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On some iterated weighted spaces

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Abstract

It is proved that the Hörmander $B^{loc}_{p,k}(\Omega_1 \times \Omega_2)$ and $B^{loc}_{p,k_1}(\Omega_1, B^{loc}_{p,k_2}(\Omega_2))$ spaces ($\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ open sets, $1 \leqslant p < \infty$, k_i Beurling–Björck weights, $k = k_1 \otimes k_2$) are isomorphic whereas the iterated spaces $B^{loc}_{p,k_1}(\Omega_1, B^{loc}_{q,k_2}(\Omega_2))$ and $B^{loc}_{q,k_2}(\Omega_2, B^{loc}_{p,k_1}(\Omega_1))$ are not if $1 < p \neq q < \infty$. A similar result for weighted L_p-spaces of entire analytic functions is also obtained. Finally a result on iterated Besov spaces is given: $B^s_{2,q}(\mathbb{R}^n, B^s_{2,q}(\mathbb{R}^m))$ and $B^s_{2,q}(\mathbb{R}^{n+m})$ are not isomorphic when $1 < q \neq 2 < \infty$.

Keywords: Beurling ultradistributions; Weighted L_p-spaces of entire analytic functions; Hörmander spaces; Besov spaces

1. Introduction and notation

Many iterated spaces of functions or distributions are isomorphic to scalar spaces of the same kind; e.g., $L_p(\mu, L_p(\nu))$ and $L_p(\mu \otimes \nu)$ ($1 \leqslant p < \infty$, μ, ν σ-finite measures), $H_p(\mathbb{D}, H_p(\mathbb{D}))$ and $H_p(\mathbb{D}^2)$ ($1 \leqslant p < \infty$, $\mathbb{D}$ unit disc), $W^s_p(\mathbb{R}^n, W^s_p(\mathbb{R}^m))$ and $W^s_p(\mathbb{R}^{n+m})$ ($1 < p < \infty$, $s = 0, 1, 2, \ldots$) or $D'(\Omega_1, D'(\Omega_2))$ and $D'(\Omega_1 \times \Omega_2)$ ($\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ open sets) are isomorphic. On the contrary, $L_\infty(\mathbb{R}^n, L_\infty(\mathbb{R}^m))$ and $L_\infty(\mathbb{R}^{n+m})$, $\mathrm{BMO}(\mathbb{T}, \mathrm{BMO}(\mathbb{T}))$ and $\mathrm{BMO}(\mathbb{T}^2)$ or $D(\Omega_1, D(\Omega_2))$ and $D(\Omega_1 \times \Omega_2)$ are never isomorphic (see, e.g., [4,6] and [7,12] and [5], respectively). In this paper we extend slightly the kernel theorem for Beurling ultradistributions (see [18, Theorem 2.3]) and as a consequence we obtain results of the former kind for Hörmander $B_{p,k}$ and $B^{loc}_{p,k}(\Omega)$ spaces in the sense of Beurling–Björck [3] (these spaces play a crucial role in the theory of linear partial differential operators, see, e.g., [3,14] and [16]), for weighted L_p-spaces of entire analytic functions $L^K_{p,\rho}$ (these spaces are the building blocks of the corresponding Besov spaces, see [27,30,32] and [24]) and for Besov spaces $B^s_{p,q}$.

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector-valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we show that $D'_\omega(\Omega_1 \times \Omega_2)$ is canonically isomorphic to $L_b(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$ for some weights ω_1, ω_2 and ω

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(see Theorem 3.2). In Section 4 we prove that the restriction of the previous canonical isomorphism to Hörmander–Beurling local space $B^{\mathrm{loc}}_{p,k}(\Omega_1 \times \Omega_2)$ is an isomorphism of this space onto the iterated space $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{p,k_2}(\Omega_2))$ (Theorem 4.5) and that the iterated spaces $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{q,k_2}(\Omega_2))$ and $B^{\mathrm{loc}}_{q,k_2}(\Omega_2, B^{\mathrm{loc}}_{p,k_1}(\Omega_1))$ are not isomorphic if $1 < p \neq q < \infty$ (Theorem 4.9). We also propose the following question: For which weights k_1, k_2 and $q \in]1, \infty]$ the iterated spaces $B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n, B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m))$ and $B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m, B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n))$ are not isomorphic? Are the Banach spaces $l_1(l_\infty)$ and $l_\infty(l_1)$ not isomorphic? In the last section we present a similar result to Theorem 4.5 for weighted L_p-spaces of entire analytic functions. We also give a result on iterated Besov spaces: $B^s_{2,q}(\mathbb{R}^n, B^s_{2,q}(\mathbb{R}^m))$ and $B^s_{2,q}(\mathbb{R}^{n+m})$ are not isomorphic when $-\infty < s < \infty$ and $1 < q \neq 2 < \infty$.

Notation. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $L_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The dual of E is denoted by E' and is given the strong topology so that $E' = L_b(E, \mathbb{C})$. $E^{\mathbb{N}}$ is the topological product of a countable number of copies of E. $\mathcal{B}_b(E, F)$ is the locally convex space of all continuous bilinear forms on $E \times F$ equipped with the bibounded topology. If E or F is sequentially complete, $\mathcal{B}^s_b(E, F)$ denotes the locally convex space of all separately continuous bilinear forms on $E \times F$ with the bibounded topology (see, e.g., [19, p. 167]). $E \hat{\otimes}_\varepsilon F$ (respectively $E \hat{\otimes}_\pi F$) is the completion of the injective (respectively projective) tensor product of E and F. If E and F are (topologically) isomorphic we put $E \simeq F$. If E is isomorphic to a complemented subspace of F we write $E < F$. We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F). If $(E_n)_{n=1}^\infty$ is a sequence of locally convex spaces, $\bigoplus_{n=1}^\infty E_n$ ($E^{(\mathbb{N})}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n. The Fréchet space defined by the projective sequence of Banach spaces E_n and linking maps A_n

$$\cdots \to E_{n+1} \xrightarrow{A_n} E_n \to \cdots \xrightarrow{A_2} E_2 \xrightarrow{A_1} E_1$$

will be denoted by $\mathrm{proj}(E_n, A_n)$.

Let $0 < p \leqslant \infty$, $k : \mathbb{R}^n \to (0, \infty)$ a Lebesgue measurable function, and E a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = (\int_{\mathbb{R}^n} \|f(x)\|^p\,dx)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \mathrm{cs}(E)$ (see, e.g., [11]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|f\|_{p,k} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \mathrm{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$ if $p \geqslant 1$. If $E = \mathrm{proj}(E_i, A_i)$ and $p \geqslant 1$, then $L_{p,k}(E)$ is isomorphic to $\mathrm{proj}(L_{p,k}(E_i), \bar{A}_i)$ via the operator $f \to (P_i \circ f)_{i=1}^\infty$ (P_i is the ith canonical projection from E into E_i and $\bar{A}_i : L_{p,k}(E_{i+1}) \to L_{p,k}(E_i) : g \to A_i \circ g$). When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi x}\,dx$. If f is a function on $\mathbb{R}^n$, then $\tilde{f}(x) = f(-x)$, $(\tau_h f)(x) = f(x-h)$ for $x, h \in R^n$, and B_b is the closed ball $\{x\colon |x| \leqslant b\}$ in $\mathbb{R}^n$. The letter C will always denote a positive constant, not necessarily the same at each occurrence.

Finally we recall the definition of A^*_p functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A^*_p provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|}\int_R \omega\,dx\right)\left(\frac{1}{|R|}\int_R \omega^{-p'/p}\,dx\right)^{p/p'} < \infty,$$

where R runs over all bounded n-dimensional intervals. The basic properties of these functions can be found in [10, Chapter IV].

2. Spaces of vector-valued (Beurling) ultradistributions

In this section we collect some basic facts about vector-valued (Beurling) ultradistributions and we recall the definitions of the vector-valued Hörmander–Beurling spaces and the weighted L_p-spaces of vector-valued entire analytic functions. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [3, 13,17,18] and [19]. Our notations are based on [3] and [27, pp. 14–19].

Let $\mathcal{M}_n$ be the set of all functions ω on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:

(i) $\sigma(0) = 0$,
(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2}\,dt < \infty$ (Beurling's condition),
(iii) there exist a real number a and a positive number b such that

