On multiple Fourier integrals of piecewise smooth functions with discontinuities of the second kind

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A B S T R A C T

It is well known that the Riesz means of eigenfunction expansions of piecewise smooth functions of order \( s > (n - 3)/2 \) converge uniformly on compacts where these functions are smooth. In 2000 L. Brandolini and L. Colzani considered eigenfunction expansions of piecewise smooth functions with discontinuities of the second kind across smooth surfaces. They showed that the Riesz means of these functions of order \( s > (n - 3)/2 \) may diverge even at certain points where these functions are smooth. Here it is argued that this effect depends on the measure of the singularity area, i.e. we consider functions with singularities across more limited areas and prove that the Riesz means of their eigenfunction expansions of order \( s > (n - 3)/2 \) converge uniformly on compacts where these functions are continuous.

1. Introduction

We consider a homogeneous elliptic differential operator of order \( m \) with constant coefficients

\[ A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha. \]

**Definition 1.** We say that a function \( f(x) \) is an elementary piecewise smooth function if this function is of the form

\[ f(x) = \chi_\Omega(x)g(x), \quad g(x) \in C^\infty(R^N), \]

where \( \chi_\Omega(x) \) is a characteristic function of an open set \( \Omega \) with smooth boundary \( \partial \Omega \).

According to the spectral theorem, every function \( f(x) \in L^2(R^N) \) can be associated with its expansion defined by means of a partition of unity of \( A(D); \) in our case the expansion has the form of the Fourier inversion

\[ E_\lambda f(x) = (2\pi)^{-N/2} \int_{A(\xi) < \lambda} \hat{f}(\xi)e^{i\xi x} d\xi, \]

where

\[ \hat{f}(\xi) = (2\pi)^{-N/2} \int_{R^N} f(x)e^{-i\xi x} dx, \]

and \( A(\xi) \) is the symbol of \( A(D) \) operator which has the form

\[ A(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha. \]
We also define the Riesz means \( E^s f(x) \) with real \( s \) in the following way:
\[
E^s f(x) = (2\pi)^{-N} \int_{\|\xi\| < \lambda} \left( 1 - \frac{A(\xi)}{\lambda} \right) \hat{f}(\xi) e^{i\xi \cdot x} d\xi.
\]
Eigenfunction expansions of piecewise smooth functions associated with an elliptic operator of second order have been intensively investigated by many authors for the last ten years. Using various methods it can be shown that in the case of a Laplace operator the Riesz means of order \( s > (n - 3)/2 \) for piecewise smooth functions uniformly converge on every compact set separated from the surface of discontinuity (for details see e.g. [1]).

In most papers, piecewise smooth functions are defined either as elementary piecewise smooth functions or as a linear combination of such functions with corresponding normalization on the surface of discontinuity. However, in the work [2], Brandolini and Colzani, studying the question of the convergence for the Riesz means associated with an elliptic operator of the second order, investigated functions from a broader space \( X^n \), whose definition allows more complicated singularities rather than simple jumps (as in Definition 1). Namely, for functions considered, passing through the surface of discontinuity, may have integrable singularities of order \( \alpha > -1 \). The authors revealed that the convergence of the Riesz means \( E^s f \) for any function \( f \) from \( X^n \) can take place if and only if \( s > (n - 3)/2 - \alpha \). The need to increase order \( s \) can be observed, for example, considering the function \( \phi(x) = (1 - |x|^2)^\alpha \), where \( -1 < \alpha < 0 \). \( E^s \phi(0) \) diverges, if \( s \leq (n - 3)/2 - \alpha \). It should be noted that the discontinuity of \( \phi(x) \) takes place all along \( \partial \Omega \) (in the present case \( \partial \Omega = \{ x : |x| = 1 \} \). Therefore the following question naturally arises: can the order \( s > (n - 3)/2 \) be a sufficient condition for the convergence of the Riesz means for functions with singularities of the second kind on \( \partial \Omega \), if these singularities take place in less broad areas?

In this work we shall consider a class of functions with singularities on the boundary \( \partial \Omega \) of \( \Omega \in \mathbb{R}^n \), \( n \geq 3 \), but the condition \( s > (n - 3)/2 \) for these functions remains sufficient for the convergence of their Riesz means of order \( s \).

