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}$ is given by the formula*

$$(x \otimes_{A^{\mathrm{co}\,H}} m) \cdot b = \sum_i x l_i\big(S^{-1}(b_{[1]})\big) \otimes_{A^{\mathrm{co}\,H}} \big(r_i\big(S^{-1}(b_{[1]})\big) \otimes b_{[0]}\big)m. \tag{17}$$

Proof. Observe first that the action (17) is well defined, since $M^{\mathrm{co}\,H} \in {}_{A \square B^{\mathrm{op}}}\mathcal{M}$, and by Lemma 6.1. For the sake of simplicity, we introduce the following notation: for $\sum_i a_i \otimes b_i \in A \,\square_H\, B^{\mathrm{op}}$ and $m \in M^{\mathrm{co}\,H}$, we write

$$\Big(\sum_i a_i \otimes b_i\Big) \cdot m = \sum_i a_i m b_i.$$

We have to show that φ is right H-linear. Indeed,

$$\begin{aligned}
\varphi\big((x \otimes_{A^{\mathrm{co}\,H}} m) \cdot b\big) &= \sum_i x l_i\big(S^{-1}(b_{[1]})\big) \otimes_A r_i\big(S^{-1}(b_{[1]})\big) m b_{[0]} \\
&= \sum_i x l_i\big(S^{-1}(b_{[1]})\big) r_i\big(S^{-1}(b_{[1]})\big) \otimes_A m b_{[0]} \\
&\overset{(13)}{=} \sum_i x \varepsilon\big(S^{-1}(b_{[1]})\big) \otimes_A m b_{[0]} = x \otimes_A mb. \qquad \square
\end{aligned}$$

Consider the setting of Theorem 5.10: $(A, B, M, N, \alpha, \beta)$ is a strict H-Morita context connecting the right faithfully flat H-Galois extensions A and B, and $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}, \alpha_1, \beta_1)$ is the corresponding Morita context connecting $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$. For $X, Y \in \mathcal{M}_A$, we have an isomorphism

$$\phi : \mathrm{Hom}_{A^{\mathrm{co}\,H}}(X, Y) \to \mathrm{Hom}_{B^{\mathrm{co}\,H}}\big(X \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}, Y \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}\big), \tag{18}$$

given by $\phi(f) = f \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}$. It follows from Lemma 6.2 that $\mathrm{Hom}(X \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}, Y \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H})$ is a (B, B)-bimodule, and we can consider the Miyashi–Ulbrich action on $\mathrm{Hom}_{B^{\mathrm{co}\,H}}(X \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H}, Y \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H})$.

Proposition 6.3. *With notation as above, the map ϕ from* (18) *preserves the Miyashita–Ulbrich action.*

Proof. We will use the notation

$$\gamma_B(h) = \sum_j k_j(h) \otimes_{B^{\mathrm{co}\,H}} q_j(h) \in B \otimes_{B^{\mathrm{co}\,H}} B.$$

We have to show that

$$\phi(f) \leftharpoonup h = \phi(f \leftharpoonup h),$$

for all right $A^{\mathrm{co}\,H}$-linear $f : X \to Y$ and $h \in H$. For $x \in X$ and $m \in M^{\mathrm{co}\,H}$, we compute

$$
\begin{aligned}
&\big(\phi(f)\leftharpoonup h\big)(x\otimes_{A^{\mathrm{co}H}}m)\\
&\quad=\sum_j \phi(f)\big((x\otimes_{A^{\mathrm{co}H}}m)k_j(h)\big)q_j(h)\\
&\quad=\sum_{i,j}\big[f\big[xl_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\big]\otimes_{A^{\mathrm{co}H}} r_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)mk_j(h)_{[0]}\big]q_j(h)\\
&\quad=\sum_{i,j,p} f\big[xl_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\big]l_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)\\
&\qquad\otimes_{A^{\mathrm{co}H}} r_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)r_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)mk_j(h)_{[0]}q_j(h)_{[0]}\\
&\quad\overset{(16)}{=}\sum_{i,j,p} f\big[xl_i\big(S^{-1}\big(S(h_{(1)})\big)\big)\big]l_p\big(S^{-1}(h_{(3)})\big)\\
&\qquad\otimes_{A^{\mathrm{co}H}} r_p\big(S^{-1}(h_{(3)})\big)r_i\big(S^{-1}\big(S(h_{(1)})\big)\big)mk_j(h_{(2)})q_j(h_{(2)})\\
&\quad\overset{(13)}{=}\sum_{i,p} f\big(xl_i(h_{(1)})\big)l_p\big(S^{-1}(h_{(2)})\big)\otimes_{A^{\mathrm{co}H}} r_p\big(S^{-1}(h_{(2)})\big)r_i(h_{(1)})m1_B\\
&\quad\overset{(*)}{=}\sum_{i,p} f\big(xl_i(h_{(1)})\big)l_p\big(S^{-1}(h_{(2)})\big)r_p\big(S^{-1}(h_{(2)})\big)r_i(h_{(1)})\otimes_{A^{\mathrm{co}H}} m\\
&\quad\overset{(13)}{=}\sum_{i} f\big(xl_i(h)\big)r_i(h)\otimes_{A^{\mathrm{co}H}} m\\
&\quad=(f\leftharpoonup h)(x)\otimes_{A^{\mathrm{co}H}} m=\big(\phi(f\leftharpoonup h)\big)(x\otimes_{A^{\mathrm{co}H}}m).
\end{aligned}
$$