$$\sigma(t) \geqslant a + b\log(1+t) \quad \text{for all } t \geqslant 0.$$

The assumption (ii) is essentially the Denjoy–Carleman non-quasianalyticity condition (see [3, Section 1.5]). The two most prominent examples of functions $\omega \in \mathcal{M}_n$ are given by $\omega(x) = \log(1+|x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}_n$ and E is a Fréchet space, we denote by $D_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(\xi)\| e^{\lambda\omega(\xi)}\,d\xi < \infty$ for all $\lambda > 0$ and for all $\|\cdot\| \in \mathrm{cs}(E)$. For each compact subset K of $\mathbb{R}^n$, $D_\omega(K, E) = \{f \in D_\omega(E)\colon \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda\colon \|\cdot\| \in \mathrm{cs}(E),\ \lambda > 0\}$, is a Fréchet space and $D_\omega(E) = \operatorname{ind}_{\underset{K}{\rightarrow}} D_\omega(K, E)$ becomes a strict (LF)-space. If Ω is any open set in $\mathbb{R}^n$, $D_\omega(\Omega, E)$ is the subspace of $D_\omega(E)$ consisting of all functions f with $\operatorname{supp} f \subset \Omega$. $D_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology: $D_\omega(\Omega, E) = \operatorname{ind}_{\underset{K\subset\Omega}{\rightarrow}} D_\omega(K, E)$. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}\|\partial^\alpha f(x)\| < \infty$ and $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)}\|\partial^\alpha \hat{f}(x)\| < \infty$ for all multi-indices α, all positive numbers λ and all $\|\cdot\| \in \mathrm{cs}(E)$. $S_\omega(E)$ with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $S_\omega(E)$. If $E = \mathbb{C}$, then $D_\omega(E)$ and $S_\omega(E)$ coincide with the spaces D_ω and S_ω (see [3]). Let us recall that, by Beurling's condition, the space D_ω is non-trivial and the usual procedure of the resolution of unity can be established with D_ω-functions (see [3, Theorem 1.3.7]). Furthermore, $D_\omega \overset{d}{\hookrightarrow} D$ (see [3, Theorem 1.3.18]) and D_ω is nuclear [34, Corollary 7.5]. On the other hand, $D_\omega = D \cap S_\omega$, $D_\omega \overset{d}{\hookrightarrow} S_\omega \overset{d}{\hookrightarrow} S$ (see [3, Proposition 1.8.6, Theorem 1.8.7]) and S_ω is nuclear (see [13, p. 320]). If $\mathcal{E}_\omega$ is the set of multipliers on D_ω, i.e., the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi f \in D_\omega$ for all $\varphi \in D_\omega$, then $\mathcal{E}_\omega$ with the topology generated by the seminorms $\{f \to \|\varphi f\|_\lambda = \int_{\mathbb{R}^n} |\widehat{\varphi f}(\xi)| e^{\lambda\omega(\xi)}\,d\xi\colon \lambda > 0,\ \varphi \in D_\omega\}$ becomes a nuclear Fréchet space (see [34, Corollary 7.5]) and $D_\omega \overset{d}{\hookrightarrow} \mathcal{E}_\omega$. Using the above results and [19, Theorem 1.12] we can identify $S_\omega(E)$ with $S_\omega \hat{\otimes}_\varepsilon E$. However, though $D_\omega \otimes E$ is dense in $D_\omega(E)$, in general $D_\omega(E)$ is not isomorphic to $D_\omega \hat{\otimes}_\varepsilon E$ (cf., e.g., [12, Chapter II, p. 83]). A continuous linear operator from D_ω into E is said to be a (Beurling) ultradistribution with values in E. We write $D'_\omega(E)$ for the space of all E-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $D'_\omega(E) = L_b(D_\omega, E)$. $D'_\omega(\Omega, E) = L_b(D_\omega(\Omega), E)$ is the space of all (Beurling) ultradistributions on Ω with values in E. A continuous linear operator from S_ω into E is said to be an E-valued tempered ultradistribution. $S'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology, i.e., $S'_\omega(E) = L_b(S_\omega, E)$. The Fourier transformation $\mathcal{F}$ is an automorphism of $S'_\omega(E)$.

If $\omega \in \mathcal{M}_n$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a positive constant N such that $k(x+y) \leqslant e^{N\omega(x)}k(y)$ for all x and y in $\mathbb{R}^n$ [3, Definition 2.1.1] (when $\omega(x) = \log(1+|x|)$ the functions k of the corresponding class $\mathcal{K}_\omega$ are called temperate weight functions, see [14, Definition 10.1.1]). If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number, then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [3, Theorem 2.1.3]). If $u \in L_1^{\mathrm{loc}}$ and $\int_{\mathbb{R}^n} \varphi(x)u(x)\,dx = 0$ for all $\varphi \in D_\omega$, then $u = 0$ a.e. (see [3]). This result, the Hahn–Banach theorem and [9, Chapter II, Corollary 7] prove that if $k \in \mathcal{K}_\omega$, $p \in [1,\infty]$ and E is a Fréchet space, we can identify $f \in L_{p,k}(E)$ with the E-valued tempered ultradistribution $\varphi \to \langle\varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)\,dx$, $\varphi \in S_\omega$, and $L_{p,k}(E) \hookrightarrow S'_\omega(E)$. If $\omega \in \mathcal{M}_n$, $k \in \mathcal{K}_\omega$, $p \in [1,\infty]$ and E is a Fréchet space, we denote by $B_{p,k}(E)$ the set of all E-valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle\varphi, \widehat{T}\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)\,dx$, $\varphi \in S_\omega$. $B_{p,k}(E)$ with the seminorms $\{\|T\|_{p,k} = ((2\pi)^{-n}\int_{\mathbb{R}^n} \|k(x)\widehat{T}(x)\|^p\,dx)^{1/p}\colon \|\cdot\| \in \mathrm{cs}(E)\}$ (usual modification if $p = \infty$), becomes a Fréchet space isomorphic to $L_{p,k}(E)$. Spaces $B_{p,k}(E)$ are called Hörmander–Beurling spaces with values in E (see [3,14,16] for the scalar case and [24,25,33] for the vector-valued case). We denote by $B_{p,k}^{\mathrm{loc}}(\Omega, E)$ (see [3,14,34] and [23,25,33]) the space of all E-valued ultradistributions $T \in D'_\omega(\Omega, E)$ such that, for every $\varphi \in D_\omega(\Omega)$, the map $\varphi T : S_\omega \to E$ defined by $\langle u, \varphi T\rangle = \langle u\varphi, T\rangle$, $u \in S_\omega$, belongs to $B_{p,k}(E)$. The space $B_{p,k}^{\mathrm{loc}}(\Omega, E)$ is a Fréchet space with the topology generated by the seminorms $\{\|\cdot\|_{p,k,\varphi}\colon \varphi \in D_\omega(\Omega),\ \|\cdot\| \in \mathrm{cs}(E)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in B_{p,k}^{\mathrm{loc}}(\Omega, E)$. We shall also use the spaces $B_{p,k}^c(\Omega, E)$ which generalize the scalar spaces $B_{p,k}^c(\Omega)$ considered by Hörmander in [14], by Vogt in [34] and by Björck in [3]. If ω, k, p, Ω and E are as above, then $B^c_{p,k}(\Omega, E) = \bigcup_{j=1}^{\infty}[B_{p,k}(E) \cap \overline{\mathcal{E}}'_\omega(K_j, E)]$ (here (K_j) is any fundamental sequence of compact subsets of Ω and $\overline{\mathcal{E}}'_\omega(K_j, E)$ denotes the set of all $T \in D'_\omega(E)$ such that $\operatorname{supp} T \subset K_j$). Since for every compact $K \subset \Omega$, $B_{p,k}(E) \cap \overline{\mathcal{E}}'_\omega(K, E)$ is a Fréchet space with the topology induced by $B_{p,k}(E)$, it follows that $B^c_{p,k}(\Omega, E)$ becomes a strict (LF)-space: $B^c_{p,k}(\Omega, E) = \operatorname{ind}_{\underset{j}{\to}} [B_{p,k}(E) \cap \overline{\mathcal{E}}'_\omega(K_j, E)]$. These spaces are studied in [23] and [25].

We conclude this section with the definition of the weighted L_p-spaces of E-valued entire analytic functions $L^K_{p,\rho}(E)$. First we state the vector-valued version of the Paley–Wiener–Schwartz theorem that we shall need (see [3, Theorem 1.8.14], [18, Theorem 1.1] and [27, pp. 18–19] for the scalar case): “Let $\omega \in \mathcal{M}_n$ and let E be a Banach space. If $T \in S'_\omega(E)$ and $\operatorname{supp} \widehat{T} \subset B_b$, then there exist an E-valued entire analytic function $U(\zeta)$ and a real number λ such that for any $\varepsilon > 0$,

$$\left\| U(\xi + i\eta) \right\| \leqslant C_\varepsilon \, e^{(b+\varepsilon)|\eta| + \lambda\omega(\xi)}$$

holds for all $\zeta = \xi + i\eta \in \mathbb{C}^n$ where C_ε depends on ε but not on ζ ($U(\zeta)$ is called an E-valued entire function of exponential type) and such that U represents to T, i.e., such that $\langle \varphi, T \rangle = \int_{\mathbb{R}^n} \varphi(x) U(x)\, dx$ for all $\varphi \in S_\omega$.” Next we recall the definition of $R(\omega)$ given in [30, Definition 1.3.1]. If $\omega \in \mathcal{M}_n$, then $R(\omega)$ denotes the collection of all Borel-measurable real functions $\rho(x)$ on $\mathbb{R}^n$ such that there exists a positive constant c with $0 < \rho(x) \leqslant c\, e^{\omega(x-y)} \rho(y)$ for all $x, y \in \mathbb{R}^n$. If $\rho \in R(\omega)$, $p \in [1, \infty]$ and E is a Banach space, we have the canonical embeddings $S_\omega(E) \hookrightarrow L_{p,\rho}(E) \hookrightarrow S'_\omega(E)$. Finally, we give the definition of the spaces $L^K_{p,\rho}(E)$. Let $\omega \in \mathcal{M}_n$, $\rho \in R(\omega)$, $p \in [1, \infty]$, K a compact set in $\mathbb{R}^n$ and E a Banach space, then

$$L^K_{p,\rho}(E) = \left\{ f \;\middle|\; f \in S'_\omega(E),\ \operatorname{supp} \hat{f} \subset K,\ \|f\|_{L^K_{p,\rho}(E)} = \|f\|_{p,\rho} < \infty \right\}.$$

With the norm $\|\cdot\|_{p,\rho}$, $L^K_{p,\rho}(E)$ becomes a Banach space. We shall write $L^K_{p,\rho}$ when $E = \mathbb{C}$. If $\rho(x) = 1$, then we put $L^K_{p,1}(E) = L^K_p(E)$. If there is a possibility of confusion, the notation $L^K_{p,\rho}(\mathbb{R}^n, E)$, $L^K_{p,\rho}(\mathbb{R}^n)$, $L^K_p(\mathbb{R}^n, E)$ will be used. We shall denote by S^K_ω the collection of all $\varphi \in S_\omega$ such that $\operatorname{supp} \hat{\varphi} \subset K$. The spaces $L^K_{p,\rho}(E)$ are studied in [27,30,32] and [24].

3. On the kernel theorem for ultradistributions

In this section we shall show that if $\omega_1 \in \mathcal{M}_n$, $\omega_2 \in \mathcal{M}_m$ and $\omega \in \mathcal{M}_{n+m}$ satisfy the condition

$$\frac{1}{c}\left[\omega_1(x) + \omega_2(y)\right] \leqslant \omega(x, y) \leqslant c\left[\omega_1(x) + \omega_2(y)\right], \quad (x, y) \in \mathbb{R}^{m+n} \tag{3.1}$$

(c is a constant > 0) and Ω_1 (respectively Ω_2) is an open set in $\mathbb{R}^n$ (respectively $\mathbb{R}^m$), then

$$L_b\big(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2)\big) \simeq D'_\omega(\Omega_1 \times \Omega_2).$$

This result extends slightly the kernel theorem for ultradistributions (see, e.g., [18, Theorem 2.3]) and will be used in the next sections.

