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General framework for pricing derivative securities

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Abstract

This article describes a general methodology that can be used for financial risk management. The approach is based on the model of Heath et al. (1992) of term structure movements but deals with the case of incomplete market. Both, domestic and foreign economies are investigated. Prices of various options are calculated using the forward measure introduced recently by El Karoui and Rochet (1989).

Keywords: Term structure models; HJM framework; Arbitrage free pricing; Martingale measures

0. Introduction

The general aim of the paper is to develop an approach to management of financial risks in an international economy. The key issue is then to control interest rate risks. This is why we start with a model of the term structure of interest rates and volatilities (Section 1.1). We identify conditions under which there are no arbitrage opportunities between rates of different maturities. Using the Girsanov transformation we construct a model of arbitrage-free economy appropriate to price and hedge claims contingent on interest rate-dependent assets. In Section 1.2 we extend our model in order to cover other risky assets. This is done using the general methodology developed in Section 1.1. Finally, in Sections 2.1 and 2.2 we analyse a model of arbitrage-free international economy. The idea is to consider foreign assets as new assets in the domestic economy.

Section 1.4 describes the concept of the forward measure recently introduced by El Karoui and Rochet. Let us point out that this idea leads to significant simplifications in the derivation of pricing formulas. Sections 1.5 and 2.3 make use of this new approach to derive pricing formulas for various options.

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1. Domestic economy

We present here a general mathematical framework for financial risk management. Our approach is based on the model of Heath et al. (1992) (HJM) of term structure movements and hence it allows for arbitrary term structure of interest rates and volatilities. It can also be seen as a generalisation to the case of incomplete market of the model presented in Musiela et al. (1993).

1.1. Domestic term structure

We begin with a financial market where investors can trade continuously over the time interval $[0, \tau]$. The traded assets consist of a continuum of default-free discount bonds with maturities time from 0 to τ. We do not assume that our market model is complete. This allows us subsequently to add additional assets as long as their price processes are adapted to the information available to investors at any time over the trading interval.

To be able to analyse mathematically financial risks we introduce a probability space $(\Omega, \boldsymbol{F}, \boldsymbol{P}_0)$ equipped with a filtration $(\boldsymbol{F}_t)$, $0 \leq t \leq \tau$ satisfying the usual conditions. The sub-σ-algebra $\boldsymbol{F}_t$ represents all information available to the market participants at time t. We assume that $\boldsymbol{F} = \boldsymbol{F}_\tau$.

The most basic fixed income security is the pure discount bond, a bond that pays a certain dollar at maturity with no other payment in the interim. The time t price of a T maturity bond is denoted by $P(t, T)$, indicating that the price of the bond is a function of the trading date t and the maturity date T. The pure discount bond is a theoretical instrument. Financial instruments that closely resemble these hypothetical bonds are US Treasury bills. We assume that for each $0 \leq t \leq T$ the price $P(t, T)$ depends only on information contained in $\boldsymbol{F}_t$, that is, the process $\{P(t, T); 0 \leq t \leq T\}$ is adapted to the filtration $(\boldsymbol{F}_t)$. We also require that $P(T, T) = 1$ for all $0 \leq T \leq \tau$, $P(t, T) > 0$ for all $0 \leq t \leq T \leq \tau$, and that $\partial \log P(t, T)/\partial T$ exists for all $0 \leq t \leq T \leq \tau$. The last condition implies that the instantaneous forward rate at time T as seen from date t is well defined. This rate, denoted $f(t, T)$, corresponds to the rate at which one could enter a contract at time t on a riskless loan over the forward period $[T, T + \mathrm{d}T]$. Obviously,

$$P(t, T) = \exp\left(- \int_t^T f(t, u)\,\mathrm{d}u \right).$$

Therefore, the process $\{f(t, T); 0 \leq t \leq T\}$ is also adapted to the filtration $(\boldsymbol{F}_t)$. The spot rate at time t, denoted $r(t)$, corresponds to the rate at which one could enter a contract at time t on a riskless loan over the period $[t, t + \mathrm{d}t]$. Clearly we have $r(t) = f(t, t)$ for all $0 \leq t \leq \tau$. Given an initial investment of one dollar at time 0, the amount generated at time t by continuously reinvesting in the instantaneous spot rate is equal to

$$B(t) = \exp\left(\int_0^t r(u)\,\mathrm{d}u \right).$$

The process $\{B(t),\ 0 \le t \le \tau\}$ is referred to as the accumulation factor. Note that $B(t)/B(T)$ represents the amount one needs to invest at time t in order to accumulate one dollar at time T. Hence it seems we can say that $P(t, T)$ and $B(t)/B(T)$ are equal. In fact they would be if we could know at time t the future behaviour of the spot rate r over the interval $[t, T]$. This is exactly where the role of information available to investors becomes important. We would not know at time t how much is $B(t)/B(T)$, that is $B(t)/B(T)$ is not $\boldsymbol{F}_t$-measurable. On the other hand, the random variable $P(t, T)$ is $\boldsymbol{F}_t$-measurable. Therefore, we can interpret $P(t, T)$ as the time t value of a bond that pays one dollar at T. Finally, note that under the assumption of known deterministic rates we have $P(t, T) = B(t)/B(T)$. In the general situation, as it will become clear later on, $P(t, T)$ can be understood as a "projection" of $B(t)/B(T)$ on the information $\boldsymbol{F}_t$ available at t. Hence, it seems clear that the amount of information and the way information is structured will be essential in any analysis of the interest rate risk. In the HJM model the filtration $(\boldsymbol{F}_t)$ is generated by two independent Brownian motions. Musiela et al. (1993) consider the case of filtration spanned by n independent Brownian motions. It should be noted, however, that these assumptions are rather technical. They imply that the corresponding market models are complete and hence also that claims contingent on interest rate-dependent assets can be generated by dynamic portfolio strategies based on the family of pure discount bonds with maturities covering the interval $[0, \tau]$.

In this paper we assume that for fixed, but arbitrary $0 \le T \le \tau$, $f(t, T)$ is given by

$$\mathrm{d}f(t, T) = \alpha(t, T)\,\mathrm{d}t + \sigma^*(t, T)\,\mathrm{d}W_0(t), \tag{1.1}$$

where $\{W_0(t);\ 0 \le t \le \tau\}$ is an n-dimensional $\boldsymbol{F}_t$-Brownian motion while the processes $\{\alpha(t, T);\ 0 \le t,\ T \le \tau\}$ and $\{\sigma(t, T);\ 0 \le t,\ T \le \tau\}$ are bounded on $[0, \tau]^2 \times \Omega$ adapted with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Note that, because $(\boldsymbol{F}_t)$ is not necessarily the filtration generated by W_0, there may exist contingent claims which cannot be hedged with a self-financing portfolio constructed from discount bonds. Therefore, our market model is not complete in general. For the vectors (matrices) x, y the symbols $|x|$, x^* and x^*y stand for the Euclidean norm, the transpose and the Euclidean scalar product, respectively.

The differential (1.1) can be written in the following integral form

$$f(t, T) = f(0, T) + \int_0^t \alpha(u, T)\,\mathrm{d}u + \int_0^t \sigma^*(u, T)\,\mathrm{d}W_0(u)$$

for all $0 \le t,\ T \le \tau$, where $\{f(0, T);\ 0 \le T \le \tau\}$ is an arbitrary bounded initial forward rate curve. The structure of volatilities is also arbitrary bounded because we only assume that $\sigma(\cdot, T)$ is bounded and adapted. The spot rate process satisfies

$$r(t) = f(0, t) + \int_0^t \alpha(u, t)\,\mathrm{d}u + \int_0^t \sigma^*(u, t)\,\mathrm{d}W_0(u)$$

for all $t > 0$. It is interesting to note that the process r is not a semimartingale in general.

Throughout the paper we assume the nonexistence of arbitrage opportunities in financial markets. Based on the results of Harrison and Kreps (1979), in the context of our model the above assumption will be satisfied if there exists a martingale measure for discounted price processes of bonds of all maturities. Mathematically speaking this means that there exists a probability measure $\boldsymbol{P}$ defined on $(\Omega, \boldsymbol{F})$ which is equivalent to $\boldsymbol{P}_0$ and such that the processes $\{Z(t, T);\ 0 \le t \le T\}$, where

$$Z(t, T) = P(t, T)/B(t)$$

are martingales under $\boldsymbol{P}$ for all $0 \le T \le \tau$.

In order to construct the measure $\boldsymbol{P}$ we shall henceforth impose the following assumption.

