Journal of Computational and Applied Mathematics 223 (2009) 901–907

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Stochastic PDIEs and backward doubly stochastic differential equations driven by Lévy processes

Yong Ren [a,*], Aihong Lin [b], Lanying Hu [a]

[a] Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China
[b] Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China

ARTICLE INFO

Article history:
Received 3 October 2007

Keywords:
Backward doubly stochastic differential equation
Stochastic partial differential integral equation
Lévy process
Teugels martingale

ABSTRACT

In this paper, a new class of backward doubly stochastic differential equations driven by Teugels martingales associated with a Lévy process satisfying some moment condition and an independent Brownian motion is investigated. We obtain the existence and uniqueness of solutions to these equations. A probabilistic interpretation for solutions to a class of stochastic partial differential integral equations is given.

1. Introduction

Backward stochastic differential equations (BSDEs in short) were first introduced in [1] in order to give a probabilistic interpretation (Feynman–Kac formula) for the solutions of semilinear parabolic PDEs, one can see [2,3]. Moreover, BSDEs have been considered with great interests in the past decade because for its connections with mathematical finance [4] as well as stochastic optimal control and stochastic games [5].

On the other hand, backward doubly stochastic differential equations (BDSDEs in short) driven by two Brownian motions were introduced in [6] in order to give a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs in short). Following this, Bally and Matoussi [7] gave a probabilistic representation of the solutions to parabolic semilinear stochastic PDEs in Sobolev spaces by means of BDSDEs. Furthermore, Matoussi and Scheutzow [8] discussed BDSDEs and applications in SPDEs, where the nonlinear noise term was given by the Itô–Kunita stochastic integral. Recently, Zhu and Shi [16] derived the existence and uniqueness of the solutions to BDSDEs driven by Brownian motions and a Poisson process.

The main tool in the theory of BSDEs is the martingale representation theorem, which is well known for martingales which are adapted to the filtration of the Brownian motion, or that of the Poisson point process [1,13] or that of a Poisson random measure [14]. Recently, Nualart and Schoutens [9] gave a martingale representation theorem associated with the Lévy process. Furthermore, they showed the existence and uniqueness of solutions to BSDEs driven by Teugels martingales in [10]. The results were important from a pure mathematical point of view as well as in the world of finance, which could be used for the purpose of option pricing in a Lévy market and the related partial differential equation which provided an analogue of the famous Black–Scholes partial differential equation. Following that, Bahlali et al. [15] generalized the above case to the BSDEs driven by Teugels martingales and an independent Brownian motion and gave the probabilistic representation for the solutions to a class of PDIEs.

* Corresponding author. Tel.: +86 553 5768036; fax: +86 553 5910639.
E-mail address: brightry@hotmail.com (Y. Ren).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2008.03.008

Then it is natural to extend BDSDEs to the case of Lévy processes. The purpose of the present paper is to consider BDSDEs driven by Lévy processes of the kind considered in [15]. This allows us to give a probabilistic interpretation for the solutions to a class of stochastic partial differential integral equations (SPDIEs in short).

The paper is organized as follows. In Section 2, we introduce some preliminaries and notation. Section 3 is devoted to the proof of the existence and uniqueness of the solutions to BDSDEs driven by Lévy processes. In Section 4, we give a probabilistic interpretation of solutions to a class of SPDIEs.

2. Preliminaries and notation

Let $T > 0$ be a fixed terminal time. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a filtered probability space, $\{B_t : t \in [0, T]\}$, $\{W_t : t \in [0, T]\}$ and $\{L_t : t \in [0, T]\}$ be three mutually independent processes. $\{B_t : t \in [0, T]\}$, $\{W_t : t \in [0, T]\}$ are two standard Brownian motions in $\mathbb{R}$ and $\{L_t : t \in [0, T]\}$ is a $\mathbb{R}$-valued Lévy process corresponding to a standard Lévy measure ν such that $\int_{\mathbb{R}}(1 \wedge y)\nu(\mathrm{d}y) < \infty$,

Let $\mathcal{N}$ denote the totality of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}^B_{t,T} \vee \mathcal{F}^W_t \vee \mathcal{F}^L_t,$$

where for any process $\{\eta_t\}$, $\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s : s \le r \le t\} \vee \mathcal{N}$, $\mathcal{F}^\eta_t = \mathcal{F}^\eta_{0,t}$.

Note that $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration.

