Fuzzy Inf. Eng. (2016) 8: 199–216

HOSTED BY

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Information and Engineering

http://www.elsevier.com/locate/fiae

Fuzzy Information and Engineering

An International Journal

ORIGINAL ARTICLE

Weak Coupled Coincidence Point Results Having a Partially Ordering in Fuzzy Metric Spaces

CrossMark

P. Saha · Binayak S. Choudhury · Pradyut Das

Received: 19 September 2014/ Revised: 20 August 2015/
Accepted: 29 December 2015/

Abstract Coupled coincidence and fixed point problems have been in the focus of the research interest for last few years. The problem was introduced in fuzzy metric spaces only very recently. In this paper, we work out a weak coupled coincidence point theorem for a compatible pair of mappings in fuzzy metric spaces. As one of the corollaries we have a weak coupled fuzzy contraction mapping theorem. The space is endowed with a partial ordering. We use a combination of analytic and order theoretic concepts in the proof of our main theorem. The result is illustrated with an example.

Keywords Fuzzy metric space · Coupled fixed point · Coupled coincidence point · Compatible mappings.

1. Introduction

Fuzzy metric spaces have been introduced in various ways by several authors over the years, for examples, in the works noted in [16, 18, 22]. Metric fixed point theory has

P. Saha (✉) · Binayak S. Choudhury (✉) · Corresponding Author: Pradyut Das (✉)
Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, West Bengal, India
email: parbati_saha@yahoo.co.in
binayak12@yahoo.co.in
pradyutdas747@gmail.com
Peer review under responsibility of Fuzzy Information and Engineering Branch of the Operations Research Society of China.

http://dx.doi.org/10.1016/j.fiae.2016.06.005

been extended to several of these spaces amongst which it has developed in a large way on the fuzzy metric space introduced by George et al. [16] who defined a fuzzy metric space by modifying the definition given by Kramosil et al. [24]. The motivation of George et al. [16] was to ensure a Hausdroff topology of the space and this is one of the reasons why the metric fixed point theory has very successfully developed in this space. References [14, 17, 28, 29, 35, 44] are instances of some important works on fixed point and related topics in the context of fuzzy metric spaces as defined by George and Veeramani [16].

In this paper our aim is to establish some coupled coincidence point results in fuzzy metric spaces described by George and Veeramani [16]. The idea of coupled fixed points was introduced by Guo et al. [19] in 1987. Later Bhaskar et al. [3] established a coupled contraction mapping theorem in partially ordered metric spaces in 2006 which was followed by a large number of papers dealing with coupled fixed and coincidence point theorems [5, 23, 26, 30, 39]. Such problems are also addressed in the settings of several other spaces which are generalizations of metric spaces, like that in cone metric spaces [23], G-metric spaces [6], probabilistic metric spaces [12], partial metric spaces [38, 40] etc.

In metric fixed point theory, there is an effort over the years to extend and generalize the Banach's contraction mapping principle. References [2, 4, 27, 42] are some examples from the large literature existing in this line of research. One such generalization is the weak contraction principle which introduces a new contraction intermediate to the Banach's contraction and the non-expansive mapping. It was first proved in Hilbert spaces by Alber et al. [1] and was adapted to metric spaces by Rhoades [37]. Using the same idea behind such weak contractions many theorems were established in a large number of works, not all of which are extensions of Banach's theorem. In fact they provide us with a much larger class of contractions, called weak contractions, in metric fixed point theory. Some instances of these works are in [7, 15, 33, 45]. Coupled weak contraction results also appeared in works like [10, 11, 13]. Many of the above mentioned results are established in partially ordered metric spaces. In fact, the result of Bhaskar et al. [3] was established in such spaces. Fixed point theory in partially ordered metric spaces has begun to develop recently in the first decade of the present century, although the initial result in this line of study was first established in the work Turinici [43] in 1986. Some instances of this works are in [31, 32, 34]. One of reasons for the widespread interest in these problems is the blending of the analytic and order theoretic approaches in the proofs of the associated theorems.

In fuzzy metric spaces, coupled fixed point theorem was proved successively by Zhu et al. [46] which was followed by works of Hu [20], Choudhury et al. [8, 9], etc. Amongst these works partially ordered fuzzy metric spaces was considered in [8]. The purpose of this paper is to establish a coupled coincidence point theorem utilizing a weak contraction inequality with the help of two control functions. The main theorem has two corollaries which are shown to be properly contained in the main theorem. An illustrative example is given.

Definition 1.1 [41] *A binary operation* $* : [0,1]^2 \longrightarrow [0,1]$ *is called a t-norm if the* *following properties are satisfied:*

(a) $*$ *is associative and commutative*;

(b) $a * 1 = a$ *for all* $a \in [0, 1]$;

(c) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$, *for each* $a, b, c, d \in [0, 1]$.

Generic examples of continuous $t-$norms are $a *_1 b = \min\{a, b\}$, $a *_2 b = \dfrac{ab}{\max\{a, b, \lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.

The following is the definition given by George and Veeramani [16].

Definition 1.2 [16] *The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space in the sense of George and Veeramani if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*:*

(a) $M(x, y, t) > 0$;

(b) $M(x, y, t) = 1$ *if and only if* $x = y$;

(c) $M(x, y, t) = M(y, x, t)$;

(d) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(e) $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ *is continuous*.

Let $(X, M, *)$ be a GV-fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, r, t)$ with center $x \in X$ is defined by

$$B(x, r, t) = \{y \in X \mid M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable [16].

A metric space (X, d) can be considered as a fuzzy metric space $(X, M, *)$ with $a * b = \min\{a, b\}$ and M defined as $M(x, y, t) = \dfrac{t}{t + d(x, y)}$.

Amongst other inequivalently defined fuzzy metric spaces, we will only consider this space and hence will refer to it simply as a fuzzy metric space.

Example 1.1 Let $X = \mathbb{R}$. Let $a * b = a \cdot b$ for all $a, b \in [0, \infty)$. For each $t \in (0, \infty)$, let

$$M(x, y, t) = e^{-\frac{|x - y|}{t}}$$

for all $x, y \in X$. Then $(\mathbb{R}, M, *)$ is a fuzzy metric space.

Definition 1.3 [16] *Let* $(X, M, *)$ *be a fuzzy metric space.*

(a) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to a point* $x \in X$ *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$.

(b) *A sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if for each* $0 < \varepsilon < 1$ *and* $t > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *for each* $n, m \geq n_0$.

(c) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

The following lemma, which was originally proved for the fuzzy metric space introduced by Kramosil and Mishilek [24] is also true in the present case.

Lemma 1.1 [17] *Let* $(X, M, *)$ *be a fuzzy metric space. Then* $M(x, y, \cdot)$ *is nondecreasing for all* $x, y \in X$.

Lemma 1.2 [36] *M is a continuous function on* $X^2 \times (0, \infty)$.