Assumption 1.1. There exists a bounded predictable process $H = \{H(t);\ 0 \le t \le \tau\}$ with values in $\mathbb{R}^n$ such that for all $0 \le t,\ T \le \tau$

$$\alpha(t, T) = \sigma^*(t, T)\left(\int_t^T \sigma(t, s)\,\mathrm{d}s - H(t)\right). \tag{1.2}$$

For all $A \in \boldsymbol{F}_\tau$, we now define a set function $\boldsymbol{P}(A)$ by the formula

$$\boldsymbol{P}(A) = \int_A \exp\left(\int_0^\tau H^*(t)\,\mathrm{d}W_0(t) - \tfrac{1}{2}\int_0^\tau |H(t)|^2\,\mathrm{d}t\right)\mathrm{d}\boldsymbol{P}_0. \tag{1.3}$$

Because H is assumed bounded it is not difficult to show that $\boldsymbol{P}$ is in fact a probability measure on $(\Omega, \boldsymbol{F})$ which is equivalent to the measure $\boldsymbol{P}_0$. Also it follows from the Girsanov theorem that under the measure $\boldsymbol{P}$ the process $\{W(t);\ 0 \le t \le \tau\}$, where

$$W(t) = W_0(t) - \int_0^t H(s)\,\mathrm{d}s \tag{1.4}$$

is an n-dimensional $\boldsymbol{F}_t$ Brownian motion. This observation helps to describe the arbitrage free behaviour of forward rates and bond prices.

Proposition 1.1. *We have*

(i) $$\mathrm{d}f(t, T) = \sigma^*(t, T)\int_t^T \sigma(t, u)\,\mathrm{d}u\,\mathrm{d}t + \sigma^*(t, T)\,\mathrm{d}W(t),$$

(ii) $$\mathrm{d}P(t, T) = P(t, T)\left[r(t)\,\mathrm{d}t - \int_t^T \sigma^*(t, u)\,\mathrm{d}u\,\mathrm{d}W(t)\right],$$

(iii) $$\mathrm{d}Z(t, T) = -Z(t, T)\int_t^T \sigma^*(t, u)\,\mathrm{d}u\,\mathrm{d}W(t).$$

Under the measure $\boldsymbol{P}$ the processes $\{Z(t, T);\ 0 \le t \le T\}$ are martingales for all $0 \le T \le \tau$. The spot rate process is square integrable and satisfies

(iv) $$r(t) = f(0, t) + \int_0^t \sigma^*(u, t)\int_u^t \sigma(u, s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sigma^*(u, t)\,\mathrm{d}W(u).$$

Proof. Note that

$$
\begin{aligned}
\mathrm{d}f(t,u) &= \alpha(t,u)\,\mathrm{d}t + \sigma^*(t,u)\,\mathrm{d}W_0(t)\\
&= \sigma^*(t,u)\left(\int_t^u \sigma(t,s)\,\mathrm{d}s - H(t)\right)\mathrm{d}t + \sigma^*(t,u)\,\mathrm{d}W_0(t)\\
&= \sigma^*(t,u)\int_t^u \sigma(t,s)\,\mathrm{d}s\,\mathrm{d}t + \sigma^*(t,u)\,\mathrm{d}W(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_t^T \mathrm{d}f(t,u)\,\mathrm{d}u\\
&\quad= \int_t^T \sigma^*(t,u)\int_t^u \sigma(t,s)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t + \int_t^T \sigma^*(t,u)\,\mathrm{d}u\,\mathrm{d}W(t)\\
&\quad= \frac{1}{2}\left|\int_t^T \sigma(t,u)\,\mathrm{d}u\right|^2 \mathrm{d}t + \int_t^T \sigma^*(t,u)\,\mathrm{d}u\,\mathrm{d}W(t),
\end{aligned}
$$

and, because

$$
\int_t^T \mathrm{d}f(t,u)\,\mathrm{d}u - f(t,t)\,\mathrm{d}t = \mathrm{d}\left(\int_t^T f(t,u)\,\mathrm{d}u\right) = -\,\mathrm{d}\log P(t,T),
$$

it follows that

$$
\begin{aligned}
\mathrm{d}P(t,T) &= P(t,T)\left(\mathrm{d}\log P(t,T) + \frac{1}{2}\,\mathrm{d}\langle \log P(\cdot,T)\rangle(t)\right)\\
&= P(t,T)\left(f(t,t)\,\mathrm{d}t - \frac{1}{2}\left|\int_t^T \sigma(t,u)\,\mathrm{d}u\right|^2 \mathrm{d}t - \int_t^T \sigma^*(t,u)\,\mathrm{d}u\,\mathrm{d}W(t)\right.\\
&\qquad \left. + \frac{1}{2}\left|\int_t^T \sigma(t,u)\,\mathrm{d}u\right|^2 \mathrm{d}t\right) = P(t,T)\left(r(t)\,\mathrm{d}t - \int_t^T \sigma^*(t,u)\,\mathrm{d}u\,\mathrm{d}W(t)\right).
\end{aligned}
$$

This finishes proofs of (i) and (ii). A straightforward application of the Ito formula plus boundness of $\sigma(t,T)$ leads to (iii). Finally, (iv) follows from the integral representation of (i). □

1.2. Returns on risky assets

One of the building blocks of the arbitrage-free pricing theory is the assumption that traded assets can be classified as nonrisky and risky. Geometric Brownian motion is frequently assumed as a simple model of the risky asset price dynamics. This is justified by the belief that the corresponding return process satisfies the so-called random walk hypothesis. However, there is growing practical and statistical evidence that this assumption should be seen only as a first approximation.

For example, hedging over the life of an instrument is often done using forward contracts on the underlying asset and requires parameters implied by the market. This may lead to some inconsistencies with the model assumptions particularly if one is looking at more exotic products like average rate or compound options. Implied volatilities will be in general different for different maturities of the forward contracts used for hedging while geometric Brownian motion assumes constant volatility. Here dealing room practice appears to be in contradiction with the theory. Statistical evidence which contradicts the random walk hypothesis is provided by empirical studies of returns on risky assets. For example, daily returns on US stocks indicate abnormally small returns (from the model point of view) on Mondays. Corresponding analysis of some European markets seems to confirm the existence of an analogous phenomenon. Studies of currency exchange rates (Fama and Roll, 1971; McFarland et al., 1982; So, 1987) have shown conclusively that they are abnormal. This abnormality has been seen for all major currency exchange rates, where the observed distributions have been invariably described as very peaked and thick tailed.

To be able to understand risks related to such anomalies we propose the following nonhomogeneous model of the asset price dynamics:

$$\mathrm{d}R(t) = R(t)(\mu(t)\,\mathrm{d}t + \eta^*(t)\,\mathrm{d}W_0(t)), \tag{1.5}$$

where $R(t)$ denotes the asset price at time t, $\{W_0(t);\ 0 \le t \le \tau\}$ is, as before, the n-dimensional $\boldsymbol{F}_t$-Brownian motion on $(\Omega, \boldsymbol{F}, \boldsymbol{P}_0)$ while the processes $\{\mu(t); 0 \le t \le \tau\}$ and $\{\eta(t);\ 0 \le t \le \tau\}$ are adapted with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively, $R(0)$ is a constant. The process, η is bounded on $[0, \tau] \times \Omega$ while μ satisfies

$$\boldsymbol{P}_0\left(\int_0^\tau |\mu(t)|\,\mathrm{d}t < \infty\right) = 1.$$

Note that such a model allows to price options on the asset under stochastic interest rates and with an arbitrary structure of market volatilities. It has also a potential to explain the "day of the week" effect as well as the abnormality of currency exchange rates.

Consider now the economy $\boldsymbol{E}$ which consists of the money market account B, the family $\{P(\cdot, T); 0 \le T \le \tau\}$ of pure discount bonds of all maturities and of the family $\{R(\cdot, i); i \in I\}$ of risky assets which satisfy the hypotheses (1.5), of course with different processes $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ but with the same Brownian motion W_0. The set I represents here an index set which could be finite or infinite. Assume that for each $i \in I$ and each $0 \le t \le \tau$, the variable $R(t, i)$ denotes the price at time t of the stock i which pays no dividends. We already know that under the measure $\boldsymbol{P}$ there are no arbitrage opportunities between B and the family $\{P(\cdot, T); 0 \le T \le \tau\}$. The following assumption, valid throughout the paper, will help to eliminate any possible arbitrage between B and the family $\{R(\cdot, i); i \in I\}$ as well.

Assumption 1.2. For all $0 \le t \le \tau$ and all $i \in I$

$$\mu(t, i) = r(t) - \eta^*(t, i)\,H(t). \tag{1.6}$$

It follows from (1.4) and (1.5) that

$$\mathrm{d}R(t, i) = R(t, i)((\mu(t, i) + \eta^*(t, i)\,H(t))\,\mathrm{d}t + \eta^*(t, i)\,\mathrm{d}w(t)).$$

Hence we can formulate the following proposition.

