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Theoretical Computer Science 170 (1996) 277–296

Theoretical Computer Science

An axiomatization of the algebra of Petri net concatenable processes

Vladimiro Sassone

BRICS[1], *Computer Science Department, University of Aarhus, Ny Munkegade, Bldng 540, DK-8000, Aarhus, Denmark*

Received August 1994; revised July 1995
Communicated by G. Rozenberg

Abstract

The concatenable processes of a Petri net N can be characterized abstractly as the arrows of a symmetric monoidal category $\mathscr{P}(N)$. However, this is only a partial axiomatization, since it is based on a concrete, ad hoc chosen, category of symmetries Sym_N.

In this paper we give a completely abstract characterization of the category of concatenable processes of N, thus yielding an axiomatic theory of the noninterleaving behaviour of Petri nets.

0. Introduction

Concatenable processes of Petri nets have been introduced in [3] to account, as their name indicates, for the issue of process concatenation. Let us briefly reconsider the ideas which led to their definition.

The development of theory Petri nets, focusing on the noninterleaving aspects of concurrency, brought to the foreground various notions of process, e.g. [14, 5, 2, 12, 3]. Generally speaking, Petri net processes – whose standard version is given by the Goltz–Reisig *nonsequential processes* [5] – are structures needed to account for the *causal relationships* which rule the occurrence of events in computations. Thus, ideally, processes are simply computations in which explicit information about such causal connections is added. More precisely, since it is a well-established idea that, as far as the theory of computation is concerned, causality can be faithfully described by means of partial orderings – though interesting 'heretic' ideas appear sometimes – abstractly, the processes of a net N are ordered sets whose elements are labelled by transitions of N. Concretely, in order to describe exactly which multisets of transitions are processes,

[1] Basic Research in Computer Science, Centre of the Danish National Research Foundation. The author was supported by EU Human Capital and Mobility grant ERBCHBGCT920005. Work partly carried out during the author's doctorate at Università di Pisa, Italy.

PII S0304-3975(96)00009-6

one defines a process of N to be a map $\pi: \Theta \to N$ which maps transitions to transitions and places to places respecting the 'bipartite graph structure' of nets. Here Θ is a *finite deterministic occurrence net*, i.e., roughly speaking, a finite, conflict-free, 1-safe, acyclic net. The role of π is to 'label' the places and the (partially ordered) transitions of Θ with places and transitions of N in a way compatible with the structure of N.

Given this definition, one can assign the correct *source* and *target* states to a process $\pi : \Theta \to N$ by considering the multisets of places of N which are the image via π of the places of Θ with, respectively, empty preset and empty postset (henceforth referred to as *minimal* and *maximal* places of Θ). Now, the simple minded attempt to concatenate a process $\pi_1 : \Theta_1 \to N$ with source u to a process $\pi_0: \Theta_0 \to N$ with target u by merging the maximal places of Θ_0 with the minimal places of Θ_1 in a way which preserves the labellings fails immediately. In fact, if more than one place of u is labelled by a single place of N, there are many ways to put in one-to-one correspondence the maximal places of Θ_0 and the minimal places of Θ_1 respecting the labels, i.e., there are many possible concatenations of π_0 and π_1, each of which gives a possibly different process of N. In other words, as the above argument shows, process concatenation has to do with *merging tokens*, i.e., instances of places, rather than *merging places.*

Therefore, any attempt to deal with process concatenation must disambiguate the *identity* of each token in a process. This is exactly the idea of *concatenable processes*, which are simply Goltz–Reisig processes in which the minimal and maximal places carrying the same label are linearly ordered. This yields immediately an operation of concatenation, since the ambiguity about the identity of tokens is resolved using the additional information given by the orderings. Moreover, the existence of concatenation leads easily to the definition of the category of concatenable processes of N. It turns out that such a category is a *symmetric monoidal category* whose tensor product is provided by the parallel composition of processes [3]. The relevance of this result is that it describes Petri net behaviours as *algebras* in a remarkably smooth way.

Naturally linked to the fact that they are algebraic structures, concatenable processes are amenable to abstract descriptions. In [3] the authors deal with this issue by associating to each net N a symmetric monoidal category $\mathscr{P}(N)$ isomorphic to the category of concatenable processes of N; such a characterization, however, is not completely abstract and it provides only a partial axiomatization of the algebra of concatenable processes of N, since in the cited work $\mathscr{P}(N)$ is built on a concrete, ad hoc constructed, category Sym_N.

In this paper we show that Sym_N can be characterized axiomatically, thus yielding a *purely algebraic* and *completely abstract* axiomatization of the category of concatenable processes of N. In particular, we shall describe $\mathscr{P}(N)$ in terms of *universal* constructions. Namely, we shall prove that it is the *free symmetric strict monoidal* category on the net N modulo two simple additional axioms. [1] This result complements the investigation of [3] on the structure of net computations by showing that they can be described by an *essentially algebraic theory* (whose models are symmetric monoidal

[1] We remark that the existence of a similar axiomatization was conjectured also in [6].

categories), which, in our opinion, is a remarkable fact. In addition, our axiomatization of $\mathscr{P}(N)$ naturally provides a *term algebra* and an *equational theory* of concatenable processes of N, by means of which one can 'compute' with and 'reason' about them. The relevance of this is evident when one thinks of N as modelling a complex system whose behaviour is to be analysed.

Concerning the organization of the paper, Section 1 recalls the needed definitions; the reader acquainted with [12, 3] and with monoidal categories can safely skip it. In Section 2 we prove our result. An extended abstract version of this paper appears as [16].

1. Monoidal categories and concatenable processes

The notion of *monoidal category* dates back to [1] (see [11] for an easy thorough introduction and [4] for advanced topics). In this paper we shall be concerned only with a particular kind of symmetric monoidal categories, namely those which are *strict monoidal* and whose objects form a *free commutative monoid*. Remarkably, a very similar kind of categories have appeared as distinguished algebraic structures also in [10], where they are called PROP's (for Product and Permutation categories), and in [8]. The difference between the categories we use and PROP's is that the monoid of objects of the latter have a single generator, i.e., it is the monoid of natural numbers with addition.

A *symmetric strict monoidal category* (SSMC in the following) is a structure $(\underline{C}, \otimes, e, \gamma)$, where $\underline{C}$ is a category, e is an object of $\underline{C}$, called the *unit* object, $\otimes : \underline{C} \times \underline{C} \to \underline{C}$ is a functor, called the *tensor product*, subject to the following equations

$$\otimes \circ \langle \otimes \times 1_{\underline{C}} \rangle = \otimes \circ \langle 1_{\underline{C}} \times \otimes \rangle, \tag{1}$$

$$\otimes \circ \langle \underline{e}, 1_{\underline{C}} \rangle = 1_{\underline{C}}, \tag{2}$$

$$\otimes \circ \langle 1_{\underline{C}}, \underline{e} \rangle = 1_{\underline{C}}, \tag{3}$$

where $\underline{e} : \underline{C} \to \underline{C}$ is the constant functor which associate e and id_e, respectively, to each object and each morphism of $\underline{C}$, $\langle \_, \_ \rangle$ is the pairing of functors induced by the cartesian product, and[2] $\gamma : \_1 \otimes \_2 \xrightarrow{\sim} \_2 \otimes \_1$ is a natural isomorphism, called the *symmetry* of $\underline{C}$, subject to the following Kelly–MacLane *coherence* axioms [9, 7]:

$$(\gamma_{x,z} \otimes id_y) \circ (id_x \otimes \gamma_{y,z}) = \gamma_{x \otimes y, z}, \tag{4}$$

$$\gamma_{y,x} \circ \gamma_{x,y} = id_{x \otimes y}. \tag{5}$$

Clearly, Eq. (1) states that the tensor is associative on both objects and arrows, while (2) and (3) state that e and id_e are, respectively, the unit object and the unit arrow for $\otimes$. Concerning the coherence axioms, axiom (5) says that $\gamma_{y,x}$ is the inverse of $\gamma_{x,y}$, while (4), the *real key* of symmetric monoidal categories, links the symmetry at composed objects to the symmetry at the components.

[2] We use $\_n$ for $n \in \omega$ as placeholdes and $x, y, z, \ldots$ as variables for objects.

Remark. Adapting the general definition of monoidal category to the special case of SSMC's, one finds that there is a further axiom to state, namely $\gamma_{e,x} = id_x$. Observe however that it follows from the others. In fact, by (2) we have that $e \otimes e = e$ and thus $\gamma_{e,x} = \gamma_{e\otimes e,x}$, which by (4) is equal to $(\gamma_{e,x} \otimes id_e) \circ (id_e \otimes \gamma_{e,x})$. Now, by (2) and (3) we have that $\gamma_{e,x} = \gamma_{e,x} \circ \gamma_{e,x}$ and thus, multiplying both terms by $\gamma_{x,e}$ and exploiting (5), we have $\gamma_{e,x} = id_{e\otimes x} = id_x$.

A *symmetry* s in a symmetric monoidal category $\underline{C}$ is any arrow obtained as composition and tensor of *identities* and *components* of γ. We use $Sym_{\underline{C}}$ to denote the subcategory of the symmetries of $\underline{C}$.