Let $\mathcal{H}^2$ denote the space of real-valued, square integrable and $\mathcal{F}_t$-progressively measurable processes $\varphi = \{\varphi_t : t \in [0, T]\}$ such that

$$|\varphi|^2 = E\int_0^T |\varphi_t|^2\mathrm{d}t < \infty$$

and denote by $\mathcal{P}^2$ the subspace of $\mathcal{H}^2$ formed by the predictable processes.

Let l^2 be the space of real-valued sequences $(x_n)_{n\ge0}$ such that $\sum_{i=0}^\infty x_i^2$ is finite. We shall denote by $\mathcal{H}^2(l^2)$ and $\mathcal{P}^2(l^2)$ the corresponding spaces of l^2-valued processes equipped with the norm

$$\|\varphi\|^2 = \sum_{i=0}^\infty E\int_0^T |\varphi_t^{(i)}|^2\mathrm{d}t.$$

Finally, set $\mathscr{H}^2_T = \mathcal{H}^2 \times \mathcal{P}^2 \times \mathcal{P}^2(l^2)$.

Let $\mathcal{S}^2$ be the set of real-valued, $\mathscr{F}_t$-measurable processes $\varphi = \{\varphi_t : t \in [0, T]\}$ such that

$$E(\sup_{0\le s\le T} |\varphi_s|^2) < \infty.$$

We denote by $\mathcal{E} = \mathcal{S}^2 \times \mathcal{P}^2 \times \mathcal{P}^2(l^2)$ the set of $\mathbb{R} \times \mathbb{R} \times l^2$-valued processes (Y, U, Z) defined on $\mathbb{R}_+ \times \Omega$ which are $\mathcal{F}_t-$adapted and such that

$$\|(Y, U, Z)\|^2 = E\left(\sup_{0\le t\le T} |Y_t|^2 + \int_0^T |U_s|^2\mathrm{d}s + \int_0^T \|Z_s\|^2\mathrm{d}s\right) < +\infty.$$

Then, the couple $(\mathcal{E}, \|\cdot\|)$ is a Banach space.

We denote by $(H^{(i)})_{i\ge1}$ the Teugels martingale associated with the Lévy process $\{L_t : t \in [0, T]\}$ (see [9,10,15]) which is given by

$$H^{(i)}_t = c_{i,i}Y^{(i)}_t + c_{i,i-1}Y^{(i-1)}_t + \cdots + c_{i,1}Y^{(1)}_t,$$

where $Y^{(i)}_t = L^{(i)}_t - E[L^{(i)}_t] = L^{(i)}_t - tE[L^{(i)}_1]$ for all $i \ge 1$ and $L^{(i)}_t$ are power-jump processes. That is, $L^{(1)}_t = L_t$ and $L^{(i)}_t = \sum_{0<s\le t}(\triangle L_t)^i$ for $i \ge 2$, where it was shown in [13] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(\mathrm{d}x) = x^2\nu(\mathrm{d}x) + \sigma^2\delta_0(\mathrm{d}x)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x.$$

Remark 1. If $\nu = 0$, we are in the classic Brownian case and all non-zero-degree polynomials $q_i(x)$ will vanish, giving $H^{(i)}_t = 0, i = 2, 3, \ldots$. If μ only has mass at 1, we are in the Poisson case; here also $H^{(i)}_t = 0, i = 2, 3, \ldots$. Those two cases are degenerate in this Lévy framework.

Definition 2. By definition, a solution to a BDSDE (ξ, f, g) is a triple $(Y, U, Z) \in \mathcal{E}$ such that for any $0 \le t \le T$

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, U_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_{s-}, U_s, Z_s)\mathrm{d}B_s - \int_t^T U_s\mathrm{d}W_s - \sum_{i=1}^\infty \int_t^T Z^{(i)}_s\mathrm{d}H^{(i)}_s. \tag{1}$$

Here the integral with respect to $\{B_t\}$ is the classical backward Itô integral (see [11]) and the integral with respect to $\{W_t\}$ and $\{H^{(i)}_t\}$ are two standard forward Itô integrals (see [12]).

