Sensitivity Analysis for Link Capacity Based on Elastic Demand Equilibrium with Queuing

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Abstract

Link capacity enhancement is often used for relieving traffic congestion. However, this measure may not be efficient because it may move the congestion from one road to another. Therefore, an efficient scheme should take into account the effects of the changed network flow pattern to achieve a global optimal solution. In this paper, elastic demand network equilibrium with queuing assignment model is established to describe drivers’ route choice behavior under both queuing and congestion conditions. The improved inner penalty function method is put forward to solve the equilibrium model. Then the nondegenerate extreme point of path flow equilibrium solution is derived. Finally, sensitivity analysis is performed to derive the explicit expressions of equilibrium parameters with respect to changed link capacities. This sensitivity information can accurately check whether enhancing road capacities would be helpful to relieve traffic congestion by reducing traffic delays in saturated links. The derivative information also allows to estimate a nearby solution for small changes in the link capacity parameters once an equilibrium solution has been obtained. A numerical example is presented to explicitly demonstrate the sensitivity analysis method and its application to the evaluation of network capacities expansion schemes.

1. Introduction

Traffic problems such as congestion have become a prime concern of the whole world. Over the past few decades, traffic engineers have put forward many measures to relieve the congestion among which link capacity expansions are quietly often used. However, it has been point out by Meng and Yang (2002) that link capacity expansions may not reduce the travel cost for it just remove the congestion from one road to another road in some
situation. In their given example, the OD demands are fixed, it can suppose that the situation would be even worse when OD demands are elastic for the additional attracted flows. Link capacity expansion is often discussed as Conventional Network Design Problem (CNDP), who concerns about how to determine the set of link capacity expansions and the corresponding equilibrium flows for which the measures of performance index for the network is optimal (Chiou, 2005). Usually bilevel programming models are established for CNDP and the system performance index is often defined as the sum of total travel times and investment cost (Loukas and Antony, 2008; Meng et al., 2001; Wang and Lo, 2010). However seldom papers of CNDP concern about the function of relieving traffic congestion in specific roads. Generally, when link travel flows approach to its capacity, this road will be very congested and will cause large numbers of stops and delays thus becomes a bottleneck in the whole road network. As a result, the government cares more about whether the capacity enhancement on saturated road can effectively relieve the congestion situation. In this paper, an elastic demand equilibrium model with queuing is established and sensitivity analysis is introduced to evaluate whether capacity enhancement on saturated road can effectively relieve the traffic delays.

Equilibrium models are often used in transportation network analysis, which is based on Wardrop first principles that each user is assigned to the least cost path between his origin and destination. The solution of the equilibrium models is the traffic volume on each link in the network, and the associated system performance. Because the realized travel demand and the resultant flow pattern depend heavily upon a variety parameters in demand and supply functions of transportation networks. Thus the traffic assignment problem with elastic demand is a more sound behavioral foundation and is general used in transportation system analysis, where the OD demand is assumed to be influenced by the level of service on the network (Gartner, 1980). In peak hours, queues may form in saturated roads and queuing delays is increased when the link flow is approach its physical capacity. In this situation, the drivers may choose the other routes for the increased queuing times in the intersections since they are assumed to have sufficient and perfect knowledge of traffic condition. The elastic demand equilibrium model with queuing introduced this paper can well describe drivers’ route choice behavior under conditions of both queuing and congestion.

Sensitivity analysis is very useful for the evaluation the robustness of the equilibrium model when parameters in link performance functions perturbed and can help identifying those parameters to which the equilibrium flows are most sensitive. Additionally, sensitivity analysis can also be used to find an equilibrium pattern with respect to small changes in the model parameters without the need to reanalyze the time consuming network equilibrium model. This manipulation would require the calculation of derivatives of decision variables with respect to perturbed parameters. In this paper, explicit expressions of the derivatives of equilibrium link flows, OD travel cost and queuing times with respect to capacity parameters are derived by implementing sensitivity analysis for the queuing network equilibrium problem with elastic demand. This derivative information can help judges whether expanding the capacity on saturated road deserve the most consideration to improve the current operation of the congested traffic systems. The derivative information is also used to estimate a nearby solution for small changes in the link capacity parameters once an equilibrium solution has been calculated.