A *symmetric strict monoidal functor* from $(\underline{C}, \otimes, e, \gamma)$ to $(\underline{D}, \otimes', e', \gamma')$, is a functor $\mathrm{F}\colon \underline{C} \to \underline{D}$ which preserves the monoidal structure, i.e., such that

$$\mathrm{F}(e) = e', \tag{6}$$

$$\mathrm{F}(x \otimes y) = \mathrm{F}(x) \otimes' \mathrm{F}(y), \tag{7}$$

$$\mathrm{F}(\gamma_{x,y}) = \gamma'_{\mathrm{F}x,\mathrm{F}y}. \tag{8}$$

Let $\underline{\text{SSMC}}$ be the category of SSMC's and symmetric strict monoidal functors and let $\underline{\text{SSMC}}^{\oplus}$ be the full subcategory consisting of the monoidal categories whose objects form *free* commutative monoids.

We recall now the definitions of Petri nets and their (concatenable) processes.

Notation. We denote by $S^{\oplus}$ the *free commutative monoid* on S, i.e., the monoid of *finite multisets* of S. Recall that a finite multiset is a functions from S to ω which yields nonzero values at most on finitely many arguments. We represent $u \in S^{\oplus}$ as a formal sum $\oplus_i u(a_i) \cdot a_i$ where only the $a_i \in S$ such that $u(a_i) > 0$ appear; the empty multiset will be denoted by 0.

A *Petri net* is a structure $N = (\partial^0_N, \partial^1_N \colon T_N \to S^{\oplus}_N)$, where T_N is a set of *transitions*, S_N is a set of *places*, and ∂^0_N and ∂^1_N are functions which assign to each transition, respectively, a *source* and a *target* multiset of places. For $t \in T_N$, we write $t\colon u \to v$ to indicate that $\partial^0_N(t) = u$ and $\partial^1_N(t) = v$. A *morphism* of nets $f\colon N_0 \to N_1$ consists of a pair of functions $\langle f_t\colon T_{N_0} \to T_{N_1}, f_p\colon S^{\oplus}_{N_0} \to S^{\oplus}_{N_1}\rangle$, where the place component f_p is a *monoid homomorphism*, which respect source and target, i.e., the two diagrams below commute.

$$\begin{array}{ccc} T_{N_0} & \xrightarrow{\partial^0_{N_0}} & S^{\oplus}_{N_0} \\ {\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle f_p} \\ T_{N_1} & \xrightarrow[\partial^0_{N_1}]{} & S^{\oplus}_{N_1} \end{array} \qquad\qquad \begin{array}{ccc} T_{N_0} & \xrightarrow{\partial^1_{N_0}} & S^{\oplus}_{N_0} \\ {\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle f_p} \\ T_{N_1} & \xrightarrow[\partial^1_{N_1}]{} & S^{\oplus}_{N_1} \end{array}$$

The data above define the category $\underline{\text{Petri}}$ of Petri nets.

A *process net* is a finite, acyclic net Θ such that for all $t \in T_\Theta$, $\partial^0_\Theta(t)$ and $\partial^1_\Theta(t)$ are sets (as opposed to multisets), and for all $t_0 \neq t_1 \in T_\Theta$, $\partial^i_\Theta(t_0) \cap \partial^i_\Theta(t_1) = \emptyset$, for $i = 0, 1$. Given $N \in \underline{\text{Petri}}$, a *process* of N is a morphism $\pi\colon \Theta \to N$, where Θ is a process net and π is a net morphism which maps places to places (as opposed to morphisms which map places to markings).

A *concatenable process* of N is a triple $(\pi\colon \Theta \to N, \{<_a\}_{a\in S_N}, \{\ll_a\}_{a\in S_N})$, where π is a process, and $<_a$ and $\ll_a$ are linear orderings of, respectively, the set of minimal and the set of maximal places of Θ contained in $\pi_p^{-1}(a)$ (cf. Fig. 1). In order to abstract from the details concerning the underlying process nets, concatenable processes are considered up to isomorphisms. Formally, two concatenable processes, say with underlying processes $\pi_0\colon \Theta_0 \to N$ and $\pi_1 : \Theta_1 \to N$, are identified if there exists an isomorphism $\varphi\colon \Theta_0 \to \Theta_1$ which preserves all the orderings and such that $\pi_1 \circ \varphi = \pi_0$.

Concatenable processes allow the operations of *sequential* and *parallel* composition (see Figs. 2 and 3, and consult [3] for further examples). Let CP_0 and CP_1 be concatenable processes of N, and let $\pi_0 : \Theta_0 \to N$ and $\pi_1 : \Theta_1 \to N$ denote their underlying processes. The parallel composition CP_0 Par CP_1 is the concatenable process of N whose underlying process is the disjoint union of π_0 and π_1, i.e., $\pi_0 + \pi_1 : \Theta_0 + \Theta_1 \to N$, where $+$ denotes the coproduct in Petri, and whose orderings extend those of CP_0

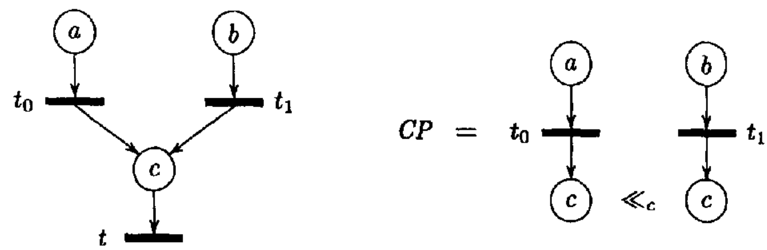


Fig. 1. A net and one of its two concatenable processes $CP\colon a \oplus b \to 2c$

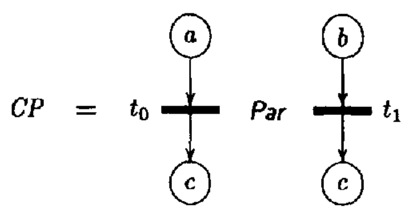


Fig. 2. CP of Fig. 1 as the parallel composition of two simpler processes.

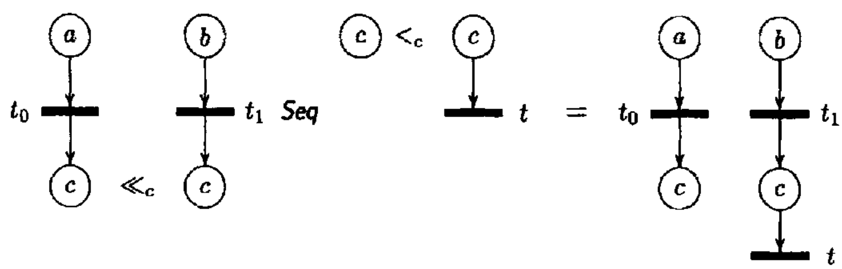


Fig. 3. Sequential composition (concatenation) of concatenable processes.

and CP_1 by making all the places of Θ_0 precede all the places of Θ_1. The sequential composition, or concatenation, $CP = CP_0 \,\mathsf{Seq}\, CP_1$ is defined if and only if the state reached by CP_0 coincide with the source state of CP_1. In this case, CP is obtained by glueing together π_0 and π_1, identifying injectively each maximal place of Θ_0 with a minimal place of Θ_1 in the *unique* way compatible with the orderings $\ll_a$ on Θ_0 and $<_a$ on Θ_1 for all $a \in S_N$.

Next, we recall the construction of the symmetric strict monoidal category $\mathcal{P}(N)$. We start by introducing the *vectors of permutations* (*vperms*) of N,[3] which will provide the symmetry isomorphism of $\mathcal{P}(N)$.

Remark. A *permutation* of n elements is an *automorphism* of the segment of the first n positive natural numbers. The set $\Pi(n)$ of the $n!$ permutations of n elements is a group under the operation of function composition called the *symmetric group* on n elements, or of order $n!$. The unit of $\Pi(n)$ is the identity function on $\{1,\dots,n\}$ and the inverse of $\sigma \in \Pi(n)$ is its inverse function σ^{-1}. Due to its triviality, the notion of permutation of zero elements is never considered; however, to simplify notation, we shall assume that the empty function $\emptyset\colon \emptyset \to \emptyset$ is the (unique) permutation of zero elements. As a notation, when $\sigma \in \Pi(n)$, we write $|\sigma|$ for n. We use sometime a graphical representation of permutations according to which σ is depicted by drawing a line from i to $\sigma(i)$ (see, for example, Figs. 4 and 5).

We say that $\sigma \in \Pi(n)$ is a *transposition* if it is a '*swapping*' of adjacent elements, i.e., if there exists $i < n$ such that $\sigma(i) = i+1$, $\sigma(i+1) = i$, and $\sigma(k) = k$ elsewhere. We shall denote such a σ as $(i\ i{+}1)$ or as τ_i. Transpositions are a relevant kind of permutations, since each permutation can be written as composition of them.

