Research Note

Nonmonotonic reasoning, conditional objects and possibility theory

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Received January 1996; revised January 1997

Abstract

This short paper relates the conditional object-based and possibility theory-based approaches for reasoning with conditional statements pervaded with exceptions, to other methods in nonmonotonic reasoning which have been independently proposed: namely, Lehmann's preferential and rational closure entailments which obey normative postulates, the infinitesimal probability approach, and the conditional (modal) logics-based approach. All these methods are shown to be equivalent with respect to their capabilities for reasoning with conditional knowledge although they are based on different modeling frameworks. It thus provides a unified understanding of nonmonotonic consequence relations. More particularly, conditional objects, a purely qualitative counterpart to conditional probabilities, offer a very simple semantics, based on a 3-valued calculus, for the preferential entailment, while in the purely ordinal setting of possibility theory both the preferential and the rational closure entailments can be represented. © 1997 Elsevier Science B.V.

Keywords: Plausible reasoning; Nonmonotonic reasoning; Possibility theory; Infinitesimal probabilities; Conditional object

1. Introduction

In the last ten years, many works in nonmonotonic reasoning have focused on the determination of natural properties for a nonmonotonic consequence relation that are likely to achieve a satisfactory treatment of plausible reasoning in the presence of incomplete information [24,26,32]. Further, Pearl [41] has suggested that Adams'
logic of infinitesimal probabilities [1] is a good basis for nonmonotonic reasoning, and indeed the core properties of a nonmonotonic consequence relation are present in this logic. These properties constitute the basis of the inference System P (P for preferential) proposed by Kraus, Lehmann and Magidor [32], which provides a very cautious inference system. In order to get a less conservative inference, Lehmann (see [36,38]) and Pearl [42] have proposed to add an inference rule, first suggested by Makinson, called rational monotony, and a particular entailment (named “rational closure entailment” [38]) has been defined which satisfies rational monotony. This is restated in Section 2.

It is noteworthy that Adams’ logic of infinitesimal probabilities can be expressed in terms of conditional objects [20]. A conditional object $\beta|\alpha$ can be seen as a purely symbolic counterpart to the conditional probability $\text{Prob}(\beta|\alpha)$ [31]. Thus it shows that numerical probabilities do not play a crucial role in the modeling of preferential entailment, since no probability degrees, whether infinitesimal or not, are necessary when conditional objects are used. These can be easily handled in terms of a 3-valued semantics much simpler than the preferential semantics [32], as explained in Section 3.

Developed independently of the main stream of nonmonotonic logic research, possibilistic logic [16], handles pairs of the form $(\phi, w)$ made of a classical logic formula $\phi$ and of a certainty weight $w$ belonging to a totally ordered set (usually taken as the unit interval). A pair $(\phi, w)$ expresses that $\phi$ is more certain than $\neg\phi$, the latter being all the more impossible as $w$ becomes higher. The certainty weight is dually related to a possibility measure [45]. Possibilistic logic copes with partial inconsistency, and its inference machinery turns out to express a preferential entailment à la Shoham [43]. Indeed, the semantics of possibilistic logic can be expressed in terms of a complete ordering of the interpretations. A conditional knowledge base made of default rules of the form “generally, if $\alpha_i$ then $\beta_i$” can be viewed as a set of constraints stating that $\alpha_i \land \beta_i$ is strictly more possible than $\alpha_i \land \neg\beta_i$. The default rules can then be turned into possibilistic logic formulas $(\neg\alpha_i \lor \beta_i, w_i)$ where the weight $w_i$ reflects a rule priority, computed from the least informed possibility measure compatible with the set of constraints. Under this ranking of rules, possibilistic logic entailment is equivalent to rational closure entailment. When we consider all the possibility measures compatible with the set of constraints induced by the conditional assertions instead of the least informed one, System P [32] is recovered. This is the topic of Sections 4, 5 and 6.

2. A review of existing plausible inference systems

In the following, Greek letters denote formulas of a finite propositional language. $\Omega$ is the set of all corresponding interpretations. $\top$ represents the tautology and $\bot$ any inconsistent formula. By a conditional assertion (we also call it a default rule) we mean a generic rule of the form “generally, if $\alpha$ then $\beta$”, possibly having some exceptions. These rules are denoted by “$\alpha \rightarrow \beta$” where $\rightarrow$ is a nonclassical arrow relating two classical formulas. A default base, or a conditional knowledge base, is a set $\Delta = \{\alpha_i \rightarrow \beta_i \mid i = 1, \ldots, n\}$ of default rules.
This paper is not intended to give an overview of default reasoning systems; complementary overviews are provided in [9,35]. However the approaches considered in our paper are not covered by these two overviews. We start this review with the probabilistic approach. Pearl [41], after Adams [1], models a default rule \( \alpha_i \rightarrow \beta_i \) by the constraint \( P(\beta_i|\alpha_i) \geq 1 - \varepsilon \) where \( \varepsilon \) denotes an arbitrary small positive real number and \( P \) denotes a probability measure. Roughly speaking, given a default base \( \Delta \), the consequence relation is then defined as: \( \beta \) is a consequence of \( \alpha \) with respect to \( \Delta \) if the conditional probability \( P(\beta|\alpha) \) is very high whenever the conditional probability attached to each default rule in \( \Delta \) is also very high. More formally, let

\[
\Lambda_{\varepsilon} = \{P \mid P(\beta_i|\alpha_i) \geq 1 - \varepsilon \text{ and } \alpha_i \rightarrow \beta_i \in \Delta\}.
\]

**Definition 1.** A formula \( \psi \) is said to be an \( \varepsilon \)-consequence of a formula \( \phi \) (with respect to \( \Delta \)), denoted by \( \phi \models_{\varepsilon} \psi \), if and only if for each probability measure \( P \) in \( \Lambda_{\varepsilon} \) there exists a function \( O \), such that \( \lim_{\varepsilon \to 0} O(\varepsilon) = 0 \) and \( P(\psi|\phi) \geq 1 - O(\varepsilon) \).

Adams [1] has characterized \( \varepsilon \)-consequence relations by a set of postulates which are the core of System P [32]; for an extensive discussion see [27]. A nonmonotonic consequence relation \( \models \) obeying System P satisfies the six following properties:

1. **Reflexivity** (axiom schema): \( \alpha \models \alpha \).
2. **Left Logical Equivalence** (LLE): from \( \alpha \equiv \alpha' = \top \) and \( \alpha \models \beta \) deduce \( \alpha' \models \beta \).
3. **Right Weakening** (RW): from \( \beta \models \beta' \) and \( \alpha \models \beta \) deduce \( \alpha \models \beta' \).
4. **OR:** from \( \alpha \models \gamma \) and \( \beta \models \gamma \) deduce \( \alpha \lor \beta \models \gamma \).
5. **Cautious Monotony** (CM): from \( \alpha \models \beta \) and \( \alpha \models \gamma \) deduce \( \alpha \land \beta \models \gamma \).
6. **Cut:** from \( \alpha \lor \beta \models \gamma \) and \( \alpha \models \beta \) deduce \( \alpha \models \gamma \).

Remarkable consequences of System P are the following rules [20]:

7. **Quasi-Conjunction** (QC): from \( \alpha \models \gamma \) and \( \beta \models \delta \) deduce \( \alpha \lor \beta \models (\neg \alpha \lor \gamma) \land (\neg \beta \lor \delta) \), which also appears in [1].
8. **AND:** from \( \alpha \models \beta \) and \( \alpha \models \gamma \) deduce \( \alpha \models \beta \land \gamma \).