Application of sensitivity analysis for equilibrium network flows was first introduced by Tobin and Friesz (1988). After that, this method is extended to a number of topics of interest and is widely used for solving bilevel traffic network design problems (Chiou, 1999; Wong and Yang, 1997; Gao et al., 2004; Poorzahedy and Abulghasemi, 2005; Poorzahedy and Rouhani, 2007). Gao et al. (2004) applied sensitivity analysis method to study the relationship between vehicle price and passenger volume. Stephen and David (2002) used this method for probit-based stochastic user equilibrium assignment. Wong et al. (2006) introduced sensitivity analysis for a continuum traffic equilibrium problem. Yang et al. (1995a, 1995b) used the sensitivity analysis method for signal control in saturated traffic system and queuing network. However, in those two papers, the OD demand is assumed to be fixed, which is not appropriate for the real condition since the improved network will attract more traffic flows. Through Yang (1997a) introduced the sensitivity analysis method for elastic-demand network equilibrium problem, queuing delays in oversaturated roads and the capacity constraint was not considered, as a
result, the equilibrium link flows may be a few times of its capacity. This paper rectifies these faults by establishing traffic assignment model with elastic demand and queues which basic on the assumption that the O-D demand is a function of network service level and queuing is formed on saturated links. Derivatives of the O-D demand, consumers’ surplus and total travel cost is then calculated to tell whether the link capacity enhancement is effect to the network.

Following introduction, the next section will introduce the model of the elastic demand equilibrium with queuing after the given the general assumptions and considerations. In Section 3, a sensitivity analysis method for the elastic demand equilibrium with queuing problem is presented to obtain the explicit expressions of the derivatives of travel demand, equilibrium flow and various system performances with respect to specific link capacity. In Section 4, a numerical example drawn from network design is presented to illustrate the effectiveness of the sensitivity analysis method for link capacity expansions. General conclusions are presented in Section 5.

2. Elastic Demand Network Equilibrium with Queuing

In this section the basic assumptions and considerations for the model formulation are presented. Then the model of elastic demand equilibrium with queuing is established in view of the basic assumptions and considerations. Finally, the inner penalty Newton based method is presented to solve model.

2.1 Basic Considerations

Throughout this paper, the traffic assignment problem is discussed by extending the conventional UE principle for networks with queues and elastic demand. The proposed model is aimed to be used mainly for the traffic assignments by taking into account the effects of elastic demand and queues. Therefore, the effects can be incorporated into the traffic assignment model for traffic forecasting. The following basic assumptions are made:

(1) The demand between every origin-destination (OD) pair is not fixed and is influenced by the level of service on the network, that is, OD demands is increased if OD travel cost is diminished and will reduced if the OD travel cost is increased. (2) Yang (1995a) point out that the queuing time of a road is less if the link is not saturated and the queuing time will increase sharply when the link flow is over its capacity. In tradition user equilibrium traffic assignment, traffic flows of some links may be a few double times of their capacities, which will lead to large amount of queuing time delay and may block upstream intersections, thus is not appropriate for the real condition. Consequently, this paper assumes that the equilibrium link flow shouldn’t exceed it physical capacity by inducing capacity restraint. (3) Queuing time is treated as a variable and determined from queuing network equilibrium conditions. The queuing delay is assumed only formed when capacity is reached, below capacity link travel times only will depend on flows. (4) The travel time on a link is equal to the sum of a flow-dependent running time and queuing delay at the link exit, both link flows and equilibrium queuing delays can be obtained from a convex programming problem. (5) Drivers are all have sufficient knowledge of the traffic situation via all routes and make routing decisions in a user-optimal manner for any given traffic condition.

2.2 Queue Network Equilibrium Model with Elastic Demand

Given a transport network \( G(A,N) \), where \( A \) and \( N \) are the sets of links and nodes respectively, and each link \( a \in A \) is associated with a travel time \( t_a(v_a) \) as a function of link flow \( v_a \). The general assumptions and considerations in the last section can be formulated as the following nonlinear mathematical optimization program
\[
\begin{align*}
\text{minimize} & \quad \sum_{q,v} \int_{a=0}^{v} t_a(x)dx - \sum_{w} \int_{0}^{q_w} D^{-1}(x)dx \\
\text{Subject to} & \quad \sum_{r \in R_w} f_r = q_w, \quad w \in W \\
& \quad \sum_{r \in R_w} f_r \delta_w = v_a, \quad a \in A \\
& \quad \sum_{r \in R_w} f_r \delta_w \leq C_a, \quad a \in A \\
& \quad f_r \geq 0, \quad r \in R \\
\end{align*}
\]

Where:

- \( R \): the set of all routes in the network
- \( R_w \): the set of routes between O-D pair \( w \in W \)
- \( W \): the set of all O-D pairs in the network
- \( f_r \): the flow on route \( r \in R \)
- \( C_a \): the capacity of link \( a \in A \)
- \( D_a(u_w) \): travel demand between O-D pair \( w \in W \) as a function of O-D travel cost \( u_w \)
- \( D^{-1}(q_w) \): the inverse of demand function
- \( q_w \): the demand between O-D pair \( w \in W \)
- \( \delta_w \): 1 if route \( r \) use link and 0 otherwise.

