SOME LIMIT THEOREMS FOR WALSH-HARMONIZABLE
DYADIC STATIONARY SEQUENCES

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This paper deals with a Walsh-harmonizable dyadic stationary sequence \{X(k): k = 0, 1, 2, \ldots\} which is represented as \(X(k) = \int_0^1 \psi_k(\lambda) d\zeta(\lambda)\), where \(\psi_k(\lambda)\) is the \(k\)-th Walsh function and \(\zeta(\lambda)\) is a second-order process with orthogonal increments. One of the aims is to express the process \{\zeta(\lambda): \lambda \in [0, 1]\} in terms of the Walsh-Stieltjes series \(\sum X(k)\psi_k(\lambda)\) of the original sequence \(X(k)\). In order to do this a Littlewood’s Tauberian theorem for a series of random variables is introduced. A finite Walsh series expression of \(X(k)\) is derived by introducing an approximate Walsh series of \(X(k)\). Also derived is a strong law of large numbers for the dyadic stationary sequences.

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Dyadic stationary processes * Walsh–Stieltjes series * inversion formula * approximate Walsh series * strong law of large numbers

1. Introduction

Let \(I_+\) be the set of nonnegative integers, and \(\oplus\) be the dyadic addition on \(I_+\) (cf. [3]). Let \(\{X(k): k \in I_+\}\) be a sequence of random variables with zero mean and finite variance. If
\[
EX(j \oplus h)X(k \oplus h) - EX(j)X(k) \quad (h, j, k \in I_+)
\]
is satisfied then \(\{X(k): k \in I_+\}\) (or simply \(X(k)\)) is said to be a dyadic stationary (DS) sequence [8, 11–13]. Putting \(r(j, k) = EX(j)X(k)\), we know that \(r(j, k) = r(j \oplus k, 0)\), i.e., the covariance function of a DS sequence depends only on the dyadic shift \(j \oplus k\), so we may write \(r(j \oplus k) = r(j, k)\). It is easy to see that
\[
r(k) = r(k), \quad |r(k)| \leq r(0)
\]
and \(r(k)\) is a \(W\)-positive definite function, i.e., for any set of complex numbers \(\{c_1, \ldots, c_m\}\)
\[
\sum_{j=1}^m \sum_{k=1}^m c_j \bar{c}_k r(j \oplus k) \geq 0.
\]
If a $W$-positive definite function $r(k)$ is representable as
\[ r(k) = \int_0^1 \psi_k(\lambda) \, dF(\lambda), \tag{1.4} \]
where $F(\lambda)$ is a finite nondecreasing function on $[0, 1)$ and $\psi_k(\lambda)$ is the (Paley ordered) Walsh function (e.g. \cite{3, 14, 17}), then we call it $W$-harmonizable. We remark that the $k$-th Walsh function is expressed \cite{4} by
\[ \psi_k(\lambda) = \exp \left\{ \pi i \sum_{n=-N}^{\infty} \lambda_n k_{1-n} \right\}, \]
where $\lambda_n$ and $k_n$ are the coefficients of dyadic expansions of $\lambda \in [0, 1)$ and $k \in I_+$ respectively;
\[ \lambda = \sum_{n=-\infty}^{\infty} \lambda_n 2^{-n} \quad \text{and} \quad k = \sum_{n=-M}^{0} k_n 2^{-n}. \]
It follows that
\[ |\psi_k(\lambda)| = 1 \tag{1.5} \]
and
\[ \psi_{j \otimes k}(\lambda) = \psi_j(\lambda) \psi_k(\lambda). \tag{1.6} \]
The set of the Walsh functions $\{ \psi_k(\lambda) \colon k = 0, 1, 2, \ldots \}$ constitutes an orthonormal system on the unit interval $[0, 1)$. It is known \cite{2} that a $W$-positive definite function $r(n)$ is $W$-harmonizable if and only if
\[ \lim_{n \to \infty} 2^{-r} \sum_{n=0}^{2^r-1} r(n) \psi_n(\lambda -) = 0 \tag{1.7} \]
for every dyadic rational $\lambda \in (0, 1)$. Hence a covariance function of a DS sequence is $W$-harmonizable if and only if (1.7) is satisfied. If the covariance function of a DS sequence is $W$-harmonizable, then the DS sequence is representable as
\[ X(k) = \int_0^1 \psi_k(\lambda) \, d\xi(\lambda), \tag{1.8} \]
where $\{\xi(\lambda) \colon \lambda \in [0, 1)\}$ is a process with orthogonal increments such that
\[ E|d\xi(\lambda)|^2 = dF(\lambda). \tag{1.9} \]
If a DS sequence assumes such a representation, then it is said to be $W$-harmonizable. It is straightforward that if a DS sequence is $W$-harmonizable then its covariance function is $W$-harmonizable. The condition (1.7), therefore, is necessary and sufficient for $W$-harmonizability of a DS sequence. It must be remarked that a DS sequence is not necessarily $W$-harmonizable \cite{2}, though some authors \cite{8, 11, 15} state that every DS sequence is $W$-harmonizable.
In the theory of stationary sequences the exponentials play the key role. Since the \( e^{i\lambda} \) are positive definite, their linear combinations \( \sum_k f_k \exp \{i\lambda_k\} \) with positive coefficients are positive definite. The Stieltjes integral

