Chains of Factorizations and Sets of Lengths

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1. INTRODUCTION

The theory of non-unique factorizations has its origin in algebraic number theory. A motivating problem was to describe, or even to characterize, the arithmetic of the ring of integers in an algebraic number field by arithmetical invariants (cf. Narkiewicz's book [Na, Chap. 9 and problems 29–32]). Sets of lengths turned out to be a crucial invariant. There is even some evidence for the conjecture that the system of sets of lengths completely determines the arithmetic of such a domain (cf. the end of Section 3). A main result on sets of lengths is their structure theorem: sets of lengths in a ring of integers (or more generally, in a Krull domain with finite divisor class group) are almost arithmetical progressions (cf. [Ge1, Satz 1]).

In arbitrary noetherian domains (even in the one-dimensional case) sets of lengths alone are too weak to describe the arithmetic of the domain (cf. the example in Section 3). This was one reason for the investigation of the catenary degree, a subtle arithmetical invariant dealing directly with factorizations and not only with their lengths. A recent result states that the catenary degree is finite for noetherian weakly Krull domains $R$ with finite-r-class group and whose integral closure is a finite $R$-module [Ge5, Theorem 5.4]. This class of domains includes orders in global fields and hence domains with infinite elasticity.

In this paper we derive a result, valid for the above mentioned class of weakly Krull domains, which implies the structure theorem on sets of lengths and the finiteness of the catenary degree (see Theorem 6.2). It

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states that there is some constant $C \in \mathbb{N}_+$ such that for all elements $a$ of the domain $R$ and each two factorizations $z, z'$ of $a$ there is a $C$-chain of factorizations from $z$ to $z'$ whose lengths form an almost arithmetical progression. This result is new even for rings of integers in algebraic number fields. For non-principal orders in algebraic number fields even its corollary, a structure theorem on sets of lengths, is new.

The paper is written entirely in the setting of weakly Krull monoids as introduced in [HK1]. Their relationship to ring theory is discussed in detail in [HK1, Ge4, Ge5]. In the examples following our main results (Theorem 5.1 and Theorem 6.2) we briefly remind the reader of these ring theoretical applications, which in fact are the motivation for our whole investigation.

We start with two introductory sections. The main conceptional tool used in the proof of Theorem 6.2 is the notion of local tameness of factorizations. It will be treated in Section 4. The structure of sets of lengths is especially simple in two cases: firstly, for large elements, this situation will be studied in Section 5; secondly, in the case of local domains, which we discuss briefly at the beginning of Section 6. Then we formulate the main result in Theorem 6.2. Its proof rests on the finiteness of the catenary degree shown in [Ge5, Theorem 5.4] and on the already mentioned concept of local tameness.

2. PRELIMINARIES

Throughout this paper, a monoid is a multiplicatively written, commutative, and cancellative semigroup with unit element. For a set $\mathcal{P}$ we denote by $\mathcal{P}(\mathcal{P})$ for the free abelian monoid with basis $\mathcal{P}$. Then every $a \in \mathcal{P}(\mathcal{P})$ has a unique representation

$$a = \prod_{p \in \mathcal{P}} p^{v_p(a)}$$

with $v_p(a) \in \mathbb{N}$ and $v_p(a) = 0$ for almost all $p \in \mathcal{P}$. Furthermore,

$$\sigma(a) = \sum_{p \in \mathcal{P}} v_p(a) \in \mathbb{N}$$

is called the size of $a$.

Let $D$ be a monoid. Then $D^\times$ denotes the group of invertible elements of $D$. $D$ is called reduced if $D^\times = \{1\}$. $\mathcal{C}(D)$ denotes a quotient group of $D$, and we always assume $D \subseteq \mathcal{C}(D)$. A subset $D' \subseteq D$ is called divisor closed, if for all $a, b \in D$ with $a \mid b$ and $b \in D'$ we have $a \in D'$. For a
family $(D_i)_{i \in \Omega}$ of monoids we denote by $\prod_{i \in \Omega} D_i$ their direct product and by
\[ \prod_{i \in \Omega} D_i = \left\{ (a_i)_{i \in \Omega} \in \prod_{i \in \Omega} D_i \mid a_i = 1 \text{ for almost all } i \in \Omega \right\} \]
their coproduct.

Let $H$ be a submonoid of some monoid $D$. We define the congruence modulo $H$ is $\equiv$ $\mod H$ if $xH = yH$. The factor monoid of $D$ with respect to the congruence modulo $H$ is denoted by $D/H$. For $a \in D$, $[a] \in D/H$ denotes the class containing $a$. In particular, we set $D_{\text{red}} = D/D^\times$. The quotient group $\#(D/H)$ of $D/H$ is called the class group of $H \subseteq D$. It will be written additively. $H \subseteq D$ is called saturated, if $H = [1] \in D/H$ (equivalently, $H = \mathcal{C}(H) \cap D$).

A monoid $D$ is said to be primary, if $D \neq D^\times$ and if $a, b \in D$, $b \not\in D^\times$ implies that $a \mid b^n$ for some $n \in \mathbb{N}_+$. $D$ is said to be finitely primary (of rank $s \in \mathbb{N}_+$ and of exponent $\alpha \in \mathbb{N}_+$), if it is a submonoid of a factorial monoid $F$ containing exactly $s$ mutually non-associated prime elements $p_1, \ldots, p_s$:
\[ D \subseteq F = [p_1, \ldots, p_s] \times F^\times \]
such that the following two conditions hold:
(a) $D^\times = D \cap F^\times$.
(b) For any $a = \varepsilon p_1^{k_1} \cdots p_s^{k_s} \in F$ we have
(i) if $a \in D \setminus D^\times$, then $k_1 \geq 1, \ldots, k_s \geq 1$,
(ii) if $k_1 \geq \alpha, \ldots, k_s \geq \alpha$, then $a \in D$.

In fact, $F$ is just the complete integral closure $\hat{D}$ of $D$. If $a \in F = \hat{D}$ as above, we set $v_{p_i}(a) = k_i$ for $1 \leq i \leq s$.

Clearly, finitely primary monoids are primary. For more information on finitely primary monoids the reader is referred to [Ge4].

A monoid homomorphism $\varphi : H \to D$ is said to be
(a) a divisor homomorphism, if $a, b \in H$ and $\varphi(a) \mid \varphi(b)$ implies $a \mid b$,
(b) a weak divisor theory, if $D$ is a coproduct of reduced primary monoids $D_p$, $D = \coprod_{p \in \mathcal{P}} D_p$, and if for all $a \in D$ there exists $u_1, \ldots, u_m \in H$ such that $a$ is a strict greatest common divisor of $\varphi(u_1), \ldots, \varphi(u_m)$.
(c) a divisor theory, if it is a weak divisor theory and $D_p = (\mathbb{N}_+ +)$ for all $p \in \mathcal{P}$.

A monoid $H$ is called a (weakly) Krull monoid, if it admits a (weak) divisor theory $\varphi : H \to D$. In this case

$$G = D/\varphi(H) = \mathcal{E}(D/\varphi(H))$$

just depends on $H$. It is called the (divisor) class group of $H$.

If $\varphi : H \to D$ is a divisor homomorphism, then the induced homomorphism $\varphi_{\text{red}} : H_{\text{red}} \to D_{\text{red}}$ is injective and $\varphi(H) \subseteq D$ is saturated. So when studying the arithmetic of a monoid $H$ with divisor homomorphism $\varphi : H \to D$ it means no restriction to suppose that $H \subseteq D$ is a saturated submonoid. For general information on monoid homomorphisms see [G-HK] and for weak divisor theories consult [H-K1].

Let $H \subseteq D = \amalg_{i \in \Omega} D_i$ be a submonoid of a coproduct of finitely primary monoids. Then $H$ is a BF-monoid, i.e., $H$ is atomic and all $L \in \mathcal{L}(H)$ are finite (cf. [Ge5, Proposition 5.3]). Whenevwe consider such a situation we use the following notation. We suppose that $D_i$ is finitely primary of rank $s_i \in \mathbb{N}_+$ and that $p_{i,1}, \ldots, p_{i,s_i}$ are pairwise non-associated primes of $D_i$. Every $a \in D$ has a unique decomposition of the form

$$a = \prod_{i \in \Omega} a_i$$

with all $a_i \in D_i$ and $a_i = 1$ for all but finitely many $i \in \Omega$. For all $i \in \Omega$ and all $1 \leq \nu \leq s_i$ we set

$$v_{p_{i,\nu}}(a) = v_{p_{i,\nu}}(a_i).$$

Let $J \subseteq \Omega$ be a subset and $M \in \mathbb{N}_+$. We view

$$D_J = \amalg_{j \in J} D_j$$

as a submonoid of $D$ and set

$$D_{j,M} = \{a \in D_j \mid \min\{v_{p_{i,\nu}}(a) \mid 1 \leq \nu \leq s_i\} \leq M \text{ for all } j \in J\}.$$

Clearly, $D_\emptyset = D_{\emptyset,M} = \{1\}$, $D_{j,M} = D_{j,M} \cap D$, and we write $D_j$ instead of $D_{j,j}$ for all $j \in J$. Furthermore, we set

$$H_j = H \cap D_j.$$

If $\Omega = I \cup J$, then every $a \in D$ has a unique decomposition of the form

$$a = a_I a_J$$

with $a_I \in D_I$ and $a_J \in D_J$. 
3. ON THE ARITHMETIC OF MONOIDS

Let $H$ be a monoid. We denote by $\mathcal{U}(H)$ the set of irreducible elements of $H$. The factorization monoid $\mathcal{Z}(H)$ of $H$ is defined as the free abelian monoid with basis $\mathcal{U}(H_{\text{red}})$. Let $\pi : \mathcal{Z}(H) = \mathcal{U}(H_{\text{red}}) \to H_{\text{red}}$ be the canonical homomorphism. We say that $H$ is atomic, if $\pi$ is surjective. For an element $a \in H$ the elements of $
abla H \pi^{-1}(aH^*) \subseteq \mathcal{Z}(H)$ are called factorizations of $a$.

Let $H$ be an atomic monoid and $H' \subseteq H$ a subset. Most arithmetical invariants deal with lengths of factorizations. For some $z \in \mathcal{Z}(H)$, $\sigma(z) \in \mathbb{N}$ is called the length of the factorization $z$. For $a \in H'$

$L(a) = \{ \sigma(z) \mid z \in \mathcal{Z}(a) \} \subseteq \mathbb{N}$

denotes the set of lengths of $a$ and

$\mathcal{Z}(H') = \{ L(a) \mid a \in H' \}$

is the system of sets of lengths of $H'$. The elasticity $\rho(H')$ of $H'$ is defined as

$\rho(H') = \sup \left( \frac{\sup L}{\min L} \mid L \in \mathcal{Z}(H') \right)$.