The concept of coupled fixed point was introduced by Guo et al. [19]. Bhaskar et al. [3] proved a coupled contraction mapping theorem in partially ordered metric spaces. Coupled coincidence point results were proved by Lakshmikantham et al. [25] for two commuting mappings and by Chaudhury et al. [5] for compatible pair of mappings. There are several results in this direction of research in metric spaces. Some of these are noted in [26, 30, 39].

It is our purpose in this paper to prove a coupled coincidence point theorem for two mappings in complete fuzzy metric spaces.

Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$ be a mapping from X to itself. The mapping F is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$ [3].

Definition 1.4 [3] *Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$ *be a mapping. The mapping* F *is said to have the mixed monotone property if* F *is non-decreasing in its first argument and is non-increasing in its second argument, that is, if for all* $x_1, x_2 \in X$, $x_1 \preceq x_2$ *implies* $F(x_1, y) \preceq F(x_2, y)$ *for fixed* $y \in X$ *and if for all* $y_1, y_2 \in X$, $y_1 \preceq y_2$ *implies* $F(x, y_1) \succeq F(x, y_2)$ *for fixed* $x \in X$.

Definition 1.5 [25] *Let* $(X, \preceq)$ *be a partially ordered set and* $F : X \times X \to X$ *and* $g : X \to X$ *be two mappings. The mapping* F *is said to have the mixed g-monotone property if* F *is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, if for all* $x_1, x_2 \in X$, $gx_1 \preceq gx_2$ *implies* $F(x_1, y) \preceq F(x_2, y)$ *for all* $y \in X$ *and if for all* $y_1, y_2 \in X$, $gy_1 \preceq gy_2$ *implies* $F(x, y_1) \succeq F(x, y_2)$ *for any* $x \in X$.

Definition 1.6 [3] *An element* $(x, y) \in X \times X$ *is called a coupled fixed point of the mapping* $F : X \times X \to X$ *if*

$$F(x, y) = x, \ F(y, x) = y.$$

Further Lakshmikantham and Ćirić introduced the concept of coupled coincidence point.

Definition 1.7 [25] *An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = gx, \ F(y, x) = gy.$$

Definition 1.8 [5] *Let (X, d) be a metric space. The mappings F and g where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

In fuzzy metric spaces coupled fixed point results were first successfully proved by Zhu et al. [46]. After that coupled coincidence point and coupled fixed point results in this space have appeared in works of Hu [20], Choudhury et al. [8], Jain et al. [21]. In particular compatibility was defined by Hu [20] as the fuzzy counterpart of the concept introduced in Choudhury et al. [5].

Definition 1.9 [8, 20] *Let $(X, M, *)$ be a fuzzy metric space. The mappings F and g where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if for all $t > 0$*

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

We note that a compatible pair (g, F) is also commuting, that is, also satisfies $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

In the following, we prove two lemmas which we use in the proof of our main theorem in the next section.

Lemma 1.3 *If $*$ is a continuous t-norm, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha$, $\gamma_n \to \gamma$ as $n \to \infty$, then $\overline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim_{k\to\infty}} \beta_k * \gamma$ and $\underline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim_{k\to\infty}} \beta_k * \gamma$.*

Proof From the definition of limit supremum, there exists $\{\beta_{n(p)}\} \subset \{\beta_n\}$ such that

$$\lim_{p\to\infty} \beta_{n(p)} = \overline{\lim_{k\to\infty}} \beta_k = \beta \text{ (say)}.$$

Then

$$\begin{aligned} \alpha * \overline{\lim_{k\to\infty}} \beta_k * \gamma &= \lim_{p\to\infty} \alpha_{n(p)} * \lim_{p\to\infty} \beta_{n(p)} * \lim_{p\to\infty} \gamma_{n(p)} \\ &= \lim_{p\to\infty} (\alpha_{n(p)} * \beta_{n(p)} * \gamma_{n(p)}) \\ &\qquad \text{(by the continuity property of } *) \\ &\le \overline{\lim_{k\to\infty}} (\alpha_k * \beta_k * \gamma_k). \end{aligned} \tag{1}$$

We now show that the equality in (1) must hold. If not, then there exists a sequence of natural $\{n(q)\}$ such that

$$\begin{aligned}\alpha * \overline{\lim_{k\to\infty}}\beta_k * \gamma < \overline{\lim_{k\to\infty}}(\alpha_k * \beta_k * \gamma_k) &= \lim_{q\to\infty}(\alpha_{n(q)} * \beta_{n(q)} * \gamma_{n(q)})\\ &= \lim_{q\to\infty}\alpha_{n(q)} * \lim_{q\to\infty}\beta_{n(q)} * \lim_{q\to\infty}\gamma_{n(q)}\\ &\qquad\text{(by the continuity property of } *)\\ &= \alpha * \lim_{q\to\infty}\beta_{n(q)} * \gamma.\end{aligned}$$

By the monotone property of $*$ we have that

$$\overline{\lim_{k\to\infty}}\beta_k < \overline{\lim_{q\to\infty}}\beta_{n(q)}, \text{ which is contradiction.}$$

Therefore, we conclude that $\alpha * \overline{\lim_{k\to\infty}}\beta_k * \gamma = \overline{\lim_{k\to\infty}}(\alpha_k * \beta_k * \gamma_k)$.

The other part of the lemma is similarly proved.

Lemma 1.4 *Let $\{f(k,\cdot) : (0,\infty) \to (0,1], k = 0,1,2,\cdots\}$ be a sequence of functions such that $f(k,\cdot)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim_{k\to\infty}} f(k,t)$ is a left continuous function in t and $\underline{\lim_{k\to\infty}} f(k,t)$ is a right continuous function in t.*

proof For fixed $t \in (0,\infty)$, let $g(n,t) = \sup_{p\geq n} f(p,t)$. Then $\lim_{n\to\infty} g(n,t) = \overline{\lim_{k\to\infty}} f(k,t)$. By the conditions of the lemma, the above limit exists finitely. Let $0 < \eta < t$ be arbitrary. We can find $q \geq n$ such that

$$f(q,t) > \sup_{p\geq n} f(p,t) - \eta = g(n,t) - \eta, \text{ that is, } g(n,t) < \eta + f(q,t).$$

Since $f(p,t)$ is monotone increasing in t for each p, $g(n,t) = \sup_{p\geq n} f(p,t)$ is also monotone increasing in t for each n. Then

$$\begin{aligned}0 &\leq g(n,t) - g(n,t-\eta) \leq \eta + f(q,t) - \sup_{p\geq n} f(p,t-\eta)\\ &\leq \eta + f(q,t) - f(q,t-\eta).\end{aligned}$$

Taking $n \to \infty$ in the above inequality,

$$\lim_{n\to\infty} g(n,t) - \lim_{n\to\infty} g(n,t-\eta) = \overline{\lim_{k\to\infty}} f(k,t) - \overline{\lim_{k\to\infty}} f(k,t-\eta) \to 0 \quad \text{as} \quad \eta \to 0.$$

This establishes that $\overline{\lim_{k\to\infty}} f(k,t)$ is left continuous in t.