A syntactic entailment, denoted by \( \models_p \), from \( \Delta = \{\alpha_i \rightarrow \beta_i \mid i = 1, \ldots, n\} \), can then be defined [32]: \( \Delta \models_p \phi \rightarrow \psi \) (or \( \phi \models_p \psi \) for short) iff \( \phi \models \psi \) can be derived from \( \{\alpha_i \models \beta_i \mid i = 1, \ldots, n\} \) using System P.

Lehmann and Magidor [38] have shown the following equivalence:

**Proposition 2.** \( \phi \models_{\varepsilon} \psi \) if and only if \( \phi \models_p \psi \).

The set \( \Delta^p = \{\phi \rightarrow \psi \mid \phi \models_p \psi\} \) is usually considered as the minimal set of conclusions that any reasonable nonmonotonic consequence relation, applied to default reasoning, should generate. However, System P is cautious and suffers from a so-called "irrelevance" problem: if a formula \( \delta \) is a plausible consequence of \( \alpha \), and if a formula \( \beta \) is a formula composed of propositional symbols which do not appear in the default base, then \( \delta \) cannot be deduced from \( \alpha \land \beta \). For example, from the rule "generally, birds fly" it is not possible to deduce that "red birds fly too" (when no conditional assertion
in the knowledge base deals with "red" things). This cautious behavior can be avoided if the consequence relation also satisfies the rational monotony property proposed by Makinson, and extensively discussed in [38]:

(9) Rational Monotony (RM): from \( \alpha \models \delta \) and \( \alpha \not\models \neg \beta \) deduce \( \alpha \land \beta \models \delta \).

Here \( \alpha \not\models \neg \beta \) means that \( \alpha \models \neg \beta \) does not hold. This property states that in the absence of relevant information in the conditional knowledge base \( \Delta \) leading to \( \alpha \rightarrow \neg \beta \in \Delta^p \), one can deduce the same thing from \( \alpha \) or from \( \alpha \land \beta \). One way of adding the rational monotony to \( \Delta^p \) is to use the System Z proposed by Pearl [42]. System Z builds an ordering on the default rules in \( \Delta \), based on the notion of tolerated default:

**Definition 3.** A default rule \( \alpha \rightarrow \beta \) is said to be *tolerated* by \( \Delta \) iff there exists an interpretation \( \omega \) such that \( \omega \models \alpha \land \beta \) (\( \omega \) is then said to *verify* \( \alpha \rightarrow \beta \) following Adams terminology) and for each \( \alpha_i \rightarrow \beta_i \) in \( \Delta \), \( \omega \models \neg \alpha_i \lor \beta_i \) (\( \omega \) is said to *satisfy* (i.e., \( \omega \) does not falsify) \( \alpha_i \rightarrow \beta_i \)).

This definition is the basis for producing an ordered partition \( \{ \Delta_0, \Delta_1, \ldots, \Delta_k \} \) of \( \Delta \) such that any rule in \( \Delta_i \) is tolerated by \( \Delta_i \cup \cdots \cup \Delta_k \). Then, Pearl [42] attaches the weight \( Z(d) = i \in \mathbb{N} \) to each default rule \( d \in \Delta \). This ranking of default rules induces a ranking of interpretations, namely it defines a function \( \kappa \) which characterizes a partition of \( \Delta \). The rank \( \kappa(\omega) \) of an interpretation \( \omega \) is the rank of the highest-ranked rule falsified by \( \omega \), augmented by the unit, i.e.,

\[
\kappa(\omega) = \begin{cases} 
0, & \text{if } \omega \text{ satisfies each default in } \Delta, \\
\max_i \{Z(d_i) + 1 \mid \omega \models \alpha_i \land \neg \beta_i\}, & \text{otherwise}.
\end{cases}
\]

Finally, a nonmonotonic inference relation, denoted \( \models_{\sim} \) is then defined as

\[
\phi \models_{\sim} \psi \quad \text{if and only if} \quad \kappa(\phi \land \psi) < \kappa(\phi \land \neg \psi),
\]

where \( \kappa \) is extended to formulas by \( \kappa(\phi) = \min \{ \kappa(\omega) \mid \omega \models \phi \} \). It can be shown that \( \models_{\sim} \) handles correctly the above particular irrelevance problem and satisfies all the rules of System P and the rational monotony property. As shown by Goldszmidt and Pearl [28] the inference \( \models_{\sim} \) corresponds to a particular closure of \( \Delta^p \) under the rational monotony property called by Lehmann "the rational closure" [38]. Pearl [42] has also suggested a more conservative inference, denoted by \( \models_{\sim 0} \), and defined by:

\[
\phi \models_{\sim 0} \psi \quad \text{if and only if} \quad \Delta \cup \{ \phi \rightarrow \neg \psi \} \text{ is inconsistent},
\]

where a default base \( \Delta' \) is said to be *inconsistent* iff there exists a nonempty subset \( A \) of \( \Delta' \) such that no default in \( A \) is tolerated by other defaults in \( A \). An example of an inconsistent default base is \( \Delta = \{ \alpha \rightarrow \beta, \alpha \rightarrow \neg \beta \} \). Another example of inconsistent default base is when there exists default rules with an inconsistent antecedent. From now on, we will only consider consistent default bases. \( \models_{\sim 0} \) is equivalent to \( \models_{\sim P} [42] \).

In their System \( Z^+ \), Goldszmidt and Pearl [29] have proposed to encode defaults in \( \Delta \) by means of constraints of the form \( \kappa(\alpha_i \land \beta_i) + c_i < \kappa(\alpha_i \land \neg \beta_i) \) where \( c_i \) is interpreted as the minimal cost charged to interpretations falsifying the rule \( \alpha_i \rightarrow \beta_i \);
the larger \( c_i \) the stronger is the rule. This type of constraint can be justified by viewing \( \kappa \) as a Spohn's \([44]\) function, since it is equivalent to \( \kappa(-\beta_i|\alpha_i) > c_i \) in terms of the conditionalized \( \kappa \)-function, and where, by definition, \( \kappa(-\beta_i|\alpha_i) = \kappa(\alpha_i \land -\beta_i) - \kappa(\alpha_i) \) (using \( \kappa(\alpha_i) = \min\{\kappa(\alpha_i \land \beta_i), \kappa(\alpha_i \land -\beta_i)\} \)); it thus expresses that \( \alpha_i \land -\beta_i \) is all the more impossible as \( c_i \) is large.

Lastly, let us briefly mention the conditional (modal) logic-based approach to default reasoning. The connection between conditional logics and nonmonotonic reasoning is not surprising since some properties of Lewis' logics of counterfactuals \([39]\) appear to be natural for default reasoning. Several authors like Boutilier \([7]\), Lamarre \([33]\) and Crocco \([11]\) have investigated these links. Lamarre \([33]\) works with a class of conditional logics called \( \mathcal{V} \) \([39]\). More precisely, he proposes to construct, given a default base, a system of spheres called the "big normal model" and he shows that the inference relation, denoted \( \models_{\mathcal{V}} \), given by this big normal model is equivalent to \( \models \) in System \( Z \).

