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# The epimorphism problem for Hausdorff topological groups

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**Abstract**

Given a Hausdorff topological group $G$ and a proper closed subgroup $H$, must there exist two distinct morphisms $f, g: G \to K$ of Hausdorff groups with $f \mid H = g \mid H$? Equivalently, must epimorphisms in the category of Hausdorff topological groups have dense ranges? This question, asked by K.H. Hofmann in the sixties, appears as problem 512 in "Open Problems in Topology". We answer the question in the negative.

*Key words:* Epimorphism; Uniformity; $G$-space; Free product with a subgroup amalgamated
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## 1. Introduction

A morphism $e: G \to H$ in a category $\mathscr{K}$ is an *epimorphism* if for any pair of morphisms $f, g: H \to K$ satisfying $fe = ge$ we must have $f = g$. For many categories, epimorphisms are easy to describe. For example, in the category of sets epimorphisms coincide with onto maps. The same is true for the categories of compact spaces (and continuous maps), of groups, or of modules over a given ring. For the categories of Hausdorff spaces or of Tichonoff spaces, epimorphisms coincide with maps with a dense range. The same is true for the category of Hausdorff Abelian groups.

Now consider the category $\mathscr{G}$ of all Hausdorff topological groups and continuous homomorphisms. Clearly every morphism $e$ with $\overline{e(G)} = H$ is an epimorphism

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in $\mathscr{G}$. It was an open problem since the late sixties whether the converse is true. This problem is equivalent to the following problem of K.H. Hofmann in [1]:

**Problem 512.** If $G$ is a Hausdorff topological group and $H$ a proper closed subgroup of $G$, must there exist distinct morphisms of Hausdorff topological groups $f, g: G \to K$ with $f \mid H = g \mid H$?

There is a considerable literature on this problem [3,6,7,9,10,12,14,15], where it is shown that the answer is yes for many special cases, in particular for locally compact groups [10] and for $k_\omega$-groups [7,9]. The aim of this paper is to show that in the general case the answer is no. This result was announced in [16].

**Theorem 1.1.** *Let $M$ be a compact connected manifold without boundary (either finite-dimensional or a Hilbert cube manifold). Let $G = \operatorname{Aut} M$ be the group of all self-homeomorphisms of $M$, equipped with the topology of uniform convergence, and let $H$ be the stability subgroup at some point in $M$. Then the inclusion $H \to G$ is an epimorphism in $\mathscr{G}$; in other words, for any pair of morphisms $f, g: G \to K$ to a Hausdorff group $K$ with $f \mid H = g \mid H$ we must have $f = g$.*

Theorem 1.1 can be reformulated in terms of free products with an amalgamated subgroup. Let $G$ and $H$ be Hausdorff groups with a common closed subgroup $A$. The free product $P = G *_A H$ (in the category $\mathscr{G}$ of Hausdorff groups) is characterized by the following properties: $P$ is an object in $\mathscr{G}$, canonical morphisms $g: G \to P$ and $h: H \to P$ in $\mathscr{G}$ are given which agree on $A$, and for any pair of morphisms $g_1: G \to Q$ and $h_1: H \to Q$ in $\mathscr{G}$ which agree on $A$ there exists a unique morphism $f: P \to Q$ with $g_1 = fg$ and $h_1 = fh$. Such a product always exists. It was an open problem (discussed, for example, in [4,11]) whether $G *_A H$, considered as a group without topology, coincides with the free product $G *_A H$ in the category of groups without topology. The answer is positive for some special cases: if $G$ and $H$ are equal SIN-groups [11], if $A$ is a central subgroup of $G$ and $H$ [5], if $G$ and $H$ are $k_\omega$-groups [9]. Theorem 1.1 means that in the general case the answer is negative: if $G$ and $H$ are as in Theorem 1.1, the free product $G *_H G$ (in the category $\mathscr{G}$) of two copies of $G$ with the subgroup $H$ amalgamated coincides with $G$.

The question remains open whether the canonical morphisms $g: G \to G *_A H$ and $h: H \to G *_A H$ must always be homeomorphic embeddings.