The network equilibrium problem is path-formulated due to the constraints of equation (1b)-(1e), among which equation (1b) is the O-D demand constraint, equation (1c) is the flow conservation constraint, equation (1d) is a capacity constraint and equation (1e) is a non-negativity constraint. As is mentioned before, queues are assumed to only formed on saturated links. Namely,

\[
\begin{align*}
\begin{cases}
    d_a = 0 & \text{if } v_a < C_a, \quad a \in A \\
    d_a \geq 0 & \text{if } v_a = C_a, \quad a \in A
\end{cases}
\end{align*}
\]

Where \( d_a \) is the queuing delay at the exit of link, \( a \in A \). The total link travel time is equal to be sum of a flow-dependent running time and a queuing delay at the link exit, Namely,

\[
c_a = t_a + d_a
\]

For it is assumed that each driver traveling from an origin to a destination have perfect knowledge of the travel costs and queues via all routes and will choose the route in a user-optimized manner, the equilibrium relationships in the network with queuing are thus given by

\[
\begin{align*}
\sum_{(r,w) \in R} \delta_{(r,w)} (t_a + d_a) = u_w, & \quad \text{if } f_r > 0 \quad r \in R_w, \quad w \in W \\
\sum_{(r,w) \in R} \delta_{(r,w)} (t_a + d_a) \geq u_w, & \quad \text{if } f_r = 0 \quad r \in R_w, \quad w \in W
\end{align*}
\]
2.3. The Solution Algorithm

This problem is somewhat different from the conventional network equilibrium problem by imposing the capacity constraint (1d). The similar problem have been widely studied and various methods are put forward including quadratic programming algorithm (Nguyen and Pallottino, 1988; Dembo and Tulowitzki, 1988) and Genetic Algorithm (GA) (Sadek et al., 1997; Runmei et al., 2005) and a dual ascent technique which allows bonds on the link flows. Here another algorithm called inner penalty function method is used for solving the assignment problem. Inner penalty Newton based function is effectively for solving the constraint inequality equations. The inner-penalty method is explicate explained by Dimitri (1999), Here we use it directly without prove. The penalty function for inequality restraints in question (1) is described as

$$\phi(f) = \sum_{r \in R} \frac{1}{C_a - f_r, \delta_a} + \sum_{r \in R} \frac{1}{f_r}.$$  

(5)

Then the augmented objective function is

$$F = \sum_{a \in A} \int_{0}^{q_a} t_a(x)dx - \sum_{w \in W} \int_{0}^{q_w} D^{-1}(x)dx + r \phi(f)$$  

(6)

Subject to

$$\sum_{r \in R} f_r = q_a, \quad w \in W$$  

(7)

$$\sum_{r \in R} f_r \delta_a = v_a, \quad a \in A$$  

(8)

Where \(r\) is the penalty factor. It can be seen form Equation (7) and (8) that the O-D demands and link flows are the function of path flows, namely O-D demand and link flows are dependent on the whole route flow pattern, we use \(v_a(f)\) \(a \in A\) to represent the relationship between each link flow and path flows and \(q_w(f)\) \(w \in W\) represent the relationship between each O-D demand and path flows. Where \(f\) is a vector of all route flows. Thus the variables of link flows and OD demands in equation (6) can be replaced by those functions and is rewritten as

$$F(f, r) = \sum_{a \in A} \int_{0}^{v_a(f)} t_a(x)dx - \sum_{w \in W} \int_{0}^{q_w(f)} D^{-1}(x)dx + r \phi(f)$$  

(9)

In equation (9), the variables in the target function are only route flows. Then inner-penalty Newton based method then can be used for searching the route flows to minimize the augmented objective function. The steps are described as follows:

Step 1: Choose a feasible interior point \(f^0\) and penalty factor \(r_0\). Set the convergence criteria \(\varepsilon_1\). Let \(n=0\)

Step 2: For given \(r_n\), the minimize value of link flows augmented objective function (equation (9)) can be get by Newton method demonstrated as follow steps:

Step 2.1: Determine convergence criteria \(\varepsilon_2\) for Newton method, set \(k = 1\), Let \(f^0 = f^n\).