Let us now recall that a bounded open Ω in $\mathbb{R}^n$ has the segment property if there exist open balls V_j and vectors $y^j \in \mathbb{R}^n \setminus \{0\}$, $j = 1, \ldots, N$, such that $\overline{\Omega} \subset \bigcup_{j=1}^N V_j$ and $(\overline{\Omega} \cap V_j) + ty^j \subset \Omega$ for $0 < t < 1$ and $j = 1, \ldots, N$. For instance, if Ω is convex or if $\partial\Omega \in C^{0,1}$, then Ω has the segment property. We say that a compact set K in $\mathbb{R}^n$ is regular if $K = \overline{\mathring{K}}$ and $\mathring{K}$ has the segment property (in [18, p. 614] compact regular is said compact with the cone property).

The following lemma is known (see, e.g., [17, pp. 73–75] and [3, Corollary 1.5.15, Theorem 1.5.16]).

Lemma 3.1. *If $\omega \in \mathcal{M}_n$, the set $\mathcal{P}_n$ of all polynomials in $\mathbb{R}^n$ is dense in $\mathcal{E}_\omega$.*

Theorem 3.2. *Suppose that $\omega_1 \in \mathcal{M}_n$, $\omega_2 \in \mathcal{M}_m$ and $\omega \in \mathcal{M}_{n+m}$ satisfy the condition* (3.1)*, that Ω_1* (*respectively Ω_2*) *is an open set in $\mathbb{R}^n$* (*respectively $\mathbb{R}^m$*)*, and that K_1* (*respectively K_2*) *is a regular compact in $\mathbb{R}^n$* (*respectively $\mathbb{R}^m$*)*. Then*

(1) *$D_{\omega_1}(\Omega_1) \otimes D_{\omega_2}(\Omega_2)$ is sequentially dense in $D_\omega(\Omega_1 \times \Omega_2)$.*

(2) $D_{\omega_1}(K_1)\hat{\otimes}_\varepsilon D_{\omega_2}(K_2)$ *is canonically isomorphic to* $D_\omega(K_1 \times K_2)$.
(3) $D'_\omega(\Omega_1 \times \Omega_2)$ *is canonically isomorphic to* $L_b(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$.

Proof. We are going to adapt to our context the proof given by Komatsu in [18, pp. 614–619] of the kernel theorem for ultradistributions.

(1) From (3.1) it follows that $D_{\omega_1}(\Omega_1)\otimes D_{\omega_2}(\Omega_2)$ is a linear subspace of $D_\omega(\Omega_1\times\Omega_2)$. Let then $\phi\in D_\omega(\Omega_1\times\Omega_2)$ and put $L=\operatorname{supp}\phi$, $L_1=\operatorname{proj}_{\Omega_1}L$ and $L_2=\operatorname{proj}_{\Omega_2}L$. By [3, Theorem 1.3.7] we can find functions $\varphi\in D_{\omega_1}(\Omega_1)$, $\psi\in D_{\omega_2}(\Omega_2)$ such that $\varphi\equiv 1$ in a neighborhood of L_1 and $\psi\equiv 1$ in a neighborhood of L_2. Then $\varphi\otimes\psi\in D_{\omega_1}(\Omega_1)\otimes D_{\omega_2}(\Omega_2)$ and $\varphi\otimes\psi\equiv 1$ in a neighborhood of L. Now we choose using Lemma 3.1 a sequence $P_k\in\mathcal{P}_{n+m}$ with $P_k\to\phi$ in $\mathcal{E}_\omega$. Then the functions $(\varphi\otimes\psi)P_k$ are in $D_{\omega_1}(\Omega_1)\otimes D_{\omega_2}(\Omega_2)$ and $(\varphi\otimes\psi)P_k\to(\varphi\otimes\psi)\phi=\phi$ in $D_\omega(\Omega_1\times\Omega_2)$. Thus (1) is proved.

(2) Let us denote by $D_{\omega_1}(K_1)\otimes_\omega D_{\omega_2}(K_2)$ the space $D_{\omega_1}(K_1)\otimes D_{\omega_2}(K_2)$ equipped with the topology induced by $D_\omega(K_1\times K_2)$. From (3.1) it follows that the identity $D_{\omega_1}(K_1)\otimes_\pi D_{\omega_2}(K_2)\to D_{\omega_1}(K_1)\otimes_\omega D_{\omega_2}(K_2)$ is continuous. Let us see that the identity of $D_{\omega_1}(K_1)\otimes_\omega D_{\omega_2}(K_2)$ into $D_{\omega_1}(K_1)\otimes_\varepsilon D_{\omega_2}(K_2)$ is also continuous: Let $\lambda_1,\lambda_2>0$. Let U (respectively V) be the unit ball in $D_{\omega_1}(K_1)$ (respectively $D_{\omega_2}(K_2)$) corresponding to the norm $\|\cdot\|_{\lambda_1}^{(\omega_1)}$ (respectively $\|\cdot\|_{\lambda_2}^{(\omega_2)}$). Then, by using the theorem of bipolars (cf., e.g., [15, p. 149]), we have $\|\varphi\|_{\lambda_1}^{(\omega_1)}=\sup_{u\in U^\circ}|\langle\varphi,u\rangle|$ for all $\varphi\in D_{\omega_1}(K_1)$ and $\|\psi\|_{\lambda_2}^{(\omega_2)}=\sup_{v\in V^\circ}|\langle\psi,v\rangle|$ for all $\psi\in D_{\omega_2}(K_2)$. Therefore, if $\sum_{j=1}^m\varphi_j\otimes\psi_j\in D_{\omega_1}(K_1)\otimes D_{\omega_2}(K_2)$, $u\in U^\circ$ and $v\in V^\circ$, we get by using (3.1) and the Fubini's theorem

$$\begin{aligned}\Big|\sum_j\langle\varphi_j,u\rangle\langle\psi_j,v\rangle\Big| &= \Big|\Big\langle\sum_j\langle\varphi_j,u\rangle\psi_j,v\Big\rangle\Big|\leqslant\Big\|\sum_j\langle\varphi_j,u\rangle\psi_j\Big\|_{\lambda_2}^{(\omega_2)}=\int_{\mathbb{R}^m}\Big|\sum_j\langle\varphi_j,u\rangle\hat{\psi}_j(y)\Big|e^{\lambda_2\omega_2(y)}\,dy\\ &=\int_{\mathbb{R}^m}\Big|\Big\langle\sum_j\hat{\psi}_j(y)\varphi_j,u\Big\rangle\Big|e^{\lambda_2\omega_2(y)}\,dy\leqslant\int_{\mathbb{R}^m}\Big\|\sum_j\hat{\psi}_j(y)\varphi_j\Big\|_{\lambda_1}^{(\omega_1)}e^{\lambda_2\omega_2(y)}\,dy\\ &\leqslant\int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^n}\Big|\sum_j\hat{\varphi}_j(x)\hat{\psi}_j(y)\Big|e^{\lambda_1\omega_1(x)}\,dx\Big)e^{\lambda_2\omega_2(y)}\,dy\\ &\leqslant\int_{\mathbb{R}^{n+m}}\Big|\Big(\sum_j\varphi_j\otimes\psi_j\Big)^{\wedge}(x,y)\Big|e^{c\lambda_3\omega(x,y)}\,dx\,dy\end{aligned}$$

where c is the constant of (3.1) and $\lambda_3=\max(\lambda_1,\lambda_2)$. So

$$\sup_{(u,v)\in U^\circ\times V^\circ}\Big|\sum_{j=1}^m\langle\varphi_j,u\rangle\langle\psi_j,v\rangle\Big|\leqslant\Big\|\sum_{j=1}^m\varphi_j\otimes\psi_j\Big\|_{c\lambda_3}^{(\omega)}$$

which proves the required continuity. Since the ε-topology coincides with the π-topology on $D_{\omega_1}(K_1)\otimes D_{\omega_2}(K_2)$ (by the nuclearity of the spaces $D_{\omega_i}(K_i)$, see Vogt [34, Corollary 7.5]), we conclude that $D_{\omega_1}(K_1)\hat{\otimes}_\varepsilon D_{\omega_2}(K_2)$ is a topological linear subspace of $D_\omega(K_1\times K_2)$. It remains to prove that this subspace coincides with $D_\omega(K_1\times K_2)$. In order to show this, since $D_{\omega_1}(\mathring{K}_1)\otimes D_{\omega_2}(\mathring{K}_2)$ is dense in $D_\omega(\mathring{K}_1\times\mathring{K}_2)$ (step (1)) and the canonical injection of $D_\omega(\mathring{K}_1\times\mathring{K}_2)$ into $D_\omega(K_1\times K_2)$ is continuous, it will be sufficient to prove that $D_\omega(\mathring{K}_1\times\mathring{K}_2)$ is dense in $D_\omega(K_1\times K_2)$. Let then $\phi\in D_\omega(K_1\times K_2)$. Since $K_1\times K_2$ is also a regular compact, there exist open balls V_j in $\mathbb{R}^{n+m}$ and vectors $(x^j,y^j)\in\mathbb{R}^{n+m}\setminus\{0\}$, $j=1,\dots,N$, such that $K_1\times K_2\subset\bigcup_{j=1}^N V_j$ and $(K_1\times K_2\cap V_j)+t(x^j,y^j)\subset\mathring{K}_1\times\mathring{K}_2$ for $0<t<1$ and $j=1,\dots,N$. Therefore, if $(\phi_j)_{j=1}^N$ is a D_ω-partition of unity at $K_1\times K_2$ subordinate to the covering $\{V_1,\dots,V_N\}$ (see [3, Theorem 1.3.7]), the functions $\tau_{t(x^j,y^j)}(\phi\phi_j)$ are in $D_\omega(\mathring{K}_1\times\mathring{K}_2)$ and $\sum_{j=1}^N\tau_{t(x^j,y^j)}(\phi\phi_j)\to\sum_{j=1}^N\phi\phi_j=\phi$ in $D_\omega(K_1\times K_2)$ when $t\to 0+$. This completes the proof of (2).