In order to attain the solution of Eq. (1), we propose the following conditions:

(H1) The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, P)$;

(H2) The coefficients $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times l^2 \to \mathbb{R}$ and $g : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times l^2 \to \mathbb{R}$ are progressively measurable, such that

$$f(\cdot, 0, 0, 0), g(\cdot, 0, 0, 0) \in \mathcal{H}^2;$$

(H3) There exists some constants $C > 0$ and $0 < \alpha < 1$ such that for every $(\omega, t) \in \Omega \times [0, T]$, $(y_1, u_1, z_1), (y_2, u_2, z_2) \in \mathbb{R} \times \mathbb{R} \times l^2$,

$$|f(t, y_1, u_1, z_1) - f(t, y_2, u_2, z_2)|^2 \le C(|y_1 - y_2|^2 + |u_1 - u_2|^2 + \|z_1 - z_2\|^2),$$
$$|g(t, y_1, u_1, z_1) - g(t, y_2, u_2, z_2)|^2 \le C|y_1 - y_2|^2 + \alpha(|u_1 - u_2|^2 + \|z_1 - z_2\|^2).$$

The result depends on the following extension of the well-known Itô formula. Its proof follows the same program as Lemma 1.3 of [6].

Lemma 3. *Let $\alpha \in \mathcal{S}^2$, β, γ, η and $\zeta^{(i)} \in \mathcal{H}^2$ be such that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s \mathrm{d}s + \int_0^t \gamma_s \mathrm{d}B_s + \int_0^t \eta_s \mathrm{d}W_s + \sum_{i=1}^{\infty} \zeta_s^{(i)} \int_0^t \mathrm{d}H_s^{(i)}, \quad 0 \le t \le T.$$

Then

$$|\alpha_t|^2 = |\alpha_0|^2 + 2\int_0^t \alpha_s\beta_s \mathrm{d}s + 2\int_0^t \alpha_s\gamma_s \mathrm{d}B_s + 2\int_0^t \alpha_s\eta_s \mathrm{d}W_s + 2\sum_{i=1}^{\infty}\int_0^t \alpha_s\zeta_s^{(i)}\mathrm{d}H_s^{(i)} - \int_0^t |\gamma_s|^2 \mathrm{d}s + \int_0^t |\eta_s|^2 \mathrm{d}s + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t \zeta_s^{(i)}\zeta_s^{(j)}\mathrm{d}[H^{(i)}, H^{(j)}]_s.$$

Note that $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$; we have

$$E|\alpha_t|^2 = E|\alpha_0|^2 + 2E\int_0^t \alpha_s\beta_s \mathrm{d}s - E\int_0^t |\gamma_s|^2 \mathrm{d}s + E\int_0^t |\eta_s|^2 \mathrm{d}s + \sum_{i=1}^{\infty} E\int_0^t (\zeta_s^{(i)})^2 \mathrm{d}s.$$

3. Existence and uniqueness of the solutions

Our goal in this section is to prove the following result.

Theorem 4. *Assume* (H1)–(H3) *hold. Then, there exists a unique triple $(Y, U, Z) \in \mathcal{E}$ satisfying Eq.* (1).

Proof. For $f, g \in \mathcal{H}^2$, $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, we set the filtration $\{\mathcal{G}_t, t \in [0, T]\}$

$$\mathcal{G}_t = \mathcal{F}_t^L \vee \mathcal{F}_t^W \vee \mathcal{F}_t^B$$

and the $\mathcal{G}_t$-square integrable martingale

$$M_t = E\left[\xi + \int_0^T f(s)\mathrm{d}s + \int_0^T g(s)\mathrm{d}B_s | \mathcal{G}_t\right], \quad 0 \le t \le T.$$

The predictable representation property (see [15] Proposition 2.1) yields that there exist $U \in \mathcal{P}^2$ and $Z \in \mathcal{P}^2(l^2)$ such that

$$M_t = M_0 + \int_0^t U_s \mathrm{d}W_s + \sum_{i=1}^{\infty}\int_0^t Z_s^{(i)}\mathrm{d}H_s^{(i)}.$$

Hence

$$M_T = M_t + \int_0^t U_s \mathrm{d}W_s + \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}\mathrm{d}H_s^{(i)}.$$

Let

$$\begin{aligned} Y_t &= M_t - \int_0^t f(s)\mathrm{d}s - \int_0^t g(s)\mathrm{d}B_s \\ &= M_T - \int_0^t f(s)\mathrm{d}s - \int_0^t g(s)\mathrm{d}B_s - \int_t^T U_s \mathrm{d}W_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}\mathrm{d}H_s^{(i)} \end{aligned}$$

from which we deduce that

$$Y_t = \xi + \int_t^T f(s)\mathrm{d}s + \int_t^T g(s)\mathrm{d}B_s - \int_t^T U_s \mathrm{d}W_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}\mathrm{d}H_s^{(i)}.$$