The other part of the lemma, that is, $\underline{\lim_{k\to\infty}} f(k,t)$ a right continuous function in t is similarly established.

2. Main Result

Theorem 2.1 *Let $(X,\preceq)$ be a partially ordered set and $(X,M,*)$ be a complete fuzzy metric space where $*$ is an arbitrary continuous t-norm. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property and that the following conditions are satisfied:*

(a) $F(X \times X) \subseteq gX$;

(b) *g is continuous and monotonic increasing*;

(c) *(g,F) is a compatible pair*;

(d) $\psi(M(F(x,y),F(u,v),t) * M(F(y,x),F(v,u),t))$
$$\leq \psi(M(gx,gu,t) * M(gy,gv,t)) - \phi(M(gx,gu,t) * M(gy,gv,t)) \tag{2}$$

for all $x,y,u,v \in X$, $t > 0$ *with* $gx \preceq gu$ *and* $gy \succeq gv$*, where* $\psi,\phi : (0,1] \to [0,\infty)$ *are two functions such that:*

(i) ψ *is continuous and monotone decreasing with* $\psi(s) = 0$ *if and only if* $s = 1$,

(ii) ϕ *is lower semi-continuous with* $\phi(s) = 0$ *if and only if* $s = 1$.

Also suppose that X *has the following properties:*

(a) *if a non-decreasing sequence* $\{x_n\} \to x$*, then* $x_n \preceq x$ *for all* $n \geq 0$, (3)

(b) *if a non-increasing sequence* $\{y_n\} \to y$*, then* $y_n \succeq y$ *for all* $n \geq 0$. (4)

If there exist $x_0,y_0 \in X$ *such that* $gx_0 \preceq F(x_0,y_0)$ *and* $gy_0 \succeq F(y_0,x_0)$*, then there exist* $x,y \in X$ *such that* $gx = F(x,y)$ *and* $gy = F(y,x)$*, that is,* g *and* F *have a coupled coincidence point in* X*.*

Proof According to a condition of the theorem there exist x_0,y_0 in X such that $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$. We define the sequence $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$gx_1 = F(x_0,y_0) \text{ and } gy_1 = F(y_0,x_0),$$
$$gx_2 = F(x_1,y_1) \text{ and } gy_2 = F(y_1,x_1),$$

and in general, for all $n \geq 0$,

$$gx_{n+1} = F(x_n,y_n) \text{ and } gy_{n+1} = F(y_n,x_n). \tag{5}$$

This construction is possible by the condition $F(X \times X) \subseteq gX$.

Next, we prove that for all $n \geq 0$,

$$gx_n \preceq gx_{n+1} \tag{6}$$

and

$$gy_n \succeq gy_{n+1}. \tag{7}$$

Since $gx_0 \preceq F(x_0,y_0)$ and $gy_0 \succeq F(y_0,x_0)$, in view of the fact that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$, we have $gx_0 \preceq gx_1$ and $gy_0 \succeq gy_1$. Therefore (6) and (7) hold for $n = 0$.

Let (6) and (7) hold for some $n = m$. As F has the mixed g-monotone property, $gx_m \preceq gx_{m+1}$ and $gy_m \succeq gy_{m+1}$, from (5), we get

$$gx_{m+1} = F(x_m,y_m) \preceq F(x_{m+1},y_m) \text{ and } F(y_{m+1},x_m) \preceq F(y_m,x_m) = gy_{m+1}. \tag{8}$$

Also, for the same reason, we have

$$gx_{m+2} = F(x_{m+1},y_{m+1}) \succeq F(x_{m+1},y_m),$$
$$F(y_{m+1},x_m) \succeq F(y_{m+1},x_{m+1}) = gy_{m+2}. \tag{9}$$

Then from (8) and (9),

$$gx_{m+1} \preceq gx_{m+2} \text{ and } gy_{m+1} \succeq gy_{m+2}.$$

Then, by induction, it follows that (6) and (7) hold for all $n \geq 0$.

Let for all $t > 0, n \geq 0$,

$$\delta_n(t) = M(gx_n,gx_{n+1},t) * M(gy_n,gy_{n+1},t). \tag{10}$$

By using (6) and (7), from (2) and (5), we have for all $t > 0$ and $n \geq 1$,

$$\psi(M(gx_n,gx_{n+1},t) * M(gy_n,gy_{n+1},t))$$

$$= \psi(M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) * M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t))$$
$$\leq \psi(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t))$$
$$-\phi(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)).$$

By using (10), we have

$$\psi(\delta_n(t)) \leq \psi(\delta_{n-1}(t)) - \phi(\delta_{n-1}(t)). \tag{11}$$

The above inequality implies that $\psi(\delta_n(t)) \leq \psi(\delta_{n-1}(t))$. Since ψ is a monotone decreasing function, we have that $\delta_n(t) \geq \delta_{n-1}(t)$ for all $n \geq 1$. Thus for each $t > 0$, $\{\delta_n(t); n \geq 0\}$ is an increasing sequence in $[0, 1]$ and hence tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$ for all $t > 0$. If there exists $t_0 > 0$ such that $a(t_0) < 1$, then taking limit as $n \longrightarrow \infty$ for $t = t_0$ in (11), we get $\psi(a(t_0)) \leq \psi(a(t_0)) - \phi(a(t_0))$, which is a contradiction since $\phi(a(t_0)) \neq 0$. Hence $a(t) = 1$ for every $t > 0$, that is, for all $t > 0$,

$$\lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \{M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)\} = 1. \tag{12}$$

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let, to the contrary, at least one of $\{gx_n\}$ and $\{gy_n\}$ be not a Cauchy sequence. Then there exist $\epsilon, \lambda > 0$ with $\lambda \in (0, 1)$ such that for each integer k, there are two integers $l(k)$ and $m(k)$ such that $m(k) > l(k) \geq k$ and
either

$$M(gx_{l(k)}, gx_{m(k)}, \epsilon) \leq 1 - \lambda \text{ for all } k,$$

in the case where $\{gx_n\}$ is not a Cauchy sequence
or

$$M(gy_{l(k)}, gy_{m(k)}, \epsilon) \leq 1 - \lambda \text{ for all } k,$$

in the case where $\{gy_n\}$ is not a Cauchy sequence.