3. Conditional objects in default reasoning

An approach to reasoning with conditional objects, a symbolic counterpart to conditional probabilities, has been recently developed \([17, 20]\). It turns out to be equivalent to System \( P \) but with a semantics simpler than the ones proposed by Adams or by Lehmann. A conditional object is a pair of two propositional formulas \( \alpha, \beta \) denoted by \( \beta|\alpha \) and which reads "\( \beta \) knowing \( \alpha \)". A tri-valued semantics already suggested by De Finetti \([12]\), is attached to conditional objects, where the two classical truth values \( \{T, F\} \), and a third value \( I \) which means "inapplicable", can be assigned as follows:

\[
t(\beta|\alpha) = \begin{cases} 
t(\beta), & \text{if } t(\alpha) = T, \\
I, & \text{if } t(\alpha) = F, 
\end{cases}
\]

where \( t(\alpha) \) is the truth value (in the classical sense) of the formula \( \alpha \). Notice that the 3-valued truth functions \( t \) never assign the third truth value \( I \) to classical formulas. A default rule \( \alpha \rightarrow \beta \) is then encoded by a conditional object \( \beta|\alpha \). When \( t(\beta|\alpha) = I \), the default rule \( \alpha \rightarrow \beta \) does not apply (since \( t(\alpha) = F \)). This contrasts with a modeling of the rule by the material implication. The set of conditional objects defined on a propositional language is then a superset of the language if classical formulas \( \beta \) are mapped to conditional objects \( \beta|\top \), whereby \( \beta \) is conditioned by the tautology.

The entailment \( \models_c \) between two conditional objects extends the propositional entailment. It is defined from the total ordering between truth values \( F < I < T \). Namely, \( \models_c \) is defined as \([31]\):

\[
\beta|\alpha \models_c \delta|\gamma \iff \forall t, t(\beta|\alpha) \leq t(\delta|\gamma).
\]

The ordering \( < \) comes down to interpreting \( I \) as "unknown" (i.e., \( I = \{T, F\} \)) and \( \leq \) as "not more true than". It can be checked that this is equivalent to a pair of classical entailments, namely

\[
\beta|\alpha \models_c \delta|\gamma \iff \alpha \land \beta \models \delta \land \gamma \text{ and } \gamma \land \neg \delta \models \alpha \land \neg \beta
\]
Table 1

| (QC(β|α, δ|γ)) | t(β|α) | t(δ|γ) |
|----------------|--------|--------|
|                | F      | I      | T      |
| t(β|α)          | F      | F      | F      |
| t(δ|γ)          | I      | I      | T      |
| t(β|α)          | F      | T      | T      |

(if α = γ = T, β ⊨ δ is recovered). Viewing the models of α ∧ β as the set of examples of the rule α → β (they verify it), and the models of α ∧ ¬β as the set of counterexamples (or exceptions) of α → β (they falsify it), the entailment β|α ⊨ C δ|γ means that each example of α → β is also an example of γ → δ and that each counterexample of γ → δ is also a counterexample of α → β. The above entailment ⊨ C suggests that a conditional object β|α is entirely characterized by the two formulas α ∧ β and α ∧ ¬β, similarly to the conditional probability P(β|α) which is also characterized by the two numbers P(α ∧ β) and P(α ∧ ¬β) (namely P(β|α) = P(α ∧ β)/(P(α ∧ β) + P(α ∧ ¬β)).

In order to define the inference of a conditional object from a set of conditional objects, we need to combine conditional objects. There are several possible approaches to the conjunction, the disjunction, the negation and the nesting of conditional objects [17, 20]. In the context of reasoning with rules having exceptions, only a fragment of conditional object theory is useful. It only exploits one possible definition of conjunction, corresponding to the quasi-conjunction introduced by Adams [1] in his probabilistic setting. The quasi-conjunction QC(β|α, δ|γ) of two conditional objects β|α and δ|γ is defined by the truth table in Table 1. This table can be justified in the following way: QC reduces to the usual notion of conjunction on \{T, F\} x \{T, F\}. Viewing the set \{β|α, δ|γ\} as a conditional knowledge base, if the two rules are inapplicable, the base is inapplicable (QC(T, T) = QC(F, F) = F; QC(T, I) = QC(I, T) = T). The quasi-conjunction can be equivalently expressed as the conditional object [17]:

QC(β|α, δ|γ) = (¬α ∨ β) ∧ (¬γ ∨ δ)|α ∨ γ.

Note the resemblance with property (7) of System P. The quasi-conjunction is clearly associative.

The inference, in the sense of conditional objects, of a rule from a set of defaults D = \{α_i → β_i | i = 1, ..., n\} is defined in terms of the above conjunction and consequence relation: a rule α → β is a consequence of D (denoted by D ⊨ CO α → β) iff there exists a subset S of rules such that QC(S) ⊨C β|α, where QC(S) is the quasi-conjunction of all conditional objects formed with the rules in S (changing α_i → β_i into β|α_i). When S = \emptyset, QC(S) = γ|⊥ (for an arbitrary γ) by convention, and only conditional objects

Note that the truth value of QC(β|α, δ|γ) can be simply obtained by taking the minimum of the two valuations of the components in the sense of the following ordering ⊳ between the truth values: I ⊳ T ⊳ F (not to be confused with the ordering underlying ⊨ C).
The conditional object inference relation is thus defined as:

**Definition 4.** \( \Delta \models_{CO} \alpha \rightarrow \beta \) iff \( \exists S \subseteq \Delta \) such that \( QC(S) \models C \beta | \alpha \).\(^5\)

This definition indicates that the presence in \( \Delta \) of rules of the form \( \perp \rightarrow \alpha \) does not affect the result of the inference, since \( QC(S, \alpha | \perp) = QC(S) \) while they make \( \Delta \) inconsistent in the sense of the toleration-based inference of Pearl [42]. \( \models_{CO} \) has the same deductive power as System P (in which \( \perp \models \gamma \) is valid). Namely,

**Proposition 5** (see [20]). \( \Delta \models_{CO} \alpha \rightarrow \beta \) iff \( \alpha \rightarrow \beta \in \Delta^P \).

The proof takes advantage of the fact that the quasi-conjunction rule can be derived from System P. The inference relation \( \models_{CO} \) satisfies all the rules of System P but fails to satisfy rational monotony. Nevertheless, the tri-valued semantics of conditional objects is really simpler than the preferential semantics proposed by Kraus et al. [32] for System P, and can be easily explained using the notion of applicable and inapplicable rules, of examples and counterexamples to a rule, without any need to resort to sophisticated notions such as infinitesimal probabilities.