So for each $(\overline{Y}_t, \overline{U}_t, \overline{Z}_t) \in \mathscr{H}_T^2$, there exists (Y_t, U_t, Z_t) satisfying

$$\begin{cases} -\mathrm{d}Y_t = f(t, \overline{Y}_{t-}, \overline{U}_t, \overline{Z}_t)\mathrm{d}t + g(t, \overline{Y}_{t-}, \overline{U}_t, \overline{Z}_t)\mathrm{d}B_t - \int_t^T U_s \mathrm{d}W_s - \sum_{i=1}^{\infty} Z_t^{(i)} \mathrm{d}H_t^{(i)}, \\ Y_T = \xi. \end{cases}$$

First, we give a mapping defined by

$$\Phi : \mathscr{H}_T^2 \to \mathscr{H}_T^2, \quad \Phi(\overline{U}_t') = U_t',$$

where $U_t' = (Y_t, U_t, Z_t)$, $\Phi(\overline{U}_t') = (\overline{Y}_t, \overline{U}_t, \overline{Z}_t)$.

Next, we prove that Φ is a strict contraction on $\mathscr{H}_T^2$ with the norm

$$\|(Y, U, Z)\|_\beta = \left(E\left[\int_0^T \mathrm{e}^{\beta s} (|Y_{s-}|^2 + |U_s|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)}|^2) \mathrm{d}s \right] \right)^{\frac{1}{2}},$$

for a suitable $\beta > 0$.

Let (K, G, V) and (K', G', V') be two elements of $\mathscr{H}_T^2$ and set $\Phi(K, G, V) = (Y, U, Z)$ and $\Phi(K', G', V') = (Y', U', Z')$. Denote $(\bar{K}, \bar{G}, \bar{V}) = (K - K', G - G', V - V')$ and $(\bar{Y}, \bar{U}, \bar{Z}) = (Y - Y', U - U', Z - Z')$.

Applying Itô formula to $\mathrm{e}^{\beta s}(Y_s - Y_s')^2$, from $s = t$ to $s = T$, it follows that

$$\begin{aligned} \mathrm{e}^{\beta t}(Y_t - Y_t')^2 = & -\beta \int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + 2\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')[f(s, K_{s-}, G_s, V_s) - f(s, K_{s-}', G_s', V_s')]\mathrm{d}s \\ & + 2\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')[g(s, K_{s-}, G_s, V_s) - g(s, K_{s-}', G_s', V_s')]\mathrm{d}B_s \\ & + 2\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')(Z_s - Z_s')\mathrm{d}W_s - 2\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}') \sum_{i=1}^{\infty}(Z_s^{(i)} - Z_s'^{(i)})\mathrm{d}H_s^{(i)} \\ & + \int_t^T \mathrm{e}^{\beta s}|g(s, K_{s-}, G_s, V_s) - g(s, K_{s-}', G_s', V_s')|^2 \mathrm{d}s - \int_t^T \mathrm{e}^{\beta s}(U_s - U_s')^2 \mathrm{d}s \\ & - \int_t^T \mathrm{e}^{\beta s} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(Z_s^{(i)} - Z_s'^{(i)})(Z_s^{(j)} - Z_s'^{(j)})\mathrm{d}[H^{(i)}, H^{(j)}]_s. \end{aligned}$$