For $u \in S^{\oplus}$, a *vperm* $s\colon u \to u$ is a function which assigns to each $a \in S$ a permutation $s(a) \in \Pi(u(a))$. Given $u = n_1 \cdot a_1 \oplus \cdots \oplus n_k \cdot a_k$ in $S_N^{\oplus}$, we shall represent a vperm s on u as vector of permutations, $\langle \sigma_{a_1},\dots,\sigma_{a_k}\rangle$, where $s(a_j) = \sigma_{a_j}$, whence their name. One can define the operations of sequential and parallel composition of vperms, so that they can be organized as the arrows of a SSMC. The details follow (see also Fig. 4).

Given the vperms $s = \langle \sigma_{a_1},\dots,\sigma_{a_k}\rangle : u \to u$ and $s' = \langle \sigma'_{a_1},\dots,\sigma'_{a_k}\rangle : u \to u$ their *sequential composition* $s;s' : u \to u$ is the vperm

$$\langle \sigma_{a_1};\sigma'_{a_1},\dots,\sigma_{a_k};\sigma'_{a_k}\rangle,$$

where $\sigma;\sigma'$ is the composition of permutation which we write in the diagrammatic order from left to right. Given the vperms $s = \langle \sigma_{a_1},\dots,\sigma_{a_k}\rangle : u \to u$ and $s' = \langle \sigma'_{a_1},\dots,\sigma'_{a_k}\rangle : v \to v$ (where possibly $\sigma_{a_j} = \emptyset$ for some j), their *parallel composition* $s \otimes s' : u \oplus v \to u \oplus v$ is the vperm

$$\langle \sigma_{a_1} \otimes \sigma'_{a_1},\dots,\sigma_{a_k} \otimes \sigma'_{a_k}\rangle,$$

[3] Vperms are called *symmetries* in [3]. Here, in order to avoid confusion with the general notion of symmetry in a symmetric monoidal category, we prefer to use another term.

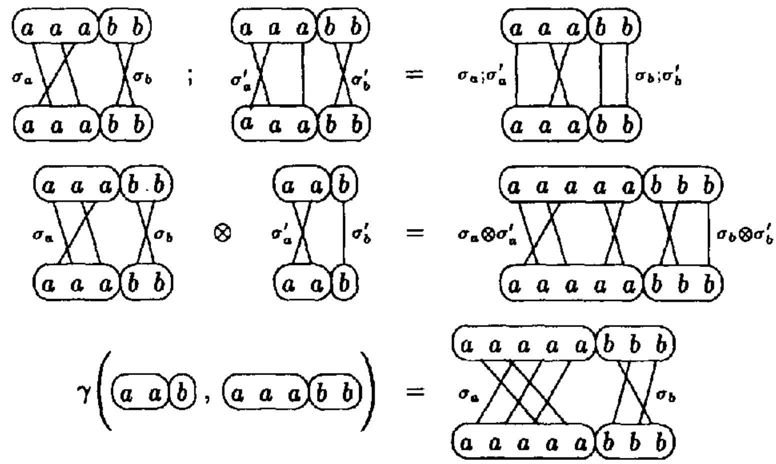


Fig. 4. The monoidal structure of vperms.

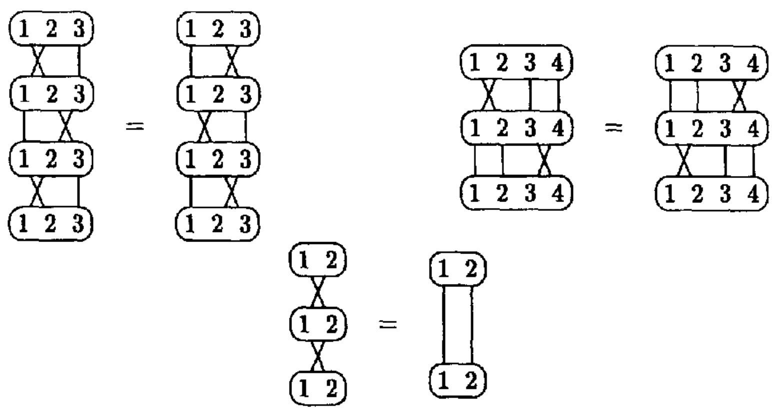


Fig. 5. Some instances of the axioms of permutations.

where

$$(\sigma \otimes \sigma')(x) = \begin{cases} \sigma(x) & \text{if } 0 < x \leqslant |\sigma|, \\ \sigma'(x - |\sigma|) + |\sigma| & \text{if } |\sigma| < x \leqslant |\sigma| + |\sigma'|. \end{cases}$$

Let γ be $(1\ 2) \in \Pi(2)$ and consider $u_i = n_1^i \cdot a_1 \oplus \cdots \oplus n_k^i \cdot a_k$, $i = 1, 2$, in $S^{\oplus}$. The *interchange vperm* $\gamma(u_1, u_2)$ is the vperm $\langle \sigma_{a_1}, \ldots, \sigma_{a_k} \rangle : u_1 \oplus u_2 \to u_1 \oplus u_2$, where

$$\sigma_{a_j}(x) = \begin{cases} x + n_j^2 & \text{if } 0 < x \leqslant n_j^1, \\ x - n_j^1 & \text{if } n_j^1 < x \leqslant n_j^1 + n_j^2. \end{cases}$$

It is immediate to verify that $\_;\_$ is associative. Moreover, for each $u \in S^{\oplus}$, the vperm $u = \langle id_{a_1}, \ldots, id_{a_n} \rangle : u \to u$, where id_{a_j} is the identity permutation, is an identity for sequential composition. Finally, writing 0 for the empty multiset on S, the (unique) vperm $s : 0 \to 0$, is a unit for parallel composition.

Now, for N a net, let Sym_N be the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are the vperms $s: u \to u$ for $u \in S_N^{\oplus}$. It is easy to show that Sym_N is a **SSMC** with respect to the given composition and tensor product, with identities and unit element as explained above, and with the symmetry natural isomorphism given by the collection $\gamma = \{\gamma(u,v)\}_{u,v \in Sym_N}$ of the interchange vperms. Observe that, although Sym_N is not strictly symmetric, it is so on the objects. More strongly, the objects form a free commutative monoid, i.e., $Sym_N \in \underline{\text{SSMC}}^{\oplus}$.

We can now define $\mathscr{P}(N)$ as the category which includes Sym_N as a subcategory and has as additional arrows those defined by the following rules:

$$\frac{t: u \to v \text{ in } T_N}{t: u \to v \text{ in } \mathscr{P}(N)}$$

$$\frac{\alpha: u \to v \text{ and } \beta: u' \to v' \text{ in } \mathscr{P}(N)}{\alpha \otimes \beta: u \oplus u' \to v \oplus v' \text{ in } \mathscr{P}(N)} \qquad \frac{\alpha: u \to v \text{ and } \beta: v \to w \text{ in } \mathscr{P}(N)}{\alpha;\beta: u \to w \text{ in } \mathscr{P}(N)}$$

plus axioms expressing the fact that $\mathscr{P}(N)$ is a **SSMC** with composition $\_;\_$, tensor $\_\otimes\_$ (extending those already defined on vperms) and symmetry isomorphism γ, and the following axioms involving transitions and vperms

$$\begin{aligned} t;s &= t, \quad \text{where } t: u \to v \text{ in } T_N \text{ and } s: v \to v \text{ in } Sym_N, \\ s;t &= t, \quad \text{where } t: u \to v \text{ in } T_N \text{ and } s: u \to u \text{ in } Sym_N. \end{aligned} \qquad (\Psi)$$

In other words, $\mathscr{P}(N)$ is built on the category Sym_N by adding the transitions of N and freely closing with respect to sequential and parallel composition of arrows, so that $\mathscr{P}(N)$ is made symmetric strict monoidal and axioms (Ψ) hold.

The relevant fact about $\mathscr{P}(N)$ is that its arrows represent exactly the concatenable processes of N, i.e., $\mathscr{P}(N)$ represents the noninterleaving behaviour of N, including its algebraic structure. (See [3] for the details.)

Theorem 1.1. ($\mathscr{P}(N)$ vs. concatenable processes [3]). *For any net N there exists a one-to-one correspondence between the arrows of $\mathscr{P}(N)$ and the concatenable processes of N such that, for each $u, v \in S_N^{\oplus}$, the arrows of type $u \to v$ correspond to the processes enabled by u and producing v, and such that sequential and parallel composition (tensor product) of processes (arrows) are respected.*

Vperms play in this correspondence an absolutely fundamental role: Sym_N accounts for the families of orderings $\{<_a\}_{a \in S_N}$ and $\{\ll_a\}_{a \in S_N}$, which are the key to concatenable processes, guaranteeing a correct treatment of sequential composition. In other words, Sym_N is an algebraic representation of the 'threads of causality' in process concatenation.

Unfortunately, the concrete definition of vperms weakens considerably the essentially axiomatic character of $\mathscr{P}(N)$ and, therefore, the results of [3]. Also, it makes $\mathscr{P}(N)$ rather uncomfortable an algebra to handle, since the laws which rule it remain partly concealed in Sym_N. An abstract characterization of Sym_N, one yielding an entirely axiomatic presentation of the concatenable processes of N, is called-for. This is what we shall do next.

2. Axiomatizing concatenable processes

This section provides a fully axiomatic description of the concatenable processes of N obtained by proving that $\mathscr{P}(N)$ is a quotient of the free **SSMC** on N. As already remarked, the key to this result will be an axiomatization of the category of vperms Sym_N. We start by showing that we can associate a free **SSMC** to each net N. Although this may not look very surprising, our proof will identify a 'minimal' description of such categories which will be useful later on.