Proposition 1.2. *Under the measure $\boldsymbol{P}$ we have*

(i) $\mathrm{d}R(t,i) = R(t,i)(r(t)\,\mathrm{d}t + \eta^*(t,i))\,\mathrm{d}W(t)),$

(ii) $\mathrm{d}Z(t,i) = Z(t,i)\eta^*(t,i)\,\mathrm{d}W(t),$

where $Z(t,i) = R(t,i)/B(t)$, $0 \le t \le \tau$, $i \in I$.

Remark 1.1. Note that under the measure $\boldsymbol{P}$ the appreciation rate on the stock i is equal to the spot rate for every $i \in I$. In this sense the measure $\boldsymbol{P}$ is "risk-neutral".

Now let for $0 \le t \le \tau$ and $T \in [0,\tau]$, $i \in I$

$$M(t,T) = -\int_0^t \int_s^T \sigma^*(s,u)\,\mathrm{d}u\,\mathrm{d}W(s),$$

$$M(t,i) = \int_0^t \eta^*(s,i)\,\mathrm{d}W(s). \tag{1.7}$$

Since W is a Brownian motion under $\boldsymbol{P}$ and the integrands are bounded adapted processes, the process M mapping t into $\{M(t,T),\ T \in [0,\tau],\ M(t,i),\ i \in I\}$, that is $M: t \to M(t,\cdot)$ is a vector valued continuous martingale. Of course, we also have

$$\langle M(\cdot,T), M(\cdot,S)\rangle(t) = \int_0^t \int_s^T \sigma(s,u)\,\mathrm{d}u \int_s^S \sigma(s,u)\,\mathrm{d}u\,\mathrm{d}s,$$

$$\langle M(\cdot,T), M(\cdot,i)\rangle(t) = -\int_0^t \int_s^T \sigma^*(s,u)\,\mathrm{d}u\,\eta(s,i)\,\mathrm{d}s, \tag{1.8}$$

$$\langle M(\cdot,i), M(\cdot,j)\rangle(t) = \int_0^t \eta^*(s,i)\eta(s,j)\,\mathrm{d}s,$$

which defines the "matrix" $\langle M\rangle$ of the joint quadratic variations of "co-ordinates"of the martingale M. In particular, in terms of $\langle M\rangle$ one can calculate the quadratic variation process for any linear combination of a finite number of "co-ordinates" of M.

Also note that using the above compact notation we can write parts (ii) and (iii) of Propositions 1.1 and 1.2, respectively, in the following form:

$$\mathrm{d}Z(t,\cdot) = Z(t,\cdot)\,\mathrm{d}M(t,\cdot). \tag{1.9}$$

This shows that the process $Z: t \to Z(t,\cdot)$ is a vector valued martingale under $\boldsymbol{P}$ and in practical terms it means that under Assumptions 1.1 and 1.2 there are no arbitrage opportunities between the money market account B, the family $\{P(\cdot,T);\ 0 \le T \le \tau\}$ of pure discount bonds and the family $\{R(\cdot,i);\ i \in I\}$ of stocks. From the pricing and hedging point of view, however, the difficulty is that the measure $\boldsymbol{P}$ is not unique.

Let then $\mathscr{P}$ be the set of probability measures $\boldsymbol{Q}$ on $(\Omega, \boldsymbol{F})$ which are equivalent to $\boldsymbol{P}$ (and hence to $\boldsymbol{P}_0$) and such that Z is a martingale under $\boldsymbol{Q}$.

Proposition 1.3. *The process M defined in (1.7) is a continuous martingale, with the quadratic variation process $\langle M\rangle$ given in (1.8), under any $Q\in\mathscr{P}$. Moreover, if there exist $T_1,\dots,T_k$ and $i_{k+1},\dots,i_n$ such that for every $0\le t\le\tau$ the matrix*

$$\Sigma(t)=\left[\int_t^{T_1}\sigma(t,u)\,\mathrm{d}u,\dots,\int_t^{T_k}\sigma(t,u)\,\mathrm{d}u,\eta(t,i_{k+1}),\dots,\eta(t,i_n)\right]$$

is nonsingular and $\Sigma^{-1}(t)$ is locally bounded then the process W defined in (1.4) is an n-dimensional F_t-Brownian motion under any $Q\in\mathscr{P}$.

Proof. The quadratic variation under Q is the same as under P because P and Q are equivalent. Also, W is a semimartingale under Q and for any x belonging to $[0,T]$ or I

$$Z(t,x)=Z(0,x)\exp(M(t,x)-\tfrac{1}{2}\langle M(\cdot,x)\rangle(t))$$

is a martingale. This implies that

$$M(t,x)=\int_0^t Z^{-1}(s,x)\,\mathrm{d}Z(s,x),\quad 0\le t\le\tau$$

is a continuous local martingale with

$$\langle M(\cdot,T)\rangle(t)=\int_0^t\left|\int_s^T\sigma(s,u)\,\mathrm{d}u\right|^2\mathrm{d}s$$

and

$$\langle M(\cdot,i)\rangle(t)=\int_0^t|\eta(s,i)|^2\,\mathrm{d}s$$

and hence, $E(\sup_{0\le t\le\tau}M(t,x))^2<\infty$. Finally, let

$$U(t)=\int_0^t\Sigma^{*-1}(s)\,\mathrm{d}N(s),$$

where

$$\begin{aligned}N(t)&=(M(t,T_1),\dots,M(t,T_k),M(t,i_{k+1}),\dots,M(t,i_n))^*\\&=\int_0^t\Sigma^{*-1}(s)\,\mathrm{d}W(s).\end{aligned}$$

It follows from the above that $U(t)$ is a continuous local martingale with the tensor quadratic variation $\langle U\rangle(t)=tI$ and hence (cf. Dellacherie and Meyer, 1980, p. 381) it is a Brownian motion in $\mathbb{R}^n$ □

Note that if the quadratic variation process $\langle M\rangle$ is deterministic then the martingale M is Gaussian seen, for example, as a two parameter random field

$$\{M(t,x);\,0\le t\le\tau,\,x\in[0,\tau]\cup I\}.$$

Consequently in this case the distribution of M, defined on the appropriate σ-algebra on $C[0,\tau]\times\mathbb{R}^{[0,\tau]\cup I}$ does not depend on the choice of $Q\in\mathscr{P}$. This allows to formulate the following proposition.

Proposition 1.4. *If the quadratic variation process $\langle M \rangle$ given in* (1.8) *is deterministic then the distribution of the process Z defined in* (1.9) *does not depend on the choice of the arbitrage free measure $Q \in \mathscr{P}$.*

Remark 1.2. Last two propositions are important from the point of view of pricing and hedging contingent claims. Proposition 1.3 will find first applications already in the next section where we will analyse some simple attainable claims. Proposition 1.4 in some sense solves the problem of incompleteness (under the assumption that $\langle M \rangle$ is deterministic). It implies that the joint distribution of the collection of processes

$$E = \{B(\cdot), P(\cdot, T), R(\cdot, i);\ T \in [0, \tau],\ i \in I\} \tag{1.10}$$

does not depend on the choice of $Q \in \mathscr{P}$. Consequently, calculations of expected values, under Q, of functions depending on these processes can be carried out under the measure P. In other words, the market model is not complete but different measures $Q \in \mathscr{P}$ produce its identical copies.

1.3. Contingent claims

The T-maturity contingent claim is an F_T-measurable random variable $C(T)$ which represents a contract equivalent to a stochastic cash flow $C(T)$ at time T. The claim $C(T)$ is called integrable if for every arbitrage free measure $Q \in \mathscr{P}$,

$$E_Q(|C(T)|/B(T)) < \infty .$$

We have the following examples.

Example 1.1. Constant contingent claims are integrable. Clearly it is sufficient to show that for every $0 \le T \le \tau$ and every $Q \in \mathscr{P}$ $E_Q(1/B(T)) < \infty$. But $1/B(T) = Z(T, T)$ and the process $Z(\cdot, T)$ is a martingale under any Q for any T. Consequently,

$$E_Q(1/B(T)) = E_Q Z(T, T) = E_Q Z(0, T) = P(0, T) < \infty .$$

Example 1.2. The time T price of the risky asset i is a T-maturity integrable claim.

Indeed $E_Q(R(T, i)/B(T)) = E_Q Z(T, i) = R(0, i) < \infty$.

The question now is how to price (and hedge) contingent claims. When the market model is complete it is well known that the unique time t price of the T-maturity integrable claim $C(T)$ is equal to

$$E(C(T)B(t)/B(T) \mid F_t),$$

where the expectation is calculated with respect to the unique arbitrage-free measure. When the market model is not complete as in our case the same statement remains valid but only for attainable integrable claims (cf. Harrison and Pliska, 1981) and the expectation is then calculated with respect to an arbitrary $Q \in \mathscr{P}$. It turns out that $C(T)$ is attainable in our model if and only if $C(T)/B(T)$ can be represented as a stochastic integral with respect to the martingale Z or, due to Proposition 1.3, with respect to the martingales M or W. Note that for the claim $C(T) = 1$ of a T-maturity pure discount bond we can write

$$C(T)/B(T) = Z(T, T) = Z(t, T) + \int_t^T \mathrm{d}Z(s, T),$$

which shows that this claim is attainable. Clearly, we also have

$$R(T, i)/B(T) = Z(T, i) = Z(t, i) + \int_t^T \mathrm{d}Z(s, i)$$

and hence, the claim $C(T) = R(T, i)$ is attainable as well. The prices of claims are listed in the proposition below where the expectation is calculated with respect to the arbitrage-free measure $\boldsymbol{P} \in \mathscr{P}$.