Taking mathematical expectation on both sides, we obtain

$$\begin{aligned} & E[\mathrm{e}^{\beta t}(Y_t - Y_t')^2] + \int_t^T \mathrm{e}^{\beta s}(U_s - U_s')^2 \mathrm{d}s + \sum_{i=1}^{\infty} E\int_t^T \mathrm{e}^{\beta s}(Z_s^{(i)} - Z_s'^{(i)})^2 \mathrm{d}s \\ & = -\beta E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + 2E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')[f(s, K_{s-}, G_s, V_s) - f(s, K_{s-}', G_s', V_s')]\mathrm{d}s \\ & \quad + E\int_t^T \mathrm{e}^{\beta s}|g(s, K_{s-}, G_s, V_s) - g(s, K_{s-}', G_s', V_s')|^2 \mathrm{d}s. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & E[\mathrm{e}^{\beta t}(Y_t - Y_t')^2] + \beta E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + \int_t^T \mathrm{e}^{\beta s}(U_s - U_s')^2 \mathrm{d}s + \sum_{i=1}^{\infty} E\int_t^T \mathrm{e}^{\beta s}(Z_s^{(i)} - Z_s'^{(i)})^2 \mathrm{d}s \\ & \leq \frac{2C}{1-\alpha} E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + \frac{1-\alpha}{2} E\int_t^T \mathrm{e}^{\beta s}(|K_{s-} - K_{s-}'|^2 + |G_s - G_s'|^2 + \|V_s - V_s'\|^2)\mathrm{d}s \\ & \quad + E\int_t^T \mathrm{e}^{\beta s}(C|K_{s-} - K_{s-}'|^2 + \alpha|G_s - G_s'|^2 + \alpha\|V_s - V_s'\|^2)\mathrm{d}s \\ & = \frac{2C}{1-\alpha} E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + \left(C + \frac{1-\alpha}{2}\right) E\int_t^T \mathrm{e}^{\beta s}|K_{s-} - K_{s-}'|^2 \mathrm{d}s \\ & \quad + \frac{1+\alpha}{2} E\int_t^T \mathrm{e}^{\beta s}|G_s - G_s'|^2 \mathrm{d}s + \frac{1+\alpha}{2} E\int_t^T \mathrm{e}^{\beta s}\|V_s - V_s'\|^2 \mathrm{d}s. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & E[\mathrm{e}^{\beta t}(Y_t - Y_t')^2] + \left(\beta - \frac{2C}{1-\alpha}\right) E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2 \mathrm{d}s + \int_t^T \mathrm{e}^{\beta s}(U_s - U_s')^2 \mathrm{d}s + E\int_t^T \mathrm{e}^{\beta s}\|Z_s - Z_s'\|^2 \mathrm{d}s \\ & \leq \left(C + \frac{1-\alpha}{2}\right) E\int_t^T \mathrm{e}^{\beta s}|K_{s-} - K_{s-}'|^2 \mathrm{d}s + \frac{1+\alpha}{2} E\int_t^T \mathrm{e}^{\beta s}|G_s - G_s'|^2 \mathrm{d}s + \frac{1+\alpha}{2} E\int_t^T \mathrm{e}^{\beta s}\|V_s - V_s'\|^2 \mathrm{d}s. \end{aligned}$$

Let $\gamma = \frac{2C}{1-\alpha}$, $\bar{C} = 2(C + \frac{1-\alpha}{2})/1 + \alpha$ and $\beta = \gamma + \bar{C}$, we get

$$E[\mathrm{e}^{\beta t}|Y_t - Y_t'|^2] + \bar{C}E\int_t^T \mathrm{e}^{\beta s}(Y_{s-} - Y_{s-}')^2\mathrm{d}s + E\int_t^T \mathrm{e}^{\beta s}|U_s - U_s'|^2\mathrm{d}s + E\int_t^T \mathrm{e}^{\beta s}\|Z_s - Z_s'\|^2\mathrm{d}s$$
$$\leq \frac{1+\alpha}{2}E\int_t^T \mathrm{e}^{\beta s}(\bar{C}|K_{s-} - K_{s-}'|^2 + |G_s - G_s'|^2 + \|V_s - V_s'\|^2)\mathrm{d}s.$$

Note that $E[\mathrm{e}^{\beta t}(Y_t - Y_t')^2] \geq 0$, we finally obtain

$$E\int_t^T \mathrm{e}^{\beta s}(\bar{C}|Y_{s-} - Y_{s-}'|^2\mathrm{d}s + |U_s - U_s'|^2 + \|Z_s - Z_s'\|^2)\mathrm{d}s \leq \frac{1+\alpha}{2}E\int_t^T \mathrm{e}^{\beta s}(\bar{C}|K_{s-} - K_{s-}'|^2 + |G_s - G_s'|^2 + \|V_s - V_s'\|^2)\mathrm{d}s.$$

That is

$$\|(Y, U, Z)\|_\beta^2 \leq \frac{1+\alpha}{2}\|(K, G, V)\|_\beta^2.$$

From this it follows that Φ is a strict contraction on $\mathscr{H}_T^2$ with the norm $\|\cdot\|_\beta$ where β is defined as above. Then, Φ has a unique fixed point $(Y, U, Z) \in \mathscr{H}_T^2$. From the Burkholder–Davis–Gundy inequality, it follows that $(Y, U, Z) \in \mathscr{E}$ which is the unique solution of Eq. (1). □

4. Application to SPDIEs

In this section, we study the link between BDSDEs driven by Lévy processes and the solution of a class of SPDIEs. Suppose that our Lévy process has the form of $L_t = \int_0^t \sigma(L_s)\mathrm{d}W_s + X_t$ where X_t is a Lévy process with Lévy measure $\nu(\mathrm{d}x)$, which takes the form $X_t = at + l_t$, where l_t is a pure jump process.

In order to attain our main result, we give a Lemma that appeared in [10].