Step 2.2: If \(\left\| \nabla, F(f^k, r^n) \right\| < \varepsilon_2\), then stop, let \(f^n = f^k\), turn to step 3, else turn to step 2.3.

Step 2.3: Calculate \([\nabla (F(f^k, r^n))]^{-1}\), Let \(p^k = -[(\nabla (F(f^k, r^n))]^{-1} \nabla, F(f^k, r^n)\).

Step 2.4: Let \(f^{k+1} = f^k + p^k\), \(k = k + 1\), turn to step 2.2.
Step 3: Let $g(f) = \sum_{x} \int_{a}^{b} t_{x}(x)dx - \sum_{x} \int_{a}^{b} D^{-1}(x)dx$, If $\frac{|g(f^n) - g(f^{n-1})|}{g(f^n)} < \varepsilon_i$, Then stop. Otherwise, let $r_{n+1} = \sigma r_n$ $(0 < \sigma < 1)$, $n = n + 1$, turn to step 2.

3. Sensitivity Analysis for Queuing Network Equilibrium Problem

3.1 approach for nondegenerate extreme point of path flows calculation

This sensitivity analysis approach taken here is to calculate the derivatives of various system performances with respect to the perturbation parameters. Note that in general, the path flows equilibrium solution is not unique even if link flow solution is unique, thus can not be used for sensitivity analysis directly. In order to overcome the nonuniqueness difficulty, Tobin and Friesz (1988) proposed a restriction approach for sensitivity analysis of equilibrium assignment problems, which is to select a nondegenerate extreme point in the feasible region of equilibrium path flows. This extreme point can be obtained easily with Frank-Wolfe method (Yang 1994). However, the paths flows calculated by the inner-penalty Newton based method are not the nondegenerate extreme point. Here the approach put forward by Tobin (1988) is used to get the nondegenerate extreme point of path flows. Assume the path flows calculated by inner-penalty Newton based method are $f_0$, then the unique link flows $v_0$ can be obtained by formula (1c) and OD demand $q_0$ can be obtained by formula (1b). Let $\Gamma = \{f \mid \Delta f = v_0, \Delta f = q_0, f \geq 0\}$

\[ \Gamma = \{f \mid \Delta f = v_0, \Delta f = q_0, f \geq 0\} \quad (10) \]

Where $\Lambda = [\delta_{uv}]$ is the origin-destination/route incidence matrix, $\Delta = [\delta_{uv}]$ is the link/route incidence matrix. The nondegenerate extreme point of paths flows $f^*$ then can be get by solving the following linear programming problem:

\[ \begin{align*}
\text{minimize} & \quad b f \\
\text{s.t.} & \quad f \in \Gamma \\
& \quad f \geq 0
\end{align*} \quad (11) \]

Where $b$ is a vector with the same size $|f|$. The exact value of $b$ is not important, but if a particular path flow solution is of interest, then $b$ can be chosen accordingly. The problem (11) is a linear programming and can be solved by simplex method.

3.2. Sensitivity Analysis for Queuing Network Equilibrium with Elastic Demand Problems

Now it is ready to implement sensitivity analysis for the perturbed queuing network equilibrium with elastic demand problem. Since the road capacities have been expanded, they will be perturbed variables which exist in the link performance functions $t(v,e)$, link exit capacities $C(e)$ and O-D demand functions $D(u,e)$. In the last section the nondegenerate extreme point $f^*$ has been obtained as equilibrium path flows at $\varepsilon = 0$. According to the theory proposed by Tobin and Friesz (1988), there exists a solution to the following system equations.

\[ \begin{align*}
\pi^T (f^*, 0) - \pi - \Lambda^T u + \Lambda^T d &= 0 \\
\pi^T f^* &= 0 \\
\Delta f^* - D(u, 0) &= 0 \\
\lambda^T (\Delta f^* - C(0)) &= 0 \\
\pi \geq 0, \quad d \geq 0, \quad C(0) - \Delta f \geq 0.
\end{align*} \quad (12a) \quad (12b) \quad (12c) \quad (12d) \quad (12e) \]
It can be shown that if only the nondegenerate extreme point of positive path flow solutions is considered, the nonnegativity constraint will be nonbinding at the optimal solution and remain so for the perturbation of \( f^* \) near \( \varepsilon = 0 \) because it is assumed that the link flows are strict positive and O-D demands are nonbinding nonnegative. Also the Lagrange multipliers associated with the non-binding constraints equal zero and remain zero near \( \varepsilon = 0 \). Thus the nonbinding constraints can be eliminated without affecting the outcome of the sensitivity calculation (Tobin and Friesz, 1988). Then the system of equations reduces to:

\[
\begin{align*}
\hat{f}^0(f^*,0) - \Lambda_{u}^0 u + \Delta_{u}^0 d^0 &= 0, \\
\Lambda_{f}^0 f^0 - D(u,0) &= 0, \\
\Delta_{f}^0 f^0 - C^0(0) &= 0.
\end{align*}
\]