Proposition 1.5. *The unique time t prices associated with the contingent claims:* 1\$ *at time T and $R(T)$ $(t \leq T)$ are given by*

$$E(B(t)/B(T) \,|\, \boldsymbol{F}_t) = P(t, T), \qquad E(B(t)R(T)/B(T) \,|\, \boldsymbol{F}_t) = R(t).$$

Remark 1.3. It seems that so far two general approaches were developed to analyse the term structure of interest rates. The first one (cf. Hull and White, 1990; Jamshidian, 1990: Artzner and Delbean, 1989) is based on a probabilistic model of the spot rate from which forward rates and bond prices are calculated using the arbitrage arguments. The second approach (cf. Heath et al., 1987; Musiela et al. 1993) assumes a model for the instantaneous forward rates of all maturities from which one needs to eliminate first any arbitrage possible. Then the spot rate as well as the forward rates and bond prices can be analysed. Continuing discussions and comparisons exhibit relative advantages and disadvantages of both methodologies and it remains an open question which approach is more natural, better adapted to give answers to many theoretical and practical problems or simply easier for the market to accept. Let us point out here that Proposition 1.5 and namely the formula

$$E\left(\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right)\middle|\,\boldsymbol{F}_t\right) = \exp\left(-\int_t^T f(t, u)\,\mathrm{d}u\right)$$

establishes a one-to-one correspondence between the spot rate and the forward rates under a relatively general set of assumptions concerning exclusively boundness of volatilities. (These assumptions could be weakened even more and replaced by some integrability conditions, we decided not to do it here for expositional clarity.) This implies that both methods are equivalent in the above sense. It also shows that the spot rate is the only parameter which "drives" the entire term structure of interest rates.

1.4. Forward contracts and forward measures

A forward contract with maturity T on a risky asset obligates its owner to purchase the asset at time T for a fixed price, called the forward price. By convention, the forward price is set to make the forward contract's value at initiation equal to zero.

The forward contract we consider here is written on the asset R which represents either a stock from the family $\{R(\cdot, i); i \in I\}$ or a zero coupon from the family $\{P(\cdot, T); T \in [0, \tau]\}$.

Hence, under the measure $\boldsymbol{P}$,

$$\mathrm{d}R(t) = R(t)(r(t)\,\mathrm{d}t + \eta^*(t)\,\mathrm{d}W(t)),$$

where $\eta(t)$ equals $\eta(t, i)$ or $-\int_t^T \sigma(t, u)\,\mathrm{d}u$ for some $i \in I$, $T \in [0, \tau]$. As before we define the martingale $Z(t, R) = R(t)/B(t)$.

Now assume that the contract is initiated at time $t < T$ and that $F(t)$ denotes the forward price. It is clear that the contract corresponds to an attainable integrable claim with a cash flow of $C(T) = R(T) - F(t)$ and hence from the above definition it follows (cf. Jarrow, 1988) that

$$E((R(T) - F(t))B(t)/B(T) \mid \boldsymbol{F}_t) = 0.$$

This and Proposition 1.5 give

$$F(t) = R(t)/P(t, T). \tag{1.11}$$

Example 1.3. The forward price at time t in a T-maturity forward contract on a discount bond with maturity τ is equal to $P(t, \tau)/P(t, T)$.

From (1.11) it follows that F is a semimartingale.

Proposition 1.6. *The semimartingale decomposition of F under $\boldsymbol{P}$ is given by*

$$\mathrm{d}F(t) = F(t)\left(\int_t^T \sigma(t, u)\,\mathrm{d}u + \eta(t)\right)^*\left(\int_t^T \sigma(t, u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t)\right).$$

Proof. The Ito formula and Propositions 1.1 and 1.2 allow to write

$$\begin{aligned} F(t) &= R(t)(P(t, T))^{-1} \\ &= (R(t)/B(t))(P(t, T)/B(t))^{-1} = Z(t, R)(Z(t, T))^{-1}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \mathrm{d}(Z(t, T))^{-1} &= -(Z(t, T))^{-2}\mathrm{d}Z(t, T) + (Z(t, T))^{-3}\mathrm{d}\langle Z(\cdot, T)\rangle(t) \\ &= (Z(t, T))^{-1}\int_t^T \sigma^*(t, u)\,\mathrm{d}u\,\mathrm{d}W(t) + (Z(t, T))^{-1}\left|\int_t^T \sigma(t, u)\,\mathrm{d}u\right|^2 \mathrm{d}t \end{aligned}$$

and hence,

$$\begin{aligned} \mathrm{d}F(t) &= (\mathrm{d}Z(t, R))(Z(t, T))^{-1} + Z(t, R)\,\mathrm{d}(Z(t, T))^{-1} \\ &\quad + \mathrm{d}\langle Z(\cdot, R), (Z(\cdot, T))^{-1}\rangle(t) = F(t)\eta^*(t)\,\mathrm{d}W(t) \\ &\quad + F(t)\left(\int_t^T \sigma^*(t, u)\,\mathrm{d}u\,\mathrm{d}W(t) + \left|\int_t^T \sigma(t, u)\,\mathrm{d}u\right|^2 \mathrm{d}t\right) \\ &\quad + F(t)\eta^*(t)\int_t^T \sigma(t, u)\,\mathrm{d}u\,\mathrm{d}t \\ &= F(t)\left(\int_t^T \sigma(t, u)\,\mathrm{d}u + \eta(t)\right)^*\left(\int_t^T \sigma(t, u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t)\right). \qquad \square \end{aligned}$$

Remark 1.4. Note that if

$$\eta(t) = -\int_t^T \sigma(t, u)\,\mathrm{d}u,$$

then $F(t) = F(0) = R(0)$. In particular the forward price at time t in a T-maturity forward contract on a T-maturity pure discount bond is equal to 1\$. This simple observation seems to support an intuitively clear more general belief that by looking into forward contracts we in some sense eliminate, or rather control through the volatility, risks related to fluctuations of interest rates. From this point there is only a small step to another Girsanov transformation developed by El Karoui and Rochet (1989) to price options on coupon bonds and then successfully used by El Karoui et al. (1991) to value other claims that are contingent on interest rate dependent assets. The idea is given below.

Let for all $A \in \boldsymbol{F}_T$

$$\boldsymbol{P}_T(A) = \int_A (P(0, T)B(T))^{-1}\,\mathrm{d}\boldsymbol{P}. \tag{1.13}$$

It was shown in Proposition 1.5 that $E(B(T))^{-1} = P(0, T)$. Hence (1.13) defines a probability measure on $(\Omega, \boldsymbol{F}_T)$. We call it here the forward measure. Note that the local density $E((P(0, T)B(T))^{-1} \mid \boldsymbol{F}_t)$ can be represented as follows:

$$\begin{aligned} E((P(0, T)B(T))^{-1} \mid \boldsymbol{F}_t) &= (P(0, T)B(t))^{-1}E(B(t)/B(T) \mid \boldsymbol{F}_t) \\ &= (P(0, T)B(t))^{-1}P(t, T) = (Z(0, T))^{-1}Z(t, T). \end{aligned}$$

This together with Proposition 1.1 (iii) implies that

$$(P(0, T)B(T))^{-1} = \exp(M(T, T) - \tfrac{1}{2}\langle M(\cdot, T)\rangle(T)),$$

where the martingale $M(\cdot, T)$ is defined in (1.7). Consequently, the Girsanov theorem asserts that the process $\{W(t, T);\ 0 \le t \le T\}$, where

$$W(t, T) = W(t) + \int_0^t \int_u^T \sigma(u, s)\,\mathrm{d}s\,\mathrm{d}u \tag{1.14}$$

is an $\boldsymbol{F}_t$-Brownian motion under $\boldsymbol{P}_T$. This together with Proposition 1.6 lead to the following proposition.