Lemma 5. *Let $c : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$|c(s, y)| \leq a_s(y^2 \wedge |y|)\ a.s.,$$

where $\{a_s, s \in [0, T]\}$ is a non-negative predictable process such that $E\int_0^T a_s^2\mathrm{d}s < \infty$. Then, for each $0 \leq t \leq T$, we have

$$\sum_{t<s\leq T} c(s, \Delta L_s) = \sum_{i=1}^{\infty}\int_t^T \langle c(s,\cdot), p_i\rangle_{L^2(\nu)}\mathrm{d}H_s^{(i)} + \int_t^T\int_{\mathbb{R}} c(s, y)\nu(\mathrm{d}y)\mathrm{d}s.$$

Consider the following BDSDEs:

$$Y_t = h(L_T) + \int_t^T f(s, Y_{s-}, U_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_{s-}, U_s, Z_s)\mathrm{d}B_s - \int_t^T U_s\mathrm{d}W_s - \sum_{i=1}^{\infty} Z_s^{(i)}\mathrm{d}H_s^{(i)}, \quad 0 \leq t \leq T, \tag{2}$$

where $E|h(L_T)|^2 < \infty$.

Define

$$u^1(t, x, y) = u(t, x + y) - u(t, x) - \frac{\partial u}{\partial x}(t, x)y,$$

where u is the solution of the following SPDIEs:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t, x) + a'\frac{\partial u}{\partial x}(t, x) + \int_{\mathbb{R}} u^1(t, x, y)\nu(\mathrm{d}y) \\ \quad + f\left(t, u(t, x), \sigma(x)\frac{\partial u}{\partial x}(t, x), \{u^{(i)}(t, x)\}_{i=1}^{\infty}\right) \\ \quad + g\left(t, u(t, x), \sigma(x)\frac{\partial u}{\partial x}(t, x), \{u^{(i)}(t, x)\}_{i=1}^{\infty}\right)\dot{B}_t\mathrm{d}t = 0, \\ u(T, x) = h(x), \end{cases} \tag{3}$$

where $a' = a + \int_{\{|y|\geq 1\}} y\nu(\mathrm{d}y)$, $\dot{B}_t$ is a white noise and

$$u^{(1)}(t, x) = \int_{\mathbb{R}} u^1(t, x, y)p_1(y)\nu(\mathrm{d}y) + \frac{\partial u}{\partial x}(t, x)\left(\int_{\mathbb{R}} y^2\nu(\mathrm{d}y)\right)^{\frac{1}{2}},$$

and for $i \geq 2$

$$u^{(i)}(t, x) = \int_{\mathbb{R}} u^1(t, x, y)p_i(y)\nu(\mathrm{d}y).$$

In order to give the meaning to $\dot{B}_t \mathrm{d}t$, we write the above SPDIEs in the following integral form:

$$u(t,x) = h(x) + \int_t^T \left[\frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(s,x) + \int_{\mathbb{R}} u^1(s,x,y)\nu(\mathrm{d}y) + a'\frac{\partial u}{\partial x}(s,x) + f\left(s, u(s,x), \sigma(x)\frac{\partial u}{\partial x}(s,x), \{u^{(i)}(s,x)\}_{i=1}^{\infty}\right)\right]\mathrm{d}s + \int_t^T g(s,u(s,x),\sigma(x)\frac{\partial u}{\partial x}(s,x),\{u^{(i)}(s,x)\}_{i=1}^{\infty})\mathrm{d}B_s. \quad (4)$$

Suppose that u is $\mathcal{C}^{1,2}$ function such that $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ is bounded by a polynomial function of x, uniformly in t. Then we have the following:

Theorem 6. *The unique adapted solution of* (2) *is given by*

$$\begin{aligned} Y_t &= u(t, L_t), \\ U_t &= \sigma(L_t)\frac{\partial u}{\partial x}(t, L_t), \\ Z_t^{(i)} &= \int_{\mathbb{R}} u^1(t, L_{t-}, y)p_i(y)\nu(\mathrm{d}y), \quad i \geq 2, \\ Z_t^{(1)} &= \int_{\mathbb{R}} u^1(t, L_{t-}, y)p_1(y)\nu(\mathrm{d}y) + \frac{\partial u}{\partial x}(t, L_{t-})\left(\int_{\mathbb{R}} y^2\nu(\mathrm{d}y)\right)^{\frac{1}{2}}. \end{aligned}$$