Where superscript ‘0’ represents the corresponding reduced vectors and matrices (for example, \( C^0 \) represents a vector of capacities for saturated links). It can be shown that equations (13) satisfies all the conditions for sensitivity analysis according to the theory.

Differentiating both sides of the system of equations (13a-13c) with respect to the perturbation parameter \( \varepsilon \), we obtain

\[
\begin{bmatrix}
\nabla_{f} \hat{f}^0 \\
\nabla_{u} u \\
\nabla_{d} d
\end{bmatrix} =
\begin{bmatrix}
\nabla_{f} \hat{f}^0(f^*,0) & -\Lambda_{u}^0 & \Delta_{u}^0 \\
\Lambda_{f}^0 & -\nabla_{D}(u,0) & 0 \\
-\Delta_{f}^0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\nabla_{\hat{f}}(f^*,0) \\
\nabla_{D}(u,0) \\
\nabla_{C^0(\varepsilon)}
\end{bmatrix}
\]

This equation can be simplified as following:

\[
\begin{bmatrix}
\nabla_{f} \hat{f}^0 \\
\nabla_{u} u \\
\nabla_{d} d
\end{bmatrix} =
\begin{bmatrix}
\nabla_{f} \hat{f}^0(f^*,0) & -M^T & 0 \\
M & z & 0
\end{bmatrix}
\begin{bmatrix}
\nabla_{\hat{f}}(f^*,0) \\
\nabla_{D}(u,0) \\
\nabla_{C^0(\varepsilon)}
\end{bmatrix}
\]

The Jacobian matrix of the system of equations (13a-13c) with respect to \((f^0,u,d^0)\) and evaluated at \( \varepsilon = 0 \) is

\[
J_{f^0,u,d^0} =
\begin{bmatrix}
\nabla_{f} \hat{f}^0(f^*,0) & -M^T & 0 \\
M & z & 0
\end{bmatrix}
\]

Suppose

\[
J_{f^0,u,d^0}^{-1} =
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
\]

It can be shown that

\[
\begin{align*}
J_{22} &= (z + M \nabla_{f} \hat{f}^0(f^*,0)^{-1} M^T)^{-1} \\
J_{12} &= \nabla_{f} \hat{f}^0(f^*,0)^{-1} M^T J_{22} \\
J_{21} &= -J_{22} M \nabla_{f} \hat{f}^0(f^*,0)^{-1} \\
J_{11} &= \nabla_{f} \hat{f}^0(f^*,0)^{-1} [E + M^T J_{21}].
\end{align*}
\]
where \( E \) is an identity matrix of appropriate dimension. Since the Jacobian matrix of the system of equations have been calculated, according to the equation (14), we can obtain the derivatives of link flows, travel cost and link delays with respect to \( \varepsilon \) at \( \varepsilon = 0 \) as

\[
\nabla_{\varepsilon} v = -\Delta^0_1 \Delta^0_1 \nabla_{\varepsilon} t(v,0)+\Delta^0_{12} \nabla_{\varepsilon} y(0)
\]

\[
\nabla_{\varepsilon} w = -\Delta^0_{21} \nabla_{\varepsilon} t(v,0)+\Delta^0_{22} \nabla_{\varepsilon} y(0)
\]

Since

\[
\nabla_{\varepsilon} v = \Delta^0 \nabla_{\varepsilon} f(0) \quad \text{and} \quad \nabla_{\varepsilon} t(v,0) = \Delta^0 \nabla_{\varepsilon} t(v,0) \Delta^0
\]

where \( J_{11}, J_{12}, J_{21}, J_{22} \) are given in equation (19a-19d).

Equation (22) is the derivatives of link flows and Equation (23) are combination of derivatives of constraint multipliers (OD travel cost and link queuing delays) with respect to capacity parameters in the network equilibrium problem.