### 4. The possibilistic uncertainty setting

#### 4.1. Basic notions

Possibility theory is based on the notion of a *possibility distribution* \( \pi \) which is a mapping from the set \( \Omega \) to the interval \([0, 1]\) and thus provides a complete ordering of interpretations where the most plausible ones get the highest value 1. Here, the unit interval is taken as a prototypical bounded totally ordered scale. \( \pi(\omega) > 0 \) means that \( \omega \) is only somewhat plausible, while \( \pi(\omega) = 0 \) means that \( \omega \) is impossible. \( \pi \) restricts the set of interpretations according to the available knowledge about the normal course of things. \( \pi(\omega) > \pi(\omega') \) means that \( \omega \) is more plausible than \( \omega' \). \( \pi \) is said to be *normal* if \( \exists \omega \) such that \( \pi(\omega) = 1 \); and any such \( \omega \) corresponds to a "normal state of facts". Note that if we choose a threshold \( a > 0 \) and consider \( \{ \omega \mid \pi(\omega) \geq a \} \) we get what Lewis [39] calls a "sphere" around the most plausible interpretations. Hence \( \pi \) encodes a system of spheres, a unique one for the whole set of interpretations. Two set functions are associated with \( \pi \):

- The possibility degree \( \Pi(\phi) = \sup \{ \pi(\omega) \mid \omega \models \phi \} \) which evaluates to what extent \( \phi \) is consistent with the available knowledge expressed by \( \pi \). \( \Pi \) satisfies the characteristic properties:

\(^5\)Note that \( \models_{CO} \) cannot be defined for a conditional knowledge base \( \Delta \) by \( QC(\Delta) \models_C \beta | \alpha \) only. Otherwise \( \Delta \) would not always entail conditional objects associated with the defaults it contains (since some defaults may be inapplicable while others are verified). Indeed if \( \Delta = \{ \alpha \rightarrow \beta, \gamma \rightarrow \delta \} \) with \( \neg \alpha \land (\gamma \land \delta) \neq \perp \) we do have \( t(\beta | \alpha) = 1 \) and \( t(\delta | \gamma) = T \) for some interpretations; hence \( t(QC(\Delta)) \leq t(\beta | \alpha) \) does not hold. With Definition 4, if \( \Delta = \{ \alpha \rightarrow \beta \} \), \( \alpha \rightarrow \beta \models_{CO} \gamma \rightarrow \delta \) is equivalent to \( \beta | \alpha \models_C \delta | \gamma \) or \( \gamma \models \delta \).
The dual necessity (or certainty) degree

\[ N(\phi) = 1 - \Pi(\neg\phi) = \inf \{ 1 - \pi(\omega) \mid \omega \models \neg\phi \} \]

which evaluates to what extent \( \phi \) is entailed by the available knowledge. We have:

\[ \forall \phi, \forall \psi, N(\phi \land \psi) = \min(N(\phi), N(\psi)). \]

Note that \( N(\phi) \) and \( N(\neg\phi) \) are only weakly related by \( \min(N(\phi), N(\neg\phi)) = 0 \) if \( \pi \) is normal. The normalization condition of \( \pi \) reflects the consistency of the available knowledge represented by \( \pi \), since otherwise we would have \( \forall \phi, N(\phi \land \neg\phi) = N(\bot) > 0 \). A systematic assumption in possibility theory is that the actual situation is normal, i.e., it is described by any \( \omega \) such that \( \pi(\omega) \) is maximal, given other known constraints. It justifies the evaluation \( N(\phi) \), and contrasts with the probabilistic evaluation of the likelihood of events. Moreover \( N(\phi) > 0 \) means that \( \phi \) holds in all the most normal interpretations. Under the assumption that the actual situation is normal, \( N(\phi) > 0 \) means that \( \phi \) is an accepted belief, i.e., one may act as if \( \phi \) were true. And indeed the set of formulas \( \{ \phi \mid N(\phi) > 0 \} \) is deductively closed. It is a belief set in the sense of Gärdenfors [25].

Possibility theory is driven by the principle of minimum specificity. A possibility distribution \( \pi \) is said to be at least as specific as another \( \pi' \) if and only if for each interpretation \( \omega \) it holds that \( \pi(\omega) \leq \pi'(\omega) \). Then, \( \pi \) is at least as restrictive and informative as \( \pi' \). Given a set of constraints restricting a feasible subset of possibility distributions, the best representative is the least specific feasible possibility distribution, which assigns the highest degree of possibility to each world, since it is also the least committed one.

An ordinal conditioning notion can be defined by means of the Bayesian-like equation

\[ \Pi(\phi \land \psi) = \min(\Pi(\psi|\phi), \Pi(\phi)) \]

defined by

\[ \Pi(\psi|\phi) = \begin{cases} 1, & \text{if } \Pi(\phi \land \psi) = \Pi(\psi) \quad \text{(i.e., if } \Pi(\phi \land \psi) \geq \Pi(\phi \land \neg\psi)\text{)} \smallskip \text{, otherwise} \quad & \Pi(\phi \land \psi), \end{cases} \]

when \( \Pi(\phi) > 0 \). If \( \Pi(\psi|\phi) \) is then defined as the greatest solution to the previous equation in accordance with the minimum specificity principle. It leads to

\[ \Pi(\psi|\phi) = \begin{cases} 1, & \text{if } \Pi(\phi \land \psi) = \Pi(\psi) \quad \text{(i.e., if } \Pi(\phi \land \psi) \geq \Pi(\phi \land \neg\psi)\text{)} \smallskip \Pi(\phi \land \psi), & \text{otherwise} \quad \text{(i.e., if } \Pi(\phi \land \psi) < \Pi(\phi \land \neg\psi)\text{)}, \end{cases} \]

when \( \Pi(\phi) > 0 \). If \( \Pi(\phi) = 0 \), then by convention \( \Pi(\psi|\phi) = 1, \forall \psi \neq \bot \). The conditional necessity measure is simply defined as \( N(\psi|\phi) = 1 - \Pi(\neg\psi|\phi) \). Thus \( N(\psi|\phi) > 0 \) means that in the context \( \phi, \psi \) is accepted. If \( \Pi(\phi) > 0 \), it can be easily

\[ \text{In fact, any order-reversing map on the unit interval will do as well instead of the function } 1 - x. \text{ Its choice does not affect the result of the approach to default reasoning.} \]
checked that $N(\psi|\phi) > 0$ iff $\Pi(\phi \wedge \psi) > \Pi(\phi \wedge \neg \psi)$, which means that accepting $\psi$ in the context $\phi$, is equivalent to saying that $\phi \wedge \psi$ is more plausible than $\phi \wedge \neg \psi$.

**Remark.** It is also possible to define a numerical conditional possibility measure using the product instead of min in the Bayesian-like relation (see [17] for justifications). As pointed out in [18], this is equivalent to the conditioning of Spohn’s [44] kappa functions through a rescaling. Note that the min-based conditional necessity measure $N(\beta|\alpha)$ only requires a purely ordinal (possibly finite) scale for assessing the certainty levels. This contrasts with Spohn’s [44] conditionalization which requires a scale where addition is meaningful.

### 4.2. Possibilistic entailment represents rational nonmonotonic inference

In the last section, the unit interval is understood as a mere ordinal scale, which means that possibility theory is a qualitative theory of uncertainty, as long as we use the min-based conditioning and not the product-based one. Therefore, to each possibility distribution $\pi$, we can associate its comparative counterpart, denoted by $>_\pi$, defined by $\omega >_\pi \omega'$ if and only if $\pi(\omega) > \pi(\omega')$, which induces the well-ordered partition [44] $\{E_1, \ldots, E_{n+1}\}$ of $\Omega$, that is, $\{E_1, \ldots, E_{n+1}\}$ is a partition of $\Omega$ such that:

$$\forall \omega \in E_i, \forall \omega' \in E_j, \pi(\omega) > \pi(\omega') \text{ iff } i < j \text{ (for } i \leq n+1, j \geq 1).$$

$E_{n+1}$ is a subset of impossible interpretations such that $\pi(\omega) = 0$, and is therefore denoted $E_\perp$. In a similar way a complete pre-order $\geq_\pi$ is defined as: $\forall \omega \in E_i, \forall \omega' \in E_j, \omega \geq_\pi \omega'$ iff $i \leq j$ (for $i \leq n+1, j \geq 1$). And $\omega =_\pi \omega'$ iff $\omega \geq_\pi \omega'$ and $\omega' \geq_\pi \omega$. By convention, $E_1$ represents the most normal states of facts. Thus, a possibility distribution partitions $\Omega$ into classes of equally possible interpretations. Note that each possibility distribution has exactly one comparative counterpart, but a given comparative possibility distribution $>_\pi$ admits an infinite number of representations on the unit interval. The use of $E_\perp$ enables a subset of impossible interpretations to be distinguished, with respect to $>_\pi$.