In either case we have, for all $k > 0$,

$$r_k(\epsilon) = M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon) \leq 1 - \lambda. \tag{13}$$

By choosing $m(k)$ to be the smallest integer exceeding $l(k)$ for which (13) holds, we have, for all $k > 0$,

$$M(gx_{l(k)}, gx_{m(k)-1}, \epsilon) * M(gy_{l(k)}, gy_{m(k)-1}, \epsilon) > 1 - \lambda. \tag{14}$$

Now, by the triangle inequality, for any s with $0 < s < \frac{\epsilon}{2}$ for all $k > 0$, we have

$$\begin{aligned} 1 - \lambda \geq r_k(\epsilon) &= M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon) \\ &\geq M(gx_{l(k)}, gx_{l(k)+1}, s) * M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon - 2s) \\ &\quad * M(gx_{m(k)+1}, gx_{m(k)}, s) * M(gy_{l(k)}, gy_{l(k)+1}, s) \\ &\quad * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon - 2s) * M(gy_{m(k)+1}, gy_{m(k)}, s) \\ &\geq \{M(gx_{l(k)}, gx_{l(k)+1}, s) * M(gy_{l(k)}, gy_{l(k)+1}, s)\} \\ &\quad * \{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon - 2s) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon - 2s)\} \\ &\quad * \{M(gy_{m(k)+1}, gy_{m(k)}, s) * M(gx_{m(k)+1}, gx_{m(k)}, s)\}. \end{aligned} \tag{15}$$

For $t > 0$, we define the function

$$h_1(t) = \overline{\lim_{k\to\infty}} \{M(gx_{l(k)+1}, gx_{m(k)+1}, t) * M(gy_{l(k)+1}, gy_{m(k)+1}, t)\}. \tag{16}$$

Taking limit supremum on both sides of (15), using (12), and by the continuity property of $*$, by Lemma 1.3, we obtain

$1 - \lambda \geq r_k(\epsilon)$

$$\geq 1 * \overline{\lim_{k\to\infty}}\{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon - 2s) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon - 2s)\} * 1$$
$$= h_1(\epsilon - 2s).$$

Since M is bounded with range in [0, 1], continuous and, by Lemma 1.1, monotone increasing in the third variable t, it follows by an application of Lemma 1.4 that h_1, as given in (16), is continuous from the left. Then letting $s \to 0$ in the above inequality, and by using (13), we obtain

$$h_1(\epsilon) = \overline{\lim_{k\to\infty}}\{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon)\} \leq 1 - \lambda. \tag{17}$$

Next, for all $t > 0$, we define the function

$$h_2(t) = \underline{\lim_{k\to\infty}}\{M(gx_{l(k)+1}, gx_{m(k)+1}, t) * M(gy_{l(k)+1}, gy_{m(k)+1}, t)\}. \tag{18}$$

Again for any $s > 0$, for all integer $k > 0$, we obtain

$$\begin{aligned} &M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon + 3s) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon + 3s)\\ &\geq \{M(gx_{l(k)+1}, gx_{l(k)}, s) * M(gy_{l(k)+1}, gy_{l(k)}, s)\}\\ &\quad * \{M(gx_{l(k)}, gx_{m(k)-1}, \epsilon) * M(gy_{l(k)}, gy_{m(k)-1}, \epsilon)\}\\ &\quad * \{M(gx_{m(k)-1}, gx_{m(k)}, s) * M(gy_{m(k)-1}, gy_{m(k)}, s)\}\\ &\quad * \{M(gx_{m(k)}, gx_{m(k)+1}, s) * M(gy_{m(k)}, gy_{m(k)+1}, s)\}.\\ &\geq \{M(gx_{l(k)+1}, gx_{l(k)}, s) * M(gy_{l(k)+1}, gy_{l(k)}, s)\}\\ &\quad * (1 - \lambda)\\ &\quad * \{M(gx_{m(k)-1}, gx_{m(k)}, s) * M(gy_{m(k)-1}, gy_{m(k)}, s)\}\\ &\quad * \{M(gx_{m(k)}, gx_{m(k)+1}, s) * M(gy_{m(k)}, gy_{m(k)+1}, s)\}. \qquad \text{(by (14))} \end{aligned} \tag{19}$$

Taking limit infimum as $k \to \infty$ on both sides of the above inequality and using (12), we obtain

$$\underline{\lim_{k\to\infty}} M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon + 3s) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon + 3s) \geq 1 * (1 - \lambda) * 1 * 1,$$

that is,

$$\begin{aligned} h_2(\epsilon + 3s) &= \underline{\lim_{k\to\infty}} M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon + 3s) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon + 3s)\\ &\geq (1 - \lambda). \end{aligned} \tag{20}$$

Since M is bounded with range in [0, 1], continuous and, by Lemma 1.1, monotone increasing in the third variable t, it follows by an application of Lemma 1.4 that h_2 as given in (18) is continuous from the right. Taking $s \to 0$ in the above inequality (20), we obtain

$$h_2(\epsilon) = \underline{\lim_{k\to\infty}}\{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon)\} \geq (1 - \lambda). \tag{21}$$

Combining (17) and (21), we have

$$\lim_{k\to\infty}\{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon)\} = (1 - \lambda). \tag{22}$$

Again, by (22),

$$\overline{\lim_{k\to\infty}}\{M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon)\} \leq 1 - \lambda. \tag{23}$$

For $t > 0$, we define the function

$$h_3(t) = \underline{\lim_{k\to\infty}}\{M(gx_{l(k)+1}, gx_{m(k)+1}, t) * M(gy_{l(k)+1}, gy_{m(k)+1}, t)\}. \tag{24}$$

For any $s > 0$,

$$\begin{aligned} &M(gx_{l(k)}, gx_{m(k)}, \epsilon + 2s) * M(gy_{l(k)}, gy_{m(k)}, \epsilon + 2s)\\ &\geq \{M(gx_{l(k)}, gx_{l(k)+1}, s) * M(gy_{l(k)}, gy_{l(k)+1}, s)\}\\ &\quad * \{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon)\}\\ &\quad * \{M(gx_{m(k)+1}, gx_{m(k)}, s) * M(gy_{m(k)+1}, gy_{m(k)}, s)\}. \end{aligned}$$

Taking limit infimum $k \to \infty$ in the above inequality and, using (12) and (13), by

Lemma 1.3, we have

$$\begin{aligned}&\underline{\lim}_{k\to\infty}\{M(gx_{l(k)}, gx_{m(k)}, \epsilon+2s) * M(gy_{l(k)}, gy_{m(k)}, \epsilon+2s)\}\\&\geq 1 * \underline{\lim}_{k\to\infty}\{M(gx_{l(k)+1}, gx_{m(k)+1}, \epsilon) * M(gy_{l(k)+1}, gy_{m(k)+1}, \epsilon)\} * 1\\&= 1-\lambda.\end{aligned}$$

Then, using (24), we have

$$\begin{aligned}h_3(\epsilon+2s) &= \underline{\lim}_{k\to\infty}\{M(gx_{l(k)}, gx_{m(k)}, \epsilon+2s) * M(gy_{l(k)}, gy_{m(k)}, \epsilon+2s)\}\\&\geq 1-\lambda.\end{aligned} \tag{25}$$

