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Analytical methods for solving the time-fractional Swift–Hohenberg (S–H) equation

Najeeb Alam Khan [a,*], Nasir-Uddin Khan [a], Muhammad Ayaz [a], Amir Mahmood [b]

[a] Department of Mathematics, University of Karachi, Karachi 75270, Pakistan

[b] Department of Mathematics, COMSATS Institute of Information Technology, Lahore, Pakistan

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ABSTRACT

In this paper, the most effective methods, the homotopy perturbation method (HPM) and the differential transform method (DTM), are applied to obtain the approximate solutions of the nonlinear time-fractional Swift–Hohenberg (S–H) equation. The basic philosophy of these methods does not involve linearization, weak nonlinearity assumptions or perturbation theory. Numerical solutions for various combinations of the parameters a (eigenvalue parameter), L (length) and α (fractional index) are obtained. The solutions of the S–H equation are useful for studies of shear thinning effects in non-Newtonian fluid flows. At the end, the solutions obtained are also presented graphically.

1. Introduction

Most problems in science are nonlinear and in most cases it is difficult to solve them, especially analytically. Many problems are modeled by difficult nonlinear differential equations. The perturbation method is one of the most well-known methods for solving nonlinear problems. These methods are based on the existence of small/large parameters, the so-called perturbation quantities. However, many nonlinear problems do not contain such perturbation quantities and, therefore, non-perturbation techniques are used to solve such problems. During the last few decades several methods have been used to solve ordinary and partial differential equations, integro-differential equations and fractional differential equations. Among these methods the Adomian decomposition method (ADM) [1], He's variational iteration method (VIM) [2], the differential transformation method (DTM) [3,4] and the homotopy perturbation method (HPM) [5–7] have gained much success.

Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to L'Hospital in 1695. The beauty of this subject is that the fractional derivative and the fractional integral of a given function are not based on the local properties of that function. On the other hand, the history and non-local distributed effects of the function are considered. In other words, this subject translates the reality of nature in more sophisticated manner. The number of scientific and engineering problems involving fractional calculus [8] is already very large and still growing, and perhaps the fractional calculus will be the calculus of the twenty-first century.

The Swift–Hohenberg (S–H) equation is

$$u_t(X,t) = au - (1+\nabla^2)^2 u - u^3; \quad X \in \Re,\ t > 0, \tag{1}$$

where $a \in \Re$ is a parameter. The S–H equation [9] was first proposed in 1977 by Swift and Hohenberg as a simple model for the Rayleigh–Bénard convective instability of roll waves [10,11]. It has since featured in a variety of problems such as

* Corresponding author. Tel.: +92 21 35040516.
E-mail address: njbalam@yahoo.com (N.A. Khan).

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that of Taylor–Couette flow [12] and in the study of lasers [13]. The S–H equation, as a model equation for a large class of higher-order parabolic equations, arises in a wide range of applications, e.g., as the extended Fisher–Kolmogorov equation in statistical mechanics. In this paper, we study the Cauchy–Dirichlet problem for the S–H equation on the interval $(0, L)$, where, in addition to the parameter a, the length L and fractional index α will turn out to be important parameters. We employ DTM and HPM to find approximate solutions of time-fractional S–H equations. Thus, writing the S–H equation in a more general form, we consider the problem with time-fractional derivative

$$D_t^{\alpha} u = -2u_{xx} - u_{xxxx} - (1-a)u - u^3, \tag{2}$$

with boundary conditions

$$u = 0, \qquad u_{xx} = 0 \quad \text{at } x \in 0, L \text{ for all } t > 0, \tag{3}$$

$$u(x,0) = u_0(x), \quad \text{for all } 0 < x < L. \tag{4}$$

2. Analytical approximation of the time-fractional S–H equation

2.1. The DTM solution

We apply the DTM to Eq. (2), with the initial condition [14,15]

$$u(x,0) = \frac{1}{10}\sin\left(\frac{\pi x}{L}\right). \tag{5}$$

The transformed version of Eq. (2) is

$$\begin{aligned} U_{\alpha,1}(k,h+1) = \frac{\Gamma(\alpha h+1)}{\Gamma((\alpha h+1)+1)}\Bigg(&-2(k+1)(k+2)U_{\alpha,1}(k+2,h) - (k+1)(k+2)(k+3)(k+4)U_{\alpha,1}(k+4,h) \\ &- (1-a)U_{\alpha,1}(k,h) - \sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s} U_{\alpha,1}(r,h-s-p)U_{\alpha,1}(n,s)U_{\alpha,1}(k-r-n,p)\Bigg). \end{aligned} \tag{6}$$