\[
\int_{-\pi}^{\pi} e^{i\lambda} \, dF(\lambda)
\]

with finite nondecreasing function \( F(\lambda) \) is such a limit, and is therefore positive definite. Bochner's theorem (e.g. [1]) states the converse part as well: A positive definite function assumes a representation as in (1.10). As the covariance function of a stationary sequence is positive definite, it therefore assumes the representation

\[
r(n) = \int_{-\pi}^{\pi} e^{i\lambda} \, dF(\lambda),
\]

where \( F(\lambda) \) is called the spectral distribution function. The stationary sequence is also expressed as

\[
X(n) = \int_{-\pi}^{\pi} e^{i\lambda} \, d\zeta(\lambda),
\]

where \( \zeta(\lambda) \) is a second-order process with orthogonal increments such that

\[
dF(\lambda) = E|d\zeta(\lambda)|^2.
\]

The shift-invariance of the sequence is expressed by the basic property of the exponentials,

\[
e^{i(m+n)\lambda} = e^{i\lambda} e^{i\lambda},
\]

in the following way:

\[
EX(m+k)X(n+k) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(m+k)\lambda} e^{-i(n+k)\mu} \, d\zeta(\lambda) \, d\zeta(\mu)
\]

\[
= \int_{-\pi}^{\pi} e^{i(m-n)\lambda} \, dF(\lambda) = r(m-n) = EX(m)X(n),
\]

where we assumed that \( EX(k) = 0 \).

The Walsh functions in the theory of DS sequences are the counterparts to the exponentials in the theory of stationary sequences. The dyadic invariance (1.1) of DS sequences is expressed reasonably by the property (1.6) of the Walsh functions. The representation (1.4) stems, as we have seen, from the W-positiveness of the covariance functions of DS sequences. The theory of the Walsh functions therefore is the basic tool for analysis of DS sequences.

Kohn, however, successfully applies the Walsh functions to stationary sequences [6].

At the end of this section we shall give an example of a DS sequence with a given spectral distribution function.
Example. Let an arbitrary finite nondecreasing function $F(\lambda)$ on $[0, 1)$ be given. Let $Y$ be a random variable with the distribution function $\Pr\{ Y < y \} = F(y)/F(1)$ $(0 \leq y < 1)$, where we have supposed that $F(0) = 0$. Let $Z$ be a random variable independent of $Y$, and satisfying $EZ = 0$ and $E|Z|^2 = F(1)$. Define $X(k) = Z\psi_k(Y)$; then $EX(k) = 0$ and $r(k) = \int_0^1 \psi_k(\lambda) \, dF(\lambda)$ $(k \in I_+)$.

2. Walsh–Stieltjes series

The representation (1.8) can be considered as the Walsh–Stieltjes coefficients of the process $\{\xi(\lambda): \lambda \in [0, 1] \}$. The series

$$d\xi(\lambda) \sim \sum_{k=0}^{\infty} X(k)\psi_k(\lambda)$$

(2.1)

can be, likewise, called the Walsh–Stieltjes series of the process $\xi(\lambda)$.

Defining a partial sum of W–S series as

$$s_n(\lambda) = \sum_{k=0}^{n-1} X(k)\psi_k(\lambda),$$

(2.2)

we have that

$$ES_m(\lambda)s_n(\lambda) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} r(j \oplus k)\psi_j(\lambda)\psi_k(\mu) = \int_0^1 D_m(\lambda \oplus u)D_n(\mu \oplus u) \, dF(u),$$

where $D_k(x) = \sum_{k=0}^{m-1} \psi_k(x)$ is the Dirichlet kernel of the Walsh system.