Since M is bounded with range in [0, 1], continuous and, by Lemma 1.1, monotone increasing in the third variable t, it follows by an application of Lemma 1.4 that h_3 as given in (24) is continuous from the right. Taking $s \to 0$ in the above inequality (25), we obtain

$$h_3(\epsilon) = \underline{\lim}_{k\to\infty}\{M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon)\} \geq (1-\lambda). \tag{26}$$

Combining (23) and (26), we have

$$\lim_{k\to\infty}\{M(gx_{l(k)}, gx_{m(k)}, \epsilon) * M(gy_{l(k)}, gy_{m(k)}, \epsilon)\} = (1-\lambda). \tag{27}$$

Now

$$\begin{aligned}&\psi(M(gx_{m(k)+1}, gx_{l(k)+1}, \epsilon) * M(gy_{m(k)+1}, gy_{l(k)+1}, \epsilon))\\&= \psi(M(F(x_{m(k)}, y_{m(k)}), F(x_{l(k)}, y_{l(k)}), \epsilon) * M(F(y_{m(k)}, x_{m(k)}), F(y_{l(k)}, x_{l(k)}), \epsilon))\\&\leq \psi(M(gx_{m(k)}, gx_{l(k)}, \epsilon) * M(gy_{m(k)}, gy_{l(k)}, \epsilon))\\&\quad -\phi(M(gx_{m(k)}, gx_{l(k)}, \epsilon) * M(gy_{m(k)}, gy_{l(k)}, \epsilon)). \qquad \text{(by using (4))}\end{aligned}$$

Taking $k \to \infty$ in the above inequality, using (22) and (23), we have

$$\psi(1-\lambda) \leq \psi(1-\lambda) - \phi(1-\lambda),$$

which is a contraction since $\phi(1-\lambda) \neq 0$.

Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \text{ and } \lim_{n\to\infty} gy_n = y. \tag{28}$$

Therefore, $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = x$, $\lim_{n\to\infty} gy_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = y$.

Since, (g, F) is a compatible pair, using continuity of g, we have

$$\lim_{n\to\infty} g(gx_{n+1}) = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n), \tag{29}$$

$$\lim_{n\to\infty} g(gy_{n+1}) = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{30}$$

By (6), (7) and (28), it follows that $\{gx_n\}$ is a non-decreasing sequence with $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence with $gy_n \to y$ as $n \to \infty$. Then by (3) and (4) we have for all $n \geq 0$,

$$gx_n \preceq x \text{ and } gy_n \succeq y.$$

Since, g is monotonic increasing, we have

$$g(gx_n) \preceq gx \text{ and } g(gy_n) \succeq gy. \tag{31}$$

Now we show that $gx = F(x, y)$ and $gy = F(y, x)$ for all $x, y \in X$. For all $t > 0$, $n \geq 0$, we have

$$\begin{aligned}&M(F(x, y), g(gx_{n+1}), t) * M(F(y, x), g(gy_{n+1}), t)\\&\quad = M(F(x, y), g(F(x_n, y_n)), t) * M(F(y, x), g(F(y_n, x_n)), t).\end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality, and using the properties of ψ and ϕ, for all $t > 0$, we obtain

$$\lim_{n\to\infty} \psi(M(F(x, y), g(gx_{n+1}), t) * M(F(y, x), g(gy_{n+1}), t))$$

$$= \lim_{n\to\infty} \psi(M(F(x,y), g(F(x_n,y_n)), t) * M(F(y,x), g(F(y_n,x_n)), t)),$$

By the continuity property of ψ, M and $*$, (29) and (30), for all $t > 0$, we have

$$\begin{aligned}
&\psi(M(F(x,y), gx, t) * M(F(y,x), gy, t)) \\
&\quad = \lim_{n\to\infty} \psi(M(F(x,y), g(F(x_n,y_n)), t) * M(F(y,x), g(F(y_n,x_n)), t)) \\
&\quad = \psi(M(F(x,y), \lim_{n\to\infty} g(F(x_n,y_n)), t) * M(F(y,x), \lim_{n\to\infty} g(F(y_n,x_n)), t)) \\
&\quad = \psi(M(\lim_{n\to\infty} F(gx_n, gy_n)), F(x,y), t) * M(\lim_{n\to\infty} F(gy_n, gx_n), F(y,x), t)) \\
&\quad = \lim_{n\to\infty} \psi(M(F(gx_n, gy_n)), F(x,y), t) * M(F(gy_n, gx_n), F(y,x), t)) \\
&\quad \le \lim_{n\to\infty} [\psi(M(g(gx_n), gx, t) * M(g(gy_n), gy, t)) \\
&\qquad - \phi(M(g(gx_n), gx, t) * M(g(gy_n), gy, t))] \quad \text{(By using (2) and (31))} \\
&\quad = \psi(M(gx, gx, t) * M(gy, gy, t)) - \phi(M(gx, gx, t) * M(gy, gy, t)) \\
&\quad = \psi(1 * 1) - \phi(1 * 1) \\
&\quad = \psi(1) - \phi(1) \\
&\quad = 0.
\end{aligned}$$

By using a property of ψ, we have $M(F(x,y), gx, t) * M(F(y,x), gy, t) = 1$, that is, $M(F(x,y), gx, t) = 1$ and $M(F(y,x), gy, t) = 1$, which implies that $gx = F(x,y)$ and $gy = F(y,x)$.

Thus we conclude that (x,y) is a coupled coincidence point of g and F.

Hence the proof is completed.

Next we state a corollary of the above theorem in which we replace the compatibility condition (iii) of the above theorem by the commuting condition.

Corollary 2.1 *Let $(X, \le)$ be a partially ordered set and $(X, M, *)$ be a complete fuzzy metric space where $*$ is an arbitrary continuous t-norm. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property and that the following conditions are satisfied:*

(a) $F(X \times X) \subseteq gX$,

(b) *g is continuous and monotonic increasing*,

(c) *(g, F) is a commuting pair*,

(d) $\psi(M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t))$
$\le \psi(M(gx, gu, t) * M(gy, gv, t)) - \phi(M(gx, gu, t) * M(gy, gv, t))$

for all $x, y, u, v \in X$, $t > 0$ with $gx \le gu$ and $gy \ge gv$, where $\psi, \phi : (0,1] \to [0,\infty)$ are two functions such that:

(i) *ψ is continuous and monotone decreasing with $\psi(s) = 0$ if and only if $s = 1$,*

(ii) *ϕ is lower semi- continuous with $\phi(s) = 0$ if and only if $s = 1$.*

Also suppose that X has the following properties:

(a) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all $n \ge 0$,*

(b) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \ge y$ for all $n \ge 0$.*

If there exist $x_0, y_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$*, then there exist* $x, y \in X$ *such that* $gx = F(x, y)$ *and* $gy = F(y, x)$*, that is,* g *and* F *have a coupled coincidence point in* X*.*

Proof Since a commuting pair is also a compatible pair, Corollary 2.1 follows from Theorem 2.1.

