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Viscosity approximations by the shrinking projection method in Hilbert spaces

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ABSTRACT

We consider viscosity approximations by using the shrinking projection method established by Takahashi, Takeuchi, and Kubota, and the modified shrinking projection method proposed by Qin, Cho, Kang, and Zhou, for finding a common fixed point of countably many nonlinear mappings, and we prove strong convergence theorems which extend some known results. We also consider semigroups of nonlinear mappings and obtain strong convergence of iterative schemes which approximate a common fixed point of the semigroup under certain conditions.

1. Introduction

Let us consider the problem of finding a fixed point of a nonlinear mapping defined on a nonempty closed convex subset C of a real Hilbert space H. The setting of this problem is so general that it includes a number of important problems such as convex minimization problems, variational inequalities, saddle point problems, and others. In particular, approximating the solutions of this problem by iterative schemes has been studied by many researchers, and various types of mappings have been considered.

Let T be a nonexpansive mapping of C into itself, that is, $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Suppose that the set $F = F(T)$ of all fixed points of T is nonempty. Halpern [1] introduced the following iteration scheme: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1)$. Wittmann [2] proved the strong convergence of this sequence $\{x_n\}$ to $P_F x$ under the assumptions $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, where P_F is the metric projection of H onto F. Moudafi [3] extended it to the following process, which is called Moudafi's viscosity approximations: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1)$ and $f : C \to C$ is a contraction. See also [4]. It was proved that this sequence converges strongly to a unique fixed point of $P_F f$ under similar conditions to those in [2]. Suzuki [5] considered the Meir–Keeler contractions, which is an extended notion of contractions, and studied the equivalency of convergence of these approximation schemes.

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On the other hand, Haugazeau [6] considered the hybrid method and proved the strong convergence of the generated iterative sequence. See also [7–10] and references therein. Recently, Takahashi et al. [11] proposed a modified hybrid method, the so-called shrinking projection method, as follows:

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$. They proved the strong convergence of this sequence to $P_F x$. See also [12,13].

Motivated by these results, we consider viscosity approximations by using the shrinking projection method and prove strong convergence theorems which extend the results in [11,12]. We also consider semigroups of nonlinear mappings which extend one-parameter nonexpansive semigroups and obtain iterative methods which approximate a common fixed point of the semigroup under certain conditions.

2. Preliminaries

Throughout this paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The set of all positive integers and the set of all real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x as $n \to \infty$.

Let C be a nonempty closed convex subset of H. We know that, for every $x \in H$, there exists a unique element $y \in C$ such that $\|x - y\| = \inf_{z \in C} \|x - z\|$. We denote y by $P_C x$, and P_C is said to be the metric projection of H onto C. It is known that, for $x \in H$ and $y \in C$, $y = P_C x$ is equivalent to $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$. We also know that P_C is nonexpansive. See [14,15] for more details.

A mapping f of a complete metric space (X, d) into itself is called a contraction with coefficient $r \in (0, 1)$ if $\|f(x) - f(y)\| \leq r\|x - y\|$ for all $x, y \in C$. It is known that f has a unique fixed point [16].

On the other hand, Meir and Keeler [17] defined the following mapping, called the Meir–Keeler contraction. A mapping $f : X \to X$ is called a Meir–Keeler contraction if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \epsilon + \delta$ implies that $d(f(x), f(y)) < \epsilon$ for all $x, y \in X$. We know that Meir–Keeler contraction is a generalization of contraction, and the following result, which extends the Banach contraction principle, is proved in [17].

Theorem 2.1 (*Meir–Keeler [17]*)**.** *A Meir–Keeler contraction defined on a complete metric space has a unique fixed point.*

We have the following result, given by Suzuki [5], for Meir–Keeler contractions defined on a Banach space.

Lemma 2.2 (*Suzuki [5]*)**.** *Let f be a Meir–Keeler contraction on a convex subset C of a Banach space E. Then, for every $\epsilon > 0$, there exists $r \in (0, 1)$ such that $\|x - y\| \geq \epsilon$ implies that $\|f(x) - f(y)\| \leq r\|x - y\|$ for $x, y \in C$.*

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. We define a subset s-$\mathrm{Li}_n C_n$ of H as follows: $x \in$ s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and such that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, a subset w-$\mathrm{Ls}_n C_n$ of H is defined by the following: $y \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and such that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $C_0 \subset H$ satisfies

$$C_0 = \operatorname*{s\text{-}Li}_n C_n = \operatorname*{w\text{-}Ls}_n C_n,$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [18], and we write $C_0 = \text{M-}\lim_n C_n$. One of the simplest examples of Mosco convergence is a decreasing sequence $\{C_n\}$ with respect to inclusion. The Mosco limit of such a sequence is $\bigcap_{n=1}^{\infty} C_n$. For more details, see [19].