Since $D_k(x)/k$ is bounded by 1, and converges to 1 at $x = 0$, and to 0 elsewhere, as $k$ goes to infinity [3], we obtain the following result.

Lemma 1

$$\lim_{m,n \to \infty} (1/mn) ES_m(\lambda)s_n(\lambda) = \delta_{\lambda,\mu}F(\{\lambda\}),$$

(2.3)

where $\delta_{\lambda,\mu} = 1$, $\lambda = \mu$; $= 0$, $\lambda \neq \mu$.

The set function $F$ in Lemma 1 is understood to be the measure generated by the point function $F$. In the sequel we sometimes use the same notation as both a set function and a point function.

Theorem 1

$$\lim_{n \to \infty} (1/n)s_n(\lambda) = \xi(\{\lambda\}).$$

(2.4)

Proof. Since

$$ES_n(\lambda)\xi(\{\lambda\}) = \sum_{k=0}^{n-1} \int_0^1 \psi_k(\lambda \oplus u)E d\xi(u)\xi(\{\lambda\}) = \sum_{k=0}^{n-1} \psi_k(0)F(\{\lambda\}) = nF(\{\lambda\}),$$
it is clear that
\[ E[(1/n)s_n(\lambda) - \zeta(\{\lambda\})]^2 = (1/n)^2 E|s_n(\lambda)|^2 - F(\{\lambda\}). \]

Hence the proof is completed using Lemma 1.

We can isolate, by Theorem 1, the discrete portion of \( \zeta \) by means of the partial sum of the W-S series. Our next step is to recover the process \( \zeta \). Before doing so, we shall extend the so-called Tauberian theorem, which is the converse of Abel’s continuity theorem [18], to the series of random variables.

**Lemma 2** (Littlewood’s Tauberian theorem for random series). A random series \( \sum_{n=0}^{\infty} X_n \) converges in the mean if it is Abel summable in the mean and \( E|X_n|^2 = O(n^{-2}) \).

**Proof.** We may suppose that \( E|X_n|^2 \leq n^{-2} \) for \( n > 0 \) and
\[
\lim_{r \to 1^-} E \left| \sum_{n=0}^{\infty} X_n r^n \right|^2 = 0.
\]

If we substitute here \( r^n \) for \( r \) (\( n = 1, 2, 3, \ldots \)), we see that for every polynomial \( P(r) \) without constant term we have
\[
\lim_{r \to 1^-} E \left| \sum_{n=0}^{\infty} X_n P(r^n) \right|^2 = 0. \tag{2.5}
\]

It will be shown that given any two numbers \( 0 < \xi' < \xi < 1 \) and a positive \( \delta \) we can find a \( P(r) \) such that
(i) \( 0 \leq P(r) \leq 1, \ r \in (0, 1) \),
(ii) \( P(r) \leq \delta r, \ r \in (0, \xi) \),
(iii) \( 1 - P(r) \leq \delta (1 - r), \ r \in (\xi, 1) \),
(cf. [18]). Given any \( 0 < r < 1 \), let \( N = N(r) \) be the greatest non-negative integer satisfying \( r^N \geq \xi \), and let \( N' = N'(r) \) be the least positive integer satisfying \( r^{N'} \leq \xi' \). Both \( N \) and \( N' \) are non-decreasing function of \( r \) taking successively all values \( 1, 2, 3, \ldots \) as \( r \to 1 \). Clearly \( N < N' \) and
\[
N = \log(1/\xi)/\log(1/r), \quad N' = \log(1/\xi')/\log(1/r). \tag{2.6}
\]

For \( P(r) \) satisfying (i)–(iii) we have
\[
E \left| \sum_{n=0}^{\infty} X_n P(r^n) - S_N \right|^2 \leq 3 \left( E \left| \sum_{n=0}^{N} X_n (P(r^n) - 1) \right|^2 + E \left| \sum_{N+1}^{N'} X_n P(r^n) \right|^2 + E \left| \sum_{N'+1}^{\infty} X_n P(r^n) \right|^2 \right) = 3(A(r) + B(r) + C(r)), \tag{2.7}
\]
say, where $S_N = \sum_{n=0}^{N} X_n$. It is obvious that

$$|A(r)| \leq \delta^2 N \sum_{i=1}^{N} (1-r^n)^2/2 \leq [\delta N(1-r)]^2,$$

$$|B(r)| \leq (N'-N)\delta^2 \sum_{N+1}^{N'} 1/n^2 \leq [\delta(N'-N)/N]^2,$$

and

$$|C(r)| \leq \left[ \sum_{N+1}^{\infty} (E|X_n|^2)^{1/2}|P(r^n)| \right] \leq [\delta/N'(1-r)]^2.$$