Proposition 2.1 ($\mathscr{F} \dashv \mathscr{U}$). *The forgetful functor* $\mathscr{U} : \underline{\mathsf{SSMC}^{\oplus}} \to \underline{\mathsf{Petri}}$ *has a left adjoint* $\mathscr{F} : \underline{\mathsf{Petri}} \to \underline{\mathsf{SSMC}^{\oplus}}$.

Proof. Consider the category $\mathscr{F}(N)$ whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules

$$\frac{u \in S_N^{\oplus}}{id_u : u \to u \text{ in } \mathscr{F}(N)} \qquad \frac{a \text{ and } b \text{ in } S_N}{c_{a,b}: a \oplus b \to b \oplus a \text{ in } \mathscr{F}(N)} \qquad \frac{t: u \to v \text{ in } T_N}{t: u \to v \text{ in } \mathscr{F}(N)}$$

$$\frac{\alpha: u \to v \text{ and } \beta: u' \to v' \text{ in } \mathscr{F}(N)}{\alpha \otimes \beta: u \oplus u' \to v \oplus v' \text{ in } \mathscr{F}(N)} \qquad \frac{\alpha: u \to v \text{ and } \beta: v \to w \text{ in } \mathscr{F}(N)}{\alpha;\beta: u \to w \text{ in } \mathscr{F}(N)}$$

modulo the axioms expressing that $\mathscr{F}(N)$ is a strict monoidal category, namely,

$$\begin{array}{rcl} \alpha; id_v = \alpha = id_u; \alpha & \text{and} & (\alpha;\beta);\gamma = \alpha;(\beta;\gamma), \\ (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) & \text{and} & id_0 \otimes \alpha = \alpha = \alpha \otimes id_0, \\ id_u \otimes id_v = id_{u\oplus v} & \text{and} & (\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta'), \end{array} \tag{9}$$

the latter whenever the right-hand term is defined, and the following axioms:

$$c_{a,b}; c_{b,a} = id_{a\oplus b}, \tag{10}$$

$$c_{u,u'}; (\beta \otimes \alpha) = (\alpha \otimes \beta); c_{v,v'} \quad \text{for} \quad \alpha: u \to v, \;\; \beta: u' \to v', \tag{11}$$

where $c_{u,v}$ for $u, v \in S_N^{\oplus}$ denote *any* term obtained from $c_{a,b}$ for $a, b \in S_N$ by applying recursively the following rules (compare with axiom (4)):

$$\begin{aligned} c_{0,u} &= c_{0,u} = id_u, \\ c_{a\oplus u,v} &= (id_a \otimes c_{u,v}); (c_{a,v} \otimes id_u), \\ c_{u,v\oplus a} &= (c_{u,v} \otimes id_a); (id_v \otimes c_{u,a}). \end{aligned} \tag{12}$$

Observe that Eq. (11), in particular, equalizes all the terms obtained from (12) for fixed u and v. In fact, let $c_{u,v}$ and $c'_{u,v}$ be two such terms and take α and β to be, respectively, the identities of u and v. Now, since $id_u \otimes id_v = id_{u \oplus v} = id_v \otimes id_u$, from (11) we have that $c_{u,v} = c'_{u,v}$ in $\mathscr{F}(N)$. Then, we claim that the collection $\{c_{u,v}\}_{u,v \in S_N^{\oplus}}$ is a symmetry natural isomorphism which makes $\mathscr{F}(N)$ into a **SSMC** and that, in addition, $\mathscr{F}(N)$ is the free **SSMC** on N.

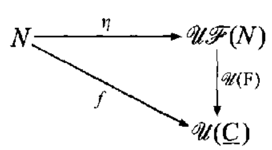
In order to show the first claim, observe that the naturality of c is expressed directly from axiom (11). We need to check that for any u and v we have $c_{u,v}; c_{v,u} = id_{u \oplus v}$, which follows easily from (10) by induction on the sum of the sizes of u and v.

Base cases: If $u = 0$ or $v = 0$, the thesis follows from the first of (12). If $|u| = |v| = 1$, then the required equation is (10).

Inductive step: Without loss of generality, assume $u = a \oplus u'$, $u' \neq 0$. Then, by (12),

$$\begin{aligned} c_{u,v}; c_{v,u} &= (id_a \otimes c_{u',v}); (c_{a,v} \otimes id_{u'}); (c_{v,a} \otimes id_{u'}); (id_a \otimes c_{v,u'}) \\ &= (id_a \otimes c_{u',v}); ((c_{a,v}; c_{v,a}) \otimes id_{u'}); (id_a \otimes c_{v,u'}) \\ &= (id_a \otimes c_{u',v}); (id_a \otimes c_{v,u'}) \\ &= id_a \otimes (c_{u',v}; c_{v,u'}) = id_a \otimes id_{u' \oplus v} = id_{u \oplus v}. \end{aligned}$$

For $\underline{\mathrm{C}}$ in $\underline{\mathrm{SSMC}}^{\oplus}$, the net $\mathscr{U}(\underline{\mathrm{C}})$ is obtained by forgetting the categorical structure of $\underline{\mathrm{C}}$. The markings and the transitions of $\mathscr{U}(\underline{\mathrm{C}})$ are, respectively, the objects and the arrows of $\underline{\mathrm{C}}$ with the given sources and targets. Similarly, for F a symmetric strict monoidal functor in $\underline{\mathrm{SSMC}}^{\oplus}$, $\mathscr{U}(\mathrm{F})$ is the net morphism whose components are the restrictions of F to, respectively, arrows and objects. Consider the net $\mathscr{U}\mathscr{F}(N)$ and the net morphism $\eta: N \to \mathscr{U}\mathscr{F}(N)$, where η_p is the identity homomorphism and η_t is the obvious injection of T_N in $T_{\mathscr{U}\mathscr{F}(N)}$. We show that η is universal, i.e., that for any $\underline{\mathrm{C}}$ in $\underline{\mathrm{SSMC}}^{\oplus}$ and for any net morphism $f: N \to \mathscr{U}(\underline{\mathrm{C}})$, there is a unique symmetric strict monoidal functor $\mathrm{F}: \mathscr{F}(N) \to \underline{\mathrm{C}}$ which makes the following diagram commute:



Let $\underline{\mathrm{C}} = (\underline{\mathrm{C}}, \otimes, 0, \gamma)$ and $f: N \to \mathscr{U}(\underline{\mathrm{C}})$ be as in the hypothesis above. In order for the diagram to commute and for F to be a symmetric strict monoidal functor, its definition on the generators of $\mathscr{F}(N)$ is compelled:

$$\mathrm{F}(u) = f_p(u), \quad \mathrm{F}(t) = f_t(t), \quad \mathrm{F}(id_u) = id_{f_p(u)}, \quad \mathrm{F}(c_{a,b}) = \gamma_{f_p(a), f_p(b)}.$$

Clearly, the extension of F to composition and tensor is also uniquely determined, namely, $\mathrm{F}(\alpha; \beta) = \mathrm{F}(\beta) \circ \mathrm{F}(\alpha)$ and $\mathrm{F}(\alpha \otimes \beta) = \mathrm{F}(\alpha) \otimes \mathrm{F}(\beta)$. Therefore, to conclude the proof we only need to show that F is a well-defined symmetric strict monoidal functor, since, then, it is necessarily the unique one such that $\mathscr{U}(\mathrm{F}) \circ \eta = f$.

To establish that F is well-defined, it is enough to prove that it preserves the axioms which generate $\mathscr{F}(N)$. Since $\underline{\mathrm{C}}$ is a strict monoidal category and $\mathrm{F}(id_u) = id_{\mathrm{F}(u)}$, axioms (9) are clearly preserved. Moreover, since $\underline{C}$ is symmetric with symmetry isomorphism γ, we have that

$$\mathrm{F}(c_{a,b}; c_{b,a}) = \gamma_{\mathrm{F}(b),\mathrm{F}(a)} \circ \gamma_{\mathrm{F}(a),\mathrm{F}(b)} = id_{\mathrm{F}(a)\oplus\mathrm{F}(b)} = id_{\mathrm{F}(a\oplus b)} = \mathrm{F}(id_{a\oplus b}),$$

i.e., F respects axiom (10). Showing that F preserves axiom (11) and it is a symmetric strict monoidal functor reduces to showing that, for each $u, v \in S_N^{\oplus}$ and for each term $c_{u,v}$ obtained from (12), we have $\mathrm{F}(c_{u,v}) = \gamma_{\mathrm{F}(u),\mathrm{F}(v)}$. In fact, this proves directly the latter claim, functoriality and axioms (6) and (7) holding by definition of F, and since γ is a natural transformation, it also proves that F preserves (11). We proceed by induction on the structure of $c_{u,v}$.