Additionally, derivatives of OD demand in perturbation capacity can be obtained as:

\[
\nabla_{\varepsilon} q = \nabla_{\varepsilon} D(u,0)+\nabla_{u} D(u,0)\nabla_{\varepsilon} u
\]

Where \( \nabla_{\varepsilon} u \) is given in equations (22).

4. Numerical Examples

This section presents an example to illustrate how to use the proposed method to find whether expand capacities in saturated roads are effective. The road network, as shown in Fig. 1, has 6 nodes, 8 links, and 2 O-D pairs (1-6 and 2-6). In this network, road 1 and 2 are access ramps used to connect the two expressways (e.t link 5 and link 6) and the rest links are arterials to provide multiple alternative routes for drivers from their origins. The following BPR (Bureau of Public Roads) link cost functions (equation (25)) are used with associated input data presented in Table 1. It can be seen in Table 1 that the two expressways are preferable links with shorter free flow time and larger capacity. However, the flows on the expressways are greatly depend on the capacity of ramp 1 and 2, so the traffic congestion may occur on the two ramps in peak time as well as excessive queuing delays. In this situation, small capacity perturbation on the two ramps may have great influence on the equilibrium link flows. Then the traffic engineer wants to widen the ramps to reduce the traffic congestion. But it is wondered that whether it is work for it may attract more traffic flows. This question makes sensitivity analysis method very meaningful for it can help find the whether widen the roads to increase the ramp capacity is effective to improve the traffic situation and how much the parameters in the road network changed by the improvements. In this example, we will present explicit process of how to carry out the sensitivity analysis and how to use the derivative information.

Table 1. Input data to the example network of Fig.1

<table>
<thead>
<tr>
<th>Link number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>free flow time ( t^*_i )</td>
<td>2</td>
<td>1.0</td>
<td>7</td>
<td>6</td>
<td>4</td>
<td>4</td>
<td>4.5</td>
<td>4</td>
</tr>
<tr>
<td>link capacity ( C_a )</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>30</td>
<td>40</td>
<td>40</td>
<td>60</td>
<td>60</td>
</tr>
</tbody>
</table>
The O-D demand functions are assumed to be the following negative exponential functions:

\[ D_{16}(u_{16}) = 300 \times \exp(-0.20u_{16}) \]
\[ D_{24}(u_{24}) = 340 \times \exp(-0.20u_{24}) \]

The paths joining the two O-D pairs are given as:

\[ p_{16} = \{p_1, p_2, p_3\}, \quad p_1 = \{3, 7\}, \quad p_2 = \{1, 5, 7\}, \quad p_3 = \{1, 6, 8\}; \quad p_{24} = \{p_4, p_5, p_6\}, \quad p_4 = \{4, 8\}, \quad p_5 = \{2, 6, 8\}, \quad p_6 = \{2, 5, 7\} \]

The inner penalty Newton based method is then used to derive the equilibrium path flows solution. The parameters used for in this method are as follows, \( \varepsilon_1 = 10^{-6}, \ v_2 = 0.5 \times 10^{-7}, \ r_0 = 20, \ \sigma = 0.7 \).

Fig 2 and Fig 3 show the value of target function and route flows in each iteration. It can be seen from the two picture that the convergence condition is satisfied after 24 iterations. The final route flows and link flows calculated by inner penalty Newton based method are as follows:

\[ v_0 = [15 \ 15 \ 14.6933 \ 24.5324 \ 1.8476] \]

Then the method put forward in 3.1 section is used to derive the nondegenerate extreme point of paths flows \( f^* \), the linear programming used is as follows:

\[
\begin{align*}
\text{minimize} & \quad \begin{bmatrix} 40 & 90 & 6 & 35 & 80 & 1 \end{bmatrix}^T \\
\text{s.t.} & \quad \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix} \\
& \quad \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix} \quad = \begin{bmatrix} 15 \\
15 \ \ 14.6993 \\
16.5409 \\
52.6846
\end{bmatrix} \\
& \quad f_1, f_2, f_3, f_4, f_5, f_6 \geq 0
\end{align*}
\]

(28)