From any comparative possibility distribution $>_\pi$, a comparative possibility (respectively necessity) relation can also be defined for any pair of formulas $\phi$ and $\psi$ as $\phi \geq_\Pi \psi$ iff $\exists \omega =_\pi \phi, \forall \omega' =_\pi \psi, \omega \geq_\pi \omega'$ iff $\Pi(\phi) \geq_\Pi \Pi(\psi)$ (respectively $\phi \geq_N \psi$ iff $\exists \omega =_\pi \neg \psi, \forall \omega' =_\pi \neg \phi, \omega \geq_\pi \omega'$ iff $N(\phi) \geq N(\psi)$). Comparative possibility relations can be equivalently defined by the following properties [14, 39]: $\forall \alpha, \forall \beta, \forall \gamma$,

(i) completeness ($\alpha \geq_\Pi \beta$ or $\beta \geq_\Pi \alpha$);

(ii) transitivity;

(iii) $\top \geq_\Pi \bot$, where $\geq_\Pi$ is the strict part of the ordering $\geq_\Pi$;

(iv) $\top \geq_\Pi \alpha \geq_\Pi \bot, \forall \alpha$;

(v) $\alpha \geq_\Pi \beta \Rightarrow \gamma \vee \alpha \geq_\Pi \gamma \vee \beta$ (characteristic property);

(vi) if $\alpha \models \beta$ then $\beta \geq_\Pi \alpha$ (syntax independence).

Comparative necessity relations are defined by duality, i.e., $\phi \geq_N \psi$ iff $\neg \psi \geq_\Pi \neg \phi$.

In the finite case, it has been shown that the only numerical counterparts to comparative necessity relations are necessity measures [14]. Comparative necessity relations are closely related to the epistemic entailment relation underlying any revision of a belief.
set in the sense of Gärdenfors [25]; see [18]. Comparative necessity and possibility relations are compatible with classical entailment in the sense that \( \phi \geq_{H} \psi \) and \( \phi \geq_{H} \psi \) whenever \( \psi \models \phi \). Now, let us introduce the notion of possibilistic entailment.

**Definition 6.** Let \( \succ_{\pi} \) be a comparative possibility distribution and \( \{E_{1}, \ldots, E_{n}, F_{\perp}\} \) be the well-ordered partition induced by \( \succ_{\pi} \). An interpretation \( \omega \) is a \( \pi \)-preferred model of a formula \( \phi \) with respect to \( \succ_{\pi} \) iff: (i) \( \omega \models \phi \), (ii) \( \omega \notin E_{\perp} \), and (iii) \( \beta \omega' \), \( \omega' \models \phi \) and \( \omega' \succ_{\pi} \omega \).

The possibilistic entailment \( \models_{\pi} \) can then be defined in the spirit of Shoham’s [43] proposal:

**Definition 7.** A formula \( \psi \) is a possibilistic consequence of \( \phi \) with respect to \( \succ_{\pi} \), denoted by \( \phi \models_{\pi} \psi \), iff \( \phi \lor \psi \succ_{H} \phi \land \neg \psi \), that is, each \( \pi \)-preferred model of \( \phi \) satisfies \( \psi \) [19].

To explain Definition 7, let \( \omega \) be a preferred model of \( \phi \) in the sense of Definition 6 (this implies that \( \phi \neq F_{\perp} \), and \( \phi \succ_{H} F_{\perp} \)). For simplicity, let \( \pi \) be a possibility distribution representing \( \succ_{H} \) and \( \Pi \) the corresponding possibility measure. Clearly \( \pi(\omega) = \Pi(\phi) > 0 \). Since each preferred model of \( \phi \) satisfies \( \psi \), it means that \( \Pi(\phi) = \Pi(\phi \land \psi) > \Pi(\phi \land \neg \psi) \); indeed no model \( \omega' \) of \( \neg \psi \) is preferred among the models of \( \phi \), and \( \Pi(\phi) = \max(\Pi(\phi \land \psi), \Pi(\phi \land \neg \psi)) \). The consequence relationship \( \models_{\pi} \) satisfies most of the remarkable properties recalled in Section 2: LLE, RW, AND, OR, CM, Cut and RM, as well as a weakened version of reflexivity and a strong form of consistency preservation [26]:

- **Restricted Reflexivity (RR):** if \( \alpha \neq F_{\perp} \), then \( \alpha \models_{\pi} \alpha \);
- **Consistency Preservation (CP):** \( \neg(\alpha \models_{\pi} F_{\perp}) \).

CP and RR are typical of the possibilistic approach. CP, RR, Cut and AND imply what could be called “Nihil ex absurdo” [4]: \( \neg(1 \models_{\pi} \alpha) \). For suppose \( 1 \models_{\pi} \alpha, \alpha \neq F_{\perp} \); then by RR, \( \neg \alpha \models_{\pi} \neg \alpha \) and since \( \alpha \land \neg \alpha \models_{\pi} \alpha \), Cut leads to \( \neg \alpha \models_{\pi} \alpha \) and to \( \neg \alpha \models_{\pi} F_{\perp} \) using AND. This result violates CP; and if \( \alpha = F_{\perp} \), then \( \neg(F_{\perp} \models_{\pi} F_{\perp}) \) is an instance of CP. Inferences from a contradictory statement (\( \alpha = F_{\perp} \)) are thus not allowed in the sense of preferential entailment; while \( \perp \) classically entails anything, it should preferentially entail nothing. As said above \( \alpha \models_{\pi} \beta \) is equivalent to \( \alpha \land \beta \succ_{H} \alpha \land \neg \beta \) where \( \succ_{H} \) is the strict possibility relation induced by \( \pi \). Conversely \( \alpha \succ_{H} \beta \) comes down to a possibilistic entailment:

**Lemma 8** (see [4]). \( \alpha \succ_{H} \beta \) is equivalent to \( \alpha \lor \beta \models_{\pi} \neg \beta \).

**Proof.** Note that the strict preference relation \( \succ_{H} \) verifies the property \( \alpha \lor \gamma \succ_{H} \beta \lor \gamma \Rightarrow \alpha \succ_{H} \beta \) (by contraposition of property (v)). Hence \( \alpha \succ_{H} \beta \) entails \( \alpha \land \neg \beta \succ_{H} \beta \) (where \( \gamma = \alpha \land \beta \)). Letting \( \phi = \alpha \lor \beta \) and \( \psi = \neg \beta \), \( \alpha \land \neg \beta \succ_{H} \beta \) reads \( \phi \land \psi \succ_{H} \phi \land \neg \psi \). This is equivalent to \( \phi \models_{\pi} \psi \), i.e., \( \alpha \lor \beta \models_{\pi} \neg \beta \). Conversely, \( \alpha \lor \beta \models_{\pi} \neg \beta \) is equivalent to \( (\alpha \lor \beta) \land \neg \beta \succ_{H} (\alpha \lor \beta) \land \beta \), i.e., \( \alpha \land \neg \beta \succ_{H} \beta \) which, along with \( \alpha \geq_{H} \alpha \land \neg \beta \) leads to \( \alpha \succ_{H} \beta \) by transitivity. \( \square \)
Lemma 8 is closely related to Gärdenfors and Makinson's [26] representation results for nonmonotonic inference by means of expectation orderings which correspond to comparative necessity relations, except that they do not request $\triangleright_N \perp$. Conversely, any nonmonotonic inference relation $\triangleright$ satisfying the above properties derives from a possibility distribution $\Omega$.