Tsukada [20] proved the following theorem for the metric projection.

Theorem 2.3 (*Tsukada [20]*)**.** *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = M\text{-}\lim_n C_n$ exists and is nonempty, then, for each $x \in H$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$.*

3. A condition for a sequence of mappings

Let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a mapping satisfying $F(T) \neq \emptyset$ and

$$\|Tx - z\|^2 \leq \|x - z\|^2 - a\|(I - T)x\|^2, \tag{1}$$

for all $x \in C$ and $z \in F(T)$, where I is the identity mapping on C and $a \in \mathbb{R}$ is a fixed coefficient. The class of mappings satisfying this condition includes a large number of important nonlinear mappings. In fact, letting $a = 1$, we can show that all firmly nonexpansive mappings such as the resolvents of a maximal monotone operator, metric projections, the convex combination of the identity mapping and a nonexpansive mapping, and others, belong to this class if it has a fixed point.

Further, we have that it is a wider class than that of quasi-nonexpansive mappings by letting $a = 0$. For more details, see [9,12].

If $a > -1$, then we know that $F(T)$ is closed and convex; see [9,12]. Moreover, we can see that there exists a mapping satisfying the inequality (1) with $a = -1$, and that its set of fixed points is neither closed nor convex.

Example 3.1. Let $u \in H$ be such that $\|u\| = 1$, and let $B = \{x \in H : \|x\| \le 1,\ \langle u, x\rangle = 0\}$. Let P_B the metric projection of H onto B, and define a mapping $T : H \to H$ by

$$Tx = \begin{cases} P_B x & (x \neq 0) \\ u & (x = 0) \end{cases}$$

for $x \in H$. It is obvious that $F(T) = B \setminus \{0\}$. For every $x \in H \setminus \{0\}$ and $z \in F(T)$, we have that

$$\|Tx - z\|^2 \le \|x - z\|^2 - \|(I - T)x\|^2 \le \|x - z\|^2 + \|(I - T)x\|^2 ,$$

since $P_B x = Tx$ and $z \in F(T) \subset F(P_B)$. Let us consider the case $x = 0$. We have that

$$\|T0 - z\|^2 = \|u - z\|^2 = \|z\|^2 + \|u\|^2 = \|0 - z\|^2 + \|(I - T)0\|^2$$

for $z \in F(T)$. Hence, for all $x \in H$ and $z \in F(T)$, we have that

$$\|Tx - z\|^2 \le \|x - z\|^2 + \|(I - T)x\|^2 = \|x - z\|^2 - a\,\|(I - T)x\|^2 ,$$

with $a = -1$. In this case, $F(T)$ is neither closed nor convex. We note that this coefficient is the greatest possible.

If $a_1 < a_2$, then condition (1) with coefficient a_2 automatically implies the same condition with a_1. Therefore, this example shows that we can always find a mapping satisfying condition (1) with coefficient $a \le -1$, whose set of fixed points is neither closed nor convex.

Next, let us consider a sequence of mappings $\{T_n\}$ satisfying $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and

$$\|T_n x - z\|^2 \le \|x - z\|^2 - a_n\,\|(I - T_n)x\|^2$$

for all $n \in \mathbb{N}$, $x \in C$, and $z \in F$, where $\{a_n\}$ is a sequence of real numbers such that $\liminf_n a_n > -1$. From the above example, some $F(T_n)$ may be neither closed nor convex, since it may hold that $a_n \le -1$ for finitely many $n \in \mathbb{N}$. Therefore, F is not guaranteed to be a closed convex set in general. However, we can prove that it is closed and convex if $\{T_n\}$ satisfies the following condition, which is considered in [12]: for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$. Indeed, let $\{z_n\}$ be a sequence in F converging strongly to $z_0 \in H$. Then, since $\{T_n z_n\}$ is identical to $\{z_n\}$, they have the same strong limit z_0. Thus, by assumption, $z_0 \in F$, and hence F is closed. On the other hand, let $z_1, z_2 \in F$, $t \in (0, 1)$, and $w = tz_1 + (1 - t)z_2$. Then, we have that