Later, by an example, we show that Corollary 2.1 is properly contained in Theorem 2.1.

Our next corollary is a weak coupled contraction mapping theorem.

Corollary 2.2 *Let* $(X, \preceq)$ *be a partially ordered set and* $(X, M, *)$ *be a complete fuzzy metric space where* $*$ *is an arbitrary continuous t-norm. Let* $F : X \times X \to X$ *be a mapping such that* F *has mixed monotone property and satisfies the following condition:*

$$\psi(M(F(x, y), F(u, v), t) * M(F(y, x), F(v, u), t)) \leq \psi(M(x, u, t) * M(y, v, t)) - \phi(M(x, u, t) * M(y, v, t)),$$

for all $x, y, u, v \in X$, $t > 0$ *with* $x \preceq u$ *and* $y \succeq v$*, where* $\psi, \phi : (0, 1] \to [0, \infty)$ *are two functions such that:*

(i) ψ is continuous and monotone decreasing with $\psi(s) = 0$ if and only if $s = 1$,

(ii) ϕ is lower semi-continuous with $\phi(s) = 0$ if and only if $s = 1$.

Also suppose that X *has the following properties:*

(a) *if a non-decreasing sequence* $\{x_n\} \to x$*, then* $x_n \preceq x$ *for all* $n \geq 0$*,*

(b) *if a non-increasing sequence* $\{y_n\} \to y$*, then* $y_n \succeq y$ *for all* $n \geq 0$*.*

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0)$ *and* $y_0 \succeq F(y_0, x_0)$*, then there exist* $x, y \in X$ *such that* $x = F(x, y)$ *and* $y = F(y, x)$*, that is,* F *has a coupled coincidence point in* X*.*

Proof The proof follows by putting $g = I$, the identity function, in Theorem 2.1.

Example 2.1 Let $(X, \preceq)$ be the partially ordered set where $X = [0,\ 1]$ and $\preceq$ be the natural ordering $\leq$ of the real numbers. Let for all $t > 0$, $x, y \in X$,

$$M(x, y, t) = e^{-\frac{|x-y|}{t}}.$$

Let $a * b = a.b$ for all $a, b \in [0,\ 1]$. Then $(X, M, *)$ is a complete fuzzy metric space. Let

$$\psi(s) = \frac{1}{s} - 1 \text{ and } \phi(s) = \frac{1}{s} - \frac{1}{\sqrt{s}} \text{ where } s \in (0, 1]. \tag{32}$$

Then ψ and ϕ satisfy the conditions given in the statement of Theorem 2.1.

Let the mapping $g : X \to X$ be defined as

$$g(x) = \frac{5}{6}x^2 \text{ for all } x \in X$$

and the mapping $F : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{6}, & \text{for } \ x \geq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly, $F(X \times X) \subseteq gX$ and F satisfies the mixed g-monotone property.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = a, \quad \lim_{n\to\infty} g(x_n) = a,$$
$$\lim_{n\to\infty} F(y_n, x_n) = b \text{ and } \lim_{n\to\infty} g(y_n) = b.$$

Now, for all $n \geq 0$,

$$g(x_n) = \frac{5}{6}x_n^2, \;\; g(y_n) = \frac{5}{6}y_n^2,$$

$$F(x_n, y_n) = \begin{cases} \dfrac{x_n^2 - y_n^2}{6}, & \text{if } x_n \geq y_n, \\ 0, & \text{otherwise} \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} 0, & \text{if } x_n \geq y_n, \\ \dfrac{y_n^2 - x_n^2}{6}, & \text{otherwise.} \end{cases}$$

Then necessarily $a = 0$ and $b = 0$. It then follows from Lemma 1.2 that, for all $t > 0$,

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n)), t) = 1.$$

Therefore the mappings g and F are compatible in X.

With the choices of ψ, ϕ as in (32) and the t-norm $*$ being given as $a * b = ab$, the inequality (2) has the form

$$\frac{1}{M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t)} - 1$$
$$\leq \frac{1}{(M(gx, gu, t) * M(gy, gv, t))} - 1 - \frac{1}{(M(gx, gu, t) * M(gy, gv, t))}$$
$$+ \frac{1}{\sqrt{(M(gx, gu, t) * M(gy, gv, t))}},$$

that is,

$$M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t) \geq \sqrt{M(gx, gu, t) * M(gy, gv, t)}. \quad (33)$$

Let $gx \leq gu$ and $gy \geq gv$, that is, $\frac{5}{6}x^2 \leq \frac{5}{6}u^2$ and $\frac{5}{6}y^2 \geq \frac{5}{6}v^2$,

that is,

$$x \leq u \text{ and } y \geq v. \quad (34)$$

We next show that the inequality (33) holds under the above condition.

The following cases may arise:

Case I $x \geq y$ and $u \geq v$.

In this case, we have $F(x,y) = \dfrac{x^2 - y^2}{6}$, $F(u,v) = \dfrac{u^2 - v^2}{6}$.

Then $|\dfrac{x^2 - y^2}{6} - \dfrac{u^2 - v^2}{6}| = \dfrac{1}{6}|(x^2 - u^2) - (y^2 - v^2)|$

$$\leq \frac{5}{24}[(x^2-u^2)|+|(y^2-v^2)]$$
$$\leq \frac{1}{4}\left|\frac{5}{6}x^2-\frac{5}{6}u^2\right|+\frac{1}{4}\left|\frac{5}{6}y^2-\frac{5}{6}v^2\right|,$$

that is, $\left|F(x,y)-F(u,v)\right| \leq \frac{1}{4}\left|g(x)-g(u)\right|+\frac{1}{4}\left|g(y)-g(v)\right|.$
Then, for all $t>0$,

$$e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq e^{-\frac{[\frac{1}{4}|g(x)-g(u)|\frac{1}{4}|g(y)-g(v)|]}{t}}$$
$$= e^{-\frac{|g(x)-g(u)|}{4t}} \cdot e^{-\frac{|g(y)-g(v)|}{4t}}$$
$$= \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}},$$

that is, $$e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \tag{35}$$

In this case $F(y,x)=0$ and $F(v,u)=0$, then trivially we have

$$0=|F(y,x)-F(v,u)| \leq \frac{1}{4}|g(y)-g(v)|+\frac{1}{4}|g(x)-g(u)|.$$

It follows that for all $t>0$,

$$1=e^{-\frac{|F(y,x)-F(v,u)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \tag{36}$$

From (35) and (36), for all $t>0$, we have

$$M(F(x,y),F(u,v),t)\cdot M(F(y,x),F(v,u),t)$$
$$\geq \sqrt[4]{M(g(x),g(u),t)\cdot M(g(y),g(v),t)}\sqrt[4]{M(g(x),g(u),t)\cdot M(g(y),g(v),t)},$$

that is,

$$M(F(x,y),F(u,v),t)*M(F(y,x),F(v,u),t)$$
$$\geq \sqrt{M(g(x),g(u),t)*M(g(y),g(v),t)}.$$

Case II $x\leq y$ and $u\geq v$.