Proof. Applying Itô formula to $u(s, L_s)$, we obtain

$$\begin{aligned} u(T, L_T) - u(t, L_t) = & \int_t^T \frac{\partial u}{\partial s}(s, L_{s-})\mathrm{d}s + \frac{1}{2}\int_t^T \sigma^2(L_s)\frac{\partial^2 u}{\partial x^2}(s, L_s)\mathrm{d}s \\ & + \int_t^T \sigma(L_s)\frac{\partial u}{\partial x}(s, L_s)\mathrm{d}W_s + \int_t^T \frac{\partial u}{\partial x}(s, L_{s-})\mathrm{d}L_s \\ & + \sum_{t<s\leq T}[u(s, L_s) - u(s, L_{s-}) - \frac{\partial u}{\partial x}(s, L_{s-})\triangle L_s]. \end{aligned} \quad (5)$$

Lemma 5 applied to $u(s, L_{s-} + y) - u(s, L_{s-}) - \frac{\partial u}{\partial x}(s, L_{s-})y$ shows

$$\begin{aligned} & \sum_{t<s\leq T}[u(s, L_s) - u(s, L_{s-}) - \frac{\partial u}{\partial x}(s, L_{s-})\triangle L_s] \\ & = \sum_{i=1}^{\infty}\int_t^T \left(\int_{\mathbb{R}} u^1(s, L_{s-}, y)p_i(y)\nu(\mathrm{d}y)\right)\mathrm{d}H_s^{(i)} + \int_t^T \int_{\mathbb{R}} u^1(s, L_{s-}, y)\nu(\mathrm{d}y)\mathrm{d}s. \end{aligned} \quad (6)$$

Note that

$$L_t = Y_t^{(1)} + tEL_1 = \left(\int_{\mathbb{R}} y^2\nu(\mathrm{d}y)\right)^{\frac{1}{2}} H_t^{(1)} + tEL_1, \quad (7)$$

where $EL_1 = a + \int_{\{|y|\geq 1\}} y\nu(\mathrm{d}y)$.

Hence, substituting (6) and (7) into (5) yields

$$\begin{aligned} h(L_T) - u(t, L_t) = & \int_t^T \left[\frac{\partial u}{\partial s}(s, L_{s-}) + \frac{1}{2}\sigma^2(L_s)\frac{\partial^2 u}{\partial x^2}(s, L_s) + a\frac{\partial u}{\partial x}(s, L_{s-}) + \int_{\{|y|\geq 1\}} y\nu(\mathrm{d}y)\right. \\ & \left. + \int_{\mathbb{R}} u^1(s, L_{s-}, y)\nu(\mathrm{d}y)\right]\mathrm{d}s + \int_t^T \sigma(L_s)\frac{\partial u}{\partial x}(s, L_s)\mathrm{d}W_s + \int_t^T \left[u^1(s, L_{s-}, y)p_1(y)\nu(\mathrm{d}y)\right. \\ & \left. + \frac{\partial u}{\partial x}(s, L_{s-})\left(\int_{\mathbb{R}} y^2\nu(\mathrm{d}y)\right)^{\frac{1}{2}}\right]\mathrm{d}H_s^{(1)} + \sum_{i=2}^{\infty}\int_t^T \left(\int_{\mathbb{R}} u^1(s, L_{s-}, y)p_i(y)\nu(\mathrm{d}y)\right)\mathrm{d}H_s^{(i)}. \end{aligned}$$

From which we get the desired result of the theorem. □

Next, we give a example of SPDIEs.

Example 7. Assume that $\nu(\mathrm{d}x) = \sum_{i=1}^{\infty}\alpha_i\delta_{\beta_i}(\mathrm{d}x)$, where $\delta_{\beta_i}(\mathrm{d}x)$ denotes the positive mass measure at $\beta_i \in \mathbb{R}$ of size 1. Furthermore, we assume that $\sum_{i=1}^{\infty}\alpha_i|\beta_i|^2 < \infty$. Then, the process L can be written as $L_t = \int_0^t \sigma(L_s)\mathrm{d}W_s + at + \sum_{i=1}^{\infty}(N_t^{(i)} - \alpha_i t)$, where $\{N^{(i)}\}_{i=1}^{\infty}$ is a sequence of independent Poisson processes with parameters $\{\alpha_i\}_{i=1}^{\infty}$. Recall that $H_t^{(1)} = \sum_{i=1}^{\infty} \frac{\beta_1}{\sqrt{\alpha_i}}(N_t^{(i)} - \alpha_1 t)$ and $H_t^{(i)} = 0,\ i \geq 2$ (see [9]). Let (Y, U, Z) be the unique solution of the following BDSDEs:

$$Y_t = h(L_T) + \int_t^T f(s, Y_{s-}, U_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_{s-}, U_s, Z_s)\mathrm{d}B_s - \int_t^T U_s \mathrm{d}W_s - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} d(N_s^{(i)} - \alpha_i s).$$