The generalized two-dimensional differential transform of the initial condition is given as

$$\begin{aligned} &U(0,0) = 0, \quad U(1,0) = \frac{\pi}{10L}, \quad U(2,0) = 0, \quad U(3,0) = \frac{-\pi^3}{60L^3}, \quad U(4,0) = 0, \\ &U(5,0) = \frac{-\pi^5}{1200L^5}, \quad U(6,0) = 0, \quad U(7,0) = \frac{-\pi^7}{50400L^7}, \quad U(8,0) = 0, \\ &U(9,0) = \frac{\pi^9}{3628800L^9}, \quad U(10,0) = 0, \quad U(11,0) = \frac{-\pi^{11}}{399168000L^{11}}. \end{aligned} \tag{7}$$

Eqs. (6) and (7) give some values of $U(k,h)$. Subsequently substituting all values of $U(k,h)$ into Eq. (6), we obtain the series form solutions of Eq. (2), with initial condition (5), as

$$\begin{aligned} u(x,t) = &\left(\frac{\pi x}{10L} + \frac{\pi x^3}{60L^3} + \frac{\pi x^5}{1200L^5} + \frac{\pi x^7}{150400L^7} + \cdots\right) \\ &+ \left(\left(\frac{(a-1)\pi}{10L} + \frac{\pi^3}{5L^3} - \frac{\pi^5}{10L^5}\right)x + \left(\frac{(47-50a)\pi^3}{3000L^3} - \frac{\pi^5}{30L^5} + \frac{\pi^7}{60L^7}\right)x^3 + \cdots\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \left(\left(\frac{(1-a)^2\pi}{10L} + \frac{(-97+100a)\pi^3}{250L^3} + \frac{5(27-10a)\pi^5}{50L^5} - \frac{2\pi^7}{5L^7} + \frac{\pi^9}{10L^9}\right)x\right. \\ &+ \left(\frac{-19\pi^3}{1500L^3} + \frac{11a\pi^3}{375L^3} - \frac{a^2\pi^3}{60L^3} + \frac{(61-100a)\pi^5}{1500L^5} + \left(\frac{-3\pi^7}{500L^7} + \frac{a\pi^7}{30L^7}\right)\right. \\ &+ \left.\left.\frac{\pi^9}{15L^9} - \frac{\pi^{11}}{60L^{11}}\right)x^3 + \cdots\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \end{aligned} \tag{8}$$

2.2. The HPM solution

Let us construct the homotopy map for Eq. (2), as follows:

$$H(v,\varepsilon) = D_t^{\alpha} v - D_t^{\alpha} u_0 + \varepsilon(D_t^{\alpha} u_0 + 2u_{xx} + u_{xxxx} + (1-a)u + u^3). \tag{9}$$