**Theorem 9** (see [4]). Given a nonmonotonic consequence relation $\triangleright$ satisfying OR, RR, CP, LLE, RW, AND and RM, the relation $\triangleright$ defined by $\alpha \triangleright \beta$ iff $\alpha \lor \beta \triangleright \lnot \beta$, is a comparative possibility relation.

**Proof.** We have to show that the relation $\triangleright$ satisfies all axioms of a comparative possibility relation:

(i) Completeness: $\alpha \triangleright \beta$ or $\beta \triangleright \alpha$, or equivalently $\lnot(\alpha \triangleright \beta$ and $\beta \triangleright \alpha$). It states that we cannot have $\alpha \lor \beta \triangleright \lnot \beta$ and $\alpha \lor \beta \triangleright \lnot \alpha$. Indeed, otherwise using "AND" leads to $\alpha \lor \beta \triangleright \lnot \alpha \land \lnot \beta$ which violates CP.

(ii) Transitivity: $\beta \triangleright \alpha$ and $\delta \triangleright \beta$ implies $\delta \triangleright \alpha$. It amounts to proving the inconsistency of $\alpha \lor \beta \lor \lnot \beta \lor \delta \lor \lnot \delta \lor \alpha \lor \delta \lor \lnot \delta$. As in Gärdenfors and Makinson [26] we first prove $\alpha \lor \beta \lor \delta \lor \lnot \delta$, applying OR to $\alpha \lor \delta \lor \lnot \delta$ and $\lnot \delta \land \beta \lor \lnot \delta$; the latter holds from RR and RW provided that $\lnot \delta \land \beta \neq \perp$. If $\lnot \delta \land \beta = \perp$ then $\delta \lor \beta = \delta$ and by LLE applied to $\alpha \lor \delta \lor \lnot \delta$, $\alpha \lor \beta \lor \delta \lor \lnot \delta$. Next, $\alpha \lor \beta \lor \delta \lor \lnot \delta \lor \delta \lor \lnot \delta$ obtains via contraposition of RM applied to $\alpha \lor \beta \lor \delta \lor \lnot \delta$ and $\beta \lor \delta \lor \lnot \delta$ (letting $\alpha' = \alpha \lor \beta \lor \delta$, $\beta' = \beta \lor \delta$, and $\delta' = \lnot \delta$). Lastly RM can again be applied to $\alpha \lor \beta \lor \delta \lor \lnot \delta$ (that follows from $\alpha \lor \beta \lor \delta \lor \lnot \delta$ and $\alpha \lor \beta \lor \delta \lor \lnot \delta$ in the same way, to conclude $\alpha \lor \beta \lor \delta \lor \lnot \delta \lor \lnot \beta \lor \lnot \delta$ and $\alpha \lor \beta \lor \delta \lor \lnot \delta$). Combining the results by the AND rule leads to $\alpha \lor \beta \lor \delta \lor \lnot \delta \lor \lnot \beta \lor \lnot \delta$ which in turn is not compatible with CP.

(iii) Nontriviality: $\top \triangleright \bot$ expresses that $\top \triangleright \top$ (RR).

(iv) $\alpha \triangleright \bot$ or equivalently $\lnot(\bot \triangleright \alpha)$, i.e., $\lnot(\alpha \triangleright \lnot \alpha)$. Indeed assume $\alpha \triangleright \lnot \alpha$ then if $\alpha \neq \bot$, using RR and AND leads to $\alpha \triangleright \bot$ which contradicts CP. $\bot \triangleright \bot$ follows from $\lnot(\bot \triangleright \top)$, with $\alpha = \top$.

(v) $\delta \lor \alpha \triangleright \delta \lor \beta$ implies $\alpha \triangleright \beta$. It amounts to prove that $\alpha \lor \beta \lor \delta \lor \lnot \delta$ implies $\alpha \lor \beta \lor \lnot \beta \lor \lnot \delta$. First note that $\alpha \lor \beta \lor \delta \lor \lnot \beta \lor \lnot \delta$ is equivalent to $\alpha \lor (\beta \lor \delta) \lor \lnot (\beta \lor \delta)$. This writes $\alpha \lor \beta \lor \delta$. Moreover we have $\delta \lor \beta \triangleright \beta$ because it expresses that $\lnot(\delta \lor \beta \lor \lnot \beta \lor \lnot \delta)$ which is due to CP and proves (vi). The result follows by transitivity.

A similar representation theorem is proved by Gärdenfors and Makinson [26]. In the finite setting, the existence of a comparative possibility relation $>_{II}$ on formulas ensures the existence of a corresponding possibility distribution $\pi$ such that $\pi(\omega) > \pi(\omega')$ if $\alpha >_{II} \beta$ where $\alpha$ and $\beta$ are the propositions whose only models are $\omega$ and $\omega'$ respectively. Lemma 8 ensures that the comparative possibility relation obtained from Theorem 9 leads to a possibilistic inference that coincides with the original nonmonotonic inference. Thus the seven postulates in Theorem 9 can be viewed as the basic properties of possibilistic inference.
5. Possibilistic semantics of conditional objects

A conditional assertion "generally α's are β's", is represented in the possibility theory framework by the following relation: "α ∧ β >_Π α ∧ ¬β", which means that the situation where α ∧ β is true is more possible than the situation where α ∧ ¬β is true [4]. Hence, a set of conditional assertions Δ = {α_i → β_i | i = 1, ..., n} can thus be viewed as a family of constraints C = {α_i ∧ β_i >_Π α_i ∧ ¬β_i | i = 1, ..., n} restricting a family of possibility distributions over the interpretations of a language. When the set of constraints is consistent, Δ only contains rules α_i → β_i such that α_i ≠ ⊥, otherwise the constraint α_i ∧ β_i >_Π α_i ∧ ¬β_i is violated. We denote by Π(Δ) all the comparative possibility distributions satisfying the set of constraints C. Then a natural definition for the entailment, first proposed in [23], is:

Definition 10. A formula ψ is a universal possibilistic consequence of ϕ, denoted by ϕ ⊨_Π ψ, if ψ is a possibilistic consequence of ϕ for each comparative possibility distribution of Π(Δ), that is, {α_i ∧ β_i >_Π α_i ∧ ¬β_i | i = 1, ..., n} implies ϕ ∧ ψ >_Π ϕ ∧ ¬ψ.

The possibilistic modeling of default rules α → β is also in accordance with conditional objects (since only α ∧ β and α ∧ ¬β are involved in both settings). Let us now develop the possibilistic semantics of conditional objects. In connecting conditional objects to probabilities, Goodman et al. [31] have proved that the entailment between conditional objects, i.e., β|α ⊨_C δ|γ ⇔ β ∧ α ⊨ δ ∧ γ and ¬α ∨ β ⊨ ¬γ ∨ δ, is equivalent to the probabilistic condition, ∨P. P(δ|α) ≤ P(δ|γ) (up to pathological cases).