Take any $\varepsilon > 0$. From (2.6) we see that if $\xi'$ and $\xi$ are sufficiently close to each other and both away from 0 and 1, we have $\limsup |B(r)| \leq \varepsilon$. Having fixed $\xi'$ and $\xi$, and observing that $N(1-r)$ and $N'(1-r)$ tend to finite nonzero limits, we obtain $\limsup |B(r)| < \varepsilon$, $\limsup |C(r)| < \varepsilon$, if $\delta$ is small enough. It follows from (2.5) that $E|S_N|^2 < 3\varepsilon$ and hence $\lim E|S_N|^2 = 0$.

Now we are at the point to prove the following result.

**Theorem 2.** If $x$ is a dyadic rational or a point of continuity of $F$, then

$$\zeta(x) - \zeta(0) = \lim_{n \to \infty} \int_{0}^{x} s_n(\lambda) \, d\lambda. \quad (2.8)$$

**Proof.** Let $1_{x}(\lambda)$ be the characteristic function of $[0, x)$, and let $1_{x}(\lambda; r)$ be its Abel sum;

$$1_{x}(\lambda; r) = \sum_{k=0}^{\infty} J_k(\lambda) \psi_k(\lambda) r^k \quad (0 \leq r < 1) \quad (2.9)$$

where $J_k(x) = \int_{0}^{x} \psi_k(\lambda) \, d\lambda$.

Under either hypothesis on $x$, $\lim_{r \to 1-} 1_{x}(\lambda; r) = 1_{x}(\lambda)$ a.e. $(F)$ and $|1_{x}(\lambda; r)| \leq 1$ (cf. [3]). Hence

$$E \left| \int_{0}^{1} 1_{x}(\lambda; r) \, d\zeta(\lambda) - \int_{0}^{1} 1_{x}(\lambda) \, d\zeta(\lambda) \right|^2 = \int_{0}^{1} |1_{x}(\lambda; r) - 1_{x}(\lambda)|^2 \, dF(\lambda) \to 0$$

as $r \to 1-$, i.e.,

$$\lim_{r \to 1-} \int_{0}^{1} 1_{x}(\lambda; r) \, d\zeta(\lambda) = \int_{0}^{1} 1_{x}(\lambda) \, d\zeta(\lambda).$$

For fixed $r < 1$, the series (2.9) converges uniformly in $\lambda$, so

$$E \left| \int_{0}^{1} 1_{x}(\lambda; r) \, d\zeta(\lambda) - \sum_{k=0}^{n-1} \frac{X(k) J_k(x) r^k}{n} \right|^2$$

$$= \int_{0}^{1} \left| 1_{x}(\lambda; r) - \sum_{k=0}^{n-1} J_k(x) \psi_k(\lambda) \right|^2 \, dF(\lambda) \to 0$$
as } n \to \infty \). Thus the series } \sum X(k)J_k(x) \text{ is Abel summable to } \xi(x) - \xi(0) - \int_0^\infty d\xi(\lambda) \text{ in the mean. Since } E|X(k)|^2 = r(0) \text{ and } |J_k(x)| \leq 1/k ((4)), E|X(k)J_k(x)|^2 = O(1/k^2). \text{ By Lemma 2 the series converges in the mean at } r = 1;

\xi(x) - \xi(0) = \lim_{n \to \infty} \sum_{k=0}^{n-1} X(k)J_k(x),

which is equivalent to (2.8).

3. Approximate Walsh series

Kawata [5] introduced an approximate Fourier series of stationary processes and studied some sample properties of the processes using the approximate series. Here we analogously consider an approximate Walsh series of DS sequences.