(3) Let $(K_j^1)_{j=1}^\infty$ (respectively $(K_j^2)_{j=1}^\infty$) be a fundamental sequence of regular compacts in Ω_1 (respectively Ω_2). Then $(K_j^1\times K_j^2)_{j=1}^\infty$ is a fundamental sequence of regular compacts in $\Omega_1\times\Omega_2$ and, by (2) and [28, Proposition 50.7], we have the canonical isomorphisms

$$\big(D_\omega\big(K_j^1\times K_j^2\big)\big)'\simeq\big(D_{\omega_1}\big(K_j^1\big)\hat{\otimes}_\varepsilon D_{\omega_2}\big(K_j^2\big)\big)'\simeq\mathcal{B}_b\big(D_{\omega_1}\big(K_j^1\big),D_{\omega_2}\big(K_j^2\big)\big).\tag{3.2}$$

Now we shall prove that the linear map

$$\begin{aligned} \iota:\quad & D'_\omega(\Omega_1 \times \Omega_2) \to \mathcal{B}^s_b\big(D_{\omega_1}(\Omega_1), D_{\omega_2}(\Omega_2)\big) \\ & \qquad u \to \iota(u)(\varphi, \psi) = \langle \varphi \otimes \psi, u \rangle \end{aligned}$$

(ι is well defined since the bilinear map $D_{\omega_1}(\Omega_1) \times D_{\omega_2}(\Omega_2) \to D_\omega(\Omega_1 \times \Omega_2) : (\varphi, \psi) \to \varphi \times \psi$ is separately continuous) is an isomorphism. That ι is one-to-one follows from (1). Now assume that $U \in \mathcal{B}^s(D_{\omega_1}(\Omega_1), D_{\omega_2}(\Omega_2))$. Then $U|_{D_{\omega_1}(K_j^1) \times D_{\omega_2}(K_j^2)} \in \mathcal{B}^s(D_{\omega_1}(K_j^1), D_{\omega_2}(K_j^2))$ and, since every separately continuous bilinear form in a product of Fréchet spaces is continuous [28, Corollary, p. 354], we can find (see (3.2)) $u_{K_j^1 \times K_j^2} \in (D_\omega(K_j^1 \times K_j^2))'$ such that $U(\varphi, \psi) = \langle \varphi \otimes \psi, u_{K_j^1 \times K_j^2} \rangle$ for all $\varphi \in D_{\omega_1}(K_j^1)$ and for all $\psi \in D_{\omega_2}(K_j^2)$. So we construct $u \in D'_\omega(\Omega_1 \times \Omega_2)$ such that $\iota(u) = U$, and ι is onto. If A (respectively B) is a bounded set in $D_{\omega_1}(\Omega_1)$ (respectively $D_{\omega_2}(\Omega_2)$), then, by [28, Proposition 14.6], there is a sufficiently large j such that A (respectively B) is contained and is bounded in $D_{\omega_1}(K_j^1)$ (respectively $D_{\omega_2}(K_j^2)$). Conversely, if M is bounded in $D_\omega(\Omega_1 \times \Omega_2)$ there exists $K_j^1 \times K_j^2$ [28, Proposition 14.6] such that M is contained and is bounded in $D_\omega(K_j^1 \times K_j^2)$. Since the spaces $D_{\omega_i}(K_j^i)$, $i = 1, 2$, are nuclear [34, Corollary 7.5], (2) and [12, Chapter II] prove that $M \subset \overline{\Gamma A \otimes B}$ being A (respectively B) a bounded set in $D_{\omega_1}(K_j^1)$ (respectively $D_{\omega_2}(K_j^2)$). It is an immediate consequence of these results that ι and ι^{-1} are continuous, that is, that ι is an isomorphism. Finally, we can argue exactly as in [18, p. 618] and obtain the canonical isomorphism $\mathcal{B}^s_b(D_{\omega_1}(\Omega_1), D_{\omega_2}(\Omega_2)) \simeq L_b(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$. □

Corollary 3.3. *If $\omega_1 \in \mathcal{M}_n$, $\omega_2 \in \mathcal{M}_m$ and $\omega \in \mathcal{M}_{n+m}$ satisfy the condition* (3.1), *then $S_{\omega_1} \otimes S_{\omega_2}$ is dense in S_ω.*

Proof. Since the canonical injection of D_ω into S_ω is continuous, it is enough to take into account that D_ω is dense in S_ω (see [3, Theorem 1.8.7]) and that $D_{\omega_1} \otimes D_{\omega_2}$ is dense in D_ω (step (1) of Theorem 3.2). □

4. Iterated Hörmander–Beurling local spaces

In this section we shall show that if Ω_1 (respectively Ω_2) is an open set in $\mathbb{R}^n$ (respectively $\mathbb{R}^m$), ω_1, ω_2 and ω are as in Section 3, $k_1 \in \mathcal{K}_{\omega_1}$, $k_2 \in \mathcal{K}_{\omega_2}$ $k = k_1 \otimes k_2$ and $1 \leqslant p < \infty$, then the restriction of the canonical isomorphism $D'_\omega(\Omega_1 \times \Omega_2) \simeq L_b(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$ (see Theorem 3.2) to Hörmander–Beurling local space $B^{\mathrm{loc}}_{p,k}(\Omega_1 \times \Omega_2)$ is an isomorphism of this space onto the iterated space $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{p,k_2}(\Omega_2))$ and that the iterated spaces $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{q,k_2}(\Omega_2))$ and $B^{\mathrm{loc}}_{q,k_2}(\Omega_2, B^{\mathrm{loc}}_{p,k_1}(\Omega_1))$ are not isomorphic if $1 < p \neq q < \infty$.

In what follows we shall denote by R the canonical isomorphism $D'_\omega(\Omega_1 \times \Omega_2) \to L_b(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2)) : u \to R(u)(\varphi)(\psi) = u(\varphi \otimes \psi)$ (Theorem 3.2). If $\Omega_1 = \mathbb{R}^n$ and $\Omega_2 = \mathbb{R}^m$, then we put R_1 instead of R. It is easily seen that the restriction of R_1 to S'_ω becomes a continuous operator from S'_ω to $L_b(S_{\omega_1}, S'_{\omega_2})$. If we denote by R_2 this restriction, we have the commutative diagram

$$\begin{array}{ccc} D'_\omega & \xrightarrow{R_1} & L_b(D_{\omega_1}, D'_{\omega_2}) \\ \uparrow & & \uparrow \\ S'_\omega & \xrightarrow{R_2} & L_b(S_{\omega_1}, S'_{\omega_2}) \end{array}$$

where the vertical arrows are the canonical injections.

Lemma 4.1. *Let ω_1, ω_2, ω, k_1, k_2, k and p as above. Then the Hörmander–Beurling space $B_{p,k}$ is isometrically isomorphic to the iterated space $B_{p,k_1}(B_{p,k_2})$ via the canonical isomorphism R_1.*

Proof. By (3.1), $k \in \mathcal{K}_\omega$. Now consider the diagram

$$\begin{array}{ccccc} B_{p,k} & & \xrightarrow{R_3} & & B_{p,k_1}(B_{p,k_2}) \\ D \downarrow & & & & \uparrow A \\ L_{p,k} & \xrightarrow{C} & L_{p,k_1}(L_{p,k_2}) & \xrightarrow{B} & B_{p,k_1}(L_{p,k_2}) \end{array}$$

where D is $(2\pi)^{-(n+m)/p}\mathcal{F}$ ($\mathcal{F}$ is the Fourier transform in S'_ω), C is defined by $Cf(x)(y) = f(x,y)$, B is $(2\pi)^{n/p}\mathcal{F}^{-1}$ (here $\mathcal{F}$ is the Fourier transform in $S'_{\omega_1}(L_{p,k_2})$), and A is defined by $A(T) = (2\pi)^{m/p}\mathcal{F}^{-1} \circ T$ ($\mathcal{F}$ being the Fourier transform in S'_{ω_2}). Since all these operators are isometrical isomorphisms, their composition R_3 is also an isometrical isomorphism. It remains to prove that the diagram

$$\begin{array}{ccc} S'_\omega & \xrightarrow{R_2} & L_b(S_{\omega_1}, S'_{\omega_2}) \\ \uparrow & & \uparrow \\ B_{p,k} & \xrightarrow{R_3} & B_{p,k_1}(B_{p,k_2}) \end{array}$$

is commutative (here the vertical arrows are the canonical injections). For this, since the canonical injections and R_2 and R_3 are continuous operators and $S_{\omega_1} \otimes S_{\omega_2}$ is dense in $B_{p,k}$ (in view of Corollary 3.3 and [3, Theorem 2.2.3]), it will be sufficient to show that $R_3(\varphi_0 \otimes \psi_0)(\varphi)(\psi) = R_2(\varphi_0 \otimes \psi_0)(\varphi)(\psi)$ for all $\varphi_0, \varphi \in S_{\omega_1}$ and for all $\psi_0, \psi \in S_{\omega_2}$,

$$\begin{aligned}
R_3(\varphi_0 \otimes \psi_0)(\varphi)(\psi) &= \big[\big(ABCD(\varphi_0 \otimes \psi_0)\big)(\psi)\big](\psi) \\
&= (2\pi)^{-(n+m)/p}\big[\big(ABC(\hat{\varphi}_0 \otimes \hat{\psi}_0)\big)(\varphi)\big](\psi) \\
&= (2\pi)^{-(n+m)/p}\big[\big(AB\big(\hat{\varphi}_0(\cdot)\hat{\psi}_0\big)\big)(\varphi)\big](\psi) \\
&= \big[\big(\mathcal{F}^{-1} \circ \big(\mathcal{F}^{-1}\big(\hat{\varphi}_0(\cdot)\hat{\psi}_0\big)\big)\big)(\varphi)\big](\psi) \\
&= \left[\mathcal{F}^{-1}\left(\int_{\mathbb{R}^n} \mathcal{F}^{-1}\varphi(x)\hat{\varphi}_0(x)\hat{\psi}_0\,dx\right)\right](\psi) \\
&= \big[\mathcal{F}^{-1}\big(\langle\varphi, \varphi_0\rangle\hat{\psi}_0\big)\big](\psi) \\
&= \big[\langle\varphi, \varphi_0\rangle\psi_0\big](\psi) \\
&= \langle\varphi, \varphi_0\rangle\langle\psi, \psi_0\rangle \\
&= \langle\varphi \otimes \psi, \varphi_0 \otimes \psi_0\rangle \\
&= R_2(\varphi_0 \otimes \psi_0)(\varphi)(\psi).
\end{aligned}$$

Thus the lemma is proved. □

Remark 4.2. In the case $p = \infty$, Lemma 4.1 is false. In fact, the spaces $B_{\infty,k}$ and $B_{\infty,k_1}(B_{\infty,k_2})$ not even are isomorphic: By virtue of [6, Theorem 5.1.5], the space $B_{\infty,k_1}(B_{\infty,k_2}) \simeq L_\infty(R^n, L_\infty(\mathbb{R}^m))$ contains a complemented copy of c_0, however the space $B_{\infty,k} \simeq L_\infty(\mathbb{R}^{n+m}) \simeq l_\infty$ has no complemented copies of c_0 by a classical result of Phillips (see, e.g., [6, Corollary 1.3.2]).