A similar result holds in possibility theory for positive possibility distributions (i.e., ν such that V'w, r(ν) > 0, or equivalently ν with E_L = 8):

Proposition 11 (see [21]). For any α, γ ≠ ⊥, β|α ⊨_C δ|γ if and only if for all positive possibility distributions such that β ∧ α >_Π ¬β ∧ α holds, δ ∧ γ >_Π ¬δ ∧ γ also holds.

The proof of the "only if" part is an easy matter using the definition of ⊨_CO and the monotonicity and the transitivity of >_Π. The "if part" of the proof is obtained by a proper choice of >_Π. Note that starting from a consistent set of constraints on a comparative possibility relation, it is not possible to predict if the corresponding possibility distribution is positive or not. In particular, a positive solution always exists. 7

In the following, we thus restrict to the set of positive possibility distributions Π^+(Δ) in Π(Δ). This assumption is also useful when comparing to the conditional object inference. Indeed, Δ ⊨_CO α → β whenever α ⊨ β in the classical sense. In particular, when Δ is empty, Δ only entails these conditional tautologies. A necessary condition for identifying the conditional object inference with the possibilistic one is then that

7 Comparative possibility relations such that E_L ≠ ⊥ are useful for modeling strict rules (with no exceptions) in conditional default bases, since, by definition, strict rules make some interpretations impossible. See Benferhat [2].
Proposition 12. \( \forall \alpha \neq \bot, \forall \beta, \alpha \models \beta \) if and only if \( \alpha \land \beta \gg_{\Pi} \alpha \land \neg \beta \) for all \( \gg_{\Pi} \) if and only if \( \alpha \models \beta \). But this is wrong if we choose \( \gg_{\Pi} \) such that \( \Pi(\alpha) = 0 \), i.e., \( \{ \omega \mid \omega \models \alpha \} \subseteq E_{\bot} \). Then:

The following theorem extends Propositions 11 and 12 and shows that the universal possibilistic consequence relation captures (and generates) exactly the same conclusions given by \( \models_{\text{co}} \), i.e.:

Theorem 13 (Soundness and completeness). If a default base \( \Delta \) is consistent, \( \forall \phi \not\models_{\text{co}} \phi \rightarrow \psi \iff \forall \gg_{\Pi} \in \Pi^+(\Delta), \phi \land \psi \gg_{\Pi} \phi \land \neg \psi \) (i.e., \( \iff \phi \models_{\Pi} \psi \)).

The complete proof of the theorem can be found in [21]. The proof is based on the fact that a set of conditional objects \( \{ \beta_{i} \mid \alpha_{i} \mid i = 1, \ldots, n \} \) is consistent (i.e., does not entail a conditional contradiction of the form \( \bot \mid \phi \), or equivalently \( \forall S \subseteq \Delta, QC(S) \) can have the truth value \( T \)), if and only if there exists \( \gg_{\Pi} \in \Pi^+(\Delta) \) such that \( \forall i, \alpha_{i} \land \beta_{i} \gg_{\Pi} \alpha_{i} \land \neg \beta_{i} \). The “only if” part is obtained by showing that if the quasi-conjunction of conditional objects is semantically equivalent to a conditional contradiction, then it leads to violate constraints \( \alpha_{i} \land \beta_{i} \gg_{\Pi} \alpha_{i} \land \neg \beta_{i} \) (using the monotonicity and the transitivity of \( \gg_{\Pi} \)). The “if part” consists in building a comparative distribution \( \gg_{\Pi} \) in a way which is similar to the Z-ranking procedure [42] based on toleration.

Note that possibility distributions in \( \Pi^+(\Delta) \) correspond to what Lehmann and Magidor [38] call “ranked models”. The above theorem is another way of stating the fact that the preferential closure of a set of conditional assertions is the intersection of all its rational extensions in accordance with the inference relation (since each comparative possibility distribution in \( \Pi^+(\Delta) \) generates a rational extension of \( \Delta \)). Farifias del Cerro et al. [23] already proved that the universal possibilistic inference \( \models_{\Pi} \) is preferential. Theorem 13 shows that any preferential consequence relation is of this form.

6. Entailment based on the least specific solution over \( \Pi(\Delta) \)

The possibilistic universal consequence is cautious since there generally exist several comparative possibility distributions compatible with a given default base (and we consider all of them in the entailment). A more adventurous entailment consists in selecting one comparative possibility distribution, the least committed one, using the minimum specificity principle.\(^8\) Namely, we select the possibility distribution compatible with the constraints which assigns to each interpretation the highest possibility value. Let \( \Delta = \{ \alpha_{i} \rightarrow \beta_{i} \mid i = 1, \ldots, n \} \) be a consistent set of conditional assertions, and \( C = \{ C_{i} \mid \alpha_{i} \land \beta_{i} \gg_{\Pi} \alpha_{i} \land \neg \beta_{i} \mid i = 1, \ldots, n \} \) be a set of constraints restricting a family of possibility distributions. An algorithm which constructs the least specific

\(^8\)In the qualitative case, \( \gg_{\Pi} = \{ E_{1}, \ldots, E_{n} \} \) is said to be less specific than \( \gg'_{\Pi} = \{ E'_{1}, \ldots, E'_{m} \} \) iff \( \forall \omega, \) if \( \omega \in E'_{j} \) then \( \omega \in E'_{j} \) with \( j \leq i \).
comparative possibility distribution is given in [4]. It is unique and denoted by \( \succ_{\pi_{LS}} \).

The idea in this algorithm is to try to assign to each world \( \omega \) the highest possibility level (in forming a well-ordered partition) without violating the constraints. Namely, first we select for \( E_1 \) the set of interpretations which do not falsify any rule in \( \Delta \). Next we remove from \( C \) constraints \( C_i \) such that there exists at least one interpretation \( \omega \) in \( E_1 \) which satisfies \( \alpha_i \land \beta_i \). We repeat this procedure to build \( E_2, E_3, \ldots \) until \( \Omega \) becomes empty. It is always possible to assume that \( \succ_{\pi_{LS}} \) is positive.

Then, the nonmonotonic consequence relation is defined by:

**Definition 14.** A formula \( \psi \) is said to be an \( LS \)-consequence of \( \phi \), denoted by \( \phi \models_{\pi_{LS}} \psi \), iff \( \psi \) is a possibilistic consequence of \( \phi \) with respect to the comparative possibility distribution \( \succ_{\pi_{LS}} \).

The ranking of models based on the minimum specificity principle is the same as the one computed by Pearl [42] from his rule-ranking algorithm:

**Proposition 15 (see [4]).** Given a set \( \Delta \) of defaults \( \{\alpha_i \rightarrow \beta_i \mid i = 1, \ldots, n\} \), then the rank ordering function \( \kappa \) on models given by System Z is such that: \( \kappa(\omega) \leq \kappa(\omega') \) iff \( \omega \succ_{\pi_{LS}} \omega' \).

This is not surprising since it is easy to verify from Section 2 that \( \kappa(\omega) = i - 1 \) iff \( \omega \in E_i \).