For } n \in I_+ \text{ put

\begin{equation}
\xi_n^k = \int_{2^{-n}k}^{2^{-n}(k+1)} d\xi(\lambda), \quad k = 0, 1, \ldots, 2^n - 1,
\end{equation}

and define an approximate Walsh series of } X(k) \text{ as

\begin{equation}
\hat{X}_n(k) = \sum_{j=0}^{2^n-1} \psi_k(2^{-n}j)\xi_n^j.
\end{equation}

Then it is easy to see that } E\hat{X}_n(k) = 0 (k \in I_+) \text{ and the covariance function of } \hat{X}_n(k) \text{ is given by

\begin{equation}
\hat{r}_n(j, k) = \sum_{h=0}^{2^n-1} \sum_{m=0}^{2^n-1} \psi_j(2^{-n}h)\psi_k(2^{-n}m) E\xi_n^{h\cap m} = \sum_{m=0}^{2^n-1} \psi_j(2^{-n}m) \int_{2^{-n}m}^{2^{-n}(m+1)} dF(\lambda) = \hat{r}(j \oplus k)
\end{equation}

which depends only on } j \oplus k \text{, and has a W-harmonizable representation with the spectral distribution function

\begin{equation}
\hat{F}_n(\lambda) = \int_0^{2^{-n}k} dF(\mu) \quad (2^{-n}(k-1) \leq \lambda < 2^{-n}k).
\end{equation}

Hence } \hat{X}_n(k) \text{ is also a W-harmonizable DS sequence. Since } \hat{r}_n(0) = r(0),

\begin{equation}
E|X(k) - \hat{X}_n(k)|^2 = 2r(0) - 2EX(k)\hat{X}_n(k).
\end{equation}

Hence, rewriting

\begin{equation}
E X(k)\hat{X}_n(k) = \sum_{h=0}^{2^n-1} \int_{2^{-n}h}^{2^{-n}(h+1)} \psi_k(2^{-n}h)\psi_k(\lambda) dF(\lambda),
\end{equation}
we have
\[
E|X(k) - \hat{X}_n(k)|^2 = 2 \sum_{h=0}^{2^n-1} \int_{2^{-n}h}^{2^{-n}(h+1)} (1 - \psi_k(2^{-n}h \ominus \lambda)) \, dF(\lambda) \\
= 2 \sum_{h=0}^{2^n-1} \int_0^{2^{-n}} (1 - \psi_k(\lambda)) \, dF(\lambda \oplus 2^{-n}h),
\]
which is equal to zero when \( k < 2^n \), since, for such \( k \), \( \psi_k(\lambda) = 1 \) when \( \lambda < 2^n \). This proves the following result.

**Theorem 3.** Let \( X(k) \) be a \( W \)-harmonizable DS sequence, and \( \hat{X}_n(k) \) be its approximate Walsh series defined by (3.2). Then for \( k < 2^n \)
\[
\hat{X}_n(k) = X(k) \quad \text{(a.s.)} \tag{3.3}
\]

The equation (3.3) states that a \( W \)-harmonizable DS sequence is expressed by a finite Walsh series:
\[
X(k) = \sum_{h=0}^{2^n-1} \psi_k(2^{-n}h) \xi_n^h \quad (k < 2^n),
\]
since \( \psi_k(2^{-n}h) = \psi_k(2^{-n}k) \) \((h, k, n \in I_+)\) (e.g. [4]).

4. **Laws of large numbers**

We know by Theorem 1 that
\[
\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} X(k) = \xi(\{0\}), \tag{4.1}
\]
because \( \psi_k(0) = 1 \) for \( k \in I_+ \) (cf. [3]). This means that a weak law of large numbers holds for a \( W \)-harmonizable DS sequence. It follows from (4.1) that a \( W \)-harmonizable DS sequence is ergodic if and only if its spectral distribution function satisfies \( F(\{0\}) = 0 \).

Next we shall consider a strong law of large numbers. Suppose in the following that a \( W \)-harmonizable DS sequence is real-valued.

**Theorem 4.** (i) If
\[
\sum_{k=0}^{\infty} k^2 \bar{r}(2^k) < \infty \tag{4.2}
\]
then
\[
\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} X(k) = 0 \quad \text{(a.s.)} \tag{4.3}
\]
where \( \bar{r}(n) = (1/n) \sum_{k=0}^{n-1} r(k) \).
(ii) If

\[ \sum_{k=0}^{\infty} \hat{r}(2^k) < \infty, \]  

then

\[ \lim_{n \to \infty} 2^{-n} \sum_{k=0}^{2^n-1} X(k) = 0 \quad (a.s.). \]  