In order to describe the structure of sets of lengths we consider for some $L = \{ l_1, \ldots, l_k \} \in \mathcal{Z}(H')$ with $l_1 < \cdots < l_k$ the set

$\Delta(L) = \{ l_i - l_{i-1} \mid 2 \leq i \leq k \} \subseteq \mathbb{N}$

of distances of successive lengths of $L$. Furthermore, we set

$\Delta(H') = \bigcup_{L \in \mathcal{Z}(H')} \Delta(L) \subseteq \mathbb{N}.$

The monoid $H$ is called half-factorial, if $\#L = 1$ for all $L \in \mathcal{Z}(H')$ (equivalently, $\rho(H) = 1$; equivalently, $\Delta(H) = \emptyset$). If $H$ is not half-factorial, then for every $n \in \mathbb{N}_+$ there exists some $L \in \mathcal{Z}(H)$ with $\#L \geq n$. This makes it worthwhile to study the structure of the sets $L \in \mathcal{Z}(H)$. Note that it may happen that $\Delta(H)$ is finite but $\rho(H) = \infty$ (cf. [Gee5, Sect. 5]). Conversely, there are monoids $H$ with finite elasticity for which the set $\Delta(H)$ is infinite (see the example at the end of this section).
The monoid $H$ is called factorial, if $\#\mathcal{Z}(a) = 1$ for all $a \in H$. If $H$ is not factorial, then for every $n \in \mathbb{N}$, there exists some $a \in H$ with $\#\mathcal{Z}(a) \geq n$ (cf. [Ge5, Lemma 3.1]). In order to investigate these sets $\mathcal{Z}(a)$ we define a distance function $d: \mathcal{Z}(H) \times \mathcal{Z}(H) \to \mathbb{N}$ by

$$d(z, z') = \max\left(\sigma\left(\frac{z}{\gcd(z, z')}\right), \sigma\left(\frac{z'}{\gcd(z, z')}\right)\right) \in \mathbb{N}$$

for two factorizations $z, z' \in \mathcal{Z}(H)$. The following properties of the distance function will be used without further quoting.

**Lemma 3.1.** Let $H$ be an atomic monoid. Then the distance function $d: \mathcal{Z}(H) \times \mathcal{Z}(H) \to \mathbb{N}$ satisfies the following properties for all $x, x', y, y', z \in \mathcal{Z}(H)$:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$,
4. $d(xy, x'y') \leq d(x, x') + d(y, y')$,
5. $d(x, y) = d(xz, yz)$.

**Proof.** Parts (1), (2), and (5) are obvious. (4) follows from (3).

(3) Let $x, y, z \in \mathcal{Z}(H)$ be given. Suppose $x = rwx', y = rwy'$ and $z = rz'$ with $r, w, x', y' \in \mathcal{Z}(H)$ such that $\gcd(x', y') = 1$ and $\gcd(z', w) = 1$. Then

$$\gcd(x, z) = rgcd(wx', z') = rgcd(x', z')$$

and

$$\gcd(y, z) = rgcd(ww', z') = rgcd(y', z').$$

Setting $s = \gcd(x', z')$ and $t = \gcd(y', z')$, we have

$$x' = sx'', y' = ty'' \text{ and } z' = stz''$$

for some $x'', y'', z'' \in \mathcal{Z}(H)$. Then we infer that

$$d(x, y) = \max\{\sigma(x''), \sigma(y'')\}$$

$$\leq \max\{\sigma(x''), \sigma(w), \sigma(t) + \sigma(z'')\}$$

$$+ \max\{\sigma(y''), \sigma(w), \sigma(s) + \sigma(z'')\}$$

$$= d(x''w, tz'') + d(y''w, sz'')$$

$$= d(x''rw, rstz'') + d(ty''rw, rstz'')$$

$$= d(x, z) + d(y, z). \qed$$
Let $H$ be an atomic monoid, $a \in H$, $z, z' \in \mathcal{Z}(a)$, and $N \in \mathbb{N} \cup \{\infty\}$. We say that there is an $N$-chain (of factorizations) from $z$ to $z'$, if there exists factorizations $z = z_0, z_1, \ldots, z_k = z' \in \mathcal{Z}(a)$ such that $d(z_{i-1}, z_i) \leq N$ for $1 \leq i \leq k$. The catenary degree

$$c_H(H') = c(H') \in \mathbb{N} \cup \{\infty\}$$

of a subset $H' \subseteq H$ is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for every $a \in H'$ and each two factorizations $z, z' \in \mathcal{Z}(a)$ there exists an $N$-chain from $z$ to $z'$ (cf. [Ge5, Sect. 3]).

The proof of the next lemma is straightforward and will be omitted.

**Lemma 3.2.** Let $D$ be an atomic monoid, $H \subseteq D$ a divisor closed submonoid, $H' \subseteq H$ a subset, and $a \in H$. Then we have

1. $H$ is an atomic monoid with $\mathcal{Z}(H) = H \cap \mathcal{Z}(D)$ and $H^\perp = H \cap D^\times$.
2. The canonical homomorphism $H/H^\perp \rightarrow D/D^\times$ is injective, and hence we may suppose $H_{\text{red}} \subseteq D_{\text{red}}$. Then $\mathcal{Z}(H) \subseteq \mathcal{Z}(D)$ and $\mathcal{Z}_H(a) = \mathcal{Z}_D(a)$.
3. $L_H(a) = L_D(a)$, $\mathcal{L}_H(H') = \mathcal{L}_D(H')$, $\rho_H(H') = \rho_D(H')$, and $c_H(H') = c_D(H')$.

The following lemma gathers some simple properties of $\Delta(H)$ and of $c(H)$.

**Lemma 3.3.** Let $H$ be an atomic monoid.

1. For $a \in H$ and $z \neq z' \in \mathcal{Z}(a)$ we have $2 + |\sigma(z) - \sigma(z')| \leq d(z, z')$.
2. For every subset $H' \subseteq H$ we have $2 + \sup \Delta(H') \leq c(H')$.
3. If $H = \bigsqcup_{i \in \Omega} H_i$, then $c(H) = \sup(c(H_i) | i \in \Omega)$.
4. If $H = \bigsqcup_{i \in \Omega} H_i$, then $\bigcup_{i \in \Omega} \Delta(H_i) \subseteq \Delta(H)$ and $\min \Delta(H) = \gcd(\bigcup_{i \in \Omega} \Delta(H_i)) = \gcd \Delta(H)$ (with the convention $\min \emptyset = \gcd \emptyset = 0$).

**Proof.** We may suppose that $H$ is reduced.

1. Let $a \in H$, $z = u_1 \ldots u_k v_1 \ldots v_r$, $z' = u_1 \ldots u_k w_1 \ldots w_s \in \mathcal{Z}(a)$ with $u_i, v_j, w_l \in \mathcal{Z}(H)$ and $(v_1, \ldots, v_r) \cap (w_1, \ldots, w_s) = \emptyset$. Since $H$ is cancellative, we have $2 \leq r$ and $2 \leq s$ which implies that $2 + |r - s| \leq \max(r, s)$.
2. If $c(H') = \infty$, nothing has to be done. Suppose $c(H') < \infty$ and let $a \in H'$ be given. Further let $l, k \in L(a)$ such that $l - k \in \Delta(H)$ and $t \notin L(a)$ for $k < t < l$. Then there are $z, z' \in \mathcal{Z}(a)$ with $\sigma(z) = k$, $\sigma(z') = l$, and a chain of factorizations $(z_i)_{i=0}^n$ from $z$ to $z'$ with $d(z_{i-1}, z_i) \leq \Delta(H)$.
\(c(H')\) for \(1 \leq i \leq m\). Then there is some \(i \in \{1, \ldots, m\}\) with \(\sigma(z_{i-1}) \leq \sigma(z) < \sigma(z_i)\). Since \(t \notin L(a)\) for \(k < t < l\), it follows that \(l \leq \sigma(z_i)\).

Hence by (1) we infer that

\[
l - k = l - \sigma(z) \leq \sigma(z_i) - \sigma(z_{i-1}) \leq d(z_{i-1}, z_i) - 2 \leq c(H') - 2.
\]

(3) This is obvious.

(4) Clearly, \(\bigcup_{i \in \Omega} \Delta(H_i) \subseteq \Delta(H)\) and from [Ge2, Lemma 3] it follows that \(\min \Delta(H) = \gcd \Delta(H)\).

For \(i \in \Omega\) we set \(d_i = \min \Delta(H_i)\), and we assert that

\[
\Delta(H) \subseteq \left\{ \sum_{i \in I} k_id_i \in \mathbb{N} \mid k_i \in \mathbb{Z}, I \subseteq \Omega \text{ finite} \right\} = E.
\]

Let \(d \in \Delta(H)\) be given. Then there exists some \(a = \prod_{i \in I} a_i \in H\) with \(I \subseteq \Omega\) finite such that \(l, l + d \in L(a)\) for some \(l \in \mathbb{N}_+\). Hence for every \(i \in I\) there are factorizations \(z_i, z'_i \in \mathcal{Z}_H(a_i)\) such that \(l = \sum_{i \in I} \sigma(z_i)\) and \(l + d = \sum_{i \in I} \sigma(z'_i)\). Since \(d_i \mid \sigma(z_i) - \sigma(z'_i)\) for all \(i \in I\), the assertion follows.

Finally we infer that

\[
\gcd(E) | \gcd(\Delta(H)) | \gcd\left( \bigcup_{i \in \Omega} \Delta(H_i) \right)
\]

and

\[
\gcd\left( \bigcup_{i \in \Omega} \Delta(H_i) \right) = \gcd\{d_i \mid i \in \Omega\} = \gcd(E),
\]

which implies that \(\gcd(\bigcup_{i \in \Omega} \Delta(H_i)) = \gcd(\Delta(H))\).

Chains of factorizations and sets of lengths in a weakly Krull monoid \(H\) may be studied in its block monoid (cf. [Ge4, Sect. 4]). We repeat the necessary definitions.

Let \(H, D\) be reduced atomic monoids such that \(H \subseteq D\) is saturated with class group \(G\). Let \(P \subseteq \mathcal{P}(D)\) be the set of prime elements of \(D\) and \(T = \{a \in D \mid p \nmid a \text{ for any } p \in P\}\). Then \(D = \mathcal{A}(P) \times T\) and we shall identify these two monoids. We set \(G_0 = \{g \in G \mid g \cap P \neq \emptyset\}\) and define a monoid homomorphism

\[
\iota: \mathcal{A}(G_0) \times T \to G
\]

by \(\iota(t) = [t] \in G\) for every \(t \in T\) and by \(\iota(g) = g\) for every \(g \in G_0\). Then

\[
\mathcal{B} = \mathcal{B}(G_0, T, \iota) = \text{Ker}(\iota) \subseteq \mathcal{A}(G_0) \times T
\]
is the \textit{block monoid} associated to $H \subseteq D$. The relationship between $H$ and $\mathcal{B}$ is established by the block homomorphism

$$\beta : \mathcal{F}(P) \times T \rightarrow \mathcal{F}(G_0) \times T$$

which is defined by $\beta(t) = t$ for all $t \in T$ and $\beta(p) = [p] \in G_0$ for all $p \in P$.

If $T = \{1\}$, then $\mathcal{B} = \mathcal{B}(G_0) \subseteq \mathcal{F}(G_0)$ is the ordinary block monoid over $G_0$. Davenport’s constant $\mathcal{D}(G)$ is defined as

$$\mathcal{D}(G) = \sup\{\sigma(U) \mid U \in \mathcal{B}(G_0) \text{ is irreducible} \} \in \mathbb{N} \cup \{\infty\}.$$

For the significance of Davenport’s constant in factorization theory the reader is referred to [Ch]. If $G_0$ is finite, then $\mathcal{B}(G_0)$ is finitely generated and hence $\mathcal{D}(G_0) < \infty$ (cf. [Ge, Proposition 2]).

In [A-A-Z, Example 4] a half-factorial domain was studied which has infinite sets $\mathcal{F}(a)$. The following example provides a half-factorial Dedekind domain having infinite catenary degree.

\textbf{Example.} Let $\rho \in \mathbb{N}_+$. We construct a Dedekind domain $R$ whose multiplicative monoid $R^*$ satisfies $c(R) = \infty$, $\rho(R) = \rho$, and $(\rho - 1)n \mid n \geq 2 \subseteq \Delta(R')$, if $\rho > 1$.