The equality $(*)$ can be justified as follows. From Lemma 6.1, we deduce that, for all i:

$$\sum_i l_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\otimes_{A^{\mathrm{co}H}} 1_A\otimes_{A^{\mathrm{co}H}} r_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\otimes k_j(h)_{[0]}$$

and

$$\sum_p 1_A\otimes_{A^{\mathrm{co}H}} l_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)\otimes_{A^{\mathrm{co}H}} r_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)\otimes q_j(h)_{[0]}$$

lie in $(A\otimes_{A^{\mathrm{co}H}}A)\otimes_{A^{\mathrm{co}H}}(A\,\square_H\,B^{\mathrm{op}})$. Consequently $(A\otimes_{A^{\mathrm{co}H}}A)\otimes_{A^{\mathrm{co}H}}(A\,\square_H\,B^{\mathrm{op}})$ also contains

$$
\begin{aligned}
&\sum_{i,p} l_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\otimes_{A^{\mathrm{co}H}} l_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)r_p\big(S^{-1}\big(q_j(h)_{[1]}\big)\big)r_i\big(S^{-1}\big(k_j(h)_{[1]}\big)\big)\\
&\qquad\otimes k_j(h)_{[0]}q_j(h)_{[0]}\\
&\quad\overset{(13),\,(16)}{=}\sum_{i,p} l_i(h_{(1)})\otimes_{A^{\mathrm{co}H}} l_p\big(S^{-1}(h_{(2)})\big)\otimes_{A^{\mathrm{co}H}} r_p\big(S^{-1}(h_{(2)})\big)r_i(h_{(1)})\otimes 1_B\\
&\quad=:Z\otimes 1_B.
\end{aligned}
$$

This means that

$$(A \otimes_{A^{\mathrm{co}\,H}} A \otimes_{A^{\mathrm{co}\,H}} \rho_A)(Z) \otimes 1_B = Z \otimes 1_H \otimes 1_B,$$

hence

$$Z \in (A \otimes_{A^{\mathrm{co}\,H}} A \otimes_{A^{\mathrm{co}\,H}} A)^{\mathrm{co}\,H} \cong A \otimes_{A^{\mathrm{co}\,H}} A \otimes_{A^{\mathrm{co}\,H}} A^{\mathrm{co}\,H},$$

since $A/A^{\mathrm{co}\,H}$ is faithfully flat. $\square$

7. Hopf subalgebras

Throughout this section, H is a Hopf algebra with bijective antipode over a field k, and K is a Hopf subalgebra of H. We assume that the antipode of K is bijective, and that H is (right) faithfully flat as a left K-module. Let $K^+ = \mathrm{Ker}(\varepsilon_K)$. It is well known, and easy to prove (see [17, Section 1]) that

$$\overline{H} = H/HK^+ \cong H \otimes_K k$$

is a left H-module coalgebra, with operations

$$h \cdot \bar{l} = \overline{hl}, \qquad \Delta_{\overline{H}}(\bar{h}) = \bar{h}_{(1)} \otimes \bar{h}_{(2)}, \qquad \varepsilon_{\overline{H}}(\bar{h}) = \varepsilon(h).$$

The class in $\overline{H}$ represented by $h \in H$ is denoted by $\bar{h}$. $\bar{1}$ is a group-like element of $\overline{H}$, and we consider coinvariants with respect to this element. A right H-comodule M is also a right $\overline{H}$-comodule, by corestriction of coscalars:

$$\rho_{\overline{H}}(m) = m_{[0]} \otimes \overline{m}_{[1]}.$$

The $\overline{H}$-coinvariants of $M \in \mathcal{M}^H$ are then

$$\begin{aligned} M^{\mathrm{co}\,\overline{H}} &= \{m \in M \mid m_{[0]} \otimes \overline{m}_{[1]} = m \otimes \bar{1}\} \\ &= \big\{m \in M \;\big|\; \rho(m) \in M \otimes K\big\} \cong M \,\square_H\, K. \end{aligned}$$

If A is a right H-comodule algebra, then $A^{\mathrm{co}\,\overline{H}}$ is a right K-comodule algebra, and $(A^{\mathrm{co}\,\overline{H}})^{\mathrm{co}\,K} = A^{\mathrm{co}\,H}$.

Proposition 7.1. *(See [16, Remark 1.8].) Let H, K and A be as above, and assume that A is a faithfully flat H-Galois extension. Then A is right faithfully flat as a right $A^{\mathrm{co}\,\overline{H}}$-module, and*

$$\mathrm{can}: A \otimes_{A^{\mathrm{co}\,\overline{H}}} A \to A \otimes \overline{H}, \qquad \mathrm{can}(a \otimes b) = ab_{[0]} \otimes \bar{b}_{[1]}$$

is bijective. The functors $(A \otimes_{A^{\mathrm{co}\,\overline{H}}} -, (-)^{\mathrm{co}\,\overline{H}})$ form a pair of inverse equivalences between the categories ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}$ and ${}_A\mathcal{M}(H)^{\overline{H}}$.

We also have an adjoint pair of functors ($F_4 = A \otimes_{A^{\mathrm{co}\,\overline{H}}} -,\ G_4 = (-)^{\mathrm{co}\,\overline{H}} \cong -\Box_H K$) between the categories ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$ and ${}_A\mathcal{M}^H$. This can be seen directly, but it is also a consequence of a more general result: we apply [7, Theorem 1.3] to the inclusion morphism between the Doi–Hopf data $(K, A^{\mathrm{co}\,\overline{H}}, K)$ and (H, A, H).