## 2. Idea of the proof

We use the notation of Theorem 1.1. Define $j: G \times G \to K$ by $j(x, y) = f(x)g(y)^{-1}$. Let $\mathscr{V} = \inf\{\mathscr{L}, \mathscr{R}\}$ (where $\mathscr{L}$ and $\mathscr{R}$ are the left and the right uniformities on $K$, respectively) be the lower uniformity on $K$ [13]. This uniformity is compatible with the topology of $K$ [13]. We shall define a uniformity $\mathscr{U}$ on

$G \times G$ for which $j$ is $(\mathscr{U}, \mathscr{V})$-uniformly continuous. Let $H$ act on the right on $G \times G$ via $(x, y) \cdot h = (xh, yh)$, and denote the orbit space of this action by $X$. Equip $X$ with the quotient uniformity of $(G \times G, \mathscr{U})$. Since $j(xh, yh) = j(x, y)$ for all $h \in H$, the map $j$ induces a uniformly continuous map $j' : X \to K$. We shall prove that the image $D$ of diag$(G \times G)$ in $X$, considered as a uniform subspace of $X$, has the trivial (coarsest) uniformity. Since $K$ is Hausdorff, $j'$ is constant on $D$ which means that $j$ is constant on diag$(G \times G)$. Thus $f = g$.

## 3. Proof of Theorem 1.1

We define the uniformity $\mathscr{U}$ on $G \times G$. Let $\mathscr{N}(G)$ denote the filter of neighbourhoods of the neutral element $e$ in $G$ (a similar notation will be used for other groups). For $U \in \mathscr{N}(G)$, let us say that two elements $(x, y), (x', y') \in G \times G$ are *U-close* if there exists $u \in U$ such that $(x', y')$ equals one of the following four:

(1) $(ux, y)$;

(2) $(x, uy)$;

(3) $(xy^{-1}uy, y)$;

(4) $(x, yx^{-1}ux)$.

If $U = U^{-1}$, which we shall assume, the relation of being $U$-close is symmetric. Define the uniformity $\mathscr{U}$ on $G \times G$ as the finest (not necessarily Hausdorff) uniformity with the following property: For every entourage $W \in \mathscr{U}$ there exists $U \in \mathscr{N}(G)$ such that any pair of $U$-close points in $G \times G$ belongs to $W$. Clearly such a uniformity exists: just take the least upper bound of all uniformities with this property. It can be shown that $\mathscr{U}$ is not compatible with the topology of $G \times G$ (this follows, for example, from the proof of Lemma 3.2 below).

**Lemma 3.1.** *The map $j : G \times G \to K$ defined by $j(x, y) = f(x)g(y)^{-1}$ is $(\mathscr{U}, \mathscr{V})$-uniformly continuous.*

Here $G$, $f$ and $g$ are as in Theorem 1.1, but the only property that we need is that $G/H$ is compact.

**Proof.** For $V \in \mathscr{N}(K)$, let us say that two points $z_1, z_2 \in V$ are *V-near* if either $z_2 \in z_1V$ or $z_2 \in Vz_1$. It suffices to prove the following assertion:

*For any $V \in \mathscr{N}(K)$ there exists $U \in \mathscr{N}(G)$ with the following property: If $(x, y), (x', y') \in G \times G$ are U-close, then $z_1 = j(x, y)$ and $z_2 = j(x', y')$ are V-near.*

This implies that the coarsest uniformity $\mathscr{W}$ on $G \times G$ for which $j$ is $(\mathscr{W}, \mathscr{V})$-uniformly continuous is coarser than $\mathscr{U}$ or, equivalently, that $j$ is $(\mathscr{U}, \mathscr{V})$-uniformly continuous.

If $(x', y')$ is $(ux, y)$ or $(x, uy)$, then $z_2z_1^{-1} = f(u)$ or $z_1^{-1}z_2 = g(u)^{-1}$, respectively, and the assertion follows from the continuity of $f$ and $g$. If $(x', y') = (xy^{-1}uy, y)$, then

$$z_1^{-1}z_2 = g(y)f(x)^{-1}f(xy^{-1}uy)g(y)^{-1} = k^{-1}f(u)k,$$

where $k = j(y, y)$. Let $F = \{j(y, y): y \in G\}$. This is a compact subset in $K$: indeed, since the map $y \mapsto j(y, y)$ is constant on left $H$-cosets, $F$ is a continuous image of the quotient space $G/H$, which can be identified with the compact manifold $M$. It follows that there exists $W \in \mathcal{N}(K)$ such that $k^{-1}Wk \subset V$ for all $k \in F$. Pick $U \in \mathcal{N}(G)$ so that $f(U) \subset W$. Then $z_1^{-1}z_2 = k^{-1}f(u)k \in k^{-1}Wk \subset V$, which means that $z_1$ and $z_2$ are $V$-near. If $(x', y') = (x, yx^{-1}ux)$, the argument is similar: in this case, $z_2 z_1^{-1} = kg(u)^{-1}k^{-1}$ with $k = j(x, x)$. □