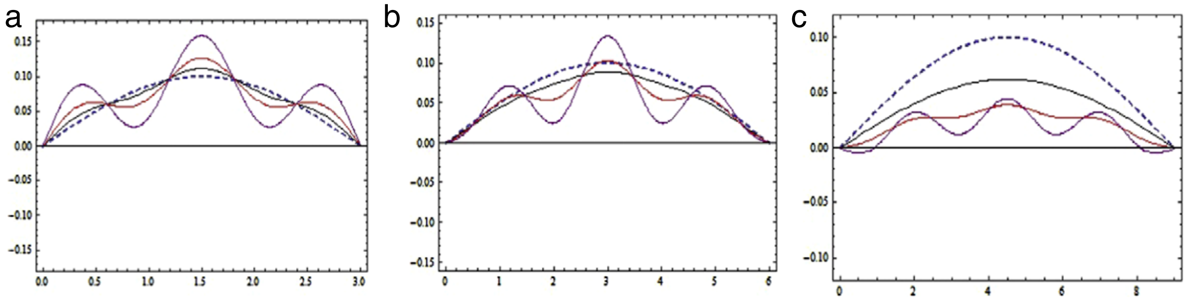

Fig. 1. Profiles for $a = 0.3$ and different choices of L: (a) $L = 3$, $\alpha = 1$, $t = 0, 0.08, 0.09, 0.1$ (dashed, black, red, purple). (b) $L = 6$, $\alpha = 1$, $t = 0, 0.6, 0.8, 0.9$ (dashed, black, red, purple). (c) $L = 9$, $\alpha = 1$, $t = 0, 1, 2.2, 2.6$ (dashed, black, red, purple). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The solution to the equation

$$H(v, \varepsilon) = 0 \tag{10}$$

is sought in the form of a power series in ε:

$$v = \sum_{j=0}^{\infty} \varepsilon^j v_j. \tag{11}$$

Substituting Eq. (11) into Eq. (9) and comparing the same powers of ε, we get

$$D_t^\alpha u_0 = 0, \tag{12}$$

$$D_t^\alpha u_i = -2\frac{\partial^2 u_{i-1}}{\partial x^2} - \frac{\partial^4 u_{i-1}}{\partial x^4} - (1-a)u_{i-1} - \sum_{m=0}^{i-1} u_{i-m-1} \sum_{j=0}^{m} u_{m-j}u_j, \quad i \geq 1. \tag{13}$$

Applying the Riemann–Liouville fractional integral operator J_t^α on both sides of Eqs. (12) and (13) and solving the ith-order deformation equations, we get the components of the homotopy series

$$u_0 = \frac{1}{10}\sin\left(\frac{\pi x}{L}\right), \tag{14}$$

$$u_1 = \frac{\left(100((1-a)L^4 - 2L^2\pi^2 + \pi^4) + L^4\sin^2\left(\frac{\pi x}{L}\right)\right)\sin\left(\frac{\pi x}{L}\right)t^\alpha}{1000L^4\Gamma(\alpha+1)}, \tag{15}$$

$$u_2 = \left(A_1\sin\left(\frac{\pi x}{L}\right) + A_2\sin\left(\frac{3\pi x}{L}\right) + A_3\sin\left(\frac{5\pi x}{L}\right)\right)\frac{t^{2\alpha}}{1600000L^8\Gamma(2\alpha+1)}, \tag{16}$$

where

$$\begin{aligned} A_1 &= 10(16483 + 160a(-203+100a)L^8 + 320(-203+200a)L^6\pi^2 \\ &\quad + 160(603-200a)L^4\pi^4 - 64000L^2\pi^6 + 16000\pi^8), \end{aligned}$$

$$A_2 = 5L^4((-323+320a)L^4 + 1920L^2\pi^2 - 6720\pi^4) \quad \text{and} \quad A_3 = 3L^8.$$

We use a nine-term approximate solution and set

$$u(x, t) = u_0 + u_1 + \cdots + u_8. \tag{17}$$

3. Conclusion

In this paper, the differential transformation and homotopy perturbation techniques are used to solve S–H equations. The analytical solution $u(x, t)$ of the time-fractional S–H equation has been obtained for the one-dimensional domain $(0, L)$. The effects of the eigenvalue parameter a, fractional index α, and the length L of the domain are also shown graphically. We show by means of the approximate solutions obtained how different values of these parameters may lead to qualitatively different profiles. In order to gain insight into the different kinds of limiting behaviors and the effects of the parameter α and the length L of the domain on the final profiles of the solution, we carried out a series of numerical simulations. In the study of the S–H equation, the profile comprises many periods of a periodic solution but here we focus on smaller domains with only a few periods. In these simulations, we keep a fixed and compare the dynamics of the profiles as the size L of the domain increases (see Fig. 1). In Fig. 2, we give results of simulations for different values of these parameters.