Take $N \in {}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$. Forgetting the K-coaction, we find that $N \in {}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}$. Then it is easy to see that the counit map $\eta_N : N \to (A \otimes_{A^{\mathrm{co}\,\overline{H}}} N^{\mathrm{co}\,\overline{H}})$ is a morphism in ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$, and coincides with the counit map from the adjunction (F_4, G_4). Since η_N is an isomorphism, the unit maps of the adjunction (F_4, G_4) are isomorphisms. In the same way, we can conclude that the counit maps are isomorphisms, and we conclude

Corollary 7.2. *Let H, K and A be as above, and assume that A is a faithfully flat H-Galois extension. Then the adjoint pair of functors ($F_4 = A \otimes_{A^{\mathrm{co}\,\overline{H}}} -,\ G_4 = (-)^{\mathrm{co}\,\overline{H}} \cong -\Box_H K$) establishes a pair of inverse equivalences between the categories ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$ and ${}_A\mathcal{M}^H$.*

Before stating our next corollary, we recall some elementary facts from category theory. If (F_1, G_1) and (F_2, G_2) are pairs of adjoint functors, respectively between categories $\mathcal{C}$ and $\mathcal{D}$, and between $\mathcal{D}$ and $\mathcal{E}$, then $(F = F_2 \circ F_1, G = G_1 \circ G_2)$ is a pair of adjoint functors between $\mathcal{C}$ and $\mathcal{E}$. If two of these three pairs are inverse equivalences, then the third one is also a pair of inverse equivalences. This follows from the following formulas, which give the relations between the units and counits of the three adjunctions: for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we have

$$\eta_C = G_1(\eta_{2,F_1(C)}) \circ \eta_{1,C}; \qquad \varepsilon_E = \eta_{2,E} \circ F_2(\varepsilon_{1,G_2(E)}).$$

This can be applied to the following situation. Assume that we are in the setting of Proposition 7.1 and Corollary 7.2. We have adjunctions

- $(F_1 = A \otimes_{A^{\mathrm{co}\,H}} -,\ G_1 = (-)^{\mathrm{co}\,H})$ between ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ and ${}_A\mathcal{M}^H$;
- $(F_3 = A^{\mathrm{co}\,\overline{H}} \otimes_{A^{\mathrm{co}\,H}} -,\ G_3 = (-)^{\mathrm{co}\,K})$ between ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$ and ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$;
- $(F_4 = A \otimes_{A^{\mathrm{co}\,\overline{H}}} -,\ G_4 = (-)^{\mathrm{co}\,\overline{H}} \cong -\Box_H K)$ between the categories ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K$ and ${}_A\mathcal{M}^H$.

It is clear that $F_1 = F_4 \circ F_3$ and $G_1 = G_3 \circ G_4$. (F_1, G_1) and (F_4, G_4) are pairs of inverse equivalences, by Theorem 1.1 and Corollary 7.2. Hence (F_3, G_3) is also a pair of inverse equivalences, and using Theorem 1.1, we obtain the following result.

Corollary 7.3. *Let H, K and A be as above, and assume that A is a faithfully flat H-Galois extension. Then $A^{\mathrm{co}\,\overline{H}}$ is a right faithfully flat K-Galois extension.*

Theorem 7.4. *Let H and K be as before: $K \subset H$ are Hopf algebras with invertible antipode over a field k, and H is faithfully flat as a left K-module. Let A and B be (right) faithfully flat right H-Galois extensions, connected by a strict H-Morita context $(A, B, M, N, \alpha, \beta)$.*

(1) *$A^{\mathrm{co}\,\overline{H}}$ and $B^{\mathrm{co}\,\overline{H}}$ are connected by a strict K-Morita context, with connecting modules $M^{\mathrm{co}\,\overline{H}}$ and $N^{\mathrm{co}\,\overline{H}}$;*

(2) *we have a pair of inverse equivalences $(M \otimes_B -, N \otimes_A -)$ between the categories ${}_B\mathcal{M}(H)^{\overline{H}}$ and ${}_A\mathcal{M}(H)^{\overline{H}}$;*

(3) *the following diagram of categories and functors commutes to within natural equivalences of functors*:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{}_A\mathcal{M}(H)^{\overline{H}} \arrow[r, shift left, "N\otimes_A -"] \arrow[d, shift left, "(-)^{\mathrm{co}\,\overline{H}}"] & {}_B\mathcal{M}(H)^{\overline{H}} \arrow[l, shift left, "M\otimes_B -"] \arrow[d, shift left, "(-)^{\mathrm{co}\,\overline{H}}"] \\
{}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M} \arrow[u, shift left, "A\otimes_{A^{\mathrm{co}\,\overline{H}}} -"] \arrow[r, shift left, "N^{\mathrm{co}\,\overline{H}}\otimes_{A^{\mathrm{co}\,\overline{H}}} -"] & {}_{B^{\mathrm{co}\,\overline{H}}}\mathcal{M}. \arrow[u, shift left, "B\otimes_{B^{\mathrm{co}\,\overline{H}}} -"] \arrow[l, shift left, "M^{\mathrm{co}\,\overline{H}}\otimes_{B^{\mathrm{co}\,\overline{H}}} -"]
\end{tikzcd}
$$