2. The main results

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \) satisfying the Lipschitz condition. We recall that according to the trace theorem (see e.g. Chapter 5.5 in [3]) there is a bounded linear operator \( T \)
\[
T : W^1_p(\Omega) \to L_p(\partial \Omega)
\]
such that \( Tf = f|_{\partial \Omega} \) for \( f \in W^1_p(\Omega) \cap C(\overline{\Omega}) \). Moreover, according to Theorem 1 in [4] the following is true:
\[
\int_{\partial \Omega} D\phi dx = \int_{\partial \Omega} f v d\sigma(x),
\]
where \( d\sigma \) is the surface Lebesgue measure, \( v(x) \) is an exterior normal of \( \partial \Omega \) at \( x \) and \( f \) is extended on \( \partial \Omega \) as \( Tf \).

**Definition 2.** We shall say that the function \( f : \mathbb{R}^n \to \mathbb{R} \) belongs to \( WP_k(\Omega) \), where \( \Omega \) is a domain in \( \mathbb{R}^n \) with a boundary \( \partial \Omega \) and \( k \) is a non-negative integer, if \( f(x) \) satisfies the following conditions:
1. \( f(x) \in C^{k+2}(\Omega) \) and \( f(x) = 0 \) if \( x \in \mathbb{R}^n \setminus \overline{\Omega} \);
2. \( f(x) \in W^{k+1}_1(\Omega) \).

We shall also agree that each function \( f \in WP_k \) on \( \partial \Omega \) coincides with its trace.

It is obvious that the class of piecewise smooth functions is contained in \( WP_k \), but as a proper subclass; the latter includes also functions with singularities at certain points on the boundary. From here on we shall only consider the class \( WP_n(k) \), where \( n_0 = \lfloor (n - 3)/2 \rfloor \).

Let \( A(\Omega) \) be defined as in (1) with symbol \( A(\xi) \) and \( \partial \Lambda = \{ \xi : A(\xi) = 1 \} \). Then the following is true.

**Theorem 1.** Let \( \partial \Omega \) be a strictly convex set. If \( s > (n - 3)/2 \), then \( \forall f(x) \in WP_n(\Omega) \),
\[
\lim_{\lambda \to \infty} E^s f(x) = f(x)
\]
uniformly on any compact \( K \subset \mathbb{R}^n \setminus \partial \Omega \).

If \( s = (n - 3)/2 \), then the Riesz means \( E^n f(x) \) are uniformly bounded on any compact \( K \subset \mathbb{R}^n \setminus \partial \Omega \).

3. Auxiliary lemma

We consider the integral
\[
\int_{\Omega} f(x) e^{P(x)} dx,
\]
where \( P(x) \), generally speaking, a complex valued function. If \( \nabla P(x) \neq 0 \), \( \forall x \in \Omega \), then, according to (3), this integral can be represented as
\[
\int_{\Omega} f(x) e^{P(x)} dx = \int_{\partial \Omega} f(x) \frac{\partial P}{|\nabla P|^2} (x) e^{P(x)} dx - \int_{\Omega} L^*(f(x)) e^{P(x)} dx,
\]
where the operator \( L(D) = \frac{1}{|\nabla P(x)|^2} \sum \frac{\partial P}{\partial x^i} \frac{\partial}{\partial x^i} \) and \( L^*(D) \) is its formal conjugate.
Now we consider $E_{\lambda} f$ itself as an integral operator
\begin{equation}
E_{\lambda} f(x) = (2\pi)^{-N/2} \int_{|\xi|<\lambda} \hat{f}(\xi)e^{i\xi x} d\xi = (2\pi)^{-N} \int_{R^n} f(y)D(x-y)dy,
\end{equation}
where kernel $D_{\lambda}(x) = \int_{|\xi|<\lambda} e^{i\xi x} d\xi$ has well-known asymptotics for $x \neq 0$ as (see for example [5])
\begin{equation}
D_{\lambda}^{ij}(x) = C_{\lambda}(n-1)/2 \left[ \frac{|K_{\lambda}(x)|}{|x|^{n+1}/2} \right] \left[ e^{i(\lambda p(x) - \frac{\pi(n-1)}{4})} (1 + \Phi_+(x;\lambda)) - e^{-i(\lambda p(x) - \frac{\pi(n-1)}{4})} (1 + \Phi_-(x;\lambda)) \right],
\end{equation}
where $C = -i/(2\pi)^{(n+1)/2}$ and $K(\xi)$ is the curvature of the surface $\partial\Lambda$ at point $\xi$ and $\Phi_{\pm}(x;\lambda) = \sum_{j=1}^{\infty} c_j^{\pm}(x)\lambda^{-j}$, with $c_j \in C^\infty$. Here $p(x) = \theta(x); \theta(x)$ is a point on $\partial\Lambda$, where an outer normal to the surface and $x$ are codirectional vectors.