It can be seen that the constrained equations in the linear programming (28) are just part of equations in \( \Gamma \), because simplex method requires that the row vectors of coefficient matrix of the equations must be irrelevant with each other and its number must be equal to the rank of \( [A | \tilde{A}] \). The nondegenerate extreme point \( f^* \) solved by the simplex method is demonstrated as follows:

\[ f^* = [14.6933 \ 0 \ 15 \ 24.5324 \ 13.1523 \ 1.8476] \]
The reduced link/path and OD/path incidence matrices thus become

\[
\Delta^0 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{bmatrix},
\]

\[
\Lambda^0 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

The Jacobians of the link cost functions and demand functions at the unique equilibrium solution are calculated as follows

\[\nabla_t(v^*,0) = \text{diag}[0.08, 0.04, 0.0164, 0.0656, 5.908 \times 10^{-6}, 0.0209, 0.0009, 0.0271]\]

\[\nabla_uD(u^*,0) = \begin{bmatrix}
-5.9386 & 0 \\
0 & -7.9064
\end{bmatrix}
\]

The Jacobian of the reduced path cost functions for the above extreme point calculated using equation (21) is

\[
\nabla_t(f^*,0) = \begin{bmatrix}
0.0174 & 0 & 0 & 0 & 0 & 0.0009 \\
0 & 0.128 & 0.0271 & 0.0480 & 0 \\
0 & 0.0271 & 0.0927 & 0.0271 & 0 \\
0 & 0.0480 & 0.0271 & 0.0880 & 0.0400 \\
0.0009 & 0 & 0 & 0.0400 & 0.0409
\end{bmatrix}
\]

Then we can acquire

\[
J_{11} = \begin{bmatrix}
5.3833 & 0 & -0.0138 & 0.1113 & -0.1113 \\
0 & 0 & 0 & 0 & 0 \\
0.1113 & 0 & -2.7097 & 2.7097 & 0 \\
-0.1113 & 0 & 2.7097 & -21.9308 & 21.9308 \\
0.9065 & -0.0017 & 0.9115 & 0.0032 & 0 \\
0 & 0 & -1 & 0 & 0
\end{bmatrix}
\]

\[
J_{12} = -J_{21}^T = \begin{bmatrix}
0.0157 & 0.0003 & 0.0149 & -0.0005 \\
0.0003 & 0.0481 & 0 & 0.0487 \\
0.0149 & 0 & 0.095 & 0.0001 \\
-0.0005 & 0.0478 & 0.0001 & 0.0884
\end{bmatrix}
\]

Once the link capacity parameters in the link cost and demand functions are specified, we can calculate the values of \(\nabla_t^d(v^*,z)\) and \(\nabla_u^d\) at the given equilibrium solutions. With these and the above matrix values, we can thus calculate various types of derivative information from equations (21)-(24), including the derivatives of
equilibrium link flows, OD travel cost and link queuing delays and OD travel demand. Furthermore, in order
determine the direction of the changes in the system performance when ramp capacities are perturbed, we also
calculate the derivatives of the consumers’ surplus (CS) and total travel cost (TC) which are given below.

\[
CS = \sum_{w \in B} \int_0^{d_w} D_w^{-1}(x)dx - \sum_{w \in B} u_w \cdot d_w
\]

\[
TC = \sum_{a \in A} t_a \cdot v_a
\]

Table 2 presents those derivatives with respect to ramp capacities, when \(C_1 = C_2 = 15\). These results represent
the variations of OD travel demand, OD travel cost, link queuing times, link flows, consumers’ surplus and travel
cost in this network when the ramp capacity at one of the ramps is increased by 1 unit. If the derivatives of the
variables are negative, then it indicates that enlarging the ramp capacity may reduce the value of the variable.
Otherwise it manifest that enlarging the ramp capacity will increase the value of the variables. Among all the
variables, the derivatives of link queuing times and OD travel cost are most important since they can tell whether
increasing ramp capacity can reduce the congestion of the network. The derivative information can not only be
useful in finding whether the measure of enlarging ramp capacity is effective, but also can make it clear that of
which the ramp capacity to equilibrium flow pattern is the most sensitive, and therefore, we can find which ramp
deserves the most consideration to improve the current operation of the corridor system. The derivative
information also allows one to estimate nearby solutions for any combination of ramp capacity changes once an
equilibrium solution has been calculated.

From Table 2, we can see that most derivatives of OD travel cost and queuing delay is negative, among which
the positive derivatives is rather small compared with the negative derivatives, meaning that widen the ramps are
effective in reducing the queuing delays, and thus are help for relieving the traffic congestion in the two ramps.
However, derivatives of consumers’ surplus and total travel cost indicate that widen the ramps will poorer the
network performance as a total. Namely the measure of broaden the ramps can reliving the traffic congestion at
the cost of the overall travel performance. Through this measure may have some negative effect, the OD travel
cost are diminished as a total, which means that the this measure can reduce the travel time of every travel unit,
thus it is a measure that worth to be considered.