Proof. (1) We have the following commutative diagram of inclusions

$$
\begin{tikzcd}
A^{\mathrm{co}\,\overline{H}} \,\Box_K\, (B^{\mathrm{op}})^{\mathrm{co}\,\overline{H}} \arrow[r, "\subset"] \arrow[d, "\subset"'] & A^{\mathrm{co}\,\overline{H}} \otimes (B^{\mathrm{op}})^{\mathrm{co}\,\overline{H}} \arrow[d, "\subset"] \\
A \,\Box_H\, B^{\mathrm{op}} \arrow[r, "\subset"] & A\otimes B^{\mathrm{op}}.
\end{tikzcd}
$$

By Theorem 5.7, we have a strict $\Box_H$-Morita context $(A^{\mathrm{co}\,H}, B^{\mathrm{co}\,H}, M^{\mathrm{co}\,H}, N^{\mathrm{co}\,H}, \alpha_1, \beta_1)$. By restriction of scalars, A is a left $A^{\mathrm{co}\,\overline{H}} \,\Box_K\, (B^{\mathrm{op}})^{\mathrm{co}\,\overline{H}}$-module. Then we can apply Theorems 5.7 and 5.9, with H replaced by K, and taking into account that $A^{\mathrm{co}\,\overline{H}}$ and $B^{\mathrm{co}\,\overline{H}}$ are right faithfully flat K-Galois extensions, by Corollary 7.3. We find that $A^{\mathrm{co}\,\overline{H}}$ and $B^{\mathrm{co}\,\overline{H}}$ are connected by a strict K-Morita context. The first connecting module is

$$
\begin{aligned}
A^{\mathrm{co}\,\overline{H}} \otimes_{A^{\mathrm{co}\,H}} M^{\mathrm{co}\,H} &= F_3G_1(M) \cong G_4F_4F_3G_1(M) \\
&\cong G_4F_1G_1(M) \cong G_4(M) = M^{\mathrm{co}\,\overline{H}}.
\end{aligned}
$$

In a similar way, we find that the second connecting module is $N^{\mathrm{co}\,\overline{H}}$.

(2) The proof is an easy adaption of the proof of Proposition 5.2.

(3) $B^{\mathrm{co}\,\overline{H}}$ is a right K-comodule algebra, and, by corestriction of coscalars, a right H-comodule algebra, so we can consider the categories ${}_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}^K_{B^{\mathrm{co}\,\overline{H}}}$ and ${}_A\mathcal{M}^H_{B^{\mathrm{co}\,\overline{H}}}$. It is then easy to see that the inverse equivalent functors of Corollary 7.2 also define a pair of inverse equivalences between these two categories of relative Hopf bimodules. Now $M \in {}_A\mathcal{M}^H_{B^{\mathrm{co}\,\overline{H}}}$, so $M \cong A\otimes_{A^{\mathrm{co}\,\overline{H}}} M^{\mathrm{co}\,\overline{H}}$ as right $B^{\mathrm{co}\,\overline{H}}$-modules. It follows that we have, for all $P \in {}_{B^{\mathrm{co}\,\overline{H}}}\mathcal{M}$,

$$
A\otimes_{A^{\mathrm{co}\,\overline{H}}} M^{\mathrm{co}\,\overline{H}} \otimes_{B^{\mathrm{co}\,\overline{H}}} P \cong M\otimes_{B^{\mathrm{co}\,\overline{H}}} P \cong M\otimes_B B\otimes_{B^{\mathrm{co}\,\overline{H}}} P.
$$

In a similar way, we can show that

$$
B\otimes_{B^{\mathrm{co}\,\overline{H}}} N^{\mathrm{co}\,\overline{H}} \otimes_{A^{\mathrm{co}\,\overline{H}}} Q \cong N\otimes_A A\otimes_{A^{\mathrm{co}\,\overline{H}}} Q,
$$

for all $Q_{A^{\mathrm{co}\,\overline{H}}}\mathcal{M}$. Finally, take $U \in {}_A\mathcal{M}(H)^{\overline{H}}$. Then

$$\begin{aligned}(N \otimes_A U)^{\mathrm{co}\,\overline{H}} &\cong \big(N \otimes_A A \otimes_{A^{\mathrm{co}\,\overline{H}}} U^{\mathrm{co}\,\overline{H}}\big)^{\mathrm{co}\,\overline{H}} \\ &\cong \big(B \otimes_{B^{\mathrm{co}\,\overline{H}}} N^{\mathrm{co}\,\overline{H}} \otimes_{A^{\mathrm{co}\,\overline{H}}} U^{\mathrm{co}\,\overline{H}}\big)^{\mathrm{co}\,\overline{H}} \cong N^{\mathrm{co}\,\overline{H}} \otimes_{A^{\mathrm{co}\,\overline{H}}} U^{\mathrm{co}\,\overline{H}},\end{aligned}$$

and, in a similar way, for $V \in {}_B\mathcal{M}(H)^{\overline{H}}$,

$$(M \otimes_B V)^{\mathrm{co}\,\overline{H}} \cong M^{\mathrm{co}\,\overline{H}} \otimes_{B^{\mathrm{co}\,\overline{H}}} V^{\mathrm{co}\,\overline{H}}. \qquad \square$$

Finally recall that if the algebras A and B are Morita equivalent, then there is a Morita equivalence between $A \otimes A^{\mathrm{op}}$ and $B \otimes B^{\mathrm{op}}$ sending A to B. In particular, this implies that the centers of A and B are isomorphic. In our context this generalizes as follows.