**Corollary 16.** \( \phi \models_{\perp} \psi \) iff \( \phi \models_{\pi_{LS}} \psi \).

It means that inferences obtained by System Z can be computed via possibilistic logic proof methods, first developed by Dubois, Lang and Prade in 1987 [15]. Possibilistic logic enables classical logic to be extended to layered sets of formulas, where layers express certainty levels. The encoding of the layers is simply achieved by assigning to each formula \( \phi \) of \( K \) a weight \( a \in [0, 1] \), which expresses a constraint of the form \( N(\phi) \geq a \). Inference in possibilistic logic can be achieved by means of the following extension of the resolution principle [15]:

\[
(R) \quad (-\alpha \lor \beta, a), (\alpha \lor \delta, b) \vdash (\beta \lor \delta, \min(a, b)).
\]

The use of this inference rule presupposes that a knowledge base \( K \) be put under clausal form, which turns out to be always possible. A knowledge base \( K \) is said to be inconsistent, if it is possible to derive the contradiction \( \bot \) with a certainty level \( a > 0 \) from \( K \) by successive applications of (R). We denote by \( Inc(K) = \sup\{a, K \vdash (\bot, a)\} \) the degree of inconsistency of \( K \). In order to prove that \( \psi \) syntactically follows from \( K \) and the fact \( \phi \), denoted \( \phi \vdash_K \psi \), an extension of the refutation method has been proposed [16]:

1. Compute \( a = Inc(K \cup \{ \phi, 1 \}) \).
2. Let \( K' = K \cup \{ \phi, 1 \} \cup \{ \neg \psi, 1 \} \). Compute \( b = Inc(K') \). If \( b > a \), then \( \phi \vdash_K \psi \).
To use the possibilistic machinery for default reasoning, we first change each conditional assertion $\alpha_i \rightarrow \beta_i$ in $\Delta$ into a possibilistic formula $\neg \alpha_i \lor \beta_i, a_i$ with $a_i = N(\neg \alpha_i \lor \beta_i)$ and $N$ is a necessity measure based on any numerical representative $\pi_{LS}$ of the least specific possibility distribution representing the default base $\Delta$. Let $K_\Delta$ be the obtained possibilistic knowledge base. Then we can show that $\phi \vdash_{K_\Delta} \psi$ iff $\phi \models_{\pi_{LS}} \psi$ due to the soundness and completeness of possibilistic logic [16,34]. Hence plausible conclusions derived from evidence $\phi$ and generic knowledge $\Delta$ can be computed via possibilistic logic proof methods. It has been shown (Lang [34]) that the possibilistic entailment $\vdash_K$ can be achieved with only $\log(n)$ satisfiability tests, where $n$ is the number of uncertainty levels in $K$. System $Z$ is a tool dedicated to conditional knowledge bases that has the same complexity (Goldszmidt and Pearl [30]), while possibilistic logic was independently developed as a generic inconsistency tolerant theorem prover.

Note that the possibilistic conclusions which are obtained from $K_\Delta$ and $\phi$ are exactly the ones which are classically provable using the consistent part of the knowledge base made of the most certain formulas, i.e., the formulas whose weights are strictly greater than the level of inconsistency of $K_\Delta \cup \{ (\phi, 1) \}$. This means that formulas whose weight is less than $\text{Inc}(K_\Delta \cup \{ (\phi, 1) \})$ are simply inhibited even if they are not involved in any inconsistency. This problem is called the “drowning problem” [3]. For example, given the default base $\Delta = \{ \alpha \rightarrow \beta, \alpha \rightarrow \neg \delta, \beta \rightarrow \delta, \alpha \rightarrow \phi, \phi \rightarrow \psi \}$. We can check that $K_\Delta = \{ (\neg \alpha \lor \beta, a), (\neg \alpha \lor \neg \delta, a), (\neg \beta \lor \delta, b), (\neg \alpha \lor \phi, a), (\neg \phi \lor \psi, b) \}$, with $1 > a > b > 0$. Then $\alpha \vdash_{K_\Delta} \psi$ is not sanctioned, since $\text{Inc}(K_\Delta \cup \{ (\alpha, 1) \}) = b$ and hence $\neg \phi \lor \psi$ is “drowned” (namely in the context $\alpha$ only the consistent subbase $\{ (\neg \alpha \lor \beta, a), (\neg \alpha \lor \neg \delta, a), (\neg \alpha \lor \phi, a) \}$ is used). A particular case of the drowning problem is called by Pearl [42] “the blocking of property inheritance problem”: a subclass cannot inherit any property of a superclass as soon as the subclass is already exceptional with respect to one property of the superclass. This problem points out the main limitation of the various forms of rational closure inference, including the $LS$-consequence.

7. Concluding discussions

Fig. 1 summarizes the connections between the different approaches surveyed in this paper. All the approaches which appear in this figure share the following points:

- From a syntactic point of view: they propose to extend the classical language by adding a new binary symbol (“$\rightarrow$” in System Z, “$|$” in the conditional object-based approach, $>_H$ in the possibilistic approach, etc.) to encode conditional information.
- From a semantic point of view, the approaches compute one or several complete pre-orderings between the classical interpretations. Each such ordering allows to select the preferred interpretations of a default base.

The main contribution of the conditional object approach is to offer a simplification of the preferential semantics proposed by Kraus et al. [32] which is based on a two level structure, i.e., a triple $(S, f, <)$ where $(S, <)$ is a partial ordered set of states and $f$ is a function from states to interpretations.
This paper has also pointed out that the main conditional approaches to nonmonotonic inference, for the purpose of reasoning from generic, exception tolerant, rule-based systems can be captured in the setting of possibility theory. Namely, each nonmonotonic inference relation satisfying acknowledged rationality postulates (including rational monotony) can be encoded in the possibilistic setting. Moreover, we have shown that the possibilistic treatment of conditional information via a principle of minimal commitment, gives the same results as the rational closure of Lehmann, or System Z of Pearl. Hence, it is possible to use generic proof methods of possibilistic logic [16], which are closely related to classical logic (refutation, resolution, semantic evaluation), to compute rational inferences from a default base.

However, all the considered approaches suffer from the same limitations regarding the drowning problem. Some systematic methods for coping with this problem have been proposed, based on refined inconsistency handling (Geffner [27], Benferhat et al. [3,6], Lehmann [37]). But they always face counterexamples and have high complexity [10,40]. A more promising way of overcoming this problem may be to add information about conditional independence of the form "in the context $\alpha$, accepting $\beta$ has no influence on accepting $\gamma$". This kind of information can be represented in the possibility theory framework by the joint constraints: $N(\gamma|\alpha) > 0$ and $N(\gamma|\alpha \land \beta) > 0$, which are of the same form as the constraints modeling default rules. Work is in progress along these lines; see [5,13] for existing proposals.

Lastly, the results of this paper only require an ordinal scale (exemplified by the well-ordered partition). Numbers used in System Z, as well as infinitesimals are not compulsory for the purpose of question-answering in a default base (modeled by one or several possibility distributions) on the basis of available pieces of evidence (modeled in propositional logic). Another issue is that of revising a default base by adding new rules (e.g., Boutilier and Goldszmidt [8]), or equivalently revising a possibility distribution by another one taken as an input (Dubois and Prade [22]). For this higher level revision problem, the question of whether purely ordinal tools are enough is still open.
Acknowledgements

The authors wish to thank Ralph Sobek who kindly helped them with the English.

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