**Proof.** We shall only prove (i), whose proof is similar to that of ordinary stationary case [16]. In order to prove (4.3) we shall show that

\[ \sum_{n=1}^{\infty} \Pr\{|(1/n)S(n)| \geq \varepsilon\} < \infty, \]  

where \( S(n) = \sum_{k=0}^{n-1} X(k) \). Dividing the sum into some parts accordingly to the dyadic expansion of \( n \) such that

\[ S(n) = S(2^{k_1}) + (S(2^{k_1} + 2^{k_2}) - S(2^{k_1})) + \cdots + (S(2^{k_1} + \cdots + 2^{k_s}) - S(2^{k_1} + \cdots + 2^{k_{s-1}})), \]

where \( n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s} \) and \( k_1 > k_2 > \cdots > k_s \geq 0 \), we have that

\[ \sum_{n=1}^{\infty} \Pr\{|(1/n)S(n)| \geq \varepsilon\} \leq \sum_{k=1}^{\infty} \Pr\{|2^{-k}S(2^k)| \geq \varepsilon/(k+1)\} + \sum_{k=1}^{\infty} \sum_{p=0}^{2^k-1} \sum_{q=0}^{2^p-1} \Pr\{|2^{-k}|S(2^k + (q+1)2^p) - S(2^k + q2^p)| \geq \varepsilon/(k+1)\}. \]  

It follows from Tchebychev's inequality that (4.7) converges if the following two series converge:

\[ \sum_{k=1}^{\infty} 2^{-2k}(k+1)^2 E[S(2^k)]^2, \]  

\[ \sum_{k=1}^{\infty} \sum_{p=0}^{2^k-1} \sum_{q=0}^{2^p-1} 2^{2k}(k+1)^2 E[S(2^k + (q+1)2^p) - S(2^k + q2^p)]^2. \]  

Hence, by the easily verified equality

\[ E[S(2^k)]^2 = 2^{2k}\hat{r}(2^k), \]  

if (4.2) holds then (4.8) converges. Similarly (4.9) converges if the following series converges:

\[ \sum_{k=1}^{\infty} \sum_{p=0}^{2^k-1} \sum_{q=0}^{2^p-1} 2^{-2k}(k+1)^2 \sum_{i=0}^{2^p-1} \sum_{j=0}^{2^q-1} r(2^k + q2^p + i, 2^k + q2^p + j). \]
Recalling that $2^k + q2^p + i = 2^k + q2^p + i$ ($k > p$, $i < 2^p$), we have

$$\sum_{i=0}^{2^p-1} \sum_{j=0}^{2^p-1} r(2^k + q2^p + i, 2^k + q2^p + j) = 2^{2^p} \bar{r}(2^p).$$

Hence, (4.11) reduces to

$$\sum_{k=1}^{2^p-1} \sum_{p=0}^{k-1} 2^{-(k-p)}(k+1)^2 \bar{r}(2^p) = \sum_{p=0}^{\infty} 2^p \bar{r}(2^p) \sum_{k=p+1}^{\infty} 2^{-k}(k+1)^2$$

$$= \sum_{p=0}^{\infty} \{(p+1)^2 + (p+1)/4 + 1/2\} \bar{r}(2^p),$$

which is majorized by (4.2) with some multiplicative constant. This completes the proof.

At the end of this section we remark that the conditions (4.2) and (4.4) can be stated in terms of the spectral distribution function using the following lemma.

**Lemma 3.** The following statements are equivalent: for $\nu \in I_+$

(i) $\sum_{k=0}^{\infty} k^\nu \bar{r}(2^k) < \infty$,

(ii) $\int_0^1 (-\log \lambda)^\nu dF(\lambda) < \infty$.

**Proof.** Clearly

$$\bar{r}(2^k) = 2^{-k} \sum_{j=0}^{2^k-1} \int_0^1 \psi_j(\lambda) dF(\lambda) = 2^{-k} \int_0^1 D_{2^k}(\lambda) dF(\lambda) = F(2^{-k}) - F(0), \quad (4.12)$$

where we used the evaluation, $D_{2^k}(\lambda) = 2^k$, $0 < \lambda < 2^{-k}$; $=0$, elsewhere [3]. Integrating by parts, we have

$$\int_0^1 (-\log \lambda)^{\nu+1} dF(\lambda) = (\nu+1) \sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} (-\log \lambda)^\nu (F(\lambda) - F(0))/\lambda d\lambda$$

$$\geq C \sum_{k=0}^{\infty} k^\nu (F(2^{-1(k+1)}) - F(0)),$$

where $C$ is a positive constant. Similarly we may have the same inequality above, except with the reverse inequality sign. Hence the proof is completed by (4.12).

The conditions (4.2) and (4.4), therefore, are replaced by the condition (ii) in Lemma 3 for $\nu = 2$ and $\nu = 0$ respectively.

**References**