Corollary 7.5. *Assume that $(A, B, M, N, \alpha, \beta)$ is a strict H-Morita context.*

(1) *Let K and L be Hopf subalgebras of H with bijective antipodes, and assume that $H \otimes H$ is faithfully flat as a right $K \otimes L$-module. Then the categories ${}_{A^{\mathrm{co}\,H/HK^+}}\mathcal{M}_{A^{\mathrm{co}\,H/HL^+}}$ and ${}_{B^{\mathrm{co}\,H/HK^+}}\mathcal{M}_{B^{\mathrm{co}\,H/HL^+}}$ are equivalent.*

(2) *There is an isomorphism*

$$C_A\big(A^{\mathrm{co}\,H}\big) \cong C_B\big(B^{\mathrm{co}\,H}\big)$$

of left H-module right H-comodule algebras, where $C_A(A^{\mathrm{co}\,H})$ denotes the centralizer in A of $A^{\mathrm{co}\,H}$.

Proof. (1) The objects $M \otimes N \in {}_{A\otimes A^{\mathrm{op}}}\mathcal{M}^{H\otimes H}_{B\otimes B^{\mathrm{op}}}$ and $N \otimes M \in {}_{B\otimes B^{\mathrm{op}}}\mathcal{M}^{H\otimes H}_{A\otimes A^{\mathrm{op}}}$ induce a Morita equivalence between $A \otimes A^{\mathrm{op}}$ and $B \otimes B^{\mathrm{op}}$. Now the assertion follows from Theorem 7.4, where we replace H by $H \otimes H$, K by $K \otimes L$, A by $A \otimes A^{\mathrm{op}}$ and B by $B \otimes B^{\mathrm{op}}$.

(2) Note that

$$C_A\big(A^{\mathrm{co}\,H}\big) \cong \mathrm{End}_{A^{\mathrm{co}\,H}\otimes A^{\mathrm{op}}}(A)$$

as H-module H-comodule algebras. Since under the equivalence of (1) (where we take $K = k$ and $L = H$), A corresponds to B, the statement follows from Proposition 6.3. $\square$

8. H-colinear equivalences

Let H be a projective Hopf algebra, and A a right H-comodule algebra. Let ${}_A\underline{\mathcal{M}}^H$ be the category with relative Hopf modules as modules; the set of morphisms between two objects M and N is ${}_A\mathrm{HOM}(M, N)$. ${}_A\underline{\mathcal{M}}^H$ is a right H-colinear category in the following sense: ${}_A\mathrm{HOM}(M, N)$ is a right H-comodule (see Proposition 4.6); the map

$$\varphi : M \otimes {}_A\mathrm{HOM}(M, N) \to N, \quad \varphi(m \otimes f) = f(m)$$

is right H-colinear (take $B = C = k$ in Proposition 4.10); if N is a third object in ${}_A\underline{\mathcal{M}}^H$, then the composition

$$\psi : {}_A\mathrm{HOM}(L, M) \otimes {}_A\mathrm{HOM}(M, N) \to {}_A\mathrm{HOM}(L, N), \quad \psi(f \otimes g) = g \circ f$$

is right H-colinear. The following result is then obvious.

Proposition 8.1. *Let H be a projective Hopf algebra. Let $(A, B, M, N, \alpha, \beta)$ be a strict H-Morita context connecting the right H-comodule algebras A and B. Then the functors $M \otimes_B -$ and $N \otimes_A -$ induce a pair of inverse right H-colinear equivalences between ${}_B\underline{\mathcal{M}}^H$ and ${}_A\underline{\mathcal{M}}^H$.*

The functors $F = M \otimes_B -$ and $G = N \otimes_A -$ are right H-colinear in the following sense: for $V, W \in {}_B\mathcal{M}^H$, the map

$$F : {}_B\mathrm{HOM}(V, W) \to {}_A\mathrm{HOM}(M \otimes_B V, M \otimes_B W), \quad F(f) = M \otimes_B f$$

is right H-colinear.

In this section, we investigate when the converse of Proposition 8.1 holds: suppose that we have a pair of inverse right H-colinear equivalences between ${}_B\underline{\mathcal{M}}^H$ and ${}_A\underline{\mathcal{M}}^H$. Is this equivalence induced by a strict H-Morita context? To this end, we will give an H-colinear version of the Eilenberg–Watts Theorem.

Proposition 8.2. *Let A and B be H-comodule algebras, and $T : {}_A\underline{\mathcal{M}}^H \to {}_B\underline{\mathcal{M}}^H$ an H-colinear functor. Then $N = T(A) \in {}_A\mathcal{M}_B^H$, and we have a natural transformation $\psi : F = N \otimes_A - \to T$, such that $\psi_A : N \otimes_A A \to T(A) = N$ is the natural isomorphism.*