(i) Let $n \geq 2$. Choose a linearly independent generating system $e_1, \ldots, e_{n-1}$ of $C_{\rho n}^{n-1}$, set $e_0 = -\sum_{i=1}^{n-1} e_i$ and $H_0^{(n)} = \{e_0, \ldots, e_{n-1}\}$. Then $\mathcal{B}(H_0^{(n)})$ has exactly $n + 1$ irreducible elements:

$$U = \prod_{i=0}^{n-1} e_i \quad \text{and} \quad V_i = e_i^{\rho n} \quad \text{for } 0 \leq i \leq n - 1.$$

Clearly,

$$U^{\rho n} = \prod_{i=0}^{n-1} V_i$$

is the only non-cancellative relation among them. Hence $c(\mathcal{B}(H_0^{(n)})) = \rho n$,

$\rho(\mathcal{B}(H_0^{(n)})) = \rho$, and $\Delta(\mathcal{B}(H_0^{(n)})) = \{\rho n - n\}$.

(ii) Set

$$G = \bigoplus_{n \geq 2} C_{\rho n}^{n-1} \quad \text{and} \quad G_0 = \bigcup_{n \geq 2} H_0^{(n)} \subseteq G.$$

Then $\mathcal{B}(G_0) = \Pi_{n \geq 2} \mathcal{B}(H_0^{(n)})$. Thus by Lemma 3.3

$$c(\mathcal{B}(G_0)) = \sup\{c(\mathcal{B}(H_0^{(n)})) \mid n \geq 2\} = \infty.$$
and
\[(\rho - 1)n \geq 2 \subseteq \Delta(\mathcal{R}(G_0))\].

Proposition 4 in [HK2] implies that
\[\rho(\mathcal{R}(G_0)) = \sup\{\rho(\mathcal{R}(H_0^n)) \mid n \geq 2\} = \rho\].

(iii) Since \(G_0\) generates \(G\), there is a Dedekind domain \(R\) with class group (isomorphic to) \(G\) and with \(G_0\) being the set of classes containing prime divisors, i.e., \((G, G_0)\) is a realizable pair (cf. [Gr, Sect. 1; Sk, Theorem 2.4]). By [Ge5, Proposition 4.2] we infer \(\Delta(R') = \Delta(\mathcal{R}(G_0))\), \(\rho(R') = \rho(\mathcal{R}(G_0))\), and \(c(R') = c(\mathcal{R}(G_0))\).

This example indicates that sets of lengths alone are a too weak measure to describe the arithmetic of noetherian domains. However, there is some evidence for the following conjecture (cf. [Ge7]). Consider the class of Krull domains with finite divisor class group where each class contains a prime divisor. We conjecture that within this class of domains the system of sets of lengths completely determines the arithmetic. More precisely, let \(R\) and \(R'\) be two Krull domains of the above type with class groups \(G\) and \(G'\). Then \#\(G \geq 4\), \(\mathcal{R}(R') = \mathcal{R}(R')\) implies \(G = G'\) (and hence \(R\) and \(R'\) are arithmetically equivalent by [HK3, Corollary 4]).

4. ON LOCAL TAMENESS

The arithmetical concept of tameness of monoids was first introduced in [Ge2] in order to prove a structure theorem for sets of lengths in finitely generated monoids. For our present purposes we need a more subtle notion which will be called local tameness.

**Definition 4.1.** Let \(H\) be an atomic monoid and \(H' \subseteq H\) a subset.

1. Let \(X \subseteq \mathcal{Z}(H)\) be a subset. The **tame degree** \(t_H(H', X)\) of \(H'\) with respect to \(X\) is the minimum of all \(N \in \mathbb{N} \cup \{\infty\}\) having the following property: if \(a \in H'\), \(z \in \mathcal{Z}(a)\), and \(x \in X\) such that \(\pi(z) \mid_z a\), then there exists some \(z' \in \mathcal{Z}(a)\) with \(x \mid_{\mathcal{Z}(H)} z'\) and \(d(z, z') \leq N\).

2. We say that \(H\) is **locally tame**, if
\[t_H(H, u) < \infty\]
for all \(u \in \mathcal{Z}(H_{\text{red}})\).
(3) $H'$ is called tame, if the tame degree

$$t(H') = t_H(H', \mathcal{Z}(H_{\text{red}})) < \infty.$$ 

Remarks. Let $H$ be an atomic monoid.

(1) Let $X \subseteq \mathcal{Z}(H)$, $H' \subseteq H$, and $\eta : H \rightarrow H_{\text{red}}$ the canonical epimorphism. Then, by definition, $t_H(H', X) = t_{H_{\text{red}}}(\eta(H'), X)$. Hence, for simplicity of notation, all results in this section will be formulated for reduced monoids.

(2) Let $u \in \mathcal{Z}(H_{\text{red}})$ be given. Then $u$ is prime if and only if $t_H(H, u) = 0$. Hence $H$ is factorial if and only if $t(H) = 0$.

(3) The definition of the tame degree given in [Ge2] differs slightly from the present one. If $t(H)$ denotes the tame degree in the sense of [Ge2], then we have $|t(H) - t(H)| \leq 1$.

(4) If $H_{\text{red}}$ is finitely generated, then $t(H) < \infty$ by [Ge2, Proposition 2].

**Proposition 4.2.** Let $H$ be a reduced atomic monoid, $H' \subseteq H$ a subset, and $X \subseteq \mathcal{Z}(H)$.

1. $t_H(H', X) = \sup \{t_H(H', x) : x \in X\}$.
2. $t_H(H', Z) \leq \sup \{t_H(H, a) : a \in H'\}$.
3. If $H'$ is divisor closed and $x = u_1 \ldots u_r \in X$ with $u_1, \ldots, u_r \in \mathcal{Z}(H)$, then $t_H(H', x) \leq \sum_{i=1}^r t_H(H', u_i) \leq \sigma(x)(H')$.
4. Let $a \in H$, $z \in \mathcal{Z}_H(a)$, and $x \in \mathcal{Z}(H)$ such that $\pi(x) |_H a$. Then there exists a factorization $z' \in \mathcal{Z}_H(a)$ and elements $y, w, w' \in \mathcal{Z}(H)$ such that $z = yw$ and $z' = yw'$ with $x |_{\mathcal{Z}(H)} w'$ and $\max(\sigma(w), \sigma(w')) \leq \sigma(x) \max(1, t(H))$.

**Proof.** Parts (1) and (2) are clear by definition.

(3) We proceed by induction on $r$. If $r = 1$, there is nothing to show. Suppose $r \geq 1$ and let $u_1, \ldots, u_r \in \mathcal{Z}(H)$, $a \in H'$ with $u_1 \ldots u_r | a$ and $z \in \mathcal{Z}(a)$ be given.

By the induction hypothesis there exists a factorization

$$z' = u_1 \ldots u_{r-1} y \in \mathcal{Z}(a)$$

with $d(z, z') \leq \sum_{i=1}^{r-1} t_H(H', u_i)$ and $y \in \mathcal{Z}(b)$ for some $b \in H'$. Then $u_r | b$ and there is a factorization $y' = u_r y'' \in \mathcal{Z}(b)$ such that $d(y, y') \leq t_H(H', u_r)$. We set

$$z'' = u_1 \ldots u_r y''$$

and infer that

$$d(z, z'') \leq d(z, z') + d(z', z'')$$

$$\leq d(z, z') + d(y, y') \leq \sum_{i=1}^r t_H(H', u_i).$$
(4) We first handle the special case that \( \gcd(z, x) = 1 \). By definition there exists a factorization \( z' \in \mathcal{Z}_H(a) \) with \( z |_{x(t)} z' \) and \( d(z, z') \leq t_H(H, x) \). By (3) it follows that \( t_H(H, x) \leq \sigma(x)t(H) \). Setting \( y = \gcd(z, z') \), \( w = y^{-1}z \), and \( w' = y^{-1}z' \) we obtain that \( z = yw \) and \( z' = yw' \) with \( \max(\sigma(w), \sigma(w')) = d(z, z') \leq \sigma(x)t(H) \). Since \( x | z' \) and \( \gcd(x, y) | \gcd(x, z) = 1 \), it follows that \( x | w' \).

Now we deal with the general case. Suppose \( \gcd(z, x) = v \in \mathcal{Z}(H) \). We set \( z = vz_0 \) and \( x = vx_0 \). Then \( \gcd(z_0, x_0) = 1 \) and hence there are \( y, w, w' \in \mathcal{Z}(H) \) such that \( z_0 = yw_0 \) and \( z'_0 = yw'_0 \) with \( x_0 | w_0 \) and \( \max(\sigma(w_0), \sigma(w'_0)) \leq \sigma(x_0)t(H) \). Setting \( w = w_0v \) and \( w' = w'_0v \) we obtain \( z = yw \) and \( z' = yw' \in \mathcal{Z}_H(a) \) with \( x | w' \) and

\[
\max(\sigma(w), \sigma(w')) \leq \sigma(v) + \sigma(x_0)t(H) \leq \sigma(x)\max\{1, t(H)\}.
\]

Let \( H \) be a reduced atomic monoid and \( H' \subseteq H \) a subset. We recall the definition of an arithmetical invariant studied in [Ge5, Definition 3.3] and 3.7.

For \( u \in H \), \( \omega_H(H', u) \) is the minimum of all \( N \in \mathbb{N}_+ \cup \{\infty\} \) having the following property: if \( a_1, \ldots, a_n \in H \setminus H'^\times \) with \( \prod_{i=1}^n a_i \in H' \) then there exists a subset \( J \subseteq \{1, \ldots, n\} \) with \( \#J \leq N \) and

\[
\omega_H(H', u) = \sup\{\omega_H(H', u) | u \in U \} \in \mathbb{N}_+ \cup \{\infty\}.
\]

**Lemma 4.3.** Let \( H \) be a reduced atomic monoid and \( H' \subseteq H \) a subset.

1. Let \( u \in \mathcal{Z}(H) \) be irreducible but not prime. Then \( t_H(H', u) \) is the minimal \( N \in \mathbb{N}_+ \cup \{\infty\} \) having the following property: if \( u_1, \ldots, u_n \in \mathcal{Z}(H) \setminus \{u\} \), \( \prod_{i=1}^n u_i \in H' \), and \( u |_{x(t)} \prod_{i=1}^n u_i \), then there exists a subset \( J \subseteq \{1, \ldots, n\} \) with \( \#J \leq N \) and \( u | \prod_{i \in J} u_i \) and a factorization \( y \in \mathcal{Z}(u^{-1}\prod_{i \in J} u_i) \) such that \( \sigma(uy) \leq N \).

2. Let \( u \in \mathcal{Z}(H) \) be irreducible but not prime. Then \( \omega_H(H', u) \leq t_H(H', u) \).

3. If \( H \) is not factorial and \( H' \) divisor closed, then

\[
\max\{\rho(H'), c(H')\} \leq \omega_H(H', \mathcal{Z}(H)) \leq t_H(H', \mathcal{Z}(H)).
\]

**Proof.** Parts (1) and (2) are obvious by definition. The left inequality in (3) follows from [Ge5, Propositions 3.5 and 3.7]. The right inequality is a consequence of (2).

The following lemma compares factorizations in a monoid \( D \) with factorizations in a submonoid. Let \( D \) be a monoid and \( H \subseteq D \) a submonoid. We set

\[
d(H, D) = \sup\{\sup L_D(u) | u \in \mathcal{Z}(H)\} \in \mathbb{N} \cup \{\infty\}.
\]
Furthermore, let $\mathcal{D}(H, D)$ be defined as the minimum of all $N \in \mathbb{N}_+ \cup \{\infty\}$ satisfying the following property: for all $u_1, \ldots, u_n \in \mathcal{U}(D)$ with $\prod_{i=1}^n u_i \in H$ there exists a subset $\emptyset \neq J \subseteq \{1, \ldots, n\}$ with $\#J \leq N$ such that $\prod_{j \in J} u_j \in H$.