Proposition 1.7. *The process F is a martingale under $\boldsymbol{P}_T$.*

Remark 1.5. Concept of the forward measure is relatively new and therefore probably not very well known to a wider audience. Let us try to explain it in a more intuitive way. An "ordinary" arbitrage-free measure makes discounted (hence interest rate influence) prices look like martingales. Speaking in general terms we place ourselves at time 0 and looking into the future we construct a distribution under which present values of securities, in terms of the dollar value at time O, behave like random walks. The forward measure $\boldsymbol{P}_T$ produces the same effect but in terms of the dollar value at time T. From the point of view of pricing T-maturity cash flows the measure

$\boldsymbol{P}_T$ integrates (in the mathematical and ordinary sense) risks related to interest rate movements inside of the corresponding volatility parameter. Finally, note that $\boldsymbol{P}_0 = \boldsymbol{P}$, of course, and that $\boldsymbol{P}_T = \boldsymbol{P}$ for all $0 \le T \le \tau$ if the spot rate is deterministic, because in this case

$$P(0, T) = E(B(T)^{-1}) = B(T)^{-1}$$

and hence, the density between $\boldsymbol{P}_T$ and $\boldsymbol{P}$ equals 1.

1.5. Options involving finite number of assets

Throughout this section we assume that the vectors $\sigma(t, T)$, $\eta(t, i)$, $t \in [0, \tau]$, $T \in [0, \tau]$, $i \in I$ which describe the structure of volatilities in our economy are deterministic. Let us remind (cf. Remark 1.2) that under this assumption the distribution of the collection of process $\boldsymbol{E}$ does not depend on the choice of the arbitrage-free measure $\boldsymbol{Q} \in \mathscr{P}$. Now select a finite number of pure discount bonds with maturities say $T_1, \ldots, T_l \in [0, \tau]$ and a finite number of stocks say $i_{l+1}, \ldots, i_m \in I$ from (1.10) and form the model

$$\boldsymbol{E}(m) = \{B(\cdot), P(\cdot\, T_j), j = 1, \ldots, l, R(\cdot, i_j), j = l+1, \ldots, m\},$$

which consists only of a finite number of risky assets. We will be interested here in the problem of pricing and hedging contingent claims concerning exclusively assets of the economy $\boldsymbol{E}(m)$. Note that the distribution of $\boldsymbol{E}(m)$ also does not depend on the choice of $\boldsymbol{Q} \in \mathscr{P}$, i.e. for each $\boldsymbol{P}, \boldsymbol{Q} \in \mathscr{P}$,

$$\boldsymbol{PE}(m)^{-1} = \boldsymbol{QE}(m)^{-1}.$$

This implies that on the canonical probability space of $\boldsymbol{E}(m)$ i.e.

$$(C([0, \tau]^{m+1}), B(C([0, \tau]^{m+1})), \boldsymbol{PE}(m)^{-1})$$

with the filtration $\boldsymbol{F}_t = \boldsymbol{F}_t^{E(m)}$ generated by the co-ordinate process the set of arbitrage-free measures $\boldsymbol{PE}(m)^{-1}$ is a singleton. Consequently (cf. Harrison and Pliska, 1983) the market model is complete and hence any integrable T-maturity claim $C(T)$ defined on the above probability space is attainable. Moreover, the time t price of $C(T)$ is equal to

$$V(t) = E(C(T)B(t)/B(T) \mid \boldsymbol{F}_t). \tag{1.15}$$

Let us now consider several examples of claims $C(T)$, which correspond to standard or more exotic options, and show how to use previously developed ideas to price them.

1.5.1. Options on coupon bonds

Consider a European call option with maturity T and exercise price K on the cash flows $C_1, \ldots, C_l$ occurring at times $T_1, \ldots, T_l$. One could think here of an option on a coupon bond where the nonzero C_j's are the coupon payments except for the last one which represents the coupon plus the face value. One could also think of a portfolio of bonds or in general of any cash flows, positive or negative. It is clear that the time T value of this cash flows, expressed in terms of the T_j-maturity pure discount bonds, is equal to

$$\sum_{j=1}^{l} C_j P(T, T_j)$$

and hence the European call option corresponds to the claim

$$C(T) = (\Sigma_{j=1}^{l} C_j P(T, T_j) - K)^+.$$

Now let for $j = 1, \ldots, l$

$$F(t,j) = P(t, T_j)/P(t, T)$$

denote the time t price of a T-maturity forward contract on the T_j-maturity pure discount bond (cf. Example 1.3). From the Propositions 1.6 and 1.7 it follows that

$$\mathrm{d}F(t,j) = -F(t,j)\int_T^{T_j} \sigma^*(t,u)\,\mathrm{d}u\,\mathrm{d}W(t,T),$$

where the process $W(\cdot, T)$ defined in (1.14) is a Brownian motion under the forward measure $\boldsymbol{P}_T$ and consequently,

$$\begin{aligned} P(T, T_j) &= F(T,j) \\ &= F(t,j)\exp\left(-\int_t^T \int_T^{T_j} \sigma^*(s,u)\,\mathrm{d}u\,\mathrm{d}W(s,T) \right. \\ &\quad \left. -\frac{1}{2}\int_t^T \left| \int_T^{T_j} \sigma(s,u)\,\mathrm{d}u \right|^2 \mathrm{d}s \right). \end{aligned}$$

But we also have

$$E_T(C(T)\,|\,\boldsymbol{F}_t)E(B(t)/B(T)\,|\,\boldsymbol{F}_t) = E(C(T)B(t)/B(T)\,|\,\boldsymbol{F}_t),$$

where E_T stands for the expectation under the measure $\boldsymbol{P}_T$. Using Proposition 1.5, (1.15) and the above it follows that

$$\begin{aligned} V(t) &= P(t,T)E_T((\Sigma_{j=1}^{l} C_j F(T,j) - K)^+\,|\,\boldsymbol{F}_t) \\ &= P(t,T)E_T\left(\left(\Sigma_{j=1}^{l} C_j F(t,j)\exp\left(-\int_t^T \int_T^{T_j} \sigma^*(s,u)\,\mathrm{d}u\,\mathrm{d}W(s,T) \right.\right.\right. \\ &\quad \left.\left.\left. -\frac{1}{2}\int_t^T \left| \int_T^{T_j} \sigma(s,u)\,\mathrm{d}u \right|^2 \mathrm{d}s \right) - K \right)^+ \,\middle|\, \boldsymbol{F}_t \right). \end{aligned}$$

Now define for $j = 1 \ldots l$

$$X_j = \int_t^T \int_T^{T_j} \sigma^*(s,u)\,\mathrm{d}u\,\mathrm{d}W(s,T)$$

and note that the $\boldsymbol{P}_T$-conditional distribution of the vector $X = (X_1, \ldots, X_l)^*$ given F_t is $N(0, \Gamma^*\Gamma)$ where $\Gamma = [\gamma_1, \ldots, \gamma_l]$ is a $k \times l$ matrix such that

$$\Gamma^*\Gamma = EXX^* = \left(\left(\int_t^T \int_T^{T_i} \sigma^*(s,u)\,\mathrm{d}u \int_T^{T_j} \sigma(s,u)\,\mathrm{d}u\,\mathrm{d}s\right)\right).$$

This implies that we can write the following representation of the price

$$V(t) = P(t,T)\int_{\mathbb{R}^k} (\Sigma_{j=1}^l C_j F(t,j)\exp(-\gamma_j^* x - \tfrac{1}{2}|\gamma_j|^2) - K)^+ \varphi(x)\,\mathrm{d}x,$$

where $\varphi(x) = (2\pi)^{-k/2}\exp(-\frac{1}{2}|x|^2)$ or equivalently that

$$V(t) = \int_{\mathbb{R}^k} (\Sigma_{j=1}^l C_j P(t,T_j)\varphi(x+\gamma_j) - KP(t,T)\varphi(x))^+\,\mathrm{d}x.$$

Note that if in addition $\sigma(s,u)$ dose not depend on s i.e. $\sigma(s,u) = \sigma(u)$ then we can write

$$\Gamma^*\Gamma = (T-t)\left(\left(\int_T^{T_i} \sigma^*(u)\,\mathrm{d}u \int_T^{T_j} \sigma(u)\,\mathrm{d}u\right)\right) = (T-t)\,B^*B,$$

where $B = [b_1, \ldots, b_l]$ is a $k \times l$ matrix. In particular one could take $b_j = \int_T^{T_j} \sigma(u)\,\mathrm{d}u$. Using this notation we can finally write

$$V(t) = \int_{\mathbb{R}^k} (\Sigma_{j=1}^l C_j P(t,T_j)\varphi(x + (T-t)^{1/2}b_j) - KP(t,T)\varphi(x))^+\,\mathrm{d}x. \tag{1.16}$$

It is perhaps interesting to note that the function

$$\vartheta(t, p_1, \ldots, p_l) = \int_{\mathbb{R}^k} (\Sigma_{j=1}^l p_j \varphi(x + (T-t)^{1/2}b_j) - K\varphi(x))^+\,\mathrm{d}x$$

is the unique solution to the Cauchy problem

$$\frac{\partial\vartheta}{\partial t} + \tfrac{1}{2}\Sigma_{i=1}^l \Sigma_{j=1}^l b_i^* b_j p_i p_j \frac{\partial^2\vartheta}{\partial p_i \partial p_j} = 0 \quad \text{on } [0,T]\times\mathbb{R}^l,$$

$$\vartheta(T, p_1, \ldots, p_l) = (\Sigma_{j=1}^l p_j - K)^+ \, p_j \in \mathbb{R}$$

and that

$$V(t) = P(t,T)\vartheta(t, C_1F(t,1), \ldots, C_1F(t,l)).$$

Interpretation of the parameters is also obvious. One can show that B^*B represents the matrix of volatilities of returns on forward prices of pure discount bonds with maturities T_j, $j = 1, \ldots, l$.