Proof. In the sequel, V and W will be objects in ${}_A\underline{\mathcal{M}}^H$. The fact that T is right H-colinear means that

$$T(f_{[0]}) \otimes f_{[1]} = \rho\big(T(f)\big), \tag{19}$$

for $f \in A\mathrm{HOM}(V, W)$. We claim that the map

$$\varphi_V : V \to {}_A\mathrm{HOM}(A, V), \quad \varphi_V(v)(a) = av$$

is well defined and right H-colinear. To this end, it suffices to show that

$$\varphi(v)_{[0]} \otimes \varphi(v)_{[1]} = \varphi(v_{[0]}) \otimes v_{[1]}, \tag{20}$$

for all $v \in V$. For all $a \in A$, we have

$$\begin{aligned}
&\big(\varphi_V(v)(a_{[0]})\big)_{[0]} \otimes S^{-1}(a_{[1]})\big(\varphi_V(v)(a_{[0]})\big)_{[1]} \\
&= a_{[0]}v_{[0]} \otimes S^{-1}(a_{[2]})a_{[1]}v_{[1]} \\
&= av_{[0]} \otimes v_{[1]} = \varphi_V(v_{[0]})(a) \otimes v_{[1]},
\end{aligned}$$

and (20) follows using (3). φ_V satisfies the following property:

$$\varphi_V(av) = \varphi_V(v) \circ \varphi_A(a), \tag{21}$$

for all $a \in A$ and $v \in V$. Indeed,

$$\varphi_V(av)(c) = cav = \varphi_V(v)(ca) = \big(\varphi_V(v) \circ \varphi_A(a)\big)(c).$$

On $N = T(A) \in {}_B\mathcal{M}^H$, we define a right A-action as follows:

$$na = T\big(\varphi_A(a)\big)(n),$$

for all $a \in A$ and $n \in N$. This makes N an object of ${}_B\mathcal{M}^H_A$, since

$$\begin{aligned}
n(ac) &= T\big(\varphi_A(ac)\big)(n) \overset{(21)}{=} T\big(\varphi_A(a) \circ \varphi_A(c)\big)(n) \\
&= \big(T\big(\varphi_A(a)\big) \circ T\big(\varphi_A(c)\big)\big)(n) = (na)c; \\
(bn)a &= T\big(\varphi_A(a)\big)(bn) = bT\big(\varphi_A(a)\big)(n) = b(na); \\
n_{[0]}a_{[0]} \otimes n_{[1]}a_{[1]} &= T\big(\varphi(a_{[0]})\big)(n_{[0]}) \otimes n_{[1]}a_{[1]} \\
&\overset{(20)}{=} T\big(\varphi(a)_{[0]}\big)(n_{[0]}) \otimes n_{[1]}\varphi(a)_{[1]} \\
&\overset{(19)}{=} T\big(\varphi(a)\big)_{[0]}(n_{[0]}) \otimes n_{[1]}T\big(\varphi(a)\big)_{[1]} \\
&\overset{(3)}{=} T\big(\varphi(a)\big)(n_{[0]})_{[0]} \otimes n_{[2]}S^{-1}(n_{[1]})T\big(\varphi(a)\big)(n_{[0]})_{[1]} \\
&= T\big(\varphi(a)\big)(n)_{[0]} \otimes T\big(\varphi(a)\big)(n)_{[1]} = \rho(na),
\end{aligned}$$

for all $a, c \in A$, $b \in B$ and $n \in N$.

For every $v \in V$, $\varphi_V(v) : A \to V$ is left A-linear, hence $T(\varphi_V(v)) : T(A) = N \to T(V)$ is left B-linear. By (19), (20), we also have that

$$T\big(\varphi_V(v_{[0]})\big) \otimes v_{[1]} = \rho\big(T\big(\varphi_V(v)\big)\big). \tag{22}$$

Now we define

$$\psi_V : N \otimes_A V \to T(V), \quad \psi_V(n \otimes_A v) = T\big(\varphi_V(v)\big)(n).$$

ψ_V is well defined since

$$\begin{aligned}
\psi_V(n \otimes_A av) &= T\big(\varphi_V(av)\big)(n) \overset{(21)}{=} \big(T\big(\varphi_V(v)\big) \circ T\big(\varphi_V(a)\big)\big)(n) \\
&= T\big(\varphi_V(v)\big)(na) = \psi_V(n \otimes_A v).
\end{aligned}$$

ψ_V is right H-colinear, since

$$\begin{aligned}\psi_V(n_{[0]}\otimes_A v_{[0]})\otimes n_{[1]}v_{[1]} &= T\big(\varphi_V(v_{[0]})\big)(n_{[0]})\otimes n_{[1]}v_{[1]}\\ &\overset{(22)}{=} T\big(\varphi_V(v)\big)_{[0]}(n_{[0]})\otimes n_{[1]}T\big(\varphi_V(v)\big)_{[1]}\\ &\overset{(3)}{=} \big(T\big(\varphi_V(v)\big)(n_{[0]})\big)_{[0]}\otimes n_{[2]}S^{-1}(n_{[1]})\big(T\big(\varphi_V(v)\big)(n_{[0]})\big)_{[1]}\\ &= \rho\big(T\big(\varphi_V(v)\big)(n)\big)=\rho\big(\psi_V(n\otimes_A v)\big).\end{aligned}$$

ψ_V is left B-linear, since

$$\psi_V(bn\otimes_A v)=T\big(\varphi_V(v)\big)(bm)=b\big(T\big(\varphi_V(v)\big)(m)\big)=b\psi_V(n\otimes_A v).$$