**Lemma 4.4.** Let $D$ be a reduced atomic monoid and $H \subseteq D$ an atomic submonoid with class group $G$. Let $G_0 = \{[u] \in G \mid u \in \mathcal{U}(D)\} \subseteq G$ and $a \in H$.

1. Suppose $H \subseteq D$ is saturated. Then
   \[ d(H, D) \leq \mathcal{D}(H, D) = \mathcal{D}(G_0) \]
   and $d(H, D) = \mathcal{D}(H, D)$, if $D$ is factorial.

2. $\mathcal{D}(G_0) = d(\mathcal{A}(G_0), \mathcal{A}(G_0)) = \mathcal{D}(\mathcal{A}(G_0), \mathcal{A}(G_0))$.

3. Every factorization $x \in \mathcal{I}_H(a)$ induces a factorization $y \in \mathcal{I}_D(a)$ with $\sigma(x) \leq \sigma(y) \leq \sigma(x)d(H, D)$.

4. Suppose $H \subseteq D$ is saturated. Then for each factorization $y \in \mathcal{I}_D(a)$ there is some $x \in \mathcal{I}_H(a)$ with $\mathcal{D}(H, D)^{-1}\sigma(y) \leq \sigma(x)$ and $\sup L_y(a) \leq \sigma(y)\rho(D)$. If $H \hookrightarrow D$ is a divisor theory and if there is no prime $q \in H$ dividing $a$, then $\sup L_y(a) \leq \frac{1}{\rho} \sigma(y)$.

5. $\rho_H(a) \leq d(H, D)\rho_D(a)$ and hence $\rho(D) \leq d(H, D)\rho(D)$.

**Proof.** (1) To verify that $d(H, D) \leq \mathcal{D}(H, D)$ let $u \in \mathcal{U}(H)$ be given. We consider an arbitrary factorization $u = v_1 \ldots v_n$ of $u$ in $D$ with $v_j \in \mathcal{U}(D)$. It is sufficient to show that $n \leq \mathcal{D}(H, D)$. Let $J \subseteq \{1, \ldots, n\}$ with $1 \leq \#J \leq \mathcal{D}(H, D)$ such that $\prod_{j \in J} u_j \in H$. Since $H \subseteq D$ is saturated and $u$ is irreducible in $H$, it follows that $n = \#J \leq \mathcal{D}(H, D)$.

Next we check that $\mathcal{D}(G_0)$ has the property mentioned in the definition of $\mathcal{D}(H, D)$. Let $u_1, \ldots, u_n \in \mathcal{U}(D)$ be given with $\prod_{i=1}^n u_i \in H = [1] \in G$. Hence $B = \prod_{i=1}^n [u_i] \in \mathcal{A}(G_0)$. Since $B$ is a product of irreducible blocks, there is some $U = \prod_{i \in J} [u_i] \in \mathcal{U}(G_0)$ with $1 \leq \#J \leq \mathcal{D}(G_0)$. Since $U$ is a block, we infer that $\prod_{i \in J} u_i \in H$. Thus $\mathcal{D}(H, D) \leq \mathcal{D}(G_0)$.

Conversely, let $U = \prod_{i=1}^n g_i$ be an irreducible block. Then for $u_i \in \mathcal{U}(D) \cap g_i$ we have $\prod_{i=1}^n u_i \in H$. If $\prod_{j \in J} u_j \in H$ for some $\emptyset \neq J \subseteq \{1, \ldots, n\}$, it follows that $J = \{1, \ldots, n\}$ and hence $n \leq \mathcal{D}(H, D)$.

Suppose that $D$ is factorial and let $u_1, \ldots, u_n \in \mathcal{U}(D)$ be given with $a = \prod_{i=1}^n u_i \in H$. Let $v \in \mathcal{U}(H)$ be a divisor of $a$. Hence there is a subset $J \subseteq \{1, \ldots, n\}$ with $v = \prod_{j \in J} u_j$ and $\#J \leq \mathcal{D}(H, D)$. Thus $\mathcal{D}(H, D) \leq d(H, D)$.

(2) Since $\mathcal{A}(G_0) \subseteq \mathcal{A}(G_0)$ is saturated, the assertion follows from (1).

(3) This is obvious.
(4) Let \( y \in \mathcal{D}(a) \). Then by definition of \( \mathcal{D}(H, D) \), \( y \) allows a product decomposition of the form

\[
y = \prod_{i=1}^{n} \prod_{j \in I_i} v_{i,j} \in \mathcal{D}(a)
\]

with \( \sigma(y) = \sum_{i=1}^{n} \# J_i \cdot \# I_i \leq \mathcal{D}(H, D) \), and \( \prod_{j \in I_i} v_{i,j} \in H \) for \( 1 \leq i \leq n \).

Choose \( x_i \in \mathcal{E}(\prod_{j \in I_i} v_{i,j}) \) and set \( x = \prod_{i=1}^{n} x_i \). Then

\[
\sigma(y) = n \mathcal{D}(H, D) \leq \sigma(x) \mathcal{D}(H, D).
\]

Clearly, \( \sup L_H(a) \leq \sup L_D(a) \leq \sigma(y) \rho(D) \).

Suppose \( H \rightarrow D \) is a divisor theory and \( a \) is not divisible by a prime element \( q \in H \). Set \( c = p_1 \ldots p_k \) with prime elements \( p_i \in D \). Since an element \( q \in H \) is prime in \( H \) if and only if \( q \) prime in \( D \) (cf. [HK3, Satz 10]), it follows that \( \sup L_H(a) \leq \frac{1}{k} \).

(5) Let \( x_0 \in H \) with \( \sigma(x_0) = \min L_H(a) \). By (3) there is some \( y \in \mathcal{D}(a) \) with \( \sigma(y) \leq \sigma(x_0) \mathcal{D}(H, D) \) and hence \( \min L_D(a) \leq \min L_H(a) \mathcal{D}(H, D) \). Therefore we infer that

\[
\rho_H(a) = \frac{\sup L_H(a)}{\min L_H(a)} \leq \frac{\sup L_D(a) \mathcal{D}(H, D)}{\min L_D(a)} \leq \mathcal{D}(H, D) \rho_D(a).
\]

**Proposition 4.5.** Let \( D \) be a reduced atomic monoid and \( H \subseteq D \) a saturated atomic submonoid with class group \( G \) and \( G_0 = \{ [u] \in G \mid u \in \mathcal{U}(D) \} \).

1. If \( D \) is tame and \( D(G_0) \) finite, then \( H \) is tame. More precisely, if \( D \) is not factorial, then

\[
t(H) \leq \rho(D) \mathcal{D}(H, D) (t(D) \mathcal{D}(H, D) - 1) + 1.
\]

2. If \( H \rightarrow D \) is a divisor theory, then \( t(H) \leq 1 + \frac{1}{2} \mathcal{D}(G_0) (\mathcal{D}(G_0) - 1) \). Moreover, if \( H \) is not factorial and \( G_0 = -G_0 \) then

\[
\mathcal{D}(G_0) \leq t(H) \leq 1 + \frac{1}{2} \mathcal{D}(G_0) (\mathcal{D}(G_0) - 1).
\]

**Proof.** If \( D \) is tame and \( D(G_0) \) is finite, then \( \rho(D) \leq t(D) < \infty \) by Lemma 4.3 and \( \mathcal{D}(H, D) \leq \mathcal{D}(G_0) < \infty \) by Lemma 4.4. If \( D \) is not factorial, \( H \rightarrow D \) a divisor theory, and \( G_0 = -G_0 \), then by [Ge5, Corollary 3.6], \( \mathcal{D}(G_0) = \omega_H(H, \mathcal{U}(H)) \) and hence \( \mathcal{D}(G_0) \leq t(H) \) by Lemma 4.3. Therefore it is sufficient to verify the wanted upper bounds for \( t(H) \).

Let \( u \in \mathcal{U}(H) \), \( a \in H \) with \( u \mid_H a \) and \( z = \prod_{i \in I} u_{i} \in \mathcal{E}(a) \) be given. If \( u \) is prime in \( H \), then \( t_T(H, u) = 0 \). So suppose that \( u \) is not prime. If \( H \rightarrow D \) is a divisor theory, we may assume without restriction that there is no prime \( q \in H \) dividing \( a \).
We choose factorizations of $u$ and of $u_{\lambda}$ in $D$. Say,

$$u = \prod_{\mu \in \Omega} v_{\mu} \quad \text{and} \quad u_{\lambda} = \prod_{\mu \in \Omega_{\lambda}} v_{\lambda, \mu}$$

with $v_{\mu}, v_{\lambda, \mu} \in \mathcal{Z}(D)$, $\#\Omega \leq d(H, D)$, and $\#\Omega_{\lambda} \leq d(H, D)$. Then

$$y = \prod_{\lambda \in \Lambda} \prod_{\mu \in \Omega_{\lambda}} v_{\lambda, \mu} \in \mathcal{Z}_{D}(a).$$

By Lemma 4.2.4 there exists some $y' \in \mathcal{Z}_{D}(a)$ with $\prod_{\mu \in \Omega_{\lambda}} v_{\mu} |_{\mathcal{Z}(D)} y'$ such that

$$y = \prod_{\lambda \in \Lambda} \prod_{\mu \in I_{\lambda}} v_{\lambda, \mu} \prod_{\lambda \in \Omega_{\lambda}} v_{\lambda, \mu}$$

and

$$y' = \prod_{\lambda \in \Lambda} \prod_{\mu \in I_{\lambda}} v_{\lambda, \mu} \prod_{\lambda \in \Omega_{\lambda}} v_{\lambda, \mu} \prod_{\gamma \in \Gamma} w_{\gamma},$$

where $\Omega_{\lambda} = I_{\lambda} \cup J_{\lambda}$ for all $\lambda \in \Lambda$, $\sum_{\lambda \in \Omega} #J_{\lambda}$, and $\#\Omega + \#\Gamma$ are bounded above by $\#\Omega \max(1, t(D))$.

We define a factorization

$$y'' = \prod_{\lambda \in \Lambda} \prod_{\mu \in I_{\lambda}} v_{\lambda, \mu} \prod_{\lambda \in \Omega_{\lambda}} v_{\lambda, \mu} x' \in \mathcal{Z}_{D}(a)$$

with

$$x' = \prod_{\lambda \in \Lambda} \prod_{\mu \in I_{\lambda}} v_{\lambda, \mu} \prod_{\gamma \in \Gamma} w_{\gamma}.$$

Since $H \subseteq D$ is saturated, $x'$ is the factorization of some $b \in H$.