Base cases. If $c_{u,v}$ is a generator, i.e., $|u| = |v| = 1$, the claim is proved by appealing directly to the definition of F. If it comes from (12) with $u = 0$, then $\mathrm{F}(c_{u,v}) = id_{\mathrm{F}(v)}$. However, since $\gamma_{e,x} = id_x$ holds in any SSMC, as shown in a previous remark, and since $\mathrm{F}(u)$=0, we have $\mathrm{F}(c_{u,v}) = \gamma_{\mathrm{F}(u),\mathrm{F}(v)}$ as required. A symmetric argument applies if $c_{u,v}$ is obtained from (12) for $v = 0$.

Inductive step. If $c_{u,v}$ is obtained from the second of (12) with $u = a \oplus u'$, then, exploiting the induction hypothesis, $\mathrm{F}(c_{u,v}) = (\gamma_{\mathrm{F}(a),\mathrm{F}(v)} \otimes id_{\mathrm{F}(u')}) \circ (id_{\mathrm{F}(a)} \otimes \gamma_{\mathrm{F}(u'),\mathrm{F}(v)})$ and thus, by the coherence axiom (4) of SSMC's, we have $\mathrm{F}(c_{u,v}) = \gamma_{\mathrm{F}(a)\oplus\mathrm{F}(u'),\mathrm{F}(v)}$ which is $\gamma_{\mathrm{F}(a\oplus u'),\mathrm{F}(v)}$, i.e., $\gamma_{\mathrm{F}(u),\mathrm{F}(v)}$. If instead $v = v' \oplus a$ and $c_{u,v}$ is obtained from the last of (12), then the claim is proved similarly by using the inverse of (4), i.e., $\gamma_{x,y\otimes z} = (id_y \otimes \gamma_{x,z}) \circ (\gamma_{x,y} \otimes id_z)$, which, of course, holds in any SSMC. □

Thus, establishing the adjunction $\mathscr{F} \dashv \mathscr{U}: \underline{\mathrm{Petri}} \rightharpoonup \underline{\mathrm{SSMC}}^{\oplus}$, we have identified $\mathscr{F}(N)$, the *free* SSMC on N, as a category generated, modulo appropriate equations, from the net N viewed as a graph enriched with formal arrows id_u, which play the role of the identities, and $c_{a,b}$ for $a, b \in S_N$, which generate all the needed symmetries.

Our aim is to relate $\mathscr{F}(N)$ and $\mathscr{P}(N)$. As a matter of fact, $\mathscr{F}(N)$ is positively more concrete than $\mathscr{P}(N)$ and far from being isomorphic (or equivalent) to it. For example, for $a \neq b$ in S_N, we have $c_{a,b} \neq id_{a\oplus b}$ in $\mathscr{F}(N)$, whilst $\gamma(a,b) = id_{a\oplus b}$ in $\mathscr{P}(N)$. Therefore, no symmetric monoidal functor $\mathrm{Q}: \mathscr{F}(N) \to \mathscr{P}(N)$ can be mono. Also, $\mathscr{F}(N)$ possesses no counterpart of axioms (Ψ). We shall prove that these are precisely the differences between $\mathscr{F}(N)$ and $\mathscr{P}(N)$. Namely, we shall obtain $\mathscr{P}(N)$ as a quotient of $\mathscr{F}(N)$ by enforcing the axioms outlined above. The next proposition, which is the adaptation to SSMC's of the usual notion of quotient algebras, provides the tool we shall use for this purpose.

Proposition 2.2 (Monoidal quotient categories). *For $\underline{C}$ a* SSMC, *let $\mathscr{R}$ be a function which assigns to each pair of objects a and b of $\underline{C}$ a binary relation $\mathscr{R}_{a,b}$ on the homset $\underline{C}(a,b)$. Then, there exist a* SSMC *$\underline{C}/\mathscr{R}$ and a symmetric strict monoidal functor $\mathrm{Q}_{\mathscr{R}}: \underline{C} \to \underline{C}/\mathscr{R}$ such that*

(i) *If $f\mathscr{R}_{a,b}f'$ then $\mathrm{Q}_{\mathscr{R}}(f) = \mathrm{Q}_{\mathscr{R}}(f')$;*

(ii) *For each symmetric strict monoidal* $\mathrm{H}: \underline{C} \to \underline{D}$ *such that* $\mathrm{H}(f) = \mathrm{H}(f')$ *whenever $f\mathscr{R}_{a,b}f'$, there exists a unique* $\mathrm{K}: \underline{C}/\mathscr{R} \to \underline{D}$, *which is necessarily symmetric* *strict monoidal such that the following diagram commutes*:

$$\begin{array}{ccc} \underline{\mathrm{C}} & \xrightarrow{\ \mathrm{Q}_{\mathscr{R}}\ } & \underline{\mathrm{C}}/\mathscr{R} \\ & {\scriptstyle \mathrm{H}}\searrow & \downarrow{\scriptstyle \mathrm{K}} \\ & & \underline{\mathrm{D}} \end{array}$$

Proof. Say that $\mathscr{R}$ is a *congruence* if $\mathscr{R}_{a,b}$ is an equivalence for each a and b and if $\mathscr{R}$ respects composition, i.e., whenever $f\mathscr{R}_{a,b}f'$ then, for all $h : a' \to a$ and $k : b \to b'$, we have $(k \circ f \circ h)\mathscr{R}_{a',b'}(k \circ f' \circ h)$. Clearly, if $\mathscr{R}$ is a congruence, the following definition is well-given: $\underline{\mathrm{C}}/\mathscr{R}$ is the category whose objects are those of $\underline{\mathrm{C}}$, whose homset $\underline{\mathrm{C}}/\mathscr{R}(a,b)$ is $\underline{\mathrm{C}}(a,b)/\mathscr{R}_{a,b}$, i.e., the quotient of the corresponding homset of $\underline{\mathrm{C}}$ modulo the appropriate component of $\mathscr{R}$, and whose composition of arrows is given by $[g]_{\mathscr{R}} \circ [f]_{\mathscr{R}} = [g \circ f]_{\mathscr{R}}$. In fact, since $\mathscr{R}_{a,b}$ is an equivalence $\underline{\mathrm{C}}/\mathscr{R}(a,b)$ is well-defined, and since $\mathscr{R}$ preserves the composition, so is the composition in $\underline{\mathrm{C}}/\mathscr{R}$.

Let $\underline{\mathrm{C}} = (\underline{\mathrm{C}}, \otimes, e, \gamma)$. Call $\mathscr{R}$ a $\otimes$-*congruence* if it is a congruence in the above sense and it respects tensor, i.e., if $f\mathscr{R}_{a,b}f'$ then, for all $h: a' \to b'$ and $k : a'' \to b''$, we have $(h \otimes f \otimes k)\mathscr{R}_{a'\otimes a\otimes a'',b'\otimes b\otimes b''}(h \otimes f' \otimes k)$. It is easy to check that, if $\mathscr{R}$ is a $\otimes$-congruence, then the definition $[f]_{\mathscr{R}}\otimes[g]_{\mathscr{R}} = [f\otimes g]_{\mathscr{R}}$ makes the quotient category $\underline{\mathrm{C}}/\mathscr{R}$ into a **SSMC** with symmetry isomorphism given by the natural transformation whose component at (u,v) is $[\gamma_{u,v}]_{\mathscr{R}}$ and unit object e.

Observe now that, given $\mathscr{R}$ as in the hypothesis, it is always possible to find the least $\otimes$-congruence $\mathscr{R}'$ which includes (componentwise) $\mathscr{R}$. Then, take $\underline{\mathrm{C}}/\mathscr{R}$ to be $\underline{\mathrm{C}}/\mathscr{R}'$ and $\mathrm{Q}_{\mathscr{R}}$ to be the obvious projection of $\underline{\mathrm{C}}$ into $\underline{\mathrm{C}}/\mathscr{R}$. Clearly, $\mathrm{Q}_{\mathscr{R}}$ is a symmetric strict monoidal functor.

Now, let $\mathrm{H}: \underline{\mathrm{C}} \to \underline{\mathrm{D}}$ be a monoidal functor as in the hypothesis and consider the mapping of objects and arrows of $\underline{\mathrm{C}}/\mathscr{R}$ to, respectively, objects and arrows of $\underline{\mathrm{D}}$ given by $\mathrm{K}(a) = \mathrm{H}(a)$ and $\mathrm{K}([f]_{\mathscr{R}}) = \mathrm{H}(f)$. It follows from definition of functor that the family $\{\mathscr{S}_{a,b}\}_{a,b\in\underline{\mathrm{C}}}$, where $\mathscr{S}_{a,b}$ is the relation $\{(f,g) \mid \mathrm{H}(f) = \mathrm{H}(g)\}$ on $\underline{\mathrm{C}}(a,b)$, is a congruence. Moreover, since $\mathrm{H}(f \otimes g) = \mathrm{H}(f) \otimes \mathrm{H}(g)$, we have that $\{\mathscr{S}_{a,b}\}_{a,b\in\underline{\mathrm{C}}}$ is a $\otimes$-congruence. Then, if H satisfies the condition in the hypothesis, i.e., if $\mathscr{R} \subseteq \mathscr{S}$, since $\mathscr{R}'$ is the least $\otimes$-congruence which contains $\mathscr{R}$, we have that $f\mathscr{R}'_{a,b}\, g$ implies $\mathrm{H}(f) = \mathrm{H}(g)$, i.e., K is well-defined. Moreover, since H is a functor, it follows that $\mathrm{K}([id_a]_{\mathscr{R}}) = id_{\mathrm{H}(a)} = id_{\mathrm{K}(a)}$ and $\mathrm{K}([g]_{\mathscr{R}} \circ [f]_{\mathscr{R}}) = \mathrm{H}(g) \circ \mathrm{H}(f) = \mathrm{K}([g]_{\mathscr{R}}) \circ \mathrm{K}([f]_{\mathscr{R}})$, i.e., K is a functor. One shows similarly that $\mathrm{K}([f]_{\mathscr{R}} \otimes [g]_{\mathscr{R}}) = \mathrm{K}([f]_{\mathscr{R}}) \otimes \mathrm{K}([g]_{\mathscr{R}})$. Then, since $\mathrm{K}([\gamma_{u,v}]_{\mathscr{R}}) = \mathrm{H}(\gamma_{u,v}) = \gamma'_{\mathrm{K}(u),\mathrm{K}(v)}$, where γ' is the symmetry isomorphism of $\underline{\mathrm{D}}$, one concludes that K is in <u>SSMC</u>.