Additionally, the derivatives also imply that the network performance may sacrifice more when increase the
capacity of ramp 2 by 1 unit compared with ramp 1, which suggest widen ramp 1 deserves the most consideration
to improve the current situation of the corridor system. Table 3 shows the equilibrium solution of O-D travel
demand, O-D travel cost, link queuing times, link flows total cost and consumers’ surplus when the ramp capacity
are varied at \(\delta C_1 = \delta C_2 = 2\). The estimation is made using a linear approximation based on the derivatives in Table
2. These results could allow traffic engineers to evaluate and predict the performance of alternative widen ramp
schemes, and hence, to determine capacities enhancement strategies efficiently.

Table 2. Derivatives of O-D demands, O-D costs, link queues, link flows, consumer surplus and total travel cost with respect
to ramp capacity at \(C_1 = C_2 = 15\)

<table>
<thead>
<tr>
<th>Solution variable</th>
<th>(\partial() / \partial C_1)</th>
<th>(\partial() / \partial C_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_{16})</td>
<td>0.0885</td>
<td>-0.0032</td>
</tr>
<tr>
<td>(q_{26})</td>
<td>-0.0002</td>
<td>0.3780</td>
</tr>
<tr>
<td>(u_{16})</td>
<td>-0.0950</td>
<td>0.0005</td>
</tr>
<tr>
<td>(u_{26})</td>
<td>0</td>
<td>-0.0884</td>
</tr>
<tr>
<td>(d_1)</td>
<td>-0.015</td>
<td>-0.0001</td>
</tr>
<tr>
<td>(d_2)</td>
<td>-0.0001</td>
<td>-0.0484</td>
</tr>
<tr>
<td>(v_1)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Solution variable</td>
<td>Unperturbed solution</td>
<td>Perturbed with $\delta C_1 = \delta C_2 = 2$</td>
</tr>
<tr>
<td>-------------------</td>
<td>----------------------</td>
<td>------------------------------------------</td>
</tr>
<tr>
<td>$q_{16}$</td>
<td>29.6931</td>
<td>29.8342</td>
</tr>
<tr>
<td>$q_{26}$</td>
<td>39.5322</td>
<td>40.2630</td>
</tr>
<tr>
<td>$u_{16}$</td>
<td>11.5643</td>
<td>11.5406</td>
</tr>
<tr>
<td>$u_{26}$</td>
<td>10.7592</td>
<td>10.6678</td>
</tr>
<tr>
<td>$d_1$</td>
<td>0.7604</td>
<td>0.5401</td>
</tr>
<tr>
<td>$d_2$</td>
<td>1.1052</td>
<td>0.9148</td>
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<td>17.0000</td>
</tr>
<tr>
<td>$v_2$</td>
<td>15.0000</td>
<td>17.0000</td>
</tr>
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<td>$v_3$</td>
<td>14.6931</td>
<td>12.8342</td>
</tr>
<tr>
<td>$v_4$</td>
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<td>23.2630</td>
</tr>
<tr>
<td>$v_5$</td>
<td>1.8471</td>
<td>5.1143</td>
</tr>
<tr>
<td>$v_7$</td>
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</tr>
<tr>
<td>$v_8$</td>
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</tr>
<tr>
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<td>346.1297</td>
<td>350.4775</td>
</tr>
<tr>
<td>$TC$</td>
<td>740.7287</td>
<td>749.0889</td>
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</table>
5. Conclusions

This paper established a queuing network equilibrium with elastic demand to help find how the link delays and other parameters change when road capacity enhancement measures is introduced to the network. Inner penalty Newton based method is showed in this paper to solving the elastic demand network equilibrium with queuing problem. It can be shown that this method can solve this capacity constraint traffic equilibrium very effectively. Then a method put forward by Tobin (1988) is used to get the nondegenerate extreme point of path flows equilibrium solution. The sensitivity analysis is performed explicitly to demonstrate how to derive the derivatives of equilibrium link flows and equilibrium queuing times with respect to capacity parameters. Those derivatives are very useful in finding whether the road capacities expanding are effective in reducing traffic congestion and improvement network performance. The derivative information also allows one to estimate nearby solution for small perturbation in link capacities once an equilibrium solution has been calculated. The given numerical example is presented to illustrate the sensitivity analysis method and its application to the evaluation of traffic improvement schemes. The further research is to study how to apply this method to the real road network improvement.

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References