Let us now consider some particular cases. Assume first $C_1 = 1$, $C_2 = C_3 = \cdots = C_l = 0$. Then formula (1.16) reduces to

$$V(t) = \int_{\mathbb{R}^k} (P(t,T_1)\varphi(x + (T-t)^{1/2}b_1) - KP(t,T)\varphi(x))^+\,\mathrm{d}x$$

and it is easy to see that the integral is positive if and only if

$$b_1^* x/|b_1| \leq (\log(P(t,T_1)/KP(t,T)) - \tfrac{1}{2}(T-t)|b_2|^2)/(T-t)^{1/2}|b_1|.$$

This allows to write that

$$V(t) = P(t,T_1)N(h) - KP(t,T)N(h-(T-t)_{1/2}|b_1|), \tag{1.17}$$

where

$$h = (\log(P(t,T_1)/KP(t,T)) + \tfrac{1}{2}(T-t)|b_1|^2)/(T-t)^{1/2}|b_1|$$

and one can recognize the Black–Scholes formula applied to the forward price as obtained in (Heath et al. 1992).

Assume next $k=1$ and $C_j \geq 0, j=1,\dots,l$. Then it is not difficult to see that the equation

$$\Sigma_{j=1}^{l} C_j P(t,T_j)\varphi(x+(T-t)^{1/2}b_j) = KP(t,T)\varphi(x)$$

has a unique solution in x which we call s. In terms of s we can write

$$V(t) = \Sigma_{j=1}^{l} C_j P(t,T_j)N(s+(T-t)^{1/2}b_j) - KP(t,T)N(s)$$

and hence (1.16) reduces to the formula derived by El Karoui and Rochet (1989) and Jamshidian (1990).

Finally assume $k=2$, $C_1>0$ and $C_2<0$. Then (1.16) can be expressed in terms of the bivariate normal distribution and applied to price options on spreads between forward rates.

1.5.2. Options on baskets of stocks

Consider now a European call option with maturity T and exercise price K on the basket $\{C_j, j=l+1,\dots,m\}$ of stocks $R(\cdot,i_j)$. Clearly this option corresponds to the claim

$$C(T) = (\Sigma_{j=l+1}^{m} C_j R(T,i_j) - K)^+.$$

Repeating arguments used in Section 1.5.1 one can verify that

$$\mathrm{d}F(t,j) = F(t,j)\left(\int_t^T \sigma(t,u)\,\mathrm{d}u + \eta(t,i_j)\right)^* \mathrm{d}W(t,T),$$

where for $j=l+1,\dots,m$

$$F(t,j) = R(t,i_j)/P(t,T).$$

Let for $j=l+1,\dots,m$

$$\lambda(t,j) = \int_t^T \sigma(t,u)\,\mathrm{d}u + \eta(t,i_j),$$

then we have

$$R(t,i_j) = F(T,j) = F(t,j)\exp\left(\int_t^T \lambda^*(s,j)\,\mathrm{d}W(s,T) - \frac{1}{2}\int_t^T |\lambda(s,j)|^2\,\mathrm{d}s\right).$$

The P_T-conditional distribution of the vector $Y = (X_{l+1}, \dots, X_m)^*$ given F_t, where

$$X_j = \int_t^T \lambda^*(s,j)\,\mathrm{d}W(s,T), \quad j = l+1, \dots, m$$

is $\mathrm{N}(0, \Delta^*\Delta)$ with a $k \times l$ matrix $\Delta = [\delta_1, \dots, \delta_l]$ such that

$$\Delta^*\Delta = EYY^* = \int_t^T \lambda(s,i)\lambda(s,j)\,\mathrm{d}s.$$

Consequently, we can write the following representation of the price

$$V(t) = E(C(T)B(t)/B(T)\,|\,F_t)$$

$$= \int_{\mathbb{R}^k} (\Sigma_{j=l+1}^m C_j R(t,i_j)\varphi(x-\delta_j) - KP(t,T)\varphi(x))^+\,\mathrm{d}x.$$

Again if $\lambda(t,j)$ does not depend on t i.e. $\lambda(t,j) = \lambda(j)$ then we can write

$$\Delta^*\Delta = EYY^* = (T-t)(\lambda^*(i)\lambda(j)) = (T-t)A^*A,$$

where $A = [b_{l+1}, \dots, b_m]$ is a $k \times l$ matrix. In particular one could take $b_j = \lambda(j)$. Finally, we can represent the option price as follows

$$V(t) = \int_{\mathbb{R}^k} (\Sigma_{j=l+1}^m C_j R(t,i_j)\varphi(x-(T-t)^{1/2}b_j) - KP(t,T)\varphi(x))^+\,\mathrm{d}x.$$

Interpretation of the parameters remains intuitive. The B^*B matrix represents volatilities of returns on forward prices of stocks. Moreover, the transformation $x \to -x$ leads to

$$V(t) = \int_{\mathbb{R}^k} (\Sigma_{j=l+1}^m C_j R(t,i_j)\varphi(x+(T-t)^{1/2}b_j) - KP(t,T)\varphi(x))^+\,\mathrm{d}x. \tag{1.18}$$

and allows to compare formulae (1.16) and (1.18). As a consequence all particular cases discussed in Section 1.5.1 can be restated here with $R(t, i_j)$ substituting $P(t, T_j)$. For example if $C_{l+1} = 1$ and $C_{l+2} = \cdots = C_m = 0$ we obtain the formula derived in Jarrow (1988).

1.5.3. Multiple options

Consider a basket of European call options with maturity T and exercise prices K_j on stocks $R(\cdot, i_j), j = l+1, \dots, m$. A multiple option is the option which gives the right to exercise in exactly one of the stocks of the buyers choice. Clearly, the holder will exercise in the stock which gives the maximal profit and consequently the multiple option corresponds to the claim

$$C(T) = \max(R(T,i_{l+1}) - K_{l+1}, \dots, R(T,i_m) - K_m, 0)$$

$$= \max\{R(T,i_j) - K_j)^+;\ l+1 \le j \le m\}.$$

Using methods and notation of Section 1.5.2 it is not difficult to see that the price $V(t)$ is given by

$$V(t) = \int_{\mathbb{R}^k} \max(R(t, i_j)\varphi(x + (T-t)^{1/2} b_j) - K_j P(t, T)\varphi(x))^+ \, dx.$$

For $l + 1 = m$ we get the Black–Scholes formula of course.

Remark 1.6. One could think that Eqs. (1.16)–(1.19) which in principle involve integration over the space $\mathbb{R}^k$ will be very difficult to implement. In fact this is not true. The common denominator in examples of options analysed in Sections 1.5.1–1.5.3. is that in practice k will always be less than or equal to 2. This statement is based on a statistical analysis of the matrix of volatilities of forward prices. It is remarkable that in almost all cases analysed by us we found consistently only two significant eigenvalues.

2. Foreign economy

In this part of the paper we show how to expand our model into a larger economy in which the foreign money market account, foreign bonds and other foreign securities can be viewed as "domestic" assets. Our approach is based on the Amin and Jarrow (1989) model of an arbitrage-free international economy. We use the same notation and write the subscript f on the quantities defined in the first part of the paper to indicate that they represent the corresponding quantities in the foreign economy.

2.1. Foreign term structure

We assume that the foreign instantaneous forward rate $f_f(t, T)$ is given by

$$df_f(t, T) = \alpha_f(t, T)\,dt + \sigma_f^*(t, T)\,dW_0(t), \tag{2.1}$$

where W_0 is the same as before while the processes $\{\alpha_f(t, T);\ 0 \le t \le T \le \tau\}$ and $\{\sigma_f(t, T); 0 \le t \le T \le \tau\}$ are bounded on $[0, \tau]^2 \times \Omega$ adapted with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

Denominated in foreign currency are the time t price $P_f(t, T)$ of a T-maturity foreign pure discount bond and the foreign accumulation factor $B_f(t)$. Of course, we have

$$P_f(t, T) = \exp\left(- \int_t^T f_f(t, u)\,du \right)$$

and

$$B_f(t) = \exp\left(\int_0^t r_f(u)\,du \right),$$

where $r_f(t) = f_f(t, t)$ stands for the foreign instantaneous riskless rate of interest at time t.