By Lemma 4.4 there is some $z' \in \mathcal{Z}_{H}(b)$ with $\sigma(z') \leq \sigma(x')\rho(D)$ (resp. $\sigma(z') \leq \frac{1}{2} \sigma(x')$ in the case of (2)). We set

$$z'' = \prod_{\lambda \in \Lambda} u_{\lambda}uz' \in \mathcal{Z}_{H}(a)$$

and infer that

$$t_{H}(H, u) \leq d(z, z'') \leq \max\{1 + \sigma(z'), \#\{\lambda \in \Lambda | J_{\lambda} \neq \emptyset\}\}.$$
Now, first suppose that $D$ is not factorial. Then
\[
\sigma(x') \leq \sum_{A \in \mathcal{A}} \#I_A + \#\Gamma \\
\leq \#\Omega t(D)(d(H, D) - 1) + (\#\Omega t(D) - \#\Omega) \\
\leq \#\Omega t(D)d(H, D) - 1) \\
\leq d(H, D)(t(D)d(H, D) - 1),
\]
which implies that
\[
t_H(H, u) \leq 1 + \rho(D)\sigma(x') \\
\leq 1 + \rho(D)d(H, D)(t(D)d(H, D) - 1).
\]
Finally, suppose that $H \rightarrow D$ is a divisor theory. Then
\[
\sigma(x') \leq \#\Omega(d(H, D) - 1)
\]
and
\[
\sigma(z') \leq \frac{1}{2}\#\Omega(d(H, D) - 1).
\]
Hence from Lemma 4.4 it follows that
\[
t_H(H, u) \leq 1 + \frac{1}{2}\mathcal{G}(G)(\mathcal{G}(G) - 1).
\]

Remarks. (1) Proposition 4.5.1 was first shown in [Ge2, Theorem 2]. However, the proof given there contains a gap which leads to an incorrect upper bound for $t(H)$.

(2) The above result shows that Krull monoids $H$ with $\mathcal{G}(G) < \infty$ are tame. Hence they are locally tame, have catenary degree, finite elasticity, and $\Delta(H)$ is finite. For the Dedekind domain discussed in the previous section we have $\mathcal{G}(G) = \infty$. It is locally tame but not tame, since it has infinite catenary degree.

Lemma 4.6. Let $D$ be a finitely primary monoid of rank $s$ and exponent $\alpha$. Then $D$ is tame if and only if $s = 1$.

Proof. If $s \geq 2$, then $\rho(D) = \infty$ by [HK2, Theorem 4] and hence $D$ is not tame by Lemma 4.3. Suppose $s = 1$ and without restriction let
\[
D \subseteq \hat{D} = \{p\} \times \hat{D}^\times
\]
be reduced. For $u \in \mathcal{U}(D)$ we have $1 \leq v_p(u) \leq 2\alpha - 1$. Let $u \in \mathcal{U}(D)$, $a \in D$ with $u \divides a$ and $z = w_1 \ldots w_r \in \mathcal{Z}_D(a)$ with $w_i \in \mathcal{U}(D)$. Then $u \divides$
\[ \prod_{i=1}^{k} w_i \text{ for some } k \leq v_p(u) + \alpha \text{ and} \]
\[ \max L_D \left( \prod_{i=1}^{k} w_i \right) \leq v_p \left( \prod_{i=1}^{k} w_i \right) \leq (v_p(u) + \alpha)(2\alpha - 1) \]
\[ \leq (3\alpha - 1)(2\alpha - 1). \]

Hence \( t_D(D, u) \leq (3\alpha - 1)(2\alpha - 1). \]

**Proposition 4.7.** Let \( H \subseteq D = \bigcup_{i \in \Omega} D_i \) be a saturated reduced sub-monoid with bounded class group \( G \). Suppose that \( d = d(H, D) < \infty \) and that all \( D_i \) are finitely primary of some exponent \( \alpha \in \mathbb{N}_+ \). Let \( I, J \subseteq \Omega \) with \( \Omega = I \cup J \), \( M, N \in \mathbb{N}_+ \), \( H' = \{ a \in H | a_j \in D_{j, M} \} \) and \( U = \{ u \in \mathcal{Z}(H) \} \). Then
\[ t_H(H', U) \leq d^2 \max \{ M, N \}(4\alpha \exp(G) + 3\alpha - 1). \]

Furthermore, \( H \) is locally tame.

**Proof.** Let \( a \in H' \), \( u \in U \) such that \( u \mid a \) and \( z = u_1 \ldots u_r \in \mathcal{Z}_H(a) \) and \( u_i \in \mathcal{Z}(H) \). We have to find a factorization \( z' \in \mathcal{Z}_H(a) \) with \( u \mid z' \) in \( \mathcal{Z}(H) \) such that \( d(z, z') \) is bounded by the above constant.

Choose factorizations of \( u \) and all \( u_i \) in \( D \) and set
\[ u = \prod_{\lambda \in \Lambda} w_\lambda \]
with \( w_\lambda \in D_\lambda \), \( \Lambda \subseteq \Omega \) with \( \# \Lambda \leq d \) and
\[ \prod_{i=1}^{r} u_i = \prod_{\lambda \in \Lambda} \prod_{l=1}^{r_\lambda} v_{\lambda, l} \]
with \( v_{\lambda, l} \in \mathcal{Z}(D_\lambda) \), \( \sum_{\lambda \in \Lambda} r_\lambda \leq rd, \) and \( \Lambda \subseteq \Lambda \).

For every \( \lambda \in \Lambda \) there exists a subset \( \Gamma_\lambda \subseteq \{1, \ldots, r\} \) such that
\[ w_\lambda \prod_{l \in \Gamma_\lambda} u_l \text{ (in } D_\lambda \text{ and hence in } D) \]
with \( \# \Gamma_\lambda \leq r_\lambda \). If \( \lambda \in I \), we may choose \( \Gamma_\lambda \) with
\[ \# \Gamma_\lambda \leq \max \{ v_{p_\lambda, \nu}(w_\lambda) | 1 \leq \nu \leq s_\lambda \} + \alpha \leq N. \]

If \( \lambda \in J \), then \( a \in H' \) implies that \( r_\lambda \leq \min \{ v_{p_\lambda, \nu}(a) | 1 \leq \nu \leq s_\lambda \} \leq M. \)

Setting \( \Gamma = \bigcup_{\lambda \in \Lambda} \Gamma_\lambda \) we infer that
\[ \prod_{l \in \Gamma} u_l = ub \]
with \( b \in D \) and hence in \( H \). By [Ge5, Lemma 5.6], \( b \) has a factorization \( y \in \mathcal{Z}_H(b) \) with
\[ \sigma(y) < (4\alpha \exp(G) + \alpha) \# \Gamma d + \sum_{\lambda \in \Lambda} v_{p_\lambda, \nu}(b). \]
Clearly,
\[ \sum_{\lambda \in \Lambda} v_{\lambda}(b) \leq \sum_{\lambda \in \Lambda} v_{\lambda}(\prod_{l \in \Gamma} u_l) \leq (2\alpha - 1)\#\Gamma d. \]

Finally we set
\[ z' = \prod_{l \in \{1, \ldots, r\}} u_l m y \in \mathcal{E}_H(a) \]
and infer that
\[ d(z, z') \leq \max\{\#\Gamma, 1 + \sigma(y)\} \leq \#\Gamma d(4\alpha \exp(G) + \alpha + 2\alpha - 1). \tag{\ast} \]

Since
\[ \#\Gamma \leq \sum_{\lambda \in I \cap \Lambda} \#\Gamma_\lambda + \sum_{\lambda \in J \cap \Lambda} \#\Gamma_\lambda \leq d \max\{M, N\}, \]
we obtain the first assertion.

To see that \( H \) is locally tame, we set \( I = J = \Omega, M = \infty, \) and \( N = \max\{v_{\lambda}(u) | 1 \leq v \leq s_\lambda, \lambda \in \Omega\} + \alpha. \) Then \( H' = H \) and
\[ \#\Gamma \leq \sum_{\lambda \in \Lambda} \#\Gamma_\lambda = \sum_{\lambda \in I \cap \Lambda} \#\Gamma_\lambda \leq d N. \]

Hence it follows from (\ast) that
\[ t_H(H, u) \leq d^2 N(4\alpha \exp(G) + 3\alpha - 1). \]

## 5. FACTORIZATIONS OF LARGE ELEMENTS

The next result shows that in a wide class of monoids large elements (i.e., multiples of some fixed element) have especially simple factorizations: each two factorizations can be concatenated by a \( C \)-chain of such factorizations whose lengths form an arithmetical progression (with a possible exception at the beginning and at the end of the chain). Factorizations of arbitrary elements will be studied in Section 6. We formulate the result, discuss it, and do the proof at the end of the section.

**Theorem 5.1.** Let \( H \) be an atomic monoid which is locally tame and has finite catenary degree. Then there exist some element \( a^* \in H \) and some constant \( C \in \mathbb{N}_+ \) such that for all \( a \in H \) with \( a^* \mid a \) and for all \( z, z' \in \mathcal{E}(a) \)
there is a chain of factorizations \((z_i)_{i=0}^{k+1}\) with \(z_{-1} = z, z_{k+1} = z'\) satisfying the following two properties:

1. \(d(z_{i-1}, z_i) \leq C\) for all \(0 \leq i \leq k + 1\)
2. \(|\sigma(z_i) - \sigma(z_{i-1})| \in (0, \min \Delta(H))\) for \(1 \leq i \leq k\).

**Corollary 5.2.** Let \(H\) be an atomic monoid satisfying the assertion of Theorem 5.1. Then for every \(a \in H\) with \(a^k | a\)

\[
L(a) = \{l_1, \ldots, l_\alpha, m, m + d, \ldots, m + \kappa d, n_1, \ldots, n_\beta\}
\]

with \(l_1 < \cdots < l_\alpha < m < m + \kappa d < n_1 < \cdots < n_\beta\), \(d = \min \Delta(H), 0 \leq \alpha \leq C - 2, \) and \(0 \leq \beta \leq C - 2\).

**Proof.** Let \(a \in H\) with \(a^k | a\) and \(L = L(a)\). We choose factorizations \(z, z' \in \mathcal{Z}(a)\) with \(\sigma(z) = \min L\) and \(\sigma(z') = \max L\). Let \((z_i)_{i=0}^{k+1}\) be a chain of factorizations of \(a\) satisfying properties (i) and (ii) of the theorem. We set \(m = \min(\sigma(z_0), \sigma(z_k))\) and \(m + \kappa d = \max(\sigma(z_0), \sigma(z_k))\). Then, by Lemma 3.3

\[
\alpha \leq m - \min L \leq \sigma(z_0) - \sigma(z_{-1}) \leq d(z_0, z_{-1}) - 2 \leq C - 2
\]

and

\[
\beta \leq \max L - (m + \kappa d) \leq \sigma(z_{k+1}) - \sigma(z_k) \\
\leq d(z_k, z_{k+1}) - 2 \leq C - 2.
\]

**Examples.** Theorem 5.1 applies to the following classes of monoids:

1. Finitely generated monoids (by Remark 4 after Definition 4.1 and Lemma 4.3).
2. Krull monoids (hence Krull domains) with divisor class group \(G\) such that \(\mathcal{D}(G) < \infty\) where \(G_0 \subseteq G\) is the set of classes containing prime divisors (by Lemma 4.3 and Proposition 4.5).
3. Weakly Krull monoids with finite class groups and weak divisor theory \(\varphi: H \rightarrow \bigoplus_{i \in \mathbb{N}} D_i\) where all \(D_i\) are finitely primary of some exponent \(\alpha\) (by Proposition 4.7 and [Ge5, Theorem 5.4]). Besides Krull monoids this class includes the following monoids (cf. [Ge5, Sect. 7]). Let \(R\) be a noetherian weakly Krull domain whose integral closure is a finite \(R\)-module. Then the monoid \(\mathcal{J}(R)\) of integral \(i\)-invertible \(i\)-ideals belongs to the above class. So does the multiplicative monoid of the domain, if the \(i\)-class group \(\mathcal{C}_i(R)\) is finite. In particular, all one-dimensional noetherian domains \(R\) are weakly Krull. In this case \(\mathcal{J}(R)\) is just the monoid of integral invertible ideals and \(\mathcal{C}_i(R)\) coincides with the Picard group \(\text{Pic}(R)\). Hence Theorem 5.1 applies to orders in global fields.
Remarks. (1) Suppose that $H$ has a norm function such that abstract analytic number theory can be done on $H$. Then an algebraic result of the form

all multiples of some fixed element satisfy some property ($P$)

implies a quantitative result of the form

almost all elements (in a sense of density) satisfy property ($P$)

(cf. [Ge6]).