In order to show that ψ is a natural transformation, we first observe the following property. For $f:V\to W$ in ${}_A\mathcal{M}^H$, $v\in V$ and $a\in A$, we have

$$\varphi_W\big(f(v)\big)(a)=af(v)=f(av)=f\big(\varphi_V(v)\big)(a),$$

so $\varphi_W(f(v))=f\circ\varphi_V(v)$. We can now show that the diagram

$$\begin{tikzcd} N\otimes_A V \arrow[r, "\psi_V"] \arrow[d, "N\otimes_A f"'] & T(V) \arrow[d, "T(f)"] \\ N\otimes_A W \arrow[r, "\psi_W"] & T(W) \end{tikzcd}$$

commutes:

$$\begin{aligned}\big(T(f)\circ\psi_V\big)(n\otimes_A v)&=\big(T(f)\circ T\big(\varphi_V(v)\big)\big)(n)\\ &=T\big(f\circ\varphi_V(v)\big)(n)=T\big(\varphi_W\big(f(v)\big)\big)(n)=\psi_W\big(n\otimes_A f(v)\big).\end{aligned}$$

It follows that ψ is a natural transformation. Finally, it is easy to compute that the map $\psi_A:N\otimes_A\to T(A)=A$ is given by $\psi_A(n\otimes a)=T(\varphi_A(a))(n)=na$, as needed. □

We are now ready to prove the following generalization of the Eilenberg–Watts Theorem (cf. [1, II.2.3]).

Proposition 8.3. *With notation and assumptions as in Proposition* 8.2*, assume that A is a generator of ${}_A\mathcal{M}^H$, and that T, viewed as a functor ${}_A\mathcal{M}^H\to{}_B\mathcal{M}^H$, preserves cokernels and arbitrary coproducts. Then the natural transformation $\psi:F=N\otimes_A-\to T$ from Proposition* 8.2 *is a natural isomorphism.*

Proof. Let I be an index set, and $A^{(I)}$ the coproduct of copies of A indexed by I. For $i \in I$, let $r_i : A \to A^{(I)}$ be the natural inclusion. Since ψ is a natural transformation, we have a commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\psi_A"] \arrow[d, "F(r_i)"'] & T(A) \arrow[d, "T(r_i)"] \\ F(A^{(I)}) \arrow[r, "\psi_{A^{(I)}}"] & T(A^{(I)}). \end{tikzcd}$$

Let $n_i : T(A) \to T(A)^{(I)}$ be the natural inclusion. Then the diagram

$$\begin{tikzcd} F(A)^{(I)} \arrow[r, "\bigoplus_{i\in I}(n_i\circ\psi_A)"] \arrow[d, "\bigoplus_{i\in I} F(r_i)"'] & T(A)^{(I)} \arrow[d, "\bigoplus_{i\in I} T(r_i)"] \\ F(A^{(I)}) \arrow[r, "\psi_{A^{(I)}}"] & T(A^{(I)}) \end{tikzcd}$$

also commutes. The vertical maps in the diagram are isomorphisms, since F and T commute with direct sums. We have seen in Proposition 8.2 that the top horizontal map is an isomorphism, so it follows that $\psi_{A^{(I)}}$ is an isomorphism.

Now take an arbitrary $V \in {}_A\mathcal{M}^H$. Since A is a generator of ${}_A\mathcal{M}^H$, we have an exact sequence

$$A^{(J)} \xrightarrow{\pi} A^{(I)} \xrightarrow{\varphi} V \to 0$$

in ${}_A\mathcal{M}^H$. Since ψ is a natural transformation, and F and G preserve cokernels, we have the following commutative diagram with exact rows in ${}_B\mathcal{M}^H$:

$$\begin{tikzcd} F(A^{(J)}) \arrow[r, "F(\pi)"] \arrow[d, "\psi_{A^{(J)}}"] & F(A^{(I)}) \arrow[r, "F(\varphi)"] \arrow[d, "\psi_{A^{(I)}}"] & F(V) \arrow[r] \arrow[d, "\psi_V"] & 0 \\ T(A^{(J)}) \arrow[r, "T(\pi)"] & T(A^{(I)}) \arrow[r, "T(\varphi)"] & T(V) \arrow[r] & 0. \end{tikzcd}$$

We know from above that $\psi_{A^{(J)}}$ and $\psi_{A^{(I)}}$ are isomorphisms, and it follows from Lemma 5 that ψ_V is also an isomorphism. $\square$

Theorem 8.4. *Let A and B be H-module algebras, and suppose that they generate the categories ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}^H$. If (T, U) is a pair of H-linear inverse equivalences between the categories ${}_A\underline{\mathcal{M}}^H$ and ${}_B\underline{\mathcal{M}}^H$, then there exists a strict H-Morita context $(A, B, M, N, \alpha, \beta)$ such that $T \cong N \otimes_A -$ and $U \cong M \otimes_B -$.*

Proof. Since (T, U) is also a pair of inverse equivalences between ${}_A\mathcal{M}^H$ and ${}_B\mathcal{M}^H$, T and U preserve coproducts and cokernels. Applying Proposition 8.3, we find $M \in {}_A\mathcal{M}^H_B$ and $N \in {}_B\mathcal{M}^H_A$ such that $T \cong N \otimes_A -$ and $U \cong M \otimes_B -$.