Note that the definition of the uniformity $\mathcal{U}$ on $G \times G$ makes sense for any topological group $G$. If $H$ is a subgroup of $G$, then, just as above, we define $X$ to be the quotient space $(G \times G)/i(H)$, where $i: H \to G \times G$ is the diagonal embedding, $i(h) = (h, h)$. Equip $X$ with the uniformity $\mathcal{U}'$, the quotient uniformity of $\mathcal{U}$. Let $D$ be the image of the diagonal of $G \times G$ in $X$.

**Lemma 3.2.** *Suppose that for any left coset $t \in G/H$ and any infinite sequence $U_0, U_1, \dots$ of neighbourhoods of $e$ in $G$ there exist a finite sequence $g_1, \dots, g_n \in G$ and $v \in U_0$ such that*

*(1) $g_1 = e$, $g_n \in t$;*

*(2) for every $k$, $1 \leqslant k < n$, there exists $u_k \in U_k$ such that either $g_{k+1} = u_k g_k$ or $g_{k+1} = v^{-1}u_k v g_k$.*

*Then $D$, considered as a uniform subspace of $(X, \mathcal{U}')$, has the coarsest uniformity.*

**Proof.** We have to show that any $\mathcal{U}'$-uniformly continuous pseudometric on $X$ is trivial on $D$. Equivalently, we have to show that any $\mathcal{U}$-uniformly continuous pseudometric $d$ on $G \times G$ which is trivial on $H$-orbits is trivial on the diagonal. Fix $a \in G$ and $\varepsilon > 0$, and let $t = aH \in G/H$. For every $k = 0, 1, \dots$ there exists $U_k \in \mathcal{N}(G)$ such that, if $p_1, p_2 \in G \times G$ are $U_k$-close, then $d(p_1, p_2) < 2^{-k}\varepsilon$. Apply the assumption of the lemma to the sequence $U_0, U_1, \dots$ and to the coset $t$: there exist $g_1 = e, \dots, g_n$ in $G$ and $v \in U_0$ such that $g_n = ah$ for some $h \in H$ and $g_{k+1} = u_k g_k$ or $g_{k+1} = v^{-1}u_k v g_k$ for some $u_k \in U_k$. Let $p_k = (g_k, vg_k) \in G \times G$.

Let us say that $q_1, q_2 \in G \times G$ are *2-$U$-close* if, for some $q_3 \in G \times G$, $q_1$ is $U$-close to $q_3$ and $q_3$ is $U$-close to $q_2$. For example, if $q = (x, y)$ and $u \in U$, then $(xy^{-1}uy, uy)$ and $(ux, yx^{-1}ux)$ are 2-$U$-close to $q$.

We claim that $p_k$ and $p_{k+1}$ are 2-$U_k$-close. Write $p_k$ as $p_k = (g_k, vg_k) = (x, y)$. If $g_{k+1} = u_k g_k$, then $p_{k+1} = (u_k g_k, vu_k g_k) = (u_k x, yx^{-1}u_k x)$, and we have just observed that such a point is 2-$U_k$-close to $(x, y)$. If $g_{k+1} = v^{-1}u_k v g_k$, then $p_{k+1} = (v^{-1}u_k v g_k, u_k v g_k) = (xy^{-1}u_k y, u_k y)$, and the claim is verified.

By the choice of the $U_k$, we have $d(p_k, p_{k+1}) < 2^{1-k}\varepsilon$. It follows that

$$d(p_1, p_n) < 2\sum_{k=1}^{n-1} 2^{-k}\varepsilon < 2\varepsilon.$$

Since $p_1 = (e, v)$ is $U_0$-close to $(e, e)$ and $p_n = (g_n, vg_n)$ is $U_0$-close to $(g_n, g_n)$, we have $d((e, e), (g_n, g_n)) < 4\varepsilon$. Since $(g_n, g_n) = (ah, ah)$ and $d$ is trivial on $H$-orbits,

$(g_n, g_n)$ can be replaced by $(a, a)$ in the last inequality. Since $\varepsilon$ was arbitrary, $d((e, e), (a, a)) = 0$. This means that $d$ is trivial on the diagonal. □

Now let $G$ and $H$ be as in Theorem 1.1: $M$ is a compact connected manifold, $G = \text{Aut } M$, $a \in M$, $H = \{f \in G: f(a) = a\}$.