**Lemma 1.** Let $f(x) \in W_{p_0}(\Omega)$; then
\[ E_{\lambda} f(x) = O\left(\frac{\|f\|_{p_0}}{\lambda^{n+2}}\right) \]
uniformly on $K \subset R^n \setminus \partial\Omega$.

**Proof.** Let $K$ be given. We define $g(x) \in C^\infty(R^n)$ such that $g(x) \equiv 1$ when $x \in K$ and $\text{supp} \ g(x) \subset \Omega$. Then
\[ E_{\lambda} f(x) = f(x) + o(1) + (2\pi)^{-N} \int_{\Omega} (1 - g(y))f(y)D_{\lambda}(x-y)dy. \]
In order to complete the proof it is sufficient to show that $J \equiv \int_{\Omega \setminus K} h(y)D_{\lambda}(x-y)dy = O(\lambda^{n+2})$, for $h(y) \in W_{p_0}(\Omega)$. To do this we employ the asymptotics of $D_{\lambda}(x)$:
\[ J = C_1\lambda^{(n-1)/2} \int_{\Omega \setminus K} h_1(x-y)e^{i\lambda_p(x-y)}dy + C_2\lambda^{(n-1)/2} \int_{\Omega \setminus K} h_2(x-y)e^{-i\lambda_p(x-y)}dy. \]
It is not hard to show that $\nabla p(x) \neq 0$, when $x \neq 0$. Therefore using (4) one can estimate both integrals as $O(\lambda^{-1})$. \[ \Box \]

**4. Proof of Theorem 1**

We base our proof on the following Tauberian theorem of Hörmander (see [6], Theorem 2.7).

**Theorem 2** (L. Hörmander). Let $\phi$ be a locally bounded variation and have a support on $R_+$ and let $\phi(t) = O(t^k)$ for some $k$. Assume that
\[ \phi((t+h)^m) - \phi(t^m) \leq (1 + t)^a, \]
where $a \geq 0$ and $m$ is integer $> 1$, and assume that the Fourier–Laplace transform
\[ \Phi(\xi) = \int e^{it\xi}d\phi(t^m), \quad \text{Im} \xi < 0, \]
has an analytic continuation to a neighborhood $\{\xi : |\xi| < \delta\}$ of 0, such that $|\Phi(\xi)| \leq 1, |\xi| < \delta; \Phi^{m_l}(0) = 0$, for all integers $l \geq 0$.

If
\[ \phi^i(t) = \int_0^t (1 - u/t)^i d\phi(u), \]
then for $s \geq 0$
\[ |\phi^i(t)| \leq C(s)t^{(a-i)/m}, \quad t \geq 1. \]

Now we prove Theorem 1.

**Proof.** Let $K \subset R^n \setminus \partial\Omega$ and $U$ be such an open set that $K \subset U$, $\overline{U} \subset R^n \setminus \partial\Omega$. We consider a smooth function $f_0(x)$ such that $f_0(x) = f(x), x \in U$ and $E_{\lambda} f_0(x)$ converges uniformly to $f_0(x)$. Then it is obvious that uniformly on $x \in K$,
\[ E_{\lambda} f_0(x) = f(x) + o(1). \]
Now we introduce the function
\[ \phi(\lambda) = E_{\lambda} f(x) - E_{\lambda} f_0(x). \]
It follows from Lemma 1 that
\[\phi \left((t + h)^m \right) - \phi \left(t^m \right) = O \left(t^{(n-3)/2} \right) + O(1) = O \left(t^{(n-3)/2} \right), \quad t \to \infty,\]
for \(0 < h \leq 1, \ x \in K\). Satisfaction of the other conditions of Theorem 2 can be verified as in [7]. By this theorem, since the Riesz means of \(f_0(x)\) of order \(s > 0\) converge uniformly, for \(s = (n - 3)/2\) one has
\[E_s^x f(x) = \phi^x(\lambda) + E_s^x f_0(x) = O(1), \quad x \in K \subset R^n \setminus \partial \Omega.\]
If \(s > (n - 3)/2\), then for sufficiently large \(\lambda\) we have
\[\phi^x(\lambda) = o(1),\]
uniformly on \(x \in K \subset R^n \setminus \partial \Omega\). Thus according to (7) we finally obtain
\[E_s^x f(x) - f(x) = \phi^x(\lambda) - E_s^x f_0(x) - f(x) = o(1). \quad \Box\]

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