Let Ω be an open set in $\mathbb{R}^n$ and let $\omega \in \mathcal{M}_n$, $k \in \mathcal{K}_\omega$ and $1 \leqslant p \leqslant \infty$. Let $(K_j)_{j=1}^\infty$ be a fundamental sequence of compacts in Ω and, for each j, let $\varphi_j \in D_\omega(\mathring{K}_{j+1})$ such that $\varphi_j = 1$ on K_j. Let Y_j be the closure of $\{\varphi_j u\colon u \in B_{p,k}\}$ in $B_{p,k}$ and let B_j be the continuous extension to Y_{j+1} of the operator $\varphi_{j+1}u \to \varphi_j u$ (this operator is continuous since, by [3, Theorem 2.2.7], $\|\varphi_j u\|_{p,k} = \|\varphi_j(\varphi_{j+1}u)\|_{p,k} \leqslant \|\varphi_j\|_{1,M_k}\|\varphi_{j+1}u\|_{p,k}$ for all $u \in B_{p,k}$). Then the following lemma holds:

Lemma 4.3. *The map $T : B_{p,k}^{\mathrm{loc}}(\Omega) \to \operatorname{proj}(Y_j, B_j)$ defined by $T(u) = (\varphi_j u)_{j=1}^\infty$ is an isomorphism.*

Proof. If $u \in B_{p,k}^{\mathrm{loc}}(\Omega)$, then $\varphi_{j+1}u \in B_{p,k}$ and $\varphi_j u = \varphi_j(\varphi_{j+1}u) \in Y_j$. Furthermore, $B_j(\varphi_{j+1}u) = B_j[\varphi_{j+1}(\varphi_{j+2}u)] = \varphi_j(\varphi_{j+2}u) = \varphi_j u$ and so T is a well-defined operator. Moreover, since the seminorms $\|\cdot\|_{p,k,\varphi_j}$ generate the topology of $B_{p,k}^{\mathrm{loc}}(\Omega)$, T becomes an isomorphism from $B_{p,k}^{\mathrm{loc}}(\Omega)$ onto $\operatorname{Im} T$. In consequence, $\operatorname{Im} T$ is a closed subspace of $\operatorname{proj}(Y_j, B_j)$. Let us see that $\operatorname{Im} T$ coincides with $\operatorname{proj}(Y_j, B_j)$. First recall that the seminorms $\|(y_j)_1^\infty\|_N^* = \sum_{j=1}^N \|y_j\|_{p,k}$, $N = 1, 2, \ldots$, generate the topology of $\operatorname{proj}(Y_j, B_j)$ (see [20, p. 230]). Then fix $(y_j) \in \operatorname{proj}(Y_j, B_j)$ and take $\varepsilon > 0$ and $N \geqslant 1$. Put $C = 1 + \sum_{j=1}^{N-1}\prod_{l=j}^{N-1}\|\varphi_l\|_{1,M_k}$ and choose $v \in B_{p,k}$ such that $\|y_N - \varphi_N v\|_{p,k} < \frac{\varepsilon}{C}$. Then $u = v|_{D_\omega(\Omega)} \in B_{p,k}^{\mathrm{loc}}(\Omega)$ and $\varphi_j u = \varphi_j v$ for all j. Thus, using Theorem 2.2.7 of [3], we get

$$\|y_j - \varphi_j u\|_{p,k} = \big\|B_j(y_{j+1}) - B_j(\varphi_{j+1}u)\big\|_{p,k} \leqslant \|B_j\|\|y_{j+1} - \varphi_{j+1}u\|_{p,k} \leqslant \|\varphi_j\|_{1,M_k}\|y_{j+1} - \varphi_{j+1}u\|_{p,k}$$
$$\leqslant \cdots \leqslant \|\varphi_j\|_{1,M_k}\cdots\|\varphi_{N-1}\|_{1,M_k}\|y_N - \varphi_N u\|_{p,k}, \quad j = 1, \ldots, N-1,$$

and so

$$\big\|(y_j) - T(u)\big\|_N^* = \sum_{j=1}^N \|y_j - \varphi_j u\|_{p,k} < \varepsilon.$$

This proves that $\operatorname{Im} T$ is dense in $\operatorname{proj}(Y_j, B_j)$. Thus $\operatorname{Im} T = \operatorname{proj}(Y_j, B_j)$ as we required. □

Lemma 4.4. *Let X be a Banach space, Y be a closed linear subspace of X and $f \in L_1^{\mathrm{loc}}(X)$ such that $\int_{\mathbb{R}^n} \varphi(x) f(x)\,dx \in Y$ for every $\varphi \in D_\omega$ ($\omega \in \mathcal{M}_n$). Then, $f(x) \in Y$ for a.e. x.*

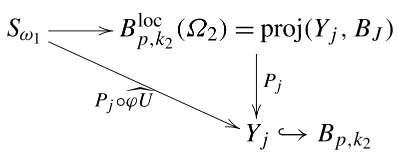
Proof. If $\pi : X \to X/Y$ is the quotient map, then $\int_{\mathbb{R}^n} \varphi(x)\pi(f(x))\,dx = \pi(\int_{\mathbb{R}^n} \varphi(x) f(x)\,dx) = 0$ for every $\varphi \in D_\omega$ and so $\int_{\mathbb{R}^n} \varphi(x)\langle\pi(f(x)), u\rangle\,dx = 0$ for all $u \in (X/Y)'$ and for all $\varphi \in D_\omega$. This implies, by [3, Theorem 1.3.18], that $u \circ (\pi \circ f) = 0$ a.e. for all $u \in (X/Y)'$. Then, applying [9, Corollary 7, p. 48], we conclude that $\pi(f(x)) = 0$ for a.e. x, i.e., that $f(x) \in Y$ for a.e. x. □

Theorem 4.5. *If Ω_1 (respectively Ω_2) is an open set in $\mathbb{R}^n$ (respectively $\mathbb{R}^m$), $\omega_1 \in \mathcal{M}_n$, $\omega_2 \in \mathcal{M}_m$ and $\omega \in \mathcal{M}_{n+m}$ satisfy (3.1), $k_1 \in \mathcal{K}_{\omega_1}$, $k_2 \in \mathcal{K}_{\omega_2}$, $k = k_1 \otimes k_2$ and $1 \leqslant p < \infty$, then the restriction of the canonical isomorphism R to $B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$ is an isomorphism of this space onto the iterated space $B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$.*

Proof. *Step* 1. We denote the restriction of R to $B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$ by R^{loc}. Let $u \in B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$ and put $U = R^{\mathrm{loc}}(u)$. Let us see that $U \in B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$. Fix $\varphi \in D_{\omega_1}(\Omega_1)$ and choose $\varphi_0 \in D_{\omega_1}(\Omega_1)$ so that $\varphi_0 = 1$ on $\operatorname{supp}\varphi$. By Theorem 3.2, $U(\varphi) \in D'_{\omega_2}(\Omega_2)$. Moreover, for every $\psi \in D_{\omega_2}(\Omega_2)$ we have (see the proof of Lemma 4.1)

$$\begin{aligned}\big[\psi U(\varphi)\big]^\wedge(\theta) &= \big[\psi U(\varphi)\big](\hat\theta) = U(\varphi)(\psi\hat\theta) = u(\varphi \otimes \psi\hat\theta) = u(\varphi\varphi_0 \otimes \psi\hat\theta) = u\big[(\varphi \otimes \psi)(\varphi_0 \otimes \hat\theta)\big]\\ &= \big[(\varphi \otimes \psi)u\big](\varphi_0 \otimes \hat\theta) = R_2\big[(\varphi \otimes \psi)u\big](\varphi_0)(\hat\theta) = \big[R_2\big[(\varphi \otimes \psi)u\big](\varphi_0)\big]^\wedge(\theta)\\ &= \big[R_3\big[(\varphi \otimes \psi)u\big](\varphi_0)\big]^\wedge(\theta)\end{aligned}$$

for all $\theta \in S_{\omega_2}$. Hence it follows that the ultradistributions $\psi U(\varphi)$ and $R_3[(\varphi \otimes \psi)u](\varphi_0)$ coincide, and so $\psi U(\varphi) \in B_{p,k_2}$. Consequently, $U(\varphi) \in B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$ and U is an operator from $D_{\omega_1}(\Omega_1)$ into $B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$. Let us see that it is continuous. Let $\phi_j \to \phi$ in $D_{\omega_1}(\Omega_1)$ and let $U(\phi_j) \to v$ in $B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$. Then $U(\phi_j) \to U(\phi)$ in $D'_{\omega_2}(\Omega_2)$, since $U \in L(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$. On the other hand, $U(\phi_j) \to v$ in $D'_{\omega_2}(\Omega_2)$ since $B_{p,k_2}^{\mathrm{loc}}(\Omega_2) \hookrightarrow D'_{\omega_2}(\Omega_2)$ [3, Theorem 2.3.5]. Therefore, $U(\phi) = v$. This proves that U is sequentially closed, and the Grothendieck's closed-graph theorem [12, Chapter I, p. 17] gives the desired continuity. Whence it follows that φU and $\widehat{\varphi U}$ are continuous operators from S_{ω_1} into $B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$. Next it will be shown that $\widehat{\varphi U} \in L_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$. To do this, we first identify $B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$ with the projective limit $\operatorname{proj}(Y_j, B_j)$ (see Lemma 4.3: if $(K_2^j)_{j=1}^\infty$ is a fundamental sequence of compacts in Ω_2 and, for each j, $\psi_j \in D_{\omega_2}(\mathring{K}_2^{j+1})$ and $\psi_j = 1$ on K_2^j, then Y_j is the closure of $\{\psi_j v\colon v \in B_{p,k_2}\}$ in B_{p,k_2}, B_j is the continuous extension to Y_{j+1} of the operator $\psi_{j+1}v \to \psi_j v$ and P_j is the jth canonical projection from $\operatorname{proj}(Y_j, B_j)$ into Y_j). Then the operator $f \to (P_j \circ f)_{j=1}^\infty$ is an isomorphism from $L_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$ onto $\operatorname{proj}(L_{p,k_1}(Y_j), \overline{B}_j)$ (see Section 1). Let us see that the operators $P_j \circ \widehat{\varphi U}$ and $[R_3[(\varphi \otimes \psi_j)u]]^\wedge$ (see Lemma 4.1)