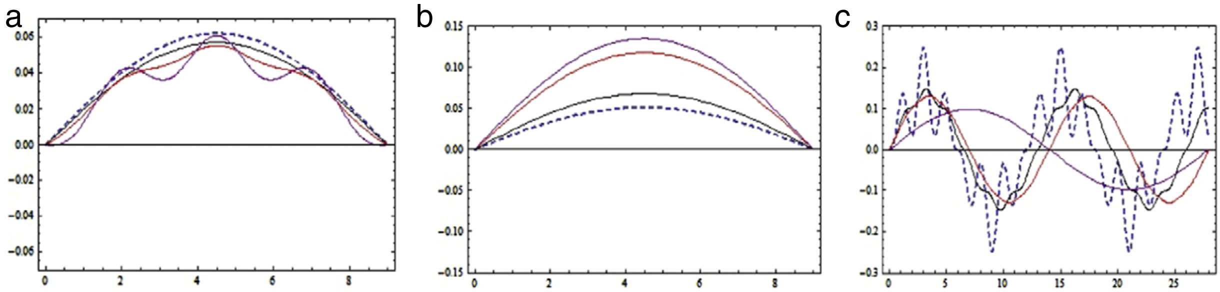

Fig. 2. Profiles for (a) $L = 9, \alpha = 1, 0.8, 0.6, 0.5, t = 1$ (dashed, black, red, purple), (b) $L = 9, \alpha = 1, t = 1.4, a = 0.3, 0.5, 0.9$ (dashed, black, red, purple), (c) $L = 6, 6.5, 7, 14, \alpha = 1, t = 1$ (dashed, black, red, purple). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

References

[1] G. Adomian, G. Rach, Modified Adomian polynomials, Math. Comput. Modelling 24 (11) (1996) 39–46.
[2] J.H. He, Approximate analytic solutions for seepage flow with fractional derivatives in porous media, Comp. Methods Appl. Mech. Eng. 167 (1998) 57–68.
[3] J.K. Zhou, Differential Transformation and its Application for Electrical Circuits, Huazhong University Press, Wuhan, China, 1986.
[4] S. Momani, Z. Odibat, V.S. Erturk, Generalized differential transform method: application to differential equations of fractional order, Appl. Math. Comput. 19 (2009) 467–477.
[5] J.H. He, Homotopy perturbation method: a new nonlinear analytical technique, Appl. Math. Comput. 135 (2003) 73–79.
[6] N.A. Khan, A. Ara, S.A. Ali, A. Mahmood, Analytical study of Navier–Stokes equation with fractional orders using He's homotopy perturbation and variation iteration method, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 1127–1134.
[7] N.A. Khan, A. Ara, A. Mahmood, Approximate solution of time-fractional chemical engineering equations: a comparative study, Int. J. of Chem. React. Eng. 8 (2010) Article A19.
[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, San Diego, 2006.
[9] J.B. Swift, P.C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A 15 (1977) 319.
[10] E. Bodenschatz, W. Pesch, G. Ahlers, Recent developments in Rayleigh–Benard convection, Ann. Rev. Fluid Mech. 32 (2000) 709–778.
[11] J. Kockelkoren, C. Storm, W. Saarloos, Evidence for slow velocity relaxation in front propagation in Rayleigh–Bénard convection, Physica D 174 (2003) 168–175.
[12] J. Lega, J.V. Moloney, A.C. Newell, Swift–Hohenberg for laser, Phys. Rev. Lett. 73 (22) (1994) 2978–2981.
[13] P.C. Hohenberg, J.B. Swift, Effects of additive noise at the onset of Rayleigh–Bénard convection, Phys. Rev. A 46 (1992) 4773–4777.
[14] L.A. Peletier, V. Rottschäfer, Pattern selection of solutions of the Swift–Hohenberg equation, Physica D 194 (2004) 95–126.
[15] F.T. Akyildiz, D.A. Siginer, K. Vajravelu, R.A.V. Gorder, Analytical and numerical results for the Swift–Hohenberg equation, Appl. Math. Comput. 216 (2010) 221–226.