$$\|T_n w - z_i\|^2 \le \|w - z_i\|^2 - a_n\,\|(I - T_n)w\|^2$$

for $i = 1, 2$. Since these are equivalent to

$$(1 + a_n)\,\|T_n w - w\|^2 \le 2\,\langle w - T_n w, w - z_i\rangle ,$$

we get that

$$(1 + a_n)\,\|T_n w - w\|^2 \le 2\,\langle w - T_n w, w - (tz_1 + (1 - t)z_2)\rangle = 0.$$

Since $1 + a_n > 0$ for sufficiently large $n \in \mathbb{N}$, we have that $\{T_n w\}$ converges strongly to w, which implies that $w \in F$. Therefore F is convex.

There are many examples of $\{T_n\}$ which satisfy the conditions above; see [9,12]. Another example which satisfies these conditions will be discussed in Section 6.

4. Convergence theorems

We deal with a sequence of mappings satisfying the conditions we observed in the previous section. We consider viscosity approximation methods converging strongly to their common fixed point.

Theorem 4.1. *Let C be a nonempty closed convex subset of H, and let $\{T_n\}$ be a sequence of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: there exists $\{a_n\} \subset \mathbb{R}$ with $\liminf_{n\to\infty} a_n > -1$ such that $\|T_n x - z\|^2 \le \|x - z\|^2 - a_n\,\|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$, and $z \in F$. Let f be a Meir–Keeler contraction of C into itself, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 - a_n\,\|x_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$. Assume that, for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

Proof. As we mentioned in the previous section, F is a closed convex subset of C. Since P_F is nonexpansive, the composed mapping $P_F f$ of C into itself is a Meir–Keeler contraction on C; see [5, Proposition 3]. By Theorem 2.1, there exists a unique fixed point $z \in F$ of $P_F f$. We have that C_n is a closed convex subset of H and $\emptyset \neq F \subset C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Thus, $\{x_n\}$ is well defined. Since the composed mapping $P_{\bigcap_{n=1}^{\infty} C_n} f$ is a Meir–Keeler contraction on C, there exists a unique fixed point $u \in \bigcap_{n=1}^{\infty} C_n$ of $P_{\bigcap_{n=1}^{\infty} C_n} f$ from Theorem 2.1. Let $z_n = P_{C_n} f(u)$ for each $n \in \mathbb{N}$. We get $\bigcap_{n=1}^{\infty} C_n = \text{M-lim}_n C_n$, since $F \subset C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. Thus, by Theorem 2.3,

$$z_n \to P_{\bigcap_{n=1}^{\infty} C_n} f(u) = u. \tag{2}$$

Next, we prove that $x_n \to u$. If this were not so, it would hold that $\limsup_{n\to\infty} \|x_n - u\| > 0$. Let $\epsilon > 0$, such that $\epsilon < \limsup_{n\to\infty} \|x_n - u\|$. By the definition of Meir–Keeler contraction, there exists $\delta > 0$ with $\epsilon + \delta < \limsup_{n\to\infty} \|x_n - u\|$ such that $\|x - y\| < \epsilon + \delta$ implies that $\|f(x) - f(y)\| < \epsilon$ for all $x, y \in C$. From Lemma 2.2, there exists $r \in (0, 1)$ such that $\|x - y\| \geq \epsilon + \delta$ implies that $\|f(x) - f(y)\| \leq r\|x - y\|$ for every $x, y \in C$. By (2), there exists $n_0 \in \mathbb{N}$ such that $\|z_n - u\| < \delta$ for each $n \geq n_0$. As in the proof of [5, Theorem 8], we consider the following two cases.

(i) There exists $n_1 \geq n_0$ such that $\|x_{n_1} - u\| < \epsilon + \delta$.
(ii) $\|x_n - u\| \geq \epsilon + \delta$ for every $n \geq n_0$.