Clearly, K renders commutative the diagram above and it is indeed the unique functor which enjoys such a property for the given H. □

In order to show that $\mathscr{P}(N)$ is a monoidal quotient of $\mathscr{F}(N)$, we need a more abstract understanding of the structure of the vperms of $\mathscr{P}(N)$. To this aim, we shall make use of the following lemma, originally proved in [13].

Lemma 2.3 (Axiomatizing $\Pi(n)$). *The symmetric group $\Pi(n)$ is (isomorphic to) the group G freely generated from the set $\{\tau_i \mid 1 \leqslant i < n\}$, modulo the equations (see also Fig. 5)*

$$\begin{aligned}\tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1},\\ \tau_i\tau_j &= \tau_j\tau_i \quad \text{if } |i-j| \geqslant 1,\\ \tau_i\tau_i &= e,\end{aligned} \tag{13}$$

where e is the unit element of G.

Proof. The proof is by induction on n. First of all, observe that for $n = 0$ and $n = 1$ the set of generators is empty and the equations are vacuous. Hence, G is the free group on the empty set of generators, i.e., the group consisting only of the unit element, which is (isomorphic to) $\Pi(0)$ and $\Pi(1)$.

Suppose now that the thesis holds for $n \geqslant 1$ and let us prove it for $n + 1$. It is immediately evident that the permutations of $n + 1$ elements are generated by the n transpositions. Moreover, the transpositions satisfy axioms (13), as a quick look to Fig. 5 shows. It follows that the order of G must be not smaller than the order of $\Pi(n + 1)$, i.e., $|G| \geqslant (n + 1)!$. Moreover, there is a group homomorphism $h: G \to \Pi(n + 1)$ which sends τ_i to the transposition $(i\ i + 1)$, and since the transpositions generate $\Pi(n+1)$, we have that h is surjective. Thus, in order to conclude the proof, we only need to show that h injective, which clearly follows if we show that $|G| = (n+1)!$.

Let H be the subgroup of G generated by $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$ and consider the $n+1$ cosets $H_1, \ldots, H_{n+1}$, where $H_i = H\tau_n \cdots \tau_i = \{x\tau_n \cdots \tau_i \mid x \in H\}$, $1 \leqslant i \leqslant n$, and $H_{n+1} = H$. Then, for $1 \leqslant i \leqslant n+1$ and $1 \leqslant j \leqslant n$, consider $H_i\tau_j$. The following cases are possible.

$i > j + 1$. By the second of axioms (13), τ_j is permutable with each of $\tau_i, \ldots, \tau_n$ and, therefore,

$$\begin{aligned}H_i\tau_j &= H\tau_n \cdots \tau_i\tau_j\\ &= H\tau_j\tau_n \cdots \tau_i\\ &= H\tau_n \cdots \tau_i = H_i.\end{aligned}$$

$i < j$. Again by the second of (13), τ_j is permutable with each of $\tau_i, \ldots, \tau_{j-2}$ and, therefore,

$$\begin{aligned}H_i\tau_j &= H\tau_n \cdots \tau_i\tau_j\\ &= H\tau_n \cdots \tau_{j+1}\tau_j\tau_{j-1}\tau_j \cdots \tau_i\\ &= H\tau_n \cdots \tau_{j+1}\tau_{j-1}\tau_j\tau_{j-1} \cdots \tau_i && \text{by the first of (13)}\\ &= H\tau_{j-1}\tau_n \cdots \tau_{j+1}\tau_j\tau_{j-1} \cdots \tau_i && \text{by the second of (13)}\\ &= H\tau_n \cdots \tau_i = H_i.\end{aligned}$$

$i = j$. Then $H_j\tau_j = H\tau_n \cdots \tau_j\tau_j$, i.e., by the third of (13), $H\tau_n \cdots \tau_{j+1} = H_{j+1}$.
$i = j+1$. Then $H_{j+1}\tau_j = H\tau_n \cdots \tau_{j+1}\tau_j = H_j$.

In other words, for $1 \leqslant j \leqslant n$, the sets $H_1 \ldots H_{n+1}$ remain all unchanged by post-multiplication by τ_j, except H_j and H_{j+1} which are exchanged with each other. Now, since each element of G is a product $\tau_{i_1} \cdots \tau_{i_k}$, it belongs to $H\tau_{i_1} \cdots \tau_{i_k}$, i.e., to one of the H_i's. Hence, G is contained in the union of the H_i's. It follows immediately that, if H is finite, we have that $|G| \leqslant (n+1) \cdot |H|$. However, by induction hypothesis, H is (isomorphic to) $\Pi(n)$, and thus H is finite and $|H| = n!$. Therefore, $|G| \leqslant (n+1)!$, which concludes the proof. □

The previous lemma is easily adapted to vperms as follows.

Lemma 2.4 (Axiomatizing Sym_N). *The arrows of Sym_N are freely generated by composition and tensor from the vperms $\gamma(a,a)\colon 2 \cdot a \to 2 \cdot a$, for $a \in S_N$, modulo the axioms (9) of strict monoidal categories and the following additional axioms:*

$$((id_a \otimes \gamma(a,a))\,;\,(\gamma(a,a) \otimes id_a))^3 = id_{3 \cdot a},$$
$$\gamma(a,a)^2 = id_{2 \cdot a}, \tag{14}$$
$$(id_b \otimes \gamma(a,a))\,;\,(\gamma(a,a) \otimes id_b) = id_{2 \cdot a \oplus b} \quad \textit{if} \quad a \neq b \in S_N,$$

where f^n indicates the composition of f with itself n times.

Proof. A vperm $p = \langle \sigma_{a_1}, \ldots, \sigma_{a_n} \rangle$ coincides with $\sigma_{a_1} \otimes \cdots \otimes \sigma_{a_n}$ which, exploiting the functoriality of $\otimes$, can be written as $(\sigma_{a_1} \otimes \cdots \otimes id_{u_n}); \cdots ; (id_{u_1} \otimes \cdots \otimes \sigma_{a_n})$. Since σ_{a_j}, as a permutation, is a composition of transpositions, and the transposition $\tau_i\colon n \cdot a \to n \cdot a$ in Sym_N can be written as $id_{(i-1)\cdot a} \otimes \gamma(a,a) \otimes id_{(n-i-1)\cdot a}$, we have that $\sigma_{a_j} = (id_{u_1} \otimes \gamma(a_j,a_j) \otimes id_{v_1}); \cdots ; (id_{u_k} \otimes \gamma(a_j,a_j) \otimes id_{v_k})$. Therefore, the vperms $\gamma(a,a)$ generate via composition and tensor all the vperms of Sym_N.

Concerning the axioms, since Sym_N is strict monoidal, it clearly validates Eqs. (9). It is easy to verify that the same happens for (14). On the other hand, suppose that two terms p and q generated from the $\gamma(a,a)$'s evaluate to the same vperm $\sigma_{c_1} \otimes \cdots \otimes \sigma_{c_k}$. We have to prove that Eqs. (9) and (14) induce $p = q$. Up to applications of axioms (9), we can assume that

$$p = (id_{u_1} \otimes \gamma(a_1,a_1) \otimes id_{v_1}); \cdots ; (id_{u_n} \otimes \gamma(a_n,a_n) \otimes id_{v_n}),$$
$$q = (id_{u'_1} \otimes \gamma(b_1,b_1) \otimes id_{v'_1}); \cdots ; (id_{u'_m} \otimes \gamma(b_m,b_m) \otimes id_{v'_m}),$$

where every a_i appearing in p and every b_i appearing in q is one of the c_i's. Observe that, by repeated applications of the third of (14) and of the functoriality of $\otimes$, viz., the last two of (9), we can reorganize p and q in such a way that all the terms involving c_1 – if any – are grouped together and immediately followed by all the terms involving c_2 – if any – and so on. Let us denote by p' and q' the terms so obtained and let us focus on the sequences p'_i and q'_i of terms involving c_i in, respectively, p' and q'. The following cases are possible.

(i) p'_i and q'_i are both empty. Then, there is nothing to show.