The exchange rate $S(t)$ of the foreign currency, denominated in the domestic currency per unit of the foreign currency, establishes the link between the two economies. We assume that for $0 \le t \le \tau$

$$dS(t) = S(t)(\beta(t)\,dt + \upsilon^*(t)\,dW_0(t)), \tag{2.2}$$

where W_0 is still the same while the processes $\{\beta(t); 0 \le t \le \tau\}$ and $\{\upsilon(t); 0 \le t \le \tau\}$ are adapted with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively, $S(0)$ is a constant, the process υ is bounded on $[0, \tau] \times \Omega$ and the process β satisfies the condition

$$\boldsymbol{P}_0\left(\int_0^\tau |\beta(t)|\,dt < \infty\right) = 1.$$

Note that for each $0 \le T \le \tau$, $P_f(t, T)S(t)$ represents the dollar value of a foreign pure discount bond. In this sense the family of processes

$$\{P_f(\cdot, T)S(\cdot); 0 \le T \le \tau\} \tag{2.3}$$

can be viewed as a new family of assets in the domestic economy. Also the dollar value of the foreign money market account $B_f(t)S(t)$ can be interpreted as a new asset. However, by bringing new assets into our economy we create new possibilities for arbitrage. To eliminate them we proceed as before. That is, first we identify a condition under which the process

$$Z_f(t) = B_f(t)S(t)/B(t) \tag{2.4}$$

is a $\boldsymbol{P}$-martingale (the measure $\boldsymbol{P}$ is the one defined in (1.3)). A straightforward application of the Ito formula to $Z_f(t)$ leads to the following differential representation

$$dZ_f(t) = Z_f(t)((r_f(t) - r(t) + \beta(t) + \vartheta^*(t)H(t))\,dt + \vartheta^*(t)\,dW(t)),$$

where the process W defined in Eq. 1.4 is Brownian motion under $\boldsymbol{P}$. This invites to formulate the following assumption.

Assumption 2.1. For all $0 \le T \le \tau$

$$\beta(t) = r(t) - r_f(t) - \vartheta^*(t)H(t). \tag{2.5}$$

The case of the family (2.3) is more involved but it can be analysed in exactly the same way. We have, as in Proposition 1.1,

$$dP_f(t, T) = P_f(t, T)(d\log P_f(t, T) + \tfrac{1}{2}d\langle \log P_f(\cdot, T)\rangle(t))$$

and

$$d\log P_f(t, T) = -d\left(\int_t^T f_f(t, u)\,du\right) = r_f(t)\,dt - \int_t^T df_f(t, u)\,du.$$

But using (2.1) it follows that

$$\int_t^T (\mathrm{d}f_{\mathrm{f}}(t,u)\,\mathrm{d}u) = \left(\int_t^T \alpha_{\mathrm{f}}(t,u)\,\mathrm{d}u\right)\mathrm{d}t + \left(\int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)\mathrm{d}W(t)$$

and hence that

$$\mathrm{d}\langle \log P_{\mathrm{f}}(\cdot,T)\rangle(t) = \left|\int_t^T \sigma_{\mathrm{f}}(t,u)\,\mathrm{d}u\right|^2 \mathrm{d}t.$$

Consequently, we have

$$\begin{aligned}\mathrm{d}P_{\mathrm{f}}(t,T) = P_{\mathrm{f}}(t,T)\Bigg(\Bigg(r_{\mathrm{f}}(t) - \int_t^T \alpha_{\mathrm{f}}(t,u)\,\mathrm{d}u \\ + \frac{1}{2}\left|\int_t^T \sigma_{\mathrm{f}}(t,u)\,\mathrm{d}u\right|^2\Bigg)\mathrm{d}t - \left(\int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)\mathrm{d}W_0(t)\Bigg)\end{aligned}$$

and therefore, together with (2.2) also

$$d\langle P_{\mathrm{f}}(\cdot,T), S(\cdot)\rangle(t) = -P_{\mathrm{f}}(t,T)S(t)\int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\,\vartheta(t)\,\mathrm{d}t.$$

All this leads to

$$\begin{aligned}\mathrm{d}(P_{\mathrm{f}}(t,T)S(t)) = P_{\mathrm{f}}(t,T)S(t)\Bigg(\Bigg(r_{\mathrm{f}}(t) - \int_t^T \alpha_{\mathrm{f}}(t,u)\,\mathrm{d}u \\ + \frac{1}{2}\left|\int_t^T \sigma_{\mathrm{f}}(t,u)\,\mathrm{d}u\right|^2 + \beta(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\,\vartheta(t)\Bigg)\mathrm{d}t \\ + \left(\vartheta^*(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)\mathrm{d}W_0(t)\Bigg).\end{aligned}$$

Now it is sufficient to analyse the process

$$Z_{\mathrm{f}}(t,T) = P_{\mathrm{f}}(t,T)S(t)/B(t). \tag{2.6}$$

Clearly, we have

$$\begin{aligned}\mathrm{d}Z_{\mathrm{f}}(t,T) &= (\mathrm{d}P_{\mathrm{f}}(t,T)S(t))/B(t) - (P_{\mathrm{f}}(t,T)S(t)r(t)/B(t))\,\mathrm{d}t \\ &= Z_{\mathrm{f}}(t,T)\Bigg(\Bigg(r_{\mathrm{f}}(t) - r(t) - \int_t^T \alpha_{\mathrm{f}}(t,u)\,\mathrm{d}u \\ &\quad + \frac{1}{2}\left|\int_t^T \sigma_{\mathrm{f}}(t,u)\,\mathrm{d}u\right|^2 + \beta(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\,\vartheta(t) \\ &\quad + \left(\vartheta^*(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)H(t)\Bigg)\mathrm{d}t \\ &\quad + \left(\vartheta^*(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)(\mathrm{d}W_0(t) - H(t)\,\mathrm{d}t)\Bigg)\end{aligned}$$

$$= Z_{\mathrm{f}}(t,T)\Bigg(\Bigg(-\int_t^T \alpha_{\mathrm{f}}(t,u)\,\mathrm{d}u + \frac{1}{2}\left|\int_t^T \sigma_{\mathrm{f}}(t,u)\,\mathrm{d}u\right|^2 - \int_t^T \sigma_f^*(t,u)\,\mathrm{d}u\,\vartheta(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\,H(t)\Bigg)\mathrm{d}t + \left(\vartheta^*(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)\mathrm{d}W(t)\Bigg)$$

and hence we need the following assumption.

Assumption 2.2. For all $0 \le t,\ T \le \tau$

$$\alpha_{\mathrm{f}}(t,T) = \sigma_{\mathrm{f}}^*(t,T)\left(\int_t^T \sigma_{\mathrm{f}}(t,s)\,\mathrm{d}s - H(t) - \vartheta(t)\right). \tag{2.7}$$

Note that assumptions (2.5) and (2.7) are consistent with assumptions (1.6) and (1.2), respectively. It seems intuitively clear that one would need to adjust the model only for the exchange rate risk.

Let us summarize the above observations.

Proposition 2.1. *Under the measure* $\boldsymbol{P}$ *the processes* $Z_{\mathrm{f}}(\cdot)$ *and* $Z_{\mathrm{f}}(\cdot, T)$, $0 \le T \le \tau$, *defined in* (2.4) *and* (2.6) *are martingales. Moreover, we have*

(i) $\mathrm{d}Z_{\mathrm{f}}(t) = Z_{\mathrm{f}}(t)\vartheta^*(t)\,\mathrm{d}W(t)$,

(ii) $\mathrm{d}Z_{\mathrm{f}}(t,T) = Z_{\mathrm{f}}(t,T)\left(\vartheta^*(t) - \int_t^T \sigma_{\mathrm{f}}^*(t,u)\,\mathrm{d}u\right)\mathrm{d}W(t)$.

We finish this section with the definitions of martingales

$$M_{\mathrm{f}}(t) = \int_0^t \vartheta^*(s)\,\mathrm{d}W(s) \tag{2.8}$$

and

$$M_{\mathrm{f}}(t,T) = \int_0^t \left(\vartheta^*(s) - \int_s^T \sigma_{\mathrm{f}}^*(s,u)\,\mathrm{d}u\right)\mathrm{d}W(s), \tag{2.9}$$

where $0 \le t \le \tau$ and $0 \le T \le \tau$.