coincide. In fact, for each $\theta \in S_{\omega_1}$, we have $(P_j \circ \widehat{\varphi U})(\theta) = \psi_j \widehat{\varphi U}(\theta) = \psi_j U(\hat{\theta}\varphi)$ and $[R_3[(\varphi \otimes \psi_j)u]]^\wedge(\theta) = R_3[(\varphi \otimes \psi_j)u](\hat{\theta})$ and then, for each $\zeta \in S_{\omega_2}$, we get $(P_j \circ \widehat{\varphi U})(\theta)(\zeta) = [R_3[(\varphi \otimes \psi_j)u]]^\wedge(\theta)(\zeta) = u(\varphi\hat{\theta} \otimes \psi_j \zeta)$ as we required. Now let f_j be the function in $L_{p,k_1}(B_{p,k_2})$ which represents to $[R_3[(\varphi \otimes \psi_j)u]]^\wedge$, that is, such that

$$(P_j \circ \widehat{\varphi U})(\theta) = \big[R_3\big[(\varphi \otimes \psi_j)u\big]\big]^\wedge(\theta) = \int_{\mathbb{R}^n} \theta(x) f_j(x)\, dx, \quad \theta \in S_{\omega_1}.$$

Then this integral lies in the subspace Y_j of B_{p,k_2} and so, by Lemma 4.4, $f_j \in L_{p,k_1}(Y_j)$. Let us check that $(f_j)_{j=1}^\infty \in \operatorname{proj}(L_{p,k_1}(Y_j), \overline{B}_j)$. For each j we have

$$\int_{\mathbb{R}^n} \theta(x) B_j\big(f_{j+1}(x)\big)\, dx = B_j\big[(P_{j+1} \circ \widehat{\varphi U})(\theta)\big] = B_j\big[\psi_{j+1} U(\hat{\theta}\varphi)\big] = \psi_j U(\hat{\theta}\varphi) = (P_j \circ \widehat{\varphi U})(\theta)$$
$$= \int_{\mathbb{R}^n} \theta(x) f_j(x)\, dx, \quad \theta \in S_{\omega_1},$$

and hence $B_j(f_{j+1}(x)) = f_j(x)$ for a.e. x, that is, $\overline{B}_j(f_{j+1}) = f_j$ by Lemma 4.4. In consequence, the function $f(x) = (f_j(x))_{j=1}^\infty$ is in $L_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$, that is, $\widehat{\varphi U} \in L_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$. Definitionnitively, $U \in B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$ and R^{loc} is an operator from $B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$ into $B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$.

Step 2. Naturally R^{loc} is one-to-one, let us see that it is onto. Let $U \in B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$. Since $B_{p,k_2}^{\mathrm{loc}}(\Omega_2) \hookrightarrow D'_{\omega_2}(\Omega_2)$, $U \in L(D_{\omega_1}(\Omega_1), D'_{\omega_2}(\Omega_2))$ and so, by Theorem 3.2, we can find $u \in D'_\omega(\Omega_1 \times \Omega_2)$ such that $U(\varphi)(\psi) = u(\varphi \otimes \psi)$ for all $\varphi \in D_{\omega_1}(\Omega_1)$ and all $\psi \in D_{\omega_2}(\Omega_2)$. We next prove that $(\varphi \otimes \psi)u \in B_{p,k}$ for each $\varphi \in D_{\omega_1}(\Omega_1)$ and each $\psi \in D_{\omega_2}(\Omega_2)$, and then, that $\phi u \in B_{p,k}$ for each $\phi \in D_\omega(\Omega_1 \times \Omega_2)$. Fix φ and ψ. Then $\varphi U \in B_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$, that is, $\widehat{\varphi U} \in L_{p,k_1}(B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$, and the function $F = M_\psi \circ \widehat{\varphi U}$ (M_ψ is the operator $v \to \psi v$ from $B_{p,k_2}^{\mathrm{loc}}(\Omega_2)$ into $B_{p,k_2}(\Omega_2)$) is in $L_{p,k_1}(B_{p,k_2})$ since it is Bochner measurable ($\widehat{\varphi U}$ is Bochner measurable and M_ψ is linear and continuous) and $\int_{\mathbb{R}^n} \|F(x)\|_{p,k_2}^p k_1^p(x)\, dx = \int_{\mathbb{R}^n} \|\psi \widehat{\varphi U}(x)\|_{p,k_2}^p k_1^p(x)\, dx = \int_{\mathbb{R}^n} \|\widehat{\varphi U}(x)\|_{p,k_2,\psi}^p k_1^p(x)\, dx < \infty$. If we prove that $[R_2[(\varphi \otimes \psi)u]]^\wedge = F$ (as elements of $L(S_{\omega_1}, S'_{\omega_2})$) then $R_2[(\varphi \otimes \psi)u] \in B_{p,k_1}(B_{p,k_2})$ and so, by Lemma 4.1, $(\varphi \otimes \psi)u \in B_{p,k}$. For all $f \in S_{\omega_1}$ and all $g \in S_{\omega_2}$ we get

$$\begin{aligned}
\big[R_2\big[(\varphi \otimes \psi)u\big]\big]^\wedge(f)(g) &= \big[R_2\big[(\varphi \otimes \psi)u\big]\big](\hat{f})(g) = \big[(\varphi \otimes \psi)u\big](\hat{f} \otimes g) = u(\varphi \hat{f} \otimes \psi g) \\
&= U(\varphi \hat{f})(\psi g) = \big[\psi U(\varphi \hat{f})\big](g) = \big[\psi(\varphi U)(\hat{f})\big](g) = \big[\psi \widehat{\varphi U}(f)\big](g) \\
&= \left[\psi \int_{\mathbb{R}^n} \widehat{\varphi U}(x) f(x)\, dx\right](g) = \left[\int_{\mathbb{R}^n} \psi \widehat{\varphi U}(x) f(x)\, dx\right](g) \\
&= \left[\int_{\mathbb{R}^n} F(x) f(x)\, dx\right](g) = F(f)(g),
\end{aligned}$$

and this establishes the required equality. To prove that $\phi u \in B_{p,k}$ for all $\phi \in D_\omega(\Omega_1 \times \Omega_2)$, we reason as follows. Given such a ϕ, let K_1, K_2 be regular compacts such that $\phi \in D_\omega(K_1 \times K_2)$ and let us see that the bilinear map $J_u : D_{\omega_1}(K_1) \times D_{\omega_2}(K_2) \to B_{p,k}$ defined by $J_u(\varphi, \psi) = (\varphi \otimes \psi)u$ is continuous. Since the $D_{\omega_i}(K_i)$ are Fréchet spaces, it will be sufficient to prove that J_u is separately continuous [28, Corollary, p. 354]. Supose that $\varphi_j \to \varphi$ in $D_{\omega_1}(K_1)$ and $(\varphi_j \otimes \psi)u \to v$ in $B_{p,k}$. Then $\varphi_j \otimes \psi \to \varphi \otimes \psi$ in $D_\omega(K_1 \times K_2)$ and $(\varphi_j \otimes \psi)u \to (\varphi \otimes \psi)u$ in S'_ω. Since $B_{p,k} \hookrightarrow S'_\omega$, it results that $v = (\varphi \otimes \psi)u$. In consequence, the map $\varphi \to (\varphi \otimes \psi)u$ is closed and therefore continuous by the closed-graph theorem [28, Corollary 4, p. 173]. The argument for the map $\psi \to (\varphi \otimes \psi)u$ is just the same. Then the linearization of J_u extends to a continuous operator $\bar{J}_u$ from $D_{\omega_1}(K_1) \hat{\otimes}_\pi D_{\omega_2}(K_2)$ into $B_{p,k}$, that is, to a continuous operator $\bar{J}_u$ from $D_\omega(K_1 \times K_2)$ into $B_{p,k}$ (see Theorem 3.2). Now it is immediate to verify that $\bar{J}_u(\phi) = \phi u$. Consequently, $\phi u \in B_{p,k}$ and $u \in B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$. Since obviously $R^{\mathrm{loc}}(u) = U$, the map R^{loc} is onto.