(2) Let $H$ satisfy the assumptions of Theorem 5.1. Furthermore, suppose that $H$ is a saturated submonoid of some monoid $D$. Let $\mathcal{B}$ denote the associated block monoid and $\beta$ the block homomorphism as defined in Section 3. By [Ge5, Sect. 4] it is sufficient to show the assertion of Theorem 5.1 for the block monoid $\mathcal{B}$. Obviously, $\mathcal{B}$ is locally tame and has finite catenary degree. Hence there exists some $a^* \in H$ such that for all $a \in H$ with $\beta(a^*) \mid \beta(a)$ the assertion of Theorem 5.1 holds.

(3) There are monoids satisfying the assertion of Theorem 5.1 and for which moreover,

$$|\sigma(z_{-1}) - \sigma(z_0)| = |\sigma(z_k) - \sigma(z_{k+1})| \in \{0, \min \Delta(H)\} \quad (\ast)$$

holds. For example, Krull monoids with finite divisor class group have this property, if each class contains a prime divisor (cf. [Ge6, Theorem 3.1] see also [Ge3, Theorem 2]). In general however, ($\ast$) does not hold as the following two examples show. For both block monoids $\mathcal{B}(G_0)$ discussed below the pairs $(G, G_0)$ are realizable (cf. the example in Section 3).

(i) Let $G = \mathbb{Z}/8\mathbb{Z}$, $G_0 = (1 + 8\mathbb{Z}, 5 = 5 + 8\mathbb{Z})$, and $H = \mathcal{B}(G_0)$. Since $\mathcal{B}(G_0)$ is finitely generated, the assertion of Theorem 5.1 holds. We verify that for all $a^* \in H$ there exists some $a \in H$ and $a^* \mid a$ such that $\min L(a) + \min \Delta(H) \neq L(a)$. Clearly,

$$U_1 = 1^3, \quad U_2 = 5^5, \quad U_3 = 5^51, \quad \text{and} \quad U_4 = 5^51^3$$

are just the irreducible elements of $H$. Since

$$U_1U_3 = U_4^3,$$

it follows that $1 = \min \Delta(H)$. For $n \in \mathbb{N}_+$ we set $B_n = (U_1U_2)^n$. For all $a^* \in H$ there is some $n \in \mathbb{N}_+$ such that $a^* \mid B_n$. Hence it is sufficient to show that for every $n \in \mathbb{N}_+$

$$\min L(B_n) + 1 \neq L(B_n).$$
Let \( n \in \mathbb{N}_+ \). Since \( \sigma(B_n) = 16n \) and \( \mathcal{D}(G_0) = 8 \), we infer that \( \min L(B_n) = 2n \). Setting

\[
B_n = \prod_{i=1}^{4} U_i^{k_i},
\]

we obtain \( 8n = 8k_1 + k_3 + 3k_4 = 8k_2 + 3k_3 + k_4 \), and hence \( \sum_{i=1}^{4} k_i - 2n = \frac{1}{2} (k_3 + k_4) \). Since \( k_3 + 3k_4 \equiv 0 \mod(8) \), it follows that

\[
\sum_{i=1}^{4} k_i - \min L(B_n) = \frac{1}{2} (k_3 + k_4) \neq 1.
\]

(ii) Let \( G = \mathbb{Z}/7\mathbb{Z}, \ G_0 = \{ \bar{1} = 1 + 7\mathbb{Z}, \bar{3} = 3 + 7\mathbb{Z}, \bar{6} = 6 + 7\mathbb{Z} \}, \) and \( H = \mathcal{D}(G_0) \). We show that for all \( a^* \in H \) there exists some \( a \in H \) with \( a^* \mid a \) such that \( \max L(a) - \min \Delta(H) \notin L(a) \).

\( H \) has the following nine irreducible elements:

\[
U_1 = \bar{1}, \quad U_2 = \bar{16}, \quad U_3 = \bar{3}^2 \bar{1}, \quad U_4 = \bar{3}^4 \bar{1}, \quad U_5 = \bar{3}^7,
\]

\[
U_6 = \bar{3}^3 \bar{6}, \quad U_7 = \bar{3}^3 \bar{6}^2, \quad U_8 = \bar{3}^6 \bar{3}, \quad U_9 = \bar{6}^7.
\]

For \( B = \bar{1}^{k_1} \bar{3}^{k_2} \bar{6}^{k_3} \in H \) we set

\[
S(B) = k_1 + 3k_2 + 6k_3 \in 7\mathbb{N}.
\]

By [Ge0, Proposition 7] it follows that

\[
\min \Delta(H) = \gcd\{\frac{1}{2} S(B) - 1 \mid B \in \mathcal{D}(H)\} = \gcd\{0, 2, 5\} = 1.
\]

Let \( N \in \mathbb{N}_+ \); we set

\[
B_n = (U_1 U_2 U_3 U_4)^n.
\]

Since \( S(B_n) = 28n \), we infer that \( \max L(B_n) = 4n \).

It is sufficient to show that \( 4n - 1 \notin L(B_n) \). Assume to the contrary, that

\[
B_n = \left( \prod_{i=1}^{k_1} A_i \right) \left( \prod_{j=1}^{k_2} C_j \right) \left( \prod_{i=1}^{k_3} D_i \right)
\]

with \( A_i, C_j, D_i \) irreducible, \( S(A_i) = 7, S(C_j) = 21, S(D_i) = 42 \), and \( k_1 + k_2 + k_3 = 4n - 1 \). Then

\[
S(B_n) = 28n = 7(k_1 + 3k_2 + 6k_3)
\]

and hence \( 1 = 2k_2 + 5k_3 \) with \( k_2, k_3 \in \mathbb{N} \), a contradiction.
For the proof of Theorem 5.1 we need the following lemma.

**Lemma 5.3.** Let H be an atomic monoid. For every \( k \in \mathbb{N}_+ \) there exists an element \( a_k \in H \) and factorizations \( x_0, x_1, \ldots, x_k \in \mathcal{Z}(a_k) \) such that \( \sigma(x_v) - \sigma(x_{v-1}) = \min \Delta(H) \) for \( 1 \leq v \leq k \).

**Proof.** Let \( k \in \mathbb{N}_+ \), \( d = \min \Delta(H) \), and \( b \in H \) such that \( m, m + d \in L(b) \) for some \( m \in \mathbb{N}_+ \). We choose two factorizations \( z, z' \in \mathcal{Z}(b) \) with \( \sigma(z) = m \) and \( \sigma(z') = m + d \). Then we set \( a_k = b^k \) and \( x_v = z^{k-v} z''^v \) for all \( 0 \leq v \leq k \). Obviously, \( x_v \in \mathcal{Z}(a_k) \) and \( \sigma(x_v) = (k - v)m + v(m + d) = km + v^2 \) for \( 0 \leq v \leq k \), which implies the assertion.

**Proof of Theorem 5.1.** By Lemma 5.3 there exists an element \( a^* \in H \) having factorizations \( x_0, x_1, \ldots, x_{c(H)} \in \mathcal{Z}(a^*) \) with \( \sigma(x_v) - \sigma(x_{v-1}) = d = \min \Delta(H) \) for \( 1 \leq v \leq c(H) \).

Let \( a \in H \) with \( a^* \mid a \). We set \( a = a^*b \) and

\[
Z = \{ x, y \in \mathcal{Z}(a) \mid 0 \leq v \leq c(H), y \in \mathcal{Z}(b) \} \subseteq \mathcal{Z}(a).
\]

Then for every \( z \in \mathcal{Z}(a) \) there exists some \( y \in \mathcal{Z}(b) \) such that \( d(z, x_0y) \leq t_{1n}(H, x_0) \). Since \( H \) is locally tame, \( t_{1n}(H, x_0) < \infty \) by Proposition 4.2.

Hence it remains to verify that for each two \( z, z' \in Z \) there is a chain of factorizations \( z = z_0, z_1, \ldots, z_k = z' \in Z \) with \( d(z_{i-1}, z_i) \leq \max L(a^*) + c(H) \) and \( |\sigma(z_{i-1}) - \sigma(z_i)| \in \{0, d\} \) for all \( 1 \leq i \leq k \).

Let \( z = x_\mu y \in Z \) and \( z' = x_\nu y' \in Z \) be given. By construction of \( x_0, x_1, \ldots, x_{c(H)} \) there is a chain from \( z \) to \( x_0y \) and a chain from \( z' \) to \( x_0y' \) satisfying the required conditions.

There is also a chain of factorizations \( (y_i)_{i=0}^{k} \) from \( y = y_0 \) to \( y_k = y' \in \mathcal{Z}(b) \) with \( d(y_{i-1}, y_i) \leq c(H) \) for all \( 1 \leq i \leq k \). Hence it remains to find a chain of factorizations from \( x_0y_{i-1} \) to \( x_0y_i \) for \( 1 \leq i \leq k \) which satisfies the above properties. Changing notation we set \( z = x_\mu y, z' = x_\nu y' \) with \( d(y, y') \leq c(H) \).

Without restriction we may suppose that \( \sigma(y') - \sigma(y) = ld \leq c(H) \) for some \( l \in \mathbb{N} \). We define \( z_v = x_vy \) for \( 0 \leq v \leq l \). Then

\[
d(z_{v-1}, z_v) = d(x_{v-1}, x_v) \leq \max L(a^*)
\]

and

\[
|\sigma(z_{v-1}) - \sigma(z_v)| = |\sigma(x_{v-1}) - \sigma(x_v)| = d.
\]

Finally, we infer that

\[
d(z, z') = d(x_\mu y, x_\nu y') \leq d(x_\mu y_0, x_\nu y) + d(y, y') \leq \max L(a^*) + c(H)
\]

and

\[
\sigma(z) - \sigma(z') = \sigma(x_\mu) - \sigma(x_\nu) + \sigma(y) - \sigma(y') = 0.
\]
6. FACTORIZATIONS IN WEAKLY KRULL MONOIDS

We present the main result of this paper. We start with the local case whose proof rests on Theorem 5.1.

**Theorem 6.1.** Let $H \subseteq D$ be a saturated submonoid of a finitely primary monoid $D$ with finite class group. Then there exists some constant $C \in \mathbb{N}_+$ such that for all $a \in H$ and all $z, z' \in \mathcal{Z}(a)$ there is a chain of factorization $(z_i)_{i=1}^{k+1}$ with $z_{-1} = z$, $z_{k+1} = z'$ satisfying the following two properties:

(i) $d(z_{i-1}, z_i) \leq C$ for $0 \leq i \leq k + 1$.

(ii) $|\sigma(z_i) - \sigma(z_{i-1})| \in \{0, \min \Delta(H)\}$ for $1 \leq i \leq k$.

**Proof.** By Proposition 4.7, $H$ is locally tame and by [Ge5, Theorem 5.4] it has finite catenary degree. Hence Theorem 5.1 implies the existence of some $a \in H$ and some $C \in \mathbb{N}_+$ such that the assertion holds for all $z, z' \in \mathcal{Z}(a)$.