**Lemma 3.3.** *If $G$ and $H$ are as in Theorem* 1.1, *the condition of Lemma* 3.2 *is satisfied.*

**Proof.** Let $U_0, U_1, \ldots \in \mathcal{N}(G)$ and $t \in G/H$ be given. For some point $b \in M$, $t = \{f \in G: f(a) = b\}$. Choose a compatible metric $\rho$ on $M$. For each $k = 0, 1, \ldots$ there exists $\varepsilon_k > 0$ such that for any two points $x, y \in M$ with $\rho(x, y) < \varepsilon_k$ there exists $u \in U_k$ with $u(x) = y$. We can choose $\varepsilon_0$ so that a stronger condition holds: If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are two $n$-tuples of distinct points in $M$ and $\rho(a_i, b_i) < \varepsilon_0$, $i = 1, \ldots, n$ (and, in case $M$ is a circle, these $n$-tuples have the same cyclic order), then there exists $u \in U_0$ with $u(a_i) = b_i$ for each $i$. Connect the points $a$ and $b$ by an $\varepsilon_0$-chain $a'_0 = a, a'_1, \ldots, a'_n = b$. Let $\delta = \min(\varepsilon_0, \ldots, \varepsilon_{2n-1})$. "Split" every point $a'_k$ $(0 < k < n)$ into two points $a_{2k}, a_{2k+1}$ to get an $\varepsilon_0$-chain of distinct points $a_1 = a, a_2, \ldots, a_{2n} = b$ with $\rho(a_{2k}, a_{2k+1}) < \delta$ $(0 < k < n)$. There exist distinct points $b_1, \ldots, b_{2n}$ such that $\rho(a_i, b_i) < \varepsilon_0$ $(i = 1, \ldots, 2n)$ and $\rho(b_{2k-1}, b_{2k}) < \delta$ $(k = 1, \ldots, n)$. (If $M$ is a circle, we may suppose that $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_{2n}$ go in the natural order on the arc $\widehat{ab}$.) Pick $v \in U_0$ so that $v(a_i) = b_i$, $i = 1, \ldots, 2n$. Pick $u_k \in U_k$ $(k = 1, \ldots, 2n - 1)$ so that $u_k(a_k) = a_{k+1}$ if $k$ is even and $u_k(b_k) = b_{k+1}$ if $k$ is odd. This is possible, since $\rho(a_k, a_{k+1}) < \delta \leqslant \varepsilon_k$ if $k$ is even and $\rho(b_k, b_{k+1}) < \delta \leqslant \varepsilon_k$ if $k$ is odd. Define a sequence $g_1, \ldots, g_{2n} \in G$ as follows: $g_1 = e$, $g_{k+1} = u_k g_k$ if $k$ is even, $g_{k+1} = v^{-1} u_k v g_k$ if $k$ is odd. Then $g_k(a) = a_k$. In particular, $g_{2n}(a) = b$, so $g_{2n} \in t$ and the sequence $g_1, \ldots, g_{2n}$ is as in Lemma 3.2. □

As explained in Section 2, Theorem 1.1 follows from Lemmas 3.1–3.3.

## 4. Morphisms of compact $G$-spaces

We now prove a criterion for an inclusion $H \to G$ to be an epimorphism in terms of morphisms of compact $G$-spaces (Theorem 4.1). A *$G$-space* is a topological space $X$ on which the group $G$ acts continuously. If $X$ is compact, then the group Aut $X$ of all self-homeomorphisms of $X$, equipped with the compact-open topology, is a topological group. Every morphism $G \to \text{Aut } X$ defines a $G$-space structure on $X$, and every such a structure can be obtained in this way.

A continuous map $f: X \to Y$ between two $G$-spaces is a *$G$-morphism* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. A *$G$-endomorphism* (respectively *$G$-automorphism*) of a $G$-space $X$ is a self-map (respectively self-homeomorphism) $f: X \to X$ which is a $G$-morphism. If $X$ is a $G$-space and $H$ is a subgroup of $G$, then $X$ can be considered as an $H$-space, so we can speak about $H$-endomorphisms and $H$-automorphisms of $X$.