Step 3. We show that R^{loc} is an isomorphism. To do this, we use the graph-closed theorem [28, Corollary 4, p. 173] again. Assume that $u_j \to u$ in $B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2)$ and $R^{\mathrm{loc}}(u_j) \to v$ in $B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$. By virtue of the embeddings $B_{p,k_1}^{\mathrm{loc}}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2)) \hookrightarrow D'_{\omega_1}(\Omega_1, B_{p,k_2}^{\mathrm{loc}}(\Omega_2))$, $B_{p,k_2}^{\mathrm{loc}}(\Omega_2) \hookrightarrow D'_{\omega_2}(\Omega_2)$ and $B_{p,k}^{\mathrm{loc}}(\Omega_1 \times \Omega_2) \hookrightarrow D'_\omega(\Omega_1 \times \Omega_2)$

we get for all $\varphi \in D_{\omega_1}(\Omega_1)$ and all $\psi \in D_{\omega_2}(\Omega_2)$

$$R^{\mathrm{loc}}(u_j)(\varphi) \to v(\varphi) \quad \text{in } B^{\mathrm{loc}}_{p,k_2}(\Omega_2),$$
$$R^{\mathrm{loc}}(u_j)(\varphi)(\psi) \to v(\varphi)(\psi),$$
$$R^{\mathrm{loc}}(u_j)(\varphi)(\psi) = u_j(\varphi \otimes \psi) \to u(\varphi \otimes \psi),$$

thus $R^{\mathrm{loc}}(u) = v$. Hence it follows, since our local spaces are Fréchet spaces, that R^{loc} is continuous. Finally, we apply the open mapping theorem [28, Theorem 17.1]. □

Using Theorem 4.5 and the natural isomorphism $B^{\mathrm{loc}}_{p,k_1\otimes k_2}(\Omega_1 \times \Omega_2) \simeq B^{\mathrm{loc}}_{p,k_2\otimes k_1}(\Omega_2 \times \Omega_1)$, one may immediately obtain the isomorphism $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{p,k_2}(\Omega_2)) \simeq B^{\mathrm{loc}}_{p,k_2}(\Omega_2, B^{\mathrm{loc}}_{p,k_1}(\Omega_1))$. Next we shall prove that if $p \neq q$, then, in general, the spaces $B^{\mathrm{loc}}_{p,k_1}(\Omega_1, B^{\mathrm{loc}}_{q,k_2}(\Omega_2))$ and $B^{\mathrm{loc}}_{q,k_2}(\Omega_2, B^{\mathrm{loc}}_{p,k_1}(\Omega_1))$ are not isomorphic.

We shall require the following simple lemma whose proof we omit.

Lemma 4.6. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}_n$, $k \in \mathcal{K}_\omega$, $1 \leqslant p \leqslant \infty$ and let $(E_j)_{j=1}^\infty$ be a sequence of Banach spaces. Then the space $B^{\mathrm{loc}}_{p,k}(\Omega, \prod_{j=1}^\infty E_j)$ is isomorphic to $\prod_{j=1}^\infty B^{\mathrm{loc}}_{p,k}(\Omega, E_j)$.*

We shall also need the following lemmata.

Lemma 4.7. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}_n$, $k \in \mathcal{K}_\omega$, $1 \leqslant p < \infty$ and let E be a Banach space whose dual E' possesses the Radon–Nykodým property. Then $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ is isomorphic to $(B^c_{p,k}(\Omega, E))'_b$.*

Proof. See Theorem 3.1 of [23]. □

In [24] we have shown that the spaces $B^c_{p,k}(\mathbb{R}^n)$ are isomorphic to $l_p^{(\mathbb{N})}$ (see [34] for $p = 1$) and the spaces $B^c_{p,k}(\mathbb{R}^n, l_2)$ are isomorphic to $(l_p(l_2))^{(\mathbb{N})}$ if $p \in (1, \infty)$ and k is a temperate weight function on $\mathbb{R}^n$ such that $k^p \in A^*_p$. By using the methods of the proof of Corollary 5.6 of [24] we have obtained in [23, Theorem 4.1] the following result.

Lemma 4.8. *Assume $1 < p, q < \infty$ and let k be a temperate weight function on $\mathbb{R}^n$ with $k^p \in A^*_p$. Then the space $B^c_{p,k}(\mathbb{R}^n, l_q)$ is isomorphic to $\bigoplus_{j=0}^\infty G_j$ where G_0 is isomorphic to $l_p(l_q)$ and G_j is isomorphic to a complemented subspace of $l_p(l_q)$ for $j = 1, 2, \ldots$.*

Theorem 4.9. *If k_1 (respectively k_2) is a temperate weight function on $\mathbb{R}^n$ (respectively $\mathbb{R}^m$) such that $k_1^p \in A^*_p$ (respectively $k_2^q \in A^*_q$) and $1 < p, q < \infty$ with $p \neq q$, then the spaces $B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n, B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m))$ and $B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m, B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n))$ are not isomorphic.*

Proof. Since $1/\tilde{k}_1$ (respectively $1/\tilde{k}_2$) is a temperate weight function on $\mathbb{R}^n$ (respectively $\mathbb{R}^m$) such that $1/\tilde{k}_1^{p'} \in A^*_{p'}$ (respectively $1/\tilde{k}_2^{q'} \in A^*_{q'}$), it follows by Lemma 4.8 that $B^c_{p',1/\tilde{k}_1}(\mathbb{R}^n, l_{q'})$ is isomorphic to $\bigoplus_{j=0}^\infty G_j$ where $G_0 \simeq l_{p'}(l_{q'})$ and $G_j < l_{p'}(l_{q'})$ for $j = 1, 2, \ldots$, and that $B^c_{q',1/\tilde{k}_2}(\mathbb{R}^m, l_{p'})$ is isomorphic to $\bigoplus_{j=0}^\infty H_j$ where $H_0 \simeq l_{q'}(l_{p'})$ and $H_j < l_{q'}(l_{p'})$ for $j = 1, 2, \ldots$. On the other hand, recall that if $(E_j)_{j=1}^\infty$ is a sequence of Banach spaces, then the space $(\bigoplus_{j=1}^\infty E_j)'_b$ is isomorphic to $\prod_{j=1}^\infty E'_j$ (see [15, p. 168]). On the basis of these results and the previous lemmata, one may derive immediately the isomorphisms

$$\begin{aligned} B^{\mathrm{loc}}_{p,k_1}\big(\mathbb{R}^n, B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m)\big) &\simeq B^{\mathrm{loc}}_{p,k_1}\big(\mathbb{R}^n, \big(B^c_{q',1/\tilde{k}_2}(\mathbb{R}^m)\big)'_b\big) \simeq B^{\mathrm{loc}}_{p,k_1}\big(\mathbb{R}^n, \big(l_{q'}^{(\mathbb{N})}\big)'_b\big) \simeq B^{\mathrm{loc}}_{p,k_1}\big(\mathbb{R}^n, l_q^{\mathbb{N}}\big) \\ &\simeq \big(B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n, l_q)\big)^{\mathbb{N}} \simeq \big(\big(B^c_{p',1/\tilde{k}_1}(\mathbb{R}^n, l_{q'})\big)'_b\big)^{\mathbb{N}} \simeq \left(\left(\bigoplus_{j=0}^\infty G_j\right)'_b\right)^{\mathbb{N}} \simeq \left(\prod_{j=0}^\infty G'_j\right)^{\mathbb{N}} \\ &< \big(l_p(l_q)^{\mathbb{N}}\big)^{\mathbb{N}} \simeq \big(l_p(l_q)\big)^{\mathbb{N}}. \end{aligned}$$

Similarly, we get

$$B^{\mathrm{loc}}_{q,k_2}\big(\mathbb{R}^m, B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n)\big) \simeq \left(\prod_{j=0}^{\infty} H'_j\right)^{\mathbb{N}} < \big(l_q(l_p)\big)^{\mathbb{N}}.$$

Suppose now that our iterated spaces are isomorphic. Then the previous isomorphisms yield that the space $l_p(l_q)$ (respectively $l_q(l_p)$) becomes isomorphic to a complemented subspace of $(l_q(l_p))^{\mathbb{N}}$ (respectively $(l_p(l_q))^{\mathbb{N}}$). Hence it follows, by [8], that there exist positive integers α, β such that $l_p(l_q) < (l_q(l_p))^{\alpha} (\simeq l_q(l_p))$ and $l_q(l_p) < (l_p(l_q))^{\beta}$ $(\simeq l_p(l_q))$. We are now in a position to apply Pelczynski's decomposition method to conclude that $l_p(l_q) \simeq l_q(l_p)$. This however contradicts the assumption that $p \neq q$ (see, e.g., [31, p. 242]). In consequence, $B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n, B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m))$ and $B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m, B^{\mathrm{loc}}_{p,k_1}(\mathbb{R}^n))$ are not isomorphic and the proof is complete. □

We do not know if the above theorem is valid for other values of p and q. We thus propose the following question.

Problem 4.10. *For which weights k_1, k_2 and $q \in\,]1, \infty]$ the iterated spaces $B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n, B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m))$ and $B^{\mathrm{loc}}_{q,k_2}(\mathbb{R}^m, B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n))$ are not isomorphic*?

By using results of Vogt [34] and [23, Theorem 3.1] we have shown (the proof will appear elsewhere) the isomorphisms $B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n, B^{\mathrm{loc}}_{\infty,k_2}(\mathbb{R}^m)) \simeq (l_1(l_\infty))^{\mathbb{N}}$ and $B^{\mathrm{loc}}_{\infty,k_2}(\mathbb{R}^m, B^{\mathrm{loc}}_{1,k_1}(\mathbb{R}^n)) \simeq (l_\infty(l_1))^{\mathbb{N}}$ for some Hörmander weights k_j, $j = 1, 2$. Hence, these iterated spaces are not isomorphic if and only if $l_1(l_\infty)$ and $l_\infty(l_1)$ are not isomorphic either. Thus we are also interested in the following question of Banach space theory.

Problem 4.11. *Are the Banach spaces $l_1(l_\infty)$ and $l_\infty(l_1)$ not isomorphic*?

5. Weighted L_p-spaces of entire analytic functions

In this last section we present a similar result to Theorem 4.5 for weighted L_p-spaces of entire analytic functions. We also give a result on iterated Besov spaces: $B^s_{2,q}(\mathbb{R}^n, B^s_{2,q}(\mathbb{R}^m))$ and $B^s_{2,q}(\mathbb{R}^{n+m})$ are not isomorphic when $-\infty < s < \infty$ and $1 < q \neq 2 < \infty$.

Theorem 5.1. *If K_1 (respectively K_2) is a regular compact in $\mathbb{R}^n$ (respectively $\mathbb{R}^m$), $K = K_1 \times K_2$, $\omega_1 \in \mathcal{M}_n$, $\omega_2 \in \mathcal{M}_m$ and $\omega \in \mathcal{M}_{n+m}$ satisfy* (3.1)*, $\rho_1 \in R(\omega_1)$, $\rho_2 \in R(\omega_2)$, $\rho = \rho_1 \otimes \rho_2$ and $1 \leqslant p < \infty$, then $L^K_{p,\rho}(\mathbb{R}^{n+m})$ is isometrically isomorphic to the iterated space $L^{K_1}_{p,\rho_1}(\mathbb{R}^n, L^{K_2}_{p,\rho_2}(\mathbb{R}^m))$.*

We shall write $L^K_{p,\rho}$ (respectively $L^{K_1}_{p,\rho_1}$, $L^{K_2}_{p,\rho_2}$, $L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})$) instead of $L^K_{p,\rho}(\mathbb{R}^{n+m})$ (respectively $L^{K_1}_{p,\rho_1}(\mathbb{R}^n)$, $L^{K_2}_{p,\rho_2}(\mathbb{R}^m)$, $L^{K_1}_{p,\rho_1}(\mathbb{R}^n, L^{K_2}_{p,\rho_2}(\mathbb{R}^m))$), and we shall denote by $S^K_\omega[L^K_{p,\rho}]$ the space S^K_ω endowed with the norm $\|\cdot\|_{p,\rho}$.