In case (i), it holds that $\|x_{n_1+1} - z_{n_1+1}\| \leq \|f(x_{n_1}) - f(u)\| < \epsilon$ since $\|x_{n_1} - u\| < \epsilon + \delta$. Thus we get

$$\|x_{n_1+1} - u\| \leq \|x_{n_1+1} - z_{n_1+1}\| + \|z_{n_1+1} - u\| < \epsilon + \delta.$$

This means that

$$\limsup_{n\to\infty} \|x_n - u\| \leq \epsilon + \delta < \limsup_{n\to\infty} \|x_n - u\|.$$

This is a contradiction. In case (ii), we have $\|f(x_n) - f(u)\| \leq r\|x_n - u\|$ for all $n \geq n_0$. Thus we get

$$\|x_{n+1} - z_{n+1}\| \leq \|f(x_n) - f(u)\| \leq r\|x_n - u\| \leq r(\|x_n - z_n\| + \|z_n - u\|)$$

for every $n \geq n_0$. By (2),

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - z_n\| &= \limsup_{n\to\infty} \|x_{n+1} - z_{n+1}\| \\ &\leq r \limsup_{n\to\infty} \|x_n - z_n\| \\ &< \limsup_{n\to\infty} \|x_n - z_n\|.\end{aligned}$$

This is a contradiction. Therefore, we get

$$\lim_{n\to\infty} x_n = u. \tag{3}$$

Since $x_{n+1} = P_{C_{n+1}} f(x_n)$, we have $\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geq 0$ for each $y \in C_{n+1}$. Using $F \subset C_{n+1}$, we get $\langle f(x_n) - x_{n+1}, x_{n+1} - y\rangle \geq 0$ for every $n \in \mathbb{N}$ and $y \in F$, which implies that

$$\langle f(u) - u,\ u - y\rangle \geq 0 \tag{4}$$

for all $y \in F$. On the other hand, since $x_{n+1} \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - a_n\|x_n - y_n\|^2;$$

that is,

$$(1 + a_n)\|y_n - x_n\|^2 \leq 2\langle y_n - x_n, x_{n+1} - x_n\rangle \leq 2\|y_n - x_n\|\|x_{n+1} - x_n\|$$

for every $n \in \mathbb{N}$. Hence, we get

$$(1 + a_n)\|y_n - x_n\| \leq 2\|x_{n+1} - x_n\|,$$

which implies that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} T_n x_n = u \tag{5}$$

by (3) and $\liminf_{n\to\infty} a_n > -1$. From (3), (5), and the assumption, we have that $u \in F$. Therefore, $u = z_0$ by (4). The proof is complete. □

We also get the following result by using the modified shrinking projection method proposed by Qin et al. [21].

Theorem 4.2. *Let C be a nonempty closed convex subset of H, and let $\{T_n\}$ be a sequence of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: there exists $\{a_n\} \subset \mathbb{R}$ with $\liminf_{n\to\infty} a_n > -1$ such that $\|T_n x - z\|^2 \le \|x - z\|^2 - a_n \|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$, and $z \in F$. Let f be a Meir–Keeler contraction of C into itself, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = T_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - a_n \|x_n - y_n\|^2\}, \\ Q_n = \begin{cases} C & (n = 1) \\ \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z\rangle \ge 0\} & (n \ge 2), \end{cases} \\ x_{n+1} = P_{C_n \cap Q_n} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$. Assume that, for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

Proof. We have that C_n and Q_n are closed convex subsets of H and $F \subset C_n$ for every $n \in \mathbb{N}$. We prove that $F \subset Q_n$ for every $n \in \mathbb{N}$ and that a sequence $\{x_n\}$ is well defined. We have $x_1 = x \in C$ and $F \subset Q_1 = C$. Assume that $x_k \in C$ and $F \subset Q_k$ for some $k \in \mathbb{N}$. Since $F \subset C_k \cap Q_k$, there exists a unique element $x_{k+1} = P_{C_k \cap Q_k} f(x_k)$, and hence $\langle f(x_k) - x_{k+1}, x_{k+1} - z\rangle \ge 0$ for all $z \in C_k \cap Q_k$, which implies that $\langle f(x_k) - x_{k+1}, x_{k+1} - z\rangle \ge 0$ for all $z \in F$. Thus we get $F \subset Q_{k+1}$. Since $P_{\bigcap_{n=1}^{\infty} Q_n} f$ is a Meir–Keeler contraction on C, there exists a unique element $u \in \bigcap_{n=1}^{\infty} Q_n$ such that $P_{\bigcap_{n=1}^{\infty} Q_n} f(u) = u$. Let $z_n = P_{Q_n}(f(u))$. Since $F \subset Q_{n+1} \subset Q_n$, it follows from Theorem 2.3 that $z_n \to u = P_{\bigcap_{n=1}^{\infty} Q_n} f(u)$. We also have $x_n = P_{Q_n}(f(x_{n-1}))$ by the definition of Q_n. Therefore, as in the proof of Theorem 4.1, we get $x_n \to u$, and we obtain that (4) holds for all $y \in F$, since $F \subset Q_n$ for $n \in \mathbb{N}$, and (5) from $\liminf_{n\to\infty} a_n > -1$. It follows that $u \in F$, and therefore we have $u = z_0$. □