(ii) Either p'_i or q'_i – without loss of generality say p'_i – is empty. Then, σ_{c_i} is the identity and since q'_i evaluates to it, by Lemma 2.3, q'_i can be proved equal to the identity permutation using axioms (13). Now notice that axioms (13) can be derived by appropriately tensoring with identities the first two of (14) instantiated to c_i and the following direct consequence of (9)

$$((\gamma(c_i, c_i) \otimes id_{n \cdot c_i}); (id_{n \cdot c_i} \otimes \gamma(c_i, c_i)))^2 = id_{(n+2) \cdot c_i} \quad \text{if } n > 1.$$

Therefore, the proof that q'_i is the identity permutation can be mimicked to prove using instances of axioms (9) and (14) that q'_i is an identity in Sym_N. Then we can drop q'_i from q'.

(iii) Both p'_i and q'_i are nonempty. Then, since they both evaluate to σ_{c_i}, they can be proved equal using axioms (13). Therefore, reasoning as in the previous case, the equality of p'_i and q'_i follows from axioms (9) and (14).

This shows that $p = q$ is induced by (9) and (14), which concludes the proof. □

We are now ready to give the promised characterization of $\mathcal{P}(N)$.

Proposition 2.5 (Axiomatizing $\mathcal{P}(N)$). *$\mathcal{P}(N)$ is the monoidal quotient of the free* SSMC *on N modulo the axioms*

$$c_{a,b} = id_{a \oplus b} \quad \textit{if} \quad a, b \in S_N \quad \textit{and} \quad a \neq b, \tag{15}$$

$$s; t; s' = t \qquad \textit{if} \quad t \in T_N \quad \textit{and } s, s' \textit{ are symmetries.} \tag{16}$$

Proof. We prove that $\mathcal{P}(N)$ is isomorphic to $\mathcal{F}(N)/\mathcal{R}$, where $\mathcal{R}$ is the $\otimes$-congruence generated from eqs. (15) and (16).

Since $\mathcal{P}(N)$ belongs to $\underline{\mathrm{SSMC}}^{\oplus}$, it follows from Proposition 2.1 that, corresponding to the net *inclusion* morphism $N \to \mathcal{U}\mathcal{P}(N)$, there is a *unique* symmetric strict monoidal functor $\mathrm{Q}\colon \mathcal{F}(N) \to \mathcal{P}(N)$ which is the identity on the places and on the transitions of N. In particular, Q is such that

$$\mathrm{Q}(c_{a,b}) = \gamma(a,b) \quad \text{for} \quad a, b \in S_N.$$

For $a \neq b \in S_N$, since $\gamma(a,b) = id_{a \oplus b}$, we have that $\mathrm{Q}(c_{a,b}) = \mathrm{Q}(id_{a \oplus b})$. Moreover, since symmetric monoidal functors map symmetries to symmetries, and since (16) holds in $\mathcal{P}(N)$, we have that $\mathrm{Q}(s; t; s') = \mathrm{Q}(s); t; \mathrm{Q}(s') = t = \mathrm{Q}(t)$ for s and s' in $Sym_{\mathcal{F}(N)}$ and $t \in T_N$. In other words, Q equalizes the pairs $\langle c_{a,b}, id_{a \oplus b} \rangle$ with $a \neq b \in S_N$ and the pairs $\langle s; t; s', t \rangle$ with s and s' symmetries and $t \in T_N$. Then, in force of Proposition 2.2 applied to Q, there is a (unique) symmetric strict monoidal functor $\mathrm{H}\colon \mathcal{F}(N)/\mathcal{R} \to \mathcal{P}(N)$ which is the identity on the objects and is such that

$$\mathrm{H}([t]_{\mathcal{R}}) = t \quad \text{for} \quad t \in T_N.$$

We shall prove that H is an isomorphism by providing its inverse $\mathscr{P}(N) \to \mathscr{F}(N)/\mathscr{R}$. To this aim, consider the mapping G of $\mathscr{P}(N)$ to $\mathscr{F}(N)/\mathscr{R}$ which acts identically on the objects and is defined on the arrows by

$$\begin{aligned} \mathrm{G}(t) &= [t]_{\mathscr{R}} \quad \text{if } t \in T_N, \\ \mathrm{G}(\gamma(a,a)) &= [c_{a,a}]_{\mathscr{R}} \quad \text{if } a \in S_N, \end{aligned}$$

extended to identities, composition and tensor by the usual laws $\mathrm{G}(id_u) = [id_u]_{\mathscr{R}}$, $\mathrm{G}(\alpha;\beta) = \mathrm{G}(\alpha);\mathrm{G}(\beta)$, and $\mathrm{G}(\alpha\otimes\beta) = \mathrm{G}(\alpha)\otimes\mathrm{G}(\beta)$. It follows from the definition of $\mathscr{P}(N)$ and from Lemma 2.4 that the equations above define G uniquely.

Suppose now for a moment that these equations yield a symmetric strict monoidal functor $\mathrm{G}: \mathscr{P}(N) \to \mathscr{F}(N)/\mathscr{R}$, and notice that $\mathrm{GH}: \mathscr{F}(N)/\mathscr{R} \to \mathscr{F}(N)/\mathscr{R}$ is the identity on the objects and that

$$\mathrm{GH}([t]_{\mathscr{R}}) = \mathrm{G}(t) = [t]_{\mathscr{R}} \quad \text{for } t \in T_N.$$

Observe further that for the universal properties of $\mathscr{F}(N)$ and $\mathscr{F}(N)/\mathscr{R}$, stated in Propositions 2.1 and 2.2, there exists a *unique* such symmetric strict monoidal functor. Therefore, it must be $\mathrm{GH} = 1_{\mathscr{F}(N)/\mathscr{R}}$. Similarly, since $\mathrm{HG}: \mathscr{P}(N) \to \mathscr{P}(N)$ is the identity on the objects and is such that

$$\mathrm{HG}(t) = \mathrm{H}([t]_{\mathscr{R}}) = t \quad \text{for } t \in T_N,$$

by the universal property of Q, it must be HGQ = Q. Then, since as an immediate corollary of Lemma 2.4 we have that Q is *epi*, we can conclude that $\mathrm{HG} = 1_{\mathscr{P}(N)}$. In other words, if G is in <u>SSMC</u>, then $\mathrm{G} = \mathrm{H}^{-1}$.

Thus, to conclude the proof we only need to prove that G is a symmetric strict monoidal functor, i.e., that it satisfies (6), (7), and (8). We start by showing that G is well-defined, which, inspecting the definition of $\mathscr{P}(N)$ and exploiting Lemma 2.4, reduces to showing that it respects axioms (14) and axioms (Ψ). The other axioms, in fact, hold for any **SSMC** and are, therefore, clearly unproblematic.

(i) From (12) we have that $(id_a \otimes c_{a,a});(c_{a,a} \otimes id_a) = c_{a\oplus a,a}$ and then from (11) we have $c_{a\oplus a,a};(id_a \otimes c_{a,a}) = (c_{a,a} \otimes id_a);c_{a\oplus a,a}$, which, again by (12), yields $(id_a \otimes c_{a,a});(c_{a,a} \otimes id_a);(id_a \otimes c_{a,a}) = (c_{a,a} \otimes id_a);(id_a \otimes c_{a,a});(c_{a,a} \otimes id_a)$, which is $((id_a \otimes c_{a,a});(c_{a,a} \otimes id_a))^3 = id_{3\cdot a}$. Then, considering the corresponding $\mathscr{R}$-classes, we have the required $[((id_a \otimes c_{a,a});(c_{a,a} \otimes id_a))^3]_{\mathscr{R}} = [id_{3\cdot a}]_{\mathscr{R}}$.

(ii) $[c_{a,a}]_{\mathscr{R}};[c_{a,a}]_{\mathscr{R}} = [id_{2\cdot a}]_{\mathscr{R}}$ follows immediately from (10).

(iii) From (12) we have that $c_{a\oplus a,b} = (id_a \otimes c_{a,b});(c_{a,b} \otimes id_a)$. If $a \neq b \in S_N$, since $[c_{a,b}]_{\mathscr{R}} = [id_{a\oplus b}]_{\mathscr{R}}$, we have that $[c_{a\oplus a,b}]_{\mathscr{R}} = [id_{2\cdot a\oplus b}]_{\mathscr{R}}$. It follows in the symmetric way that $[c_{b,a\oplus a}]_{\mathscr{R}} = [id_{2\cdot a\oplus b}]_{\mathscr{R}}$. Then, applying (11), we have that $c_{b,a\oplus a};(id_b \otimes c_{a,a}) = (c_{a,a} \otimes id_b);c_{a\oplus a,b}$ which, considering the corresponding $\mathscr{R}$-classes yields $[(id_b \otimes c_{a,a})]_{\mathscr{R}} = [(c_{a,a} \otimes id_b)]_{\mathscr{R}}$, i.e., the required $[(id_b \otimes c_{a,a})]_{\mathscr{R}};[(c_{a,a} \otimes id_b)]_{\mathscr{R}} = [id_{2\cdot a\oplus b}]_{\mathscr{R}}$.