Let $a \in H$ such that $a^s \not\mid a$ in $H$. Then $a^s \not\mid a$ in $D$. Hence there exists some $\nu \in \{1, \ldots, s\}$ such that

$$v_{p_\nu}(a) < v_{p_\nu}(a^s) + \alpha.$$ 

Hence

$$\max L_H(a) \leq \max L_D(a) \leq \min \{v_{p_\nu}(a) \mid 1 \leq \nu \leq s\}$$

$$\leq \max \{v_{p_\nu}(a^s) \mid 1 \leq \nu \leq s\} + \alpha = C''.$$

Thus for all $z, z' \in \mathcal{Z}(a)$

$$d(z, z') \leq \max L_H(a) \leq C''.$$

Therefore the assertion holds for $C = \max(C', C'')$. □

**Theorem 6.2.** Let $H \subseteq D = \bigcup_{i \in \Omega} D_i$ be a saturated submonoid with finite class group $G$. Suppose that all $D_i$ are finitely primary and that $(D_i)_{\text{red}} = (\mathbb{N}_+, \cdot)$ for all but finitely many $i \in \Omega$.

Then there exists some constant $C \in \mathbb{N}_+$ such that for all $a \in H$ and all $z, z' \in \mathcal{Z}(a)$ there is a chain of factorizations $(z_i)_{i=1}^{k+1}$ from $z = z_{-1}$ to $z' = z_{k+1}$ satisfying the following two properties:

(i) $d(z_{i-1}, z_i) \leq C$ for $0 \leq i \leq k + 1$.

(ii) there exist integers $0 = \delta_0 < \delta_1 < \cdots < \delta_k < d$ with $d \in \Delta(H)$, and some $m \in \mathbb{N}_+$ such that $L = \{\sigma(z_i) \mid 0 \leq i \leq k\} = \{\min L + \delta_i + \lambda d \mid 0 \leq \nu \leq \mu, 0 \leq \lambda \leq \kappa, \nu = 0 \text{ if } \lambda = \kappa\}$. Furthermore, if $y \in \mathcal{Z}(a)$ with $\min L \leq \sigma(y) \leq \max L$, then $\sigma(y) \in L$. 


Corollary 6.3. Let $H$ be an atomic monoid satisfying the assertion of Theorem 6.2. Then for every $a \in H$

$$L(a) = \{l_1, \ldots, l_a, m, m + \delta_1, \ldots, m + \delta_n, m + d, \ldots, m + (\kappa - 1)d + \delta_{1}, \ldots, m + \kappa d, n_1, \ldots, n_{\beta}\}$$

with $l_1 < \cdots < l_a < m < m + \delta_1 < \cdots < m + \delta_n < m + d \leq m + \kappa d < n_1 < \cdots < n_{\beta}, 0 \leq \mu < d \in \Delta(H), 0 \leq \alpha \leq C - 2$, and $0 \leq \beta \leq C - 2$.

Proof. Use Theorem 6.2 and argue as in Corollary 5.2.

Examples and Remarks. (1) Theorem 6.2 applies to noetherian weakly Krull domains $R$ with finite $\tau$-class group and whose integral closure is a finite $R$-module (cf. Example 3 in Section 5).

(2) Let $R$ be a one-dimensional local noetherian domain whose integral closure is a finite $R$-module. Then its multiplicative monoid $R^\times$ is finitely primary (cf. [Ge4, Sect. 5]). Note that the assertion of Theorem 6.1 is just the assertion of Theorem 6.2 with $\mu = 0$. So in the local case sets of lengths are extremely simple. In general, $\mu > 0$ might happen, as can be seen from the examples in [Ge8].

(3) The structure theorem for sets of lengths, as formulated in Corollary 6.3, was first proved for Krull monoids with finite class group in [Ge1, Satz 1] and then for finitely generated monoids in [Ge2, Theorem 1]. For a structure theorem for generalized sets of lengths see [HK4].

(4) In Section 3 we gave an example of a half-factorial Dedekind domain with infinite catenary degree. So it satisfies the assertion of Corollary 6.3 but not the assertion of Theorem 6.2.

We start with the preparation for the proof of Theorem 6.2. First of all, suppose without restriction that $H$ and $D$ are reduced. By [Ge5, Proposition 4.2] it is sufficient to prove the result for the associated block monoid $\mathcal{B}$. $\mathcal{B}$ is a saturated submonoid of a finite product of finitely primary monoids with class group (isomorphic to) $G$ (cf. [Ge3, Proposition 1]).

Changing notation we suppose that

$$H \subseteq D = \prod_{i \in \Omega} D_i$$

is a saturated submonoid with finite class group $G$, all $D_i$ are finitely primary of exponent $\alpha \in \mathbb{N}$, and $\Omega$ is finite. By [Ge5, Theorem 5.4], $H$ has finite catenary degree $c(H)$. We use all conventions fixed in Section 2.
Let \( I \subseteq \Omega \) be a subset. Then \( H_I = H \cap D_I \subseteq H \) is a divisor closed submonoid and all assertions of Lemma 3.2 hold. In particular, we have
\[
c(H_I) = c_{H_I}(H_I) = c_H(H_I) \leq c_H(H).
\]
Furthermore, we set
\[
d_I = \min \Delta(H_I)
\]
with \( d_I = 0 \) if \( \Delta(H_I) = \emptyset \).

**Lemma 6.4.** For all subsets \( I \subseteq \Omega \) there exist elements \( a(I) \in H_I \) having the following properties:

(i) there are factorizations \( x_1, \ldots, x_m \in \mathcal{P}_H(a(I)) \) with \( \sigma(x_v) - \sigma(x_{v-1}) = d_I \) for \( 1 \leq v \leq m = c(H_I) \);

(ii) if \( \Delta(H_I) = \emptyset \), then \( a(I) = 1 \);

(iii) if \( J \subseteq I \), then \( a(J) \mid a(I) \).

**Proof.** Let \( I \subseteq \Omega \) be a subset. By Lemma 3.2 we have \( \mathcal{P}_H(a) = \mathcal{P}_H(a) \) for every \( a \in H_I \). Therefore, by Lemma 5.3 there are elements \( a'(I) \) satisfying (i) and (ii).

If \( \Delta(H_I) = \emptyset \), then \( \Delta(H_I) = \emptyset \) for every \( J \subseteq I \). If \( \Delta(H_I) \neq \emptyset \) and \( a \in H_I \) satisfies (i), then each multiple \( ab \in H_I \) satisfies (i) too. Hence we obtain elements \( a(I) \in H_I \) for all \( I \subseteq \Omega \) having properties (i), (ii), and (iii).

Let \( \Omega = I \cup J \). For a subset \( H' \subseteq H \) we set
\[
\mu(I, H') = \max \{ \mu \in \mathbb{N} \mid \text{there are } a \in H', x \in L_H(a) \text{ and integers } 0 < \delta_1 < \cdots < \delta_\mu < d_I \text{ such that } \{x, x + \delta_1, \ldots, x + \delta_\mu, x + d_I\} \subseteq L_H(a) \}.
\]
Note that we use the convention \( \max \emptyset = 0 \). Hence we have
\[
0 \leq \mu(I, H') \leq d_I.
\]
Let \( c \in D_I \); we set
\[
H(I, c) = \{a \in H \mid a_I = c; \text{ there is some } x \in L_H(a) \text{ such that } x + d_I \in L_H(a)\}.
\]
Suppose there is some \( a \in H \) with \( a_I = c \). Then for every \( b \in aH_I \) we have \( b_I = c \) and hence \( H(I, c) \neq \emptyset \).
Lemma 6.5. Let $\Omega = I \cup J$ and $M \in \mathbb{N}_+$. Then there exists a finite set $B(I, M) \subseteq D_I$ such that for all $c \in D_{I, M}$ with $H(I, c) \neq \emptyset$ there is some $b \in B(I, M)$ such that $a = bc \in H$ and $\mu(I, a) = \mu(I, H(I, c))$.

We set

$$b(I, M) = \prod_{b \in B(I, M)} b.$$ 

Proof. We divide the proof into four steps.

1. For an element $x = \prod_{i=1}^k u_i \in \mathcal{Z}(D)$ we set $[x] = \prod_{i=1}^k [u_i] \in \mathcal{A}(G)$. Let $c \in D$. For every $z \in \mathcal{Z}_D(c)$ we consider product decompositions

$$z = \prod_{\lambda=1}^k v_{\lambda} \prod_{\lambda=1}^m w_{\lambda}$$

having the following two properties:

(i) $v_{\lambda} \in \mathcal{H}(H)$ (thus Lemma 4.4 implies that $\max \mathcal{L}_D(v_{\lambda}) \leq \mathcal{D}(G)$);

(ii) there is no $u \in \mathcal{H}(H)$ such that $u \mid w_{\lambda}$ (this implies that $\max \mathcal{L}_D(w_{\lambda}) < \mathcal{D}(G)$).

Such a decomposition gives rise to the tuple

$$(k, \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_1)] \}, \ldots, \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_m)] \}). \quad (**)$$

Let $T(c)$ denote the set of all tuples $(**)$ arising from product decompositions $(*)$.

2. Let $b \in D_I$ and $c \in D_J$ such that $bc \in H$. Then $L_H(bc)$ just depends on $T(b)$ and $T(c)$. Because, $L_H(bc)$ is the set of all $k + k' + m \in \mathbb{N}_+$ for which there exist tuples

$$(k, \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_1)] \}, \ldots, \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_m)] \}) \in T(b)$$

and

$$(k', \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_1')] \}, \ldots, \{ [[x] \in \mathcal{A}(G) \mid x \in \mathcal{Z}_D(w_m')] \}) \in T(c)$$
having the following property: there exists a bijection \( \varphi : \{1, \ldots, m\} \rightarrow \{1, \ldots, m'\} \) (hence \( m = m' \)) such that for all \( 1 \leq i \leq m \), all \( x \in \mathcal{Z}_D(w_i) \) and all \( x' \in \mathcal{Z}_D(w_{\varphi(i)}') \) \([x] \in \mathcal{U}(\mathcal{D}(G))\).

3. Let \( c, c' \in D \) with \( T(c) = T(c') \) and \( b \in D \) such that \( bc \in H \). Then \( bc' \in H \) and (2) implies that \( L_H(bc) = L_H(bc') \). Hence \( \mu(I, H(I, c)) = \mu(I, H(I, c')) \).

4. We complete the proof of Lemma 6.5. Let \( c \in D_{I, M} \). Then
\[
\max L_D(c) \leq \#J.M.
\]
For a tuple of the form \((***)\) in \( T(c) \) it follows that

(i) \( k + m \leq \max L_D(c) \leq \#J.M; \)

(ii) \( \#[x] \in \mathcal{U}(G) | x \in \mathcal{Z}_D(w_i) \) \( < \#G \) for \( 1 \leq i \leq m \).

Therefore, the set
\[
\mathcal{F} = \{ T(c) | c \in D_{I, M}, H(I, c) \neq \emptyset \}
\]
is finite, say \( \mathcal{F} = \{ T(c_1) | 1 \leq \lambda \leq r, c_\lambda \in D_{I, M} \} \).

For every \( c_\lambda \) there is some \( b_\lambda \in D_I \) such that \( b_\lambda c_\lambda \in H \) and \( \mu(I, b_\lambda c_\lambda) = \mu(I, H(I, c_\lambda)) \). We define
\[
B(I, M) = \{ b_\lambda \in D_I | 1 \leq \lambda \leq r \}.
\]
Now let \( c \in D_{I, M} \) be given with \( H(I, c) \neq \emptyset \). Then there exists some \( \lambda \in \{1, \ldots, r\} \) such that \( T(c) = T(c_\lambda) \). Since \( b_\lambda c_\lambda \in H \), it follows that \( b_\lambda c \in H \) and
\[
\mu(I, b_\lambda c) = \mu(I, b_\lambda c_\lambda) = \mu(I, H(I, c_\lambda)) = \mu(I, H(I, c)).
\]

**Lemma 6.6.** There exists a constant \( M^* \in \mathbb{N}_+ \) such that for all \( a \in H \) the following holds: there exist \( I, J \subseteq \Omega \) with \( \Omega = I \cup J \) and some \( M \in \mathbb{N}_+ \) with \( M \leq M^* \) such that \( a = a_I a_J, a(I)b(I, M) | a_I \), and \( a_J \in D_{I, M} \).