**Theorem 4.1.** *Let $H$ be a subgroup of a Hausdorff topological group $G$.*

(1) *The inclusion $i: H \to G$ is an epimorphism in $\mathscr{G}$ if and only if for any compact $G$-space $X$, every $H$-automorphism of $X$ is a $G$-automorphism.*

(2) *$H$ is dense in $G$ if and only if for any compact $G$-space, every $H$-endomorphism of $X$ is a $G$-endomorphism.*

**Proof.** Let $X$ be a compact $G$-space, and let $f: G \to \operatorname{Aut} X$ be the corresponding morphism in $\mathscr{G}$. Suppose there exists an $H$-automorphism $\sigma: X \to X$ of $X$ which is not a $G$-automorphism. Then $\sigma$ commutes with every element in $f(H)$ but not with every element in $f(G)$, so $f$ and $\sigma f \sigma^{-1}$ are distinct morphisms $G \to \operatorname{Aut} X$ which agree on $H$. This means that the inclusion $H \to G$ is not an epimorphism.

Conversely, suppose that $f, g: G \to K$ are distinct morphisms in $\mathscr{G}$ which agree on $H$. Then $h = (f, g): G \to K \times K$ maps $H$, but not $G$, into the diagonal. Let $X$ be the completion of the uniform space $(K \times K, \mathscr{R}')$, where $\mathscr{R}$ is the right uniformity on $K \times K$ and $\mathscr{R}'$ is the finest precompact uniformity which is coarser than $\mathscr{R}$. Then $X$ is a $K \times K$-space. This is a special case of the following:

**Lemma 4.2.** *Let $G$ be a topological group, $Y$ a Tichonoff $G$-space and $\mathscr{U}$ a compatible uniformity on $Y$ such that*

(1) *$\mathscr{U}$ is $G$-invariant;*

(2) *for any $\mathscr{U}$-uniform cover $\alpha$ of $Y$ there exists $U \in \mathscr{N}(G)$ such that the cover $\{Uy\colon y \in Y\}$ refines $\alpha$.*

*Let $\mathscr{U}'$ be the finest precompact uniformity on $Y$ which is coarser than $\mathscr{U}$. Then the completion of $(Y, \mathscr{U}')$ has a natural structure of a compact $G$-space.*

It is known [8] that the following are equivalent: (a) there exists a $G$-compactification of $Y$; (b) there exists a uniformity $\mathscr{U}$ for $Y$ satisfying (2).

**Proof of the lemma.** The map $G \times Y \to Y$ which defines the $G$-space structure on $Y$ is $(\mathscr{R} \times \mathscr{U}, \mathscr{U})$-uniformly continuous and hence extends to a map $G \times cY \to cY$, where $cY$ is the completion of $(Y, \mathscr{U})$. Hence $cY$ is a $G$-space. On the other hand, the uniformity $\mathscr{U}'$ also satisfies the conditions of the lemma, so the previous remark applies to $(Y, \mathscr{U}')$. □

We return to the proof of Theorem 4.1. The morphism $h: G \to K \times K$ induces a $G$-space structure on $X$. The natural involution $(x, y) \mapsto (y, x)$ on $K \times K$ extends to an involution $\sigma$ on $X$. For $g \in K \times K$, the $g$-shift of $X$ commutes with $\sigma$ iff $g$ belongs to the diagonal. It follows that the involution $\sigma$ is an $H$-automorphism of $X$ but not a $G$-automorphism.

This proves the first part of the theorem. For the second, suppose that $H$ is a dense subgroup of $G$. Then clearly every $H$-morphism of Hausdorff $G$-spaces is also a $G$-morphism. Conversely, if $H$ is not dense in $G$, we may suppose without loss of generality that $H$ is a proper closed subgroup. Applying Lemma 4.2 to the $G$-space $G/H$ and the right uniformity on $G/H$, we see that there exists a compact $G$-space $X$ such that $H$ coincides with the stabilizer of some point $p \in X$. The constant map which takes $X$ to $p$ is an $H$-morphism but not a $G$-morphism. □

Let $H$ be a proper closed subgroup of $G$. If the left uniformity on $G/H$ is Hausdorff, the inclusion $H \to G$ is not an epimorphism [10]. It follows from Theorem 4.1 that the weaker condition that this uniformity be not the coarsest is also sufficient:

**Proposition 4.3.** *If the left uniformity on $G/H$ is not trivial, the inclusion $H \to G$ is not an epimorphism in $\mathscr{G}$.*

**Proof.** In virtue of Theorem 4.1, it suffices to construct a compact $G$-space $X$ and an $H$-automorphism of $X$ which is not a $G$-automorphism. Let $Y$ be the completion of $(G, \mathscr{R}')$. This is a compact $G$-space (Lemma 4.2). Let $T$ be a circle, considered as a compact group and as a $G$-space with a trivial action. Let $X = Y \times T$. The left uniformity on $G/H$, the set of left $H$-cosets, is isomorphic to the right uniformity on the set of right $H$-cosets. By the assumption, this uniformity is not the coarsest, which means that there exists a nonconstant $\mathscr{R}$-uniformly continuous function $f: G \to T$ such that $f(hg) = f(g)$ for all $g \in G$, $h \in H$. Any $\mathscr{R}$-uniformly continuous map from $G$ to a compact space is also $\mathscr{R}'$-uniformly continuous, so $f$ extends to a map (denoted again by $f$) $Y \to T$. Define a map $\sigma: X \to X$ as follows: $\sigma(y, t) = (y, t + f(y))$. There exists an inverse map defined by $\sigma^{-1}(y, t) = (y, t - f(y))$, so $\sigma$ is a self-homeomorphism of $X$. Since $f(hy) = f(y)$ for all $h \in H$ and $y \in Y$, the map $\sigma$ is an $H$-automorphism of $X$. Since $f$ is nonconstant, $\sigma$ is not a $G$-morphism. □

## 5. Pure epimorphisms

A morphism $f: X \to Y$ in a category $\mathscr{K}$ is a *pure epimorphism* if whenever $f = gh$ and $g$ is a monomorphism, $h$ must be an epimorphism. (A morphism $m$ is a *monomorphism* if $mh_1 = mh_2$ implies $h_1 = h_2$.) Writing $f$ as $f = 1_Y f$, we see that pure epimorphisms are indeed epimorphisms. In the category of Hausdorff spaces, pure epimorphisms coincide with onto maps. The same is true for the category of Hausdorff Abelian groups. In the category $\mathscr{G}$ of Hausdorff topological groups, onto morphisms are pure epimorphisms. Conversely, if $f: H \to G$ is a pure epimorphism in $\mathscr{G}$ and $f(H)$ is a *normal* subgroup of $G$ (in the algebraic sense), then $f$ is onto. Roelcke, having learnt of the solution of Hofmann's epimorphism problem, asked whether all pure epimorphisms in $\mathscr{G}$ are onto or onto a dense subgroup. The answer is no:

**Proposition 5.1.** *Let $M$, $H$ and $G$ be as in Theorem* 1.1. *Then the inclusion $i: H \to G$ is a pure epimorphism in $\mathscr{G}$.*

**Proof.** Since the action of $G$ on $M$ is 2-transitive, $H$ is a maximal subgroup of $G$. Since $i: H \to G$ is an epimorphism in $\mathscr{G}$ (Theorem 1.1), the definitions readily imply that it suffices to prove the following: if $\mathscr{T}_0$ is the original topology on $G$ and $\mathscr{T}$ is a finer group topology on $G$ inducing the same topology on $H$, then $\mathscr{T} = \mathscr{T}_0$. For each point $q \in M$, let $H_q = \{f \in G\colon f(q) = q\}$ be the stability subgroup at $q$. For some point $p \in M$ we have $H = H_p$. Since all $H_q$ are conjugate to each other, the topologies $\mathscr{T}_0$ and $\mathscr{T}$ agree on each $H_q$. For any point $q \in M$ with $q \neq p$, the mapping $f \mapsto f(p)$ from $H_q$ to $M \setminus \{q\}$ is open. It follows that the quotient topologies $\mathscr{T}/H$ and $\mathscr{T}_0/H$ agree on $M \setminus \{q\}$ (we identify the set $G/H$ with $M$). Since this holds for all $q \in M \setminus \{p\}$, the quotient topologies $\mathscr{T}/H$ and $\mathscr{T}/H_0$ are equal. [2, Lemma 1] implies that $\mathscr{T} = \mathscr{T}_0$. $\square$

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