(T, U) is a pair of adjoint functors, and the unit η and the counit ε are natural isomorphisms. We define $\alpha = \eta_A^{-1} : M \otimes_B N \to A$. Then $\alpha \in {}_A\mathcal{M}^H$. Let us show that α is also right A-linear. For every $c \in A$, the map $f_c : A \to A$, $f_c(a) = ac$ is left A-linear. Since η is a natural transformation, the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f_c"'] & M \otimes_B N \otimes_A A \arrow[d, "M\otimes_B N\otimes_A f_c"] \\ A \arrow[r, "\eta_A"] & M \otimes_B N \otimes_A A \end{tikzcd}$$

commutes. Evaluating the diagram at 1_A, we find that $\eta_A(ac) = \eta_A(a)c$.

We define $\beta = \varepsilon_B : N \otimes_A M \to B$. Applying the above argument to the adjunction (U, T) with unit ε^{-1} and counit η^{-1}, we find that ε_B is right B-linear.

Take $W \in {}_B\mathcal{M}^H$. For every $w \in W$, we consider the left B-linear map $g_w : B \to W$, $g_w(b) = bw$. Since ε is a natural transformation, the diagram

$$\begin{tikzcd} N \otimes_A M \otimes_B B \arrow[r, "\varepsilon_B"] \arrow[d, "N\otimes_A M\otimes_B g_w"'] & B \arrow[d, "g_w"] \\ N \otimes_A M \otimes_B W \arrow[r, "\varepsilon_W"] & W \end{tikzcd}$$

commutes. Evaluating the diagram at $n \otimes_A m \otimes_B 1$, we see that $\varepsilon_W = \varepsilon_B \otimes_B W$.

From the properties of adjoint functors, we know that $\varepsilon_{T(V)} \circ T(\eta_V) = T(V)$, for all $V \in {}_A\mathcal{M}^H$. Taking $V = A$ in this formula, we see that the diagram

$$\begin{tikzcd} N\otimes_A \arrow[r, "N\otimes_A \eta_A"] \arrow[d, "\cong"'] & N \otimes_A M \otimes_B N \arrow[d, "\varepsilon_N=\varepsilon_B\otimes_B N"] \\ N \arrow[r, "\cong"] & B \otimes_B N \end{tikzcd}$$

commutes. This diagram is one of the two diagrams in the definition of a Morita context. The commutativity of the other diagram follows in a similar way. □

Corollary 8.5. *Let H be a projective Hopf algebra, and assume that the right H-comodule algebras A and B are H-Galois extensions of $A^{\mathrm{co}\,H}$ and $B^{\mathrm{co}\,H}$, respectively. If (T, U) is a pair of H-colinear inverse equivalences between the categories ${}_A\underline{\mathcal{M}}^H$ and ${}_B\underline{\mathcal{M}}^H$, then there exists a strict H-Morita context $(A, B, M, N, \alpha, \beta)$ such that $T \cong N \otimes_A -$ and $U \cong M \otimes_B -$.*

Proof. It is well known that $A^{\mathrm{co}\,H}$ is a generator of ${}_{A^{\mathrm{co}\,H}}\mathcal{M}$; since (F_1, G_1) is a pair of inverse equivalences (see Theorem 1.1), $F_1(A^{\mathrm{co}\,H}) = A$ is a generator of ${}_A\mathcal{M}^H$. In a similar way, B is a generator of ${}_B\mathcal{M}^H$, and we can apply Theorem 8.4. □

References

[1] H. Bass, Algebraic K-Theory, Benjamin, New York, 1968.
[2] N. Bourbaki, Algèbre, Hermann, Paris, 1970.
[3] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois properties, Algebr. Represent. Theory 5 (2002) 389–410.
[4] T. Brzeziński, R. Wisbauer, Corings and Comodules, London Math. Soc. Lecture Note Ser., vol. 309, Cambridge University Press, Cambridge, 2003.
[5] S. Caenepeel, Galois corings from the descent theory point of view, in: Fields Inst. Commun., vol. 43, 2004, pp. 163–186.
[6] S. Caenepeel, G. Militaru, S. Zhu, Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations, Lecture Notes in Math., vol. 1787, Springer-Verlag, Berlin, 2002.
[7] S. Caenepeel, Ş. Raianu, Induction functors for the Doi–Koppinen unified Hopf modules, in: A. Facchini, C. Menini (Eds.), Abelian Groups and Modules, Kluwer Academic Publishers, Dordrecht, 1995, pp. 73–94.
[8] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, Hopf Algebras. An Introduction, Monogr. Textbooks Pure Appl. Math., vol. 235, Marcel Dekker, New York, 2001.
[9] Y. Doi, Unifying Hopf modules, J. Algebra 153 (1992) 373–385.
[10] Y. Doi, M. Takeuchi, Hopf–Galois extensions of algebras, the Miyashita–Ulbrich action, and Azumaya algebras, J. Algebra 121 (1989) 488–516.
[11] H. Kreimer, M. Takeuchi, Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30 (1981) 675–692.
[12] A. Marcus, Equivalences induced by graded bimodules, Comm. Algebra 26 (1998) 713–731.
[13] S. Montgomery, Hopf Algebras and Their Actions on Rings, American Mathematical Society, Providence, 1993.
[14] P. Schauenburg, Hopf bimodules over Hopf–Galois extensions, Miyashita–Ulbrich actions, and monoidal center constructions, Comm. Algebra 24 (1996) 143–163.
[15] H.-J. Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990) 167–195.
[16] H.-J. Schneider, Representation theory of Hopf–Galois extensions, Israel J. Math. 72 (1990) 196–231.
[17] K.-H. Ulbrich, On modules induced or coinduced from Hopf subalgebras, Math. Scand. 67 (1990) 177–182.