**Proof.** First of all we define \( M^* \). Let \( (I_v)_{v=1}^k \) be a strictly descending sequence of subsets of \( \Omega \), i.e.,
\[
\Omega \supseteq I_1 \supset I_2 \supset \cdots \supset I_k \supset \emptyset.
\]
We set
\[
M_1 = \max \{ v_{p_i}^n(a(\Omega)) | 1 \leq v \leq s_i, i \in \Omega \} + \alpha.
\]
Lemma 6.4 implies that
\[
M_1 \geq \max \{ v_{p_i}^n(a(I_1)) | 1 \leq v \leq s_i, i \in I_1 \} + \alpha.
\]
Suppose that all \( M_\nu \) are defined for \( 1 \leq \nu \leq l < k \). Then let \( M_{l+1} \) be the maximum of \( M_1, \ldots, M_l \) and of

\[
\alpha + \max\{v_{p,a}(a(I_\nu) b(I_\nu, M_\nu)) \mid 1 \leq \nu \leq s, i \in I_\nu \setminus I_{l+1}\}.
\]

Let \( M^* \) be defined as the maximum of all \( M_k \) over all strictly descending sequences \( (I_\nu)_{\nu=1}^k \).

Let \( a \in H \) be given. We construct a strictly descending sequence of subsets \( (I_\nu)_{\nu=1}^k \) with \( a(I_\nu) \mid a_{I_\nu} \) and \( a_{I_\nu} \in D_{J_\nu, M_\nu} \) where \( J_\nu = \Omega \setminus I_\nu \) for \( 1 \leq \nu \leq k \). Furthermore, \( a(I_k) b(I_k, M_k) \mid a_{I_k} \). Hence the assertion holds for \( I = I_k \) and \( M = M_k \).

Let

\[
\Lambda = \{\Gamma \subseteq \Omega \mid a(\Gamma) \mid a_\Gamma \text{ and } a_\Gamma \notin D_{I, M_1} \text{ for all } i \in \Gamma\}.
\]

Since \( a(\emptyset) = 1 \), \( \Lambda \) is non-empty and hence there is a maximal set \( I_1 \in \Lambda \). We have to verify that \( a_{I_1} \in D_{J_1, M_1} \). Hence we have to show that \( a_j \in D_{J, M_1} \) for all \( j \in J_1 \). Let \( j \in J_1 \) and assume to the contrary that \( a_j \notin D_{J, M_1} \). By the maximality of \( I_1 \) it follows that

\[
a(I_1 \cup \{j\}) + a_{I_1 \cup \{j\}}.
\]

Hence there exists some \( k \in I_1 \cup \{j\} \) with

\[
a(I_1 \cup \{j\}) + a_k,
\]

and thus for some \( \nu \in \{1, \ldots, s_k\} \)

\[
v_{p,a}(a) \leq v_{p,a}(a(I_1 \cup \{j\})) + \alpha.
\]

Therefore,

\[
\min\{v_{p,a}(a) \mid 1 \leq \nu \leq s_k\}
\]

\[
\leq \max\{v_{p,a}(a(I_1 \cup \{j\})) \mid 1 \leq \nu \leq s_k\} + \alpha \leq M_1,
\]

i.e., \( a_k \in D_{J, M_1} \), a contradiction.

Let \( l \geq 1 \) and suppose that \( I_1, \ldots, I_l \) are already constructed. If \( a(I_\nu) b(I_\nu, M_\nu) \mid a_{I_\nu} \) we set \( k = l, I = I_l \), and we are done. So suppose that

\[
a(I_l) b(I_l, M_l) + a_{I_l}.
\]

Then

\[
\Lambda = \{i \in I_l \mid (a(I_l) b(I_l, M_l)) + a_i\} \neq \emptyset.
\]
and hence
\[ I_{i+1} = I_i \setminus \Lambda \subseteq I_i. \]

For every \( i \in \Lambda \) there is some \( \nu \in \{1, \ldots, s_i\} \) such that
\[ v_{p_{\nu,i}}(a) \leq v_{p_{\nu,i}}(a(I_i)b(I_i,M_i)) + \alpha \]
and thus
\[
\min\{v_{p_{\nu,i}}(a) \mid 1 \leq \nu \leq s_i\} \\
\leq \max\{v_{p_{\nu,i}}(a(I_i)b(I_i,M_i)) \mid 1 \leq \nu \leq s_i\} + \alpha \leq M_{i+1}.
\]

This means that \( a_\Lambda \in D_{\Lambda,M_{i+1}}. \) Since \( a_{I_i} \in D_{I_i,M_i} \subseteq D_{I_i,M_{i+1}} \) and \( J_{i+1} = J_i \cup \Lambda, \) we infer that
\[ a_{J_{i+1}} \in D_{J_{i+1},M_{i+1}}. \]

By Lemma 6.4 it follows that \( a(I_{i+1}) \mid a(I_i). \) Since \( a(I_i) \mid a_{I_i} \) it follows that \( a(I_{i+1}) \mid a_{I_i}. \)

**Lemma 6.7.** There exists a constant \( M^* \in \mathbb{N}_+ \) such that for all \( a \in H \) the following holds: there exist \( I, J \subseteq \Omega \) with \( \Omega = I \cup J \) and some \( M \in \mathbb{N}_+ \) with \( M \leq M^* \) such that \( a = a'a'' \), \( a' \in H_I, \) \( a'' \in H_J, \) \( a' \mid a'' \), \( a' = a'_Ia'_J \in H, \) \( a_I = a'_I \in D_{I,M}, \) \( a'_I \mid b(I,M), \) and \( \mu(I,a'') = \mu(I,H(I,a'_J)). \)

**Proof.** Let \( M^* \) be the constant from Lemma 6.6 and let \( a \in H \) be given. We consider a product decomposition of a
\[ a = a_Ia_J \]
with \( I, J, a_I, a_J \) and \( M \leq M^* \) having the properties of Lemma 6.6. Since \( H(I, a_J) \neq \emptyset, \) Lemma 6.5 implies the existence of an element \( a_I' \in B(I,M) \subseteq D_I \) with \( a_I' \mid b(I,M) \mid a_J \) such that \( a_I'a_J \in H \) and \( \mu(I,a_I'a_J) = \mu(I,H(I,a_J)). \)

Then we have \( a_I = a(I)a'Ic \) for some \( c \in D_I. \) We set \( a' = a(I)c \) and \( a'' = a'Ia_J. \) Since \( a = a'a'' \in H \) and \( a'' \in H, \) it follows that \( a' \in H. \)

**Proof of Theorem 6.2.** Let \( a \in H \) and \( z, z' \in \mathcal{Z}_H(a) \) be given. Our starting point is the product decomposition of \( a \) derived in Lemma 6.7. So let \( I, J, M, a', a'' \) be as in Lemma 6.7. Then
\[ a = ba(I)a'' \]
for some \( b' \in H_J. \) By construction of \( a(I) \) and \( a'' \) there exists factorizations
\[ x_0, \ldots, x_n \in \mathcal{Z}_H(a(I)a'') \]
such that
\[ \sigma(x_{(\mu+1)\ell+\nu}) = \sigma(x_{\ell}) + \lambda d_I + \delta_{\nu} \]
with \(0 \leq \nu \leq \mu\), \(0 = \delta_0 < \delta_1 < \cdots < \delta_\mu < d_I\), \(\mu = \mu(I, a^\ast) = \mu(I, H(I, a_j)), n = m(\mu + 1), 0 \leq \lambda \leq m = c(H),\) and \(\lambda = m\) implies \(\nu = 0\).

Since \(a^\ast_I|b(I, M)\), every irreducible element \(u \in \mathcal{A}(H)\) appearing in some \(x_u\) lies in the set

\[
U = \{u \in \mathcal{A}(H)| u_I| a(I)b(I, M)\}.
\]

Since \((a(I)a^\ast)_I = a^\ast_I \in D_{I, M}\), there is an upper bound for \(\max L_H(a(I)a^\ast)\) which just depends on \(M\) and on \(a(I)b(I, M)\). We define

\[
H' = \{c \in H| c_j \in D_{I, M}\}.
\]

Clearly, \(a \in H'.\) By definition of local tameness and by Proposition 4.2.3 there exists a factorization \(x_0y \in \mathcal{Z}_H(a)\) with \(y \in \mathcal{Z}_H(b)\) such that

\[
d(z, x_0y) \leq t_H(H', x_0) \leq \sigma(x_0)t_H(H', U).
\]

Proposition 4.7 gives an upper bound for \(\sigma(H', U)\) which depends on \(H, \alpha, G,\) and on \(a(I)b(I, M)\). By a similar argument we obtain a factorization \(x_ny' \in \mathcal{Z}_H(a)\) such that

\[
d(z', x_ny') \leq \sigma(x_n)t_H(H', U).
\]

Next we construct a chain of factorizations from \(z_0 = x_0y\) to \(z_k = x_ny'\) satisfying properties (i) and (ii) of Theorem 6.2. Since \(H\) has finite catenary degree \(c(H)\) (see [Ge5, Theorem 5.4]), there is a chain \((y_i)^{i=0}_{i=k}\) of factorizations of \(b \in H\) from \(y = y_0\) to \(y' = y_k\) with \(d(y_{i-1}, y_i) \leq c(H)\) for \(1 \leq i \leq k\).

We define the wanted sequence \((z_i)^{i=0}_{i=k}\) by

\[
\begin{align*}
z_0 &= x_0y_0, x_1y_0, \ldots, x_ny_0, x_ny_1, \ldots, x_0y_1, \\
x_0y_2, x_1y_2, \ldots, x_ny_l &= x_ny' = z_k = z_{(l+1)(n+1)-1}.
\end{align*}
\]

Obviously, we obtain

\[
d(z_{i-1}, z_i) \leq \max\{\sigma(x_0), \ldots, \sigma(x_n), c(H)\}.
\]

Set

\[
C(I) = \max\{\max L_H(a(I)a^\ast)t_H(H', U), \max L_H(a(I)a^\ast), c(H)\};
\]

then assertion (i) holds for \(C = \max(C(I)| I \subseteq \Omega)\).

To verify the assertion on the lengths we set

\[
L = \{\sigma(z_i) | 0 \leq i \leq (l + 1)(n + 1) - 1 = k \}
= \{\sigma(x_i) + \sigma(y_i) | 0 \leq i \leq n, 0 \leq j \leq l\}.
\]
It follows that
\[ L = \{ \sigma(x_0) + \delta_v + \lambda \alpha_l + \sigma(y_j) \mid 0 \leq j \leq l, 0 \leq \mu, 0 \leq \rho \leq m, \lambda = m \text{ implies } \nu = 0 \}. \]

Since for every \( 1 \leq j \leq l \) we have \( d_j \mid \sigma(y_j) - \sigma(y_{j-1}) \) and \( |\sigma(y_j) - \sigma(y_{j-1})| \leq c(H_t) \leq m d_j \), we obtain that
\[ L = \{ \min L + \delta_v + \lambda \alpha_l \mid 0 \leq \mu, 0 \leq \rho \leq \kappa, \lambda = \kappa \text{ implies } \nu = 0 \} \]
for some \( \kappa \in \mathbb{N}_+ \).

By definition of \( \mu(I, H(I, a)), \sigma(y) \in L \) for every \( y \in \mathcal{Z}_H(a) \) with \( \min L \leq \sigma(y) \leq \max L \).