Proof of Theorem 5.1. First we show that the natural map $N : S^K_\omega[L^K_{p,\rho}] \to L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})$ defined by $Nf(x) = f(x,\cdot)$ is well defined and is linear and norm-preserving. Let $f \in S^K_\omega$. It is easily verified that $f(x,\cdot) \in L^{K_2}_{p,\rho_2}$ and $Nf \in L_{p,\rho_1}(L^{K_2}_{p,\rho_2})$. Let us see that $\operatorname{supp}\widehat{Nf} \subset K_1$: For every $\varphi \in D_{\omega_1}(\complement K_1)$ we have

$$\langle \varphi, \widehat{Nf}\rangle = \langle \hat{\varphi}, Nf\rangle = \int_{\mathbb{R}^n} \hat{\varphi}(x) Nf(x)\,dx \quad \big(\in L^{K_2}_{p,\rho_2}\big)$$

and so, since the Dirac deltas $\delta_y \in (L^{K_2}_{p,\rho_2})'$ (see [30, p. 36]), we get

$$\begin{aligned}\big\langle \psi, \langle\varphi, \widehat{Nf}\rangle\big\rangle &= \int_{\mathbb{R}^m} \psi(y)\bigg(\int_{\mathbb{R}^n} \hat{\varphi}(x) Nf(x)\,dx\bigg)(y)\,dy = \int_{\mathbb{R}^m} \psi(y)\bigg\langle \int_{\mathbb{R}^n} \hat{\varphi}(x) Nf(x)\,dx, \delta_y\bigg\rangle dy \\ &= \int_{\mathbb{R}^m} \psi(y)\bigg(\int_{\mathbb{R}^n} \hat{\varphi}(x) f(x,y)\,dx\bigg)dy = \int_{\mathbb{R}^{n+m}} \hat{\varphi}(x)\psi(y) f(x,y)\,dx\,dy\end{aligned}$$

for all $\psi \in S_{\omega_2}$. Thus, for $\psi \in D_{\omega_2}$ we have that

$$\langle \hat{\psi}, \langle \varphi, \widehat{Nf} \rangle \rangle = \int_{\mathbb{R}^{n+m}} \varphi(x)\hat{\psi}(x) f(x,y)\,dx\,dy = \int_{\mathbb{R}^{n+m}} \varphi \otimes \psi(x,y)\hat{f}(x,y)\,dx\,dy = 0$$

since $\varphi \otimes \psi \in D_\omega(\complement K)$ in virtue of (3.1), and hence, by the denseness of $\{\hat{\psi}:\ \psi \in D_{\omega_2}\}$ in S_{ω_2} [3, Theorem 1.8.7], it follows that $\langle \varphi, \widehat{Nf} \rangle = 0$. Consequently $\operatorname{supp} \widehat{Nf} \subset K_1$ and $Nf \in L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})$. Then N is linear and preserves the norm and, since S^K_ω is dense in $L^K_{p,\rho}$ [30, p. 40], it can be extended to a norm preserving linear operator from $L^K_{p,\rho}$ into $L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})$ which will also be denoted by N. It remains to prove that N is surjective. Given $G \in L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})$, we define $f : \mathbb{R}^{n+m} \to \mathbb{C} : (x,y) \to G(x)(y)$ (we may suppose, see Section 2, that G is the restriction to $\mathbb{R}^n$ of an $L^{K_2}_{p,\rho_2}$-valued entire function of exponential type and that, for all $x \in \mathbb{R}^n$, $G(x)$ is the restriction to $\mathbb{R}^m$ of an entire function of exponential type). Let us see that $f \in L_{p,\rho}$. By virtue of the estimate $1/\rho_2(y) \leqslant C e^{\omega_2(y)}$ and the embedding $L^{K_2}_{p,\rho_2} \hookrightarrow L^{K_2}_{\infty,\rho_2}$ (see [30, p. 36]), we have that

$$\begin{aligned} \left| f(x,y) - f(x_0,y_0) \right| &= \left| G(x)(y) - G(x_0)(y_0) \right| \leqslant \left| G(x)(y) - G(x_0)(y) \right| + \left| G(x_0)(y) - G(x_0)(y_0) \right| \\ &\leqslant C e^{\omega_2(y)} \left\| G(x) - G(x_0) \right\|_{p,\rho_2} + \left| G(x_0)(y) - G(x_0)(y_0) \right| \to 0 \end{aligned}$$

when $(x,y) \to (x_0,y_0)$. Thus f is continuous, $\|f\|_{p,\rho} = \|G\|_{L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2})}$ and $f \in L_{p,\rho}$. Actually, $f \in L^K_{p,\rho}$. In fact, if we proceed as above, then

$$\langle \Phi, \hat{f} \rangle = \langle \Psi, \hat{f} \rangle = 0, \qquad \Phi \in D_{\omega_1}(\complement K_1) \otimes D_{\omega_2}, \qquad \Psi \in D_{\omega_1} \otimes D_{\omega_2}(\complement K_2),$$

and so, by Theorem 3.2(1), we get

$$\langle \Phi, \hat{f} \rangle = \langle \Psi, \hat{f} \rangle = 0, \qquad \Phi \in D_\omega\big(\complement K_1 \times \mathbb{R}^m\big), \qquad \Psi \in D_\omega\big(\mathbb{R}^n \times \complement K_2\big). \tag{5.1}$$

Hence it follows that $\langle \Phi, \hat{f} \rangle = 0$ holds for all $\Phi \in D_\omega(\complement K)$ (since given such a Φ, we have $\operatorname{supp} \Phi \subset \complement K = (\complement K_1 \times \mathbb{R}^m) \cup (\mathbb{R}^n \times \complement K_2)$ and then it suffices to take a D_ω-partition of unity at $\operatorname{supp} \Phi$ subordinate to this covering and use (5.1)). Therefore, $f \in L^K_{p,\rho}$. Finally, from the embeddings $L^{K_1}_{p,\rho_1}(L^{K_2}_{p,\rho_2}) \hookrightarrow L^{K_1}_{\infty,\rho_1}(L^{K_2}_{p,\rho_2})$ (see [24, Theorem 3.3]), $L^{K_2}_{p,\rho_2} \hookrightarrow L^{K_2}_{\infty,\rho_2}$ and $L^K_{p,\rho} \hookrightarrow L^K_{\infty,\rho}$, it follows that $Nf = G$. The proof is complete. □

The spaces L^Q_p (Q cube in $\mathbb{R}^n$) are the building blocks of the Besov spaces (see [27,30] and [31]). By using the isomorphism $L^Q_p \simeq l_p$, Triebel proves in [29] (see also [31]) that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are isomorphic to $l_q(l_p)$. Following Triebel's approach [31] it is shown in [24] the vector-valued counterpart of this result: (a) *Let $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, $-\infty < s < \infty$, let $Q \subset \mathbb{R}^n$ be a cube and let E be a Banach space with the UMD-property. Then $L^Q_p(E)$ is isomorphic to $l_p(E)$ and $B^s_{p,q}(E)$ is isomorphic to $l_q(l_p(E))$*. (For definitions, notation and basic results about vector-valued Besov spaces see [2] and [26].)

Since the spaces $l_{q_0}(l_{p_0})$ and $l_{q_1}(l_{p_1})$ are isomorphic if and only if $q_0 = q_1$ and $p_0 = p_1$ ($1 \leqslant q_0, q_1 \leqslant \infty$ and $1 < p_0, p_1 < \infty$) (see, e.g., [31, p. 242]), it follows from (a) that the spaces $L^{Q_1}_p(L^{Q_2}_q)$ and $L^{Q_2}_q(L^{Q_1}_p)$ are not isomorphic if $1 < p \neq q < \infty$ (here Q_1, Q_2 are cubes in $\mathbb{R}^n$). Another application of result (a) is the following.

Theorem 5.2. *Let $1 < q \neq 2 < \infty$ and $-\infty < s < \infty$. Then the spaces $B^s_{2,q}(\mathbb{R}^n, B^s_{2,q}(\mathbb{R}^m))$ and $B^s_{2,q}(\mathbb{R}^{n+m})$ are not isomorphic.*

Proof. The Besov space $B^s_{2,q}(\mathbb{R}^{n+m})$ is an $\mathscr{L}_q$-space since $l_q(l_2)$ is an $\mathscr{L}_q$-space (see [21, Example 8.2]) and $B^s_{2,q}(\mathbb{R}^{n+m})$ is isomorphic to $l_q(l_2)$. On the other hand, since $B^s_{2,q}(\mathbb{R}^m)$ is a UMD space ($l_q(l_2)$ is a UMD space, see, e.g., [1, Theorem 4.5.2]), we can apply (a) and obtain

$$B^s_{2,q}\big(\mathbb{R}^n, B^s_{2,q}\big(\mathbb{R}^m\big)\big) \simeq l_q\big(l_2\big(B^s_{2,q}\big(\mathbb{R}^m\big)\big)\big) \simeq l_q\big(l_2\big(l_q(l_2)\big)\big) > l_2\big(l_q(l_2)\big) > l_2(l_q).$$

Whence it follows that $B^s_{2,q}(\mathbb{R}^n, B^s_{2,q}(\mathbb{R}^m))$ is not an $\mathscr{L}_q$-space, since $l_2(l_q)$ is not an $\mathscr{L}_q$-space [21, p. 316] and a complemented subspace of an $\mathscr{L}_q$-space which is not isomorphic to a Hilbert space is an $\mathscr{L}_q$-space [22]. □

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