In this case we have $F(x,y)=0$, $F(u,v)=\frac{u^2-v^2}{6}$.

By (34), we must have $x\leq u$, it then follows that $|u^2-v^2|\leq|x^2-u^2|$,

that is, $|u^2-v^2|\leq|x^2-u^2|+|y^2-v^2|$,

that is, $\frac{|u^2-v^2|}{6}\leq\frac{5}{24}[|x^2-u^2|+|y^2-v^2|]$,

that is, $\frac{|u^2-v^2|}{6}\leq\frac{1}{4}\left[\left|\frac{5}{6}x^2-\frac{5}{6}u^2\right|+\left|\frac{5}{6}y^2-\frac{5}{6}v^2\right|\right]$,

that is, $|F(u,v)|\leq\frac{1}{4}|g(x)-g(u)|+\frac{1}{4}|g(y)-g(v)|$,

that is,

$$|F(x,y)-F(u,v)|\leq\frac{1}{4}|g(x)-g(u)|+\frac{1}{4}|g(y)-g(v)|. \text{ (since } F(x,y)=0)$$

Then, for all $t>0$,

$$e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq e^{-\frac{[\frac{1}{4}|g(x)-g(u)|+\frac{1}{4}|g(y)-g(v)|]}{t}}$$
$$= e^{-\frac{|g(x)-g(u)|}{4t}} \cdot e^{-\frac{|g(y)-g(v)|}{4t}}$$

$$= \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}},$$

that is, $$e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \quad (37)$$

In this case $F(y, x) = \frac{y^2 - x^2}{6}$ and $F(v, u) = 0$. By (34), we must have $x \leq u$, it then follows that $|y^2 - x^2| \leq |x^2 - u^2|$,

that is, $|y^2 - x^2| \leq |x^2 - u^2| + |y^2 - v^2|$,

that is, $\frac{|y^2 - x^2|}{6} \leq \frac{5}{24}[|x^2 - u^2| + |y^2 - v^2|]$,

that is, $\frac{|y^2 - x^2|}{6} \leq \frac{1}{4}\left[\left|\frac{5}{6}x^2 - \frac{5}{6}u^2\right| + \left|\frac{5}{6}y^2 - \frac{5}{6}v^2\right|\right]$,

that is,

$$|F(y,x) - F(v,u)| \leq \frac{1}{4}|g(y) - g(v)| + \frac{1}{4}|g(x) - g(u)|. \text{ (since } F(v,u) = 0)$$

Then, for all $t > 0$, we have

$$e^{-\frac{|F(y,x)-F(v,u)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \quad (38)$$

From (37) and (38), for all $t > 0$, we have

$$M(F(x,y), F(u,v), t) \cdot M(F(y,x), F(v,u), t)$$
$$\geq \sqrt[4]{M(g(x), g(u), t) \cdot M(g(y), g(v), t)}\sqrt[4]{M(g(x), g(u), t) \cdot M(g(y), g(v), t)},$$

that is,

$$M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t)$$
$$\geq \sqrt{M(g(x), g(u), t) * M(g(y), g(v), t)}.$$

Case III $x \leq y$ and $u \leq v$.

In this case, $F(x, y) = 0$, $F(u, v) = 0$. Then obviously we have

$$0 = |F(x,y) - F(u,v)| \leq \frac{1}{4}|g(x) - g(u)| + \frac{1}{4}|g(y) - g(v)|.$$

Then, for all $t > 0$,

$$1 = e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq e^{-\frac{[\frac{1}{4}|g(x)-g(u)| + \frac{1}{4}|g(y)-g(v)|]}{t}}$$
$$= e^{-\frac{|g(x)-g(u)|}{4t}} \cdot e^{-\frac{|g(y)-g(v)|}{4t}}$$
$$= \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}},$$

that is,

$$e^{-\frac{|F(x,y)-F(u,v)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \quad (39)$$

Again in this case $F(y, x) = \frac{y^2 - x^2}{6}$ and $F(v, u) = \frac{v^2 - u^2}{6}$. Then

$$\frac{1}{6}|(y^2 - v^2) - (x^2 - u^2)| \leq \frac{5}{24}[|(y^2 - v^2)| + |(x^2 - u^2)|],$$

that is, $\left|\frac{y^2 - x^2}{6} - \frac{v^2 - u^2}{6}\right| \leq \frac{1}{4}\left|\frac{5}{6}y^2 - \frac{5}{6}v^2\right| + \frac{1}{4}\left|\frac{5}{6}x^2 - \frac{5}{6}u^2\right|$,

that is, $|F(y,x) - F(v,u)| \leq \frac{1}{4}|g(y) - g(v)| + \frac{1}{4}|g(x) - g(u)|$,

$$|F(y,x) - F(v,u)| \leq \frac{1}{4}|g(y) - g(v)| + \frac{1}{4}|g(x) - g(u)|.$$

Then, for all $t > 0$,

$$e^{-\frac{|F(y,x) - F(v,u)|}{t}} \geq \sqrt[4]{e^{-\frac{|g(x)-g(u)|}{t}} \cdot e^{-\frac{|g(y)-g(v)|}{t}}}. \tag{40}$$

From (39) and (40), for all $t > 0$, we have

$$\begin{aligned} &M(F(x,y), F(u,v), t) \cdot M(F(y,x), F(v,u), t) \\ &\quad \geq \sqrt[4]{M(g(x), g(u), t) \cdot M(g(y), g(v), t)}\sqrt[4]{M(g(x), g(u), t) \cdot M(g(y), g(v), t)}, \end{aligned}$$

that is,

$$\begin{aligned} &M(F(x,y), F(u,v), t) * M(F(y,x), F(v,u), t) \\ &\quad \geq \sqrt{M(g(x), g(u), t) * M(g(y), g(v), t)}. \end{aligned}$$

The other possible choice, that is, $x \geq y$ and $u \leq v$, is not consistent with (34) except for the trivial case where $x = y$ and $u = v$ in which case the inequality (33) is trivially satisfied. Hence this possibility is excluded. Combining the above cases we see that (33) and hence (2) is satisfied under the condition $gx \leq gu$ and $gy \geq gv$. Thus all the conditions of Theorem 2.1 are satisfied. Then, by an application of Theorem 2.1, we conclude that g and F have a coupled coincidence point. Here $(0,\ 0)$ is a coupled coincidence point of g and F in X.