Then

$$\begin{aligned}
&Y_t = u(t, L_t),\\
&U_t = \sigma(L_t)\frac{\partial u}{\partial x}(t, L_t),\\
&Z_t^{(1)} = \alpha_1 u^1(t, L_{t-}, \beta_1)p_1(\beta_1) + \left(\sum_{i=1}^{\infty}\alpha_i|\beta_i|^2\right)^{\frac{1}{2}}\frac{\partial u}{\partial x}(t, L_{t-}),\\
&Z_t^{(i)} = \alpha_i u^1(t, L_{t-}, \beta_i)p_i(\beta_i), \quad i \geq 2,
\end{aligned}$$

where u is the solution of the following SPDIEs:

$$\begin{cases}
\dfrac{\partial u}{\partial t}(t, x) + \dfrac{1}{2}\sigma^2(x)\dfrac{\partial^2 u}{\partial x^2}(t, x) + \displaystyle\sum_{i=1}^{\infty}\alpha_i u^1(t, x, \beta_i) + a'\dfrac{\partial u}{\partial x}(t, x)\\
\quad + f\left(t, u(t, x), \sigma(x)\dfrac{\partial u}{\partial x}(t, x), \dfrac{\partial u}{\partial x}(t, x)\right)\\
\quad + g\left(t, u(t, x), \sigma(x)\dfrac{\partial u}{\partial x}(t, x), \dfrac{\partial u}{\partial x}(t, x)\right)\dot{B}_t \mathrm{d}t = 0,\\
u(T, x) = h(x).
\end{cases} \tag{8}$$

Acknowledgements

The work was supported by the National Natural Science Foundation of China (no. 10726075) and the Key Science and Technology Project of Ministry of Education (China no. 207047).

References

[1] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990) 55–61.
[2] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stoch. Stoch. Rep. 37 (1991) 61–74.
[3] E. Pardoux, S. Peng, Backward SDE and quasilinear parabolic PDEs, in: B.L. Rozovskii, R. Sowers (Eds.), Stochastic Partial Differential Equations and their Applications, in: Lecture Notes in Control and Information Science, vol. 176, Springer, Berlin, 1992, pp. 200–217.
[4] N. El Karoui, S. Peng, M.-C. Quenez, Backward stochastic differential equations and applications in finance, Math. Finance 7 (1997) 1–71.
[5] S. Hamadène, J.P. Lepeltier, Zero-sum stochastic differential games and BESDEs, Systems Control Lett. 4 (1995) 259–263.
[6] E. Pardoux, S. Peng, Backward doubly stochastic differential equations and systems of quasilinear SPDEs, Probab. Theory Related Fields 88 (1994) 209–227.
[7] V. Bally, A. Matoussi, Weak solutions for SPDEs and backward doubly SDEs, J. Theoret. Probab. 14 (2001) 125–164.
[8] A. Matoussi, M. Scheutzow, Stochastic PDEs driven by nonlinear noise and backward doubly SDEs, J. Theoret. Probab. 15 (2002) 1–39.
[9] D. Nualart, W. Schoutens, Chaotic and predictable representation for Lévy processes, Stochastic Process. Appl. 90 (2000) 109–122.
[10] D. Nualart, W. Schoutens, Backward stochastic differential equations and Feynman–Kac formula for Lévy processes, with applications in finance, Bernoulli 5 (2001) 761–776.
[11] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Cambridge University Press, 1990.
[12] G. Gong, An Introduction of Stochastic Differential Equations, 2nd edition, Peking University of China, Peking, 2000.
[13] S. Tang, X. Li, Necessary condition for optional control of stochastic system with random jumps, SIAM J. Control Optim. 32 (1994) 447–1475.
[14] Y. Ouknine, Reflected BSDE with jumps, Stoch. Stoch. Rep. 5 (1998) 111–125.
[15] K. Bahlali, M Eddahbi, E. Essaky, BSDE associated with Lévy processes and application to PDIE, J. Appl. Math. Stoch. Anal. 16 (2003) 1–17.
[16] Q. Zhu, Y. Shi, Forward–backward doubly stochastic differential equations with Brownian motions and Poisson process. www.paper.edu.cn, 2007.