(iv) Since G sends vperms to symmetries, for s, s' in Sym_N and $t \in T_N$, we have $[\mathrm{G}(s);t;id]_{\mathscr{R}} = [t]_{\mathscr{R}} = [id;t;\mathrm{G}(s')]_{\mathscr{R}}$, i.e., $\mathrm{G}(s;t) = \mathrm{G}(t) = \mathrm{G}(t;s')$.

Thus G is well-defined. It follows then from its own definition that it is a strict monoidal functor, i.e., a functor satisfying (6) and (7). Last, we need to prove G symmetric, i.e., that $G(\gamma(u,v)) = [c_{u,v}]_{\mathscr{R}}$. We proceed by induction on the sum of the sizes of u and v.

Base cases: If $u = 0$, then $G(\gamma(u,v)) = G(id_v) = [id_v]_{\mathscr{R}} = [c_{0,v}]_{\mathscr{R}}$. If $v = 0$, a symmetric argument applies. If $|u| = |v| = 1$, we have the following two cases:

($u = v$.) Then $G(\gamma(u,v)) = [c_{u,v}]_{\mathscr{R}}$ follows from the definition of G.

($u \neq v$.) Then $G(\gamma(u,v)) = G(id_{u\oplus v}) = [id_{u\oplus v}]_{\mathscr{R}}$ which, by definition, is $[c_{u,v}]_{\mathscr{R}}$.

Inductive step: Suppose that $u = a \oplus u'$, with $u' \neq 0$. Then, by the coherence axiom (4), $G(\gamma(u,v)) = ([id_a]_{\mathscr{R}} \otimes G(\gamma(u',v))); (G(\gamma(a,v)) \otimes [id_{u'}]_{\mathscr{R}})$ and thus, exploiting the induction hypothesis, $G(\gamma(u,v)) = ([id_a \otimes c_{u',v}]_{\mathscr{R}}); ([c_{a,v} \otimes id_{u'}]_{\mathscr{R}})$, which, again by (4), is $[c_{a\oplus u',v}]_{\mathscr{R}}$. If instead we have that $v = v' \oplus a$, $v' \neq 0$, the induction is maintained similarly by using the inverse of (4). □

The merit of this result is to describe the algebraic structure of $\mathscr{P}(N)$, and thus of the concatenable processes of N, in terms of *universal* constructions, namely the construction on the free SSMC on Petri and a quotient construction on $\underline{\text{SSMC}}^{\ominus}$, providing in this way a completely abstract view of $\mathscr{P}(N)$. It may be worth noticing in this context that (15) is actually a problematic axiom: because of its negative premise, viz., $a \neq b$, it invalidates the freeness of $\mathscr{F}(N)$ on Petri. Even worse, $\mathscr{F}(\text{-})/\mathscr{R}$ and $\mathscr{P}(\text{-})$ fail to be functors from Petri to SSMC. On the other hand, axiom (15) plays a very relevant role in capturing algebraically the essence of concatenable process, and it cannot be dispensed with easily. A detailed study of this issue and a possible solution is provided by this author in [16]. In particular, in loc. cit., a functorial and universal construction for net computations is devised, based on a refinement of the notion of concatenable processes called *strongly concatenable processes*.

Resuming our work, we give an alternative form of axiom (16).

Corollary 2.6 (Axiom (16) revisited). *Axiom* (16) *in Proposition* 2.5 *can be replaced by the axioms*

$$t; (id_u \otimes c_{a,a} \otimes id_v) = t \quad \text{if} \quad t \in T_N \text{ and } a \in S_N,$$
$$(id_u \otimes c_{a,a} \otimes id_v); t = t \quad \text{if} \quad t \in T_N \text{ and } a \in S_N. \tag{17}$$

Proof. Since $(id_u \otimes \gamma_{a,a} \otimes id_v)$ and all the identities are symmetries, axiom (16) implies the present ones. It is easy to see that, on the other hand, the axioms above, together with axiom (15), imply (16).

Let $s: u \to u$ by a symmetry of $\mathscr{F}(N)$ and suppose $s \neq id_u$. By repeated applications of (12), together the functoriality of $\otimes$, we obtain the following equality:

$$s = (id_{u_1} \otimes c_{a_1,b_1} \otimes id_{v_1}); \ldots; (id_{u_h} \otimes c_{a_h,b_h} \otimes id_{v_h})$$

for some $h \in \omega$. Moreover, by exploiting axiom (15), we can drop every term in which $a_i \neq b_i$. Thus, we have

$$s = (id_{u_1} \otimes c_{a_1,a_1} \otimes id_{v_1}); \ldots ; (id_{u_k} \otimes c_{a_k,a_k} \otimes id_{v_k})$$

for some $k \leqslant h$. Then, by this equation and by repeated applications of axioms (17), one can prove $s; t; s' = t$. □

Finally, the next corollary sums up the purely algebraic characterization of the category of concatenable processes that we proved in this paper. In particular, it identifies in algebraic terms the essential components of concatenable processes and the laws which rule their sequential and parallel composition.

Corollary 2.7 (Axiomatizing concatenable processes). *The category $\mathscr{P}(N)$ of concatenable processes of N is the category whose objects are the elements of $S_N^{\oplus}$ and whose arrows are generated by the inference rules*

$$\frac{u \in S_N^{\oplus}}{id_u : u \to u \text{ in } \mathscr{P}(N)} \qquad \frac{a \text{ in } S_N}{c_{a,a} : a \oplus a \to a \oplus a \text{ in } \mathscr{P}(N)} \qquad \frac{t : u \to v \text{ in } T_N}{t : u \to v \text{ in } \mathscr{P}(N)}$$

$$\frac{\alpha : u \to v \text{ and } \beta : u' \to v' \text{ in } \mathscr{P}(N)}{\alpha \otimes \beta : u \oplus u' \to v \oplus v' \text{ in } \mathscr{P}(N)} \qquad \frac{\alpha : u \to v \text{ and } \beta : v \to w \text{ in } \mathscr{P}(N)}{\alpha ; \beta : u \to w \text{ in } \mathscr{P}(N)}$$

modulo the axioms expressing that $\mathscr{P}(N)$ is a strict monoidal category, namely,

$$\alpha; id_v = \alpha = id_u; \alpha \quad \text{and} \quad (\alpha;\beta);\gamma = \alpha;(\beta;\gamma),$$
$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma) \quad \text{and} \quad id_0 \otimes \alpha = \alpha = \alpha \otimes id_0,$$
$$id_u \otimes id_v = id_{u \oplus v} \quad \text{and} \quad (\alpha \otimes \alpha');(\beta \otimes \beta') = (\alpha;\beta) \otimes (\alpha';\beta'),$$

the latter whenever the right-hand term is defined and the following axioms:

$$c_{a,a}; c_{a,a} = id_{a \oplus a},$$
$$t; (id_u \otimes c_{a,a} \otimes id_v) = t \qquad \text{if } t \in T_N,$$
$$(id_u \otimes c_{a,a} \otimes id_v); t = t \qquad \text{if } t \in T_N,$$
$$c_{u,u'}; (\beta \otimes \alpha) = (\alpha \otimes \beta); c_{v,v'} \quad \text{for } \alpha : u \to v,\ \beta : u' \to v',$$

where $c_{u,v}$, for $u, v \in S_N^{\oplus}$, is obtained from $c_{a,a}$ by applying recursively the rules:

$$c_{a,b} = id_{a \oplus b} \quad \text{if } a = 0 \text{ or } b = 0 \text{ or } (a, b \in S_N \text{ and } a \neq b),$$
$$c_{a \oplus u, v} = (id_a \otimes c_{u,v}); (c_{a,v} \otimes id_u),$$
$$c_{u, v \oplus a} = (c_{u,v} \otimes id_a); (id_v \otimes c_{u,a}).$$

Proof. Observe that the terms and the axioms above are obtained normalizing those of $\mathscr{F}(N)$ with respect to $c_{a,b} = id_{a \oplus b}$, for $a \neq b \in S_N$, and then adding axioms (15) and (17). The claim then follows immediately from Propositions 2.1, 2.5 and Corollary 2.6. □

3. Conclusions

The paper described the concatenable processes of a Petri net N in terms of *universal* constructions, providing in such a way an abstract, fully axiomatic presentation of their algebraic structure. In particular, Corollary 2.7 provides a *term algebra* and an *equational theory* of the concatenable processes of N.

Technically, relying on the characterization of the concatenable processes of N as the arrows of the symmetric strict monoidal category $\mathscr{P}(N)$, the result is established by proving in Proposition 2.5 that $\mathscr{P}(N)$ is the quotient of the free symmetric strict monoidal category on N modulo two simple axioms. The proof of this fact makes an essential use of the axiomatization of Sym_N, the category of *symmetries* of $\mathscr{P}(N)$, provided by Lemma 2.4. Such an axiomatization remedies to the one *weakness* of the original presentation of $\mathscr{P}(N)$: although $\mathscr{P}(N)$ captures net computations in *algebraic* terms, and as such it is a very relevant construction, its essentially axiomatic character and its manageability are spoiled by the concrete, ad hoc definition of Sym_N on which it is built.

Acknowledgements

I wish to thank José Meseguer and Ugo Montanari, who introduced me to this subject. Many thanks to Mogens Nielsen for his encouragement.

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