2.2. Foreign risky assets

Assume now that we are interested in the family $\{R_{\mathrm{f}}(\cdot, i);\ i \in I_{\mathrm{f}}\}$ of foreign stocks or other foreign assets prices of which are given by

$$\mathrm{d}R_{\mathrm{f}}(t,i) = R_{\mathrm{f}}(t,i)(\mu_{\mathrm{f}}(t,i)\,\mathrm{d}t + \eta_{\mathrm{f}}(t,i)\,\mathrm{d}W_0(t)).$$

As usual assume that the processes $\{\mu_{\mathrm{f}}(t,i);\ 0 \le t \le \tau\}$ and $\{\eta_{\mathrm{f}}(t,i);\ 0 \le t \le \tau\}$, $i \in I_{\mathrm{f}}$, are adapted with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. For every $i \in I_{\mathrm{f}}$ $\eta_{\mathrm{f}}(\cdot, i)$ is bounded on $[0, \tau] \times \Omega$ while $\mu_f(\cdot, i)$ satisfies

$$P_0\left(\int_0^\tau |\mu_f(t, i)|\,dt < \infty\right) = 1.$$

Of course for each $i \in I_f$, $R_f(t, i)S(t)$ is the dollar value of a foreign asset and again all we need to do is to find conditions under which the processes

$$Z_f(t, i) = R_f(t, i)S(t)/B(t) \tag{2.10}$$

are martingales under P. This leads to the following assumption.

Assumption 2.3. For all $0 \le t \le \tau$ and all $i \in I_f$,

$$\mu_f(t, i) = r_f(t) - \eta_f^*(t, i)H(t) - \eta_f^*(t, i)v(t). \tag{2.11}$$

It is a simple exercise to show the following proposition.

Proposition 2.2. *Under the measure P for all $i \in I_f$ the process $Z_f(\cdot, i)$ is a martingale and*

$$dZ_f(t, i) = Z_f(t, i)\,dM_f(t, i), \tag{2.12}$$

where

$$M_f(t, i) = \int_0^t (\eta_f^*(s, i) + \vartheta^*(s))\,dW(s). \tag{2.13}$$

Remark 2.1. It may be interesting to note at this stage that from Propositions 2.1(i) and 2.2 we can deduce the following intuitively obvious statement; portfolios consisting of continuously rebalanced equivalent short and long positions in two foreign assets do not carry any currency exchange risk. This is because for all $0 \le t \le \tau$,

$$\int_0^t Z_f^{-1}(s, i)\,dZ_f(s, i) - \int_0^t Z_f^{-1}(s, T)\,dZ_f(s, T)$$
$$= \int_0^t \left(\eta_f^*(s, i) + \int_s^t \sigma_f^*(s, u)\,du\right) dW(s)$$

does not depend on ϑ. As it is seen from the formula above we think here of discounted prices.

Let us remember that we treat foreign assets as new assets in the domestic economy. Therefore, we can also define new "larger" vectors M and Z. For example, the new vector M is the family of martingales formed from all martingales defined in 1.7, 2.8, 2.9 and 2.13. The new vector Z consists of martingales defined in 1.9, 2.4, 2.6 and 2.10. This allows us to restate Propositions 1.3 and 1.4 with the matrix $\Sigma(t)$ build from vectors of "volatilities" of domestic and foreign assets. In particular if the process $\langle M \rangle$ is deterministic then the joint distribution of the family of processes

$$E = \{B(\cdot), P(\cdot, T), R(\cdot, i), B_f(\cdot)S(\cdot), P_f(\cdot, T)S(\cdot), R(\cdot, j),$$
$$S(\cdot);\ T \in [0, \tau],\ i \in I,\ j \in I_f\} \tag{2.14}$$

representing our larger economy does not depend on the choice of the arbitrage free measure $Q \in \mathscr{P}$.

All results concerning forward contracts and forward measures remain valid as well. We simply need to remember to change names of vectors of volatilities.

Example 2.1. The time t price of a T-maturity forward contract on a foreign discount bond with maturity T_1 is equal to

$$F(t) = P_{\mathrm{f}}(t, T_1)S(t)/P(t, T). \tag{2.15}$$

In particular if $T = T_1$ then

$$F(t) = P_{\mathrm{f}}(t, T)S(t)/P(t, T)$$

is the value of a forward contract on foreign currency. In general when $T \neq T_1$ we can write, using Propositions 1.6 and 2.1 the following decomposition of F under P

$$\begin{aligned} \mathrm{d}F(t) = F(t)\Bigg(\bigg(\int_t^T \sigma(t, u)\,\mathrm{d}u + \vartheta(t) \\ - \int_t^{T_1} \sigma_{\mathrm{f}}(t, u)\,\mathrm{d}u\bigg)^{*}\bigg(\int_t^T \sigma(t, u)\,\mathrm{d}u\,\mathrm{d}t + \mathrm{d}W(t)\bigg)\Bigg) \end{aligned} \tag{2.16}$$

which again can be interpreted intuitively (take $T = T_1$, and $\sigma = \sigma_{\mathrm{f}}$).

2.3. Options on foreign assets

As in Section 1.5 we assume here throughout that the vectors $\sigma(t, T)$, $\eta(t, i)$, $\sigma_{\mathrm{f}}(t, T)$, $\eta_{\mathrm{f}}(t, j)$, $\vartheta(t)$, $T \in [0, \tau]$, $i \in I$, $j \in I_{\mathrm{f}}$ which define the term structure of volatilities in our economy are deterministic. This implies that finite families containing $B(\cdot)$ and other processes from (2.14) can be viewed as complete models of economies spanned by finite number of assets. We consider here several examples of such models in the context of option pricing.

2.3.1. Options on foreign discount bonds

To price a European call option with maturity T and exercise price K on a foreign discount bond with maturity T_1 we consider a finite dimensional "projection" of economy (2.14) which contains $B(\cdot)$, $P(\cdot, T)$, $B_{\mathrm{f}}(\cdot)S(\cdot)$, $P(\cdot, T_1)S(\cdot)$. The T-maturity forward contract on the asset $P(\cdot, T_1)S(\cdot)$ is priced at time t by Eq. (2.15). The semimartingale decomposition of its value $F(t)$ is given in (2.16). It allows us to identify the new volatility parameter

$$\int_t^T \sigma(t, u)\,\mathrm{d}u + \vartheta(t) - \int_t^{T_1} \sigma_{\mathrm{f}}(t, u)\,\mathrm{d}u$$

and the new γ_1 vector as any vector such that

$$|\gamma_1|^2 = \int_t^T \left| \int_s^T \sigma(s, u)\,\mathrm{d}u + \vartheta(s) - \int_t^{T_1} \sigma_{\mathrm{f}}(s, u)\,\mathrm{d}u \right|^2 \mathrm{d}s.$$

If in addition the integrand in the above formula is constant in s and equal to σ^2 then $|\gamma_1|^2 = (T-t)\sigma^2$ and we can derive from (1.16) or (1.17) the following pricing formula.

$$V(t) = P_f(t, T_1)S(t)N(h) - KP(t, T)N(h-(T-t)^{1/2}\sigma), \tag{2.17}$$

where

$$h = (\log(P_f(t, T_1)S(t)/KP(t, T)) + \tfrac{1}{2}(T-t)\sigma^2)/(T-t)^{1/2}\sigma.$$

Interpretation of the parameter σ^2 is the same as before i.e. σ^2 represents here volatility of the T-maturity forward contract on the T_1-maturity foreign discount bond.

Finally let us analyse some particular cases. If for all t, $T \in [0, \tau]$, $\sigma(t, T) = \sigma_f(t, T)$, $\vartheta(t) = 0$ and $S(0) = 1$ then Eq. (2.17) reduces to (1.17) and hence can be used to price options on "domestic" discount bonds. If $T = T_1$ then (2.17) reduces to the formula for pricing options on foreign currency under stochastic interest rates as derived by Amin and Jarrow (1989).

2.3.2. Options on spreads between the domestic and foreign discount functions

One way of thinking of a European call option with maturity T on the spread at time T_1 between the domestic and foreign discount functions is to think of an option which corresponds to the contingent claim

$$C(T) = (P_f(T, T_1)S(T) - CP(T, T_1) - K)^+,$$

where C and K are constants. To price such a claim we can follow arguments presented in Section 1.5.1. The time t value $V(t)$ of the option is given by

$$V(t) = \int_{\mathbb{R}^k} (P_f(t, T_1)S(t)\varphi(x + (T-t)^{1/2}b_1)$$
$$- CP(t, T_1)\varphi(x + (T-t)^{1/2}b_2) - KP(t, T)\varphi(x))^+ \,dx,$$

where $\varphi(x) = (2\pi)^{-k/2}\exp(-\frac{1}{2}|x|^2)$, $k \le 2$, and the matrix $(b_i^* \; b_j)$ represents the matrix of returns on prices of the corresponding forward contracts i.e. $P(\cdot, T_1)S(\cdot)/P(\cdot, T)$ and $P(\cdot, T_1)/P(\cdot, T)$.

2.3.3. Options on foreign stocks

It is a simple exercise to show that the price of a European call option with maturity T and exercise price K on the basket $\{C_j; j = 1, \ldots, m\}$ of foreign stocks $R_f(\cdot, i_j)$ can be obtained from Eq. (1.18) by substituting $R(\cdot, i_j)$ with $R_f(\cdot, i_j)S(\cdot)$.

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