Remark 2.1 In the above example, we see that (g, F) is not a commuting pair although it is a compatible pair of mappings. This shows that Corollary 2.1 can not be applied to this example. Further, in the example, g is not the identity mapping. This shows Corollary 2.2 is also properly included in Theorem 2.1.

Open problem: It remains to be investigated whether the result of Theorem 2.1 is true with conditions weaker than the compatibility between two mappings being satisfied.

3. Conclusion

The main theorem, which is Theorem 2.1, is proved with the help of two control functions of which one is continuous and the other is lower semi continuous. The proof is accomplished by application of two lemmas, that is, Lemma 1.3 and Lemma 1.4. What is remarkable about the main theorem is that, it has been possible to establish the result with arbitrary continuous t-norms. The same methodology can possibly be applied to some other problems of fuzzy fixed point theory.

Acknowledgment

The suggestions of the learned referees have been acknowledged.

References

[1] Y.I. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New Results in Operator Theory and its Applications 98 (1997) 7-22.

[2] A.D. Arvanitakis, A proof of the generalized Banach contraction conjecture, Proceedings of the American Mathematical Society 131 (2003) 3647-3656.

[3] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlin. Anal. 65 (2006) 1379-1393.

[4] D.W. Boyd, J.S.W. Wong, On nonlinear contractions, Pacific Journal of Mathematics 101(1) (1982) 41-48.

[5] B.S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlin. Anal. 73 (2010) 2524-2531.

[6] B.S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling 54 (2011) 73-79.

[7] B.S. Choudhury, P. Konar, B.E. Rhoades, N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlin. Anal. 74 (2011) 2116-2126.

[8] B.S. Choudhury, K. Das, P. Das, Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, Fuzzy Sets and Systems 222 (2013) 84-97.

[9] B.S. Choudhury, K. Das, P. Das, Coupled coincidence point results in partially ordered fuzzy metric spaces, Annals Fuzzy Math. Inform. 7 (2014) 619-628.

[10] B.S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, Fixed Point Theory & Applications 2013(1) (2013) 1-21.

[11] B.S. Choudhury, A. Kundu, Two coupled weak contraction theorems in partially ordered metric spaces, Revista De La Real Academia De Ciencias Exactas Fisicas Y Naturales Serie A Matematicas 97(2) (2003)157-175.

[12] B.S. Choudhury, P. Das, Coupled coincidence point results for compatible mappings in partially ordered probabilistic metric spaces, Asian-European Journal of Mathematics 7 (2014) doi: http://dx.doi.org/10.1142/S1793557114500090.

[13] Y.J. Cho, Z. Kadelburg, R. Saadati, W. Shatanawi, Coupled fixed point theorems under weak contractions, Discrete Dynamics in Nature and Society (2012) doi: 10.1155/2012/184534.

[14] L. Ćirić, Some new results for Banach contractions and Edelstein contractive mappings on fuzzy metric spaces, Chaos, Solitons and Fractals 42 (2009) 146-154.

[15] D. Doric, Common fixed point for generalized (ψ, φ)-weak contractions, Applied Math. Lett., 22 (2009) 1896-1900.

[16] A. George, P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and Systems 64 (1994) 395-399.

[17] M. Grabice, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385-389.

[18] V. Gregori, J.J. Miñana, S. Morillas, Some questions in fuzzy metric spaces, Fuzzy Sets and Systems 204 (2012) 71-85.

[19] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlin. Anal. 11 (1987) 623-632.

[20] X.Q. Hu, Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, Fixed Point Theory & Applications 2011(1) (2011) 1-14.

[21] M. Jain, K. Tas, S. Kumar, N. Gupta, Coupled fixed point theorems for a pair of weakly compatible maps along with CLRg property in fuzzy metric spaces, Journal of Applied Mathematics 2012(1110-757X) (2012) 1927-1936.

[22] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215-229.

[23] E. Karapinar, Coupled fixed point theorems for nonlinear contractions in cone metric spaces, Comput. Math. Appl. 59 (2010) 3656-3668.

[24] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326-334.

[25] V. Lakshmikantham and L. Ćirić,Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlin. Anal. 70 (2009) 4341-4349.

[26] N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlin. Anal. 74 (2011) 983-992.

[27] J. Merryfield, B. Rothschild, J.D. Stein Jr., An application of Ramsey's theorem to the Banach contraction principle, Proceedings of the American Mathematical Society 130 (2002) 927-933.

[28] D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 915-921.

[29] D. Mihet, On fuzzy ϵ-contractive mappings in fuzzy metric space, Fixed Point Theory Appl. (2007) Article ID 87471.

[30] H.K. Nashine, W. Shatanawi, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces, Comput. Math. Appl. 62 (2011) 1984-1993.

[31] J.J. Nieto, R. Rodriguez-Lopez, Contractive mapping theorems in partially ordered sets and applications to differential equations, Order 22 (2005) 223-239.

[32] J.J. Nieto, R. Rodriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to differential equations, Acta Math. Sin. Engl. Ser. 23 (2007) 2205-2212.

[33] O. Popescu, On a weak commutativity condition of mappings in a fixed point considerations, Publ. Inst. Math. (Beograd) 32 (1982) 149-153.

[34] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proceedings of the American Mathematical Society 132 (2004) 1435-1443.

[35] A. Razani, A contraction theorem in fuzzy metric space, Fixed Point Theory Appl. 2005 (2005) 257-265.

[36] J. Rodríguez López, S. Ramaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy Sets and Systems 147 (2004) 273-283.

[37] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlin. Anal. TMA 47 (2001) 2683-2693.

[38] S. Romaguera, Fixed point theorems for generalized contractions on partial metric spaces, Topol. Appl. 159 (2012) 194-199,

[39] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010) 4508-4517.

[40] B. Samet, C. Vetro, F. Vetro, From metric to partial metric spaces, Fixed Point Theory & Applications 2013(1) (2013) 118-134.

[41] B. Schweizer, A. Sklar, Statistical metric spaces. Pacific Journal of Mathematics 10(10) (1960) 313-334.

[42] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Amer. Math. Soc. 136 (2008) 1861-1869.

[43] M. Turinici, Product fixed points in ordered metric spaces, Mathematics (2011) arxiv: 1110.3079v1.

[44] S. Wang, S.M. Alsulami, L. Ćirić, Common fixed point theorems for nonlinear contractive mappings in fuzzy metric spaces, Fixed Point Theory & Applications 2013(1) (2013) 1-15.

[45] Q. Zhang, Y. Song, Fixed point theory for generalized ϕ-weak contractions, Applied Math. Lett. 22 (2009) 75-78.

[46] X.H. Zhu, J. Xiao, Note on coupled fixed point theorems for contractions in fuzzy metric spaces, Nonlin. Anal. 74 (2011) 5475-5479.