5. Deduced results

We will now present some convergence theorems deduced from the results in the previous section. By Theorem 4.1, we get the following result [12], which extends that in [11]. Notice that finitely many coefficients in $\{a_n\}$ can be less than or equal to -1.

Theorem 5.1 (*Kimura–Nakajo–Takahashi [12]*). *Let C be a nonempty closed convex subset of H, and let $\{T_n\}$ be a sequence of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: there exists $\{a_n\} \subset \mathbb{R}$ with $\liminf_{n\to\infty} a_n > -1$ such that $\|T_n x - z\|^2 \le \|x - z\|^2 - a_n \|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$, and $z \in F$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 - a_n \|x_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$. Assume that, for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F x$.

By Theorem 4.2, we have the following result for the modified shrinking projection method.

Theorem 5.2. *Let C be a nonempty closed convex subset of H, and let $\{T_n\}$ be a sequence of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: there exists $\{a_n\} \subset \mathbb{R}$ with $\liminf_{n\to\infty} a_n > -1$ such that $\|T_n x - z\|^2 \le \|x - z\|^2 - a_n \|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$, and $z \in F$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = T_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - a_n \|x_n - y_n\|^2\}, \\ Q_n = \begin{cases} C & (n = 1) \\ \{z \in Q_{n-1} : \langle x - x_n, x_n - z\rangle \ge 0\} & (n \ge 2), \end{cases} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for each $n \in \mathbb{N}$. Assume that, for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F x$.

Using the idea of [12, Theorem 4.6], we have the following results for a countable family of nonexpansive mappings.

Theorem 5.3. *Let C be a nonempty closed convex subset of H. Let J be a countable index set and $\{T_i : i \in J\}$ a family of nonexpansive mappings of C into itself such that $F = \bigcap_{i\in J} F(T_i) \neq \emptyset$. Let f be a Meir–Keeler contraction of C into itself, and let*

$\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T_{i(n)} x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$, where an index mapping $i : \mathbb{N} \to J$ satisfies that, for every $j \in J$, there are infinitely many $k \in \mathbb{N}$ such that $i(k) = j$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

Theorem 5.4. *Let C be a nonempty closed convex subset of H. Let J be a countable index set and $\{T_i : i \in J\}$ a family of nonexpansive mappings of C into itself such that $F = \bigcap_{i \in J} F(T_i) \neq \emptyset$. Let f be a Meir–Keeler contraction of C into itself, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = T_{i(n)} x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \begin{cases} C & (n = 1) \\ \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z\rangle \ge 0\} & (n \ge 2), \end{cases} \\ x_{n+1} = P_{C_n \cap Q_n} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$, where an index mapping $i : \mathbb{N} \to J$ satisfies that, for every $j \in J$, there are infinitely many $k \in \mathbb{N}$ such that $i(k) = j$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

6. Semigroups of quasi-pseudocontractive mappings

In this section, we discuss one-parameter Lipschitz semigroups of quasi-pseudocontractive mappings and a sequence of mappings generated by the semigroup and prove strong convergence theorems to a common fixed point of a semigroup.

Let C be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ of mappings of C into itself is called a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings with Lipschitz constants $\{L(t) : 0 \le t < \infty\}$ if it satisfies the following conditions.

(a) $T(s + t) = T(s)T(t)$ for every $s, t \ge 0$.
(b) $\|T(t)x - T(t)y\| \le L(t)\,\|x - y\|$ for all $x, y \in C$ and $t \ge 0$.
(c) $\sup_{0 \le t \le t_0} L(t) < \infty$ for some $t_0 > 0$.
(d) $\|T(t)x - T(t)z\|^2 \le \|x - z\|^2 + \|(I - T(t))x\|^2$ for every $x \in C$, $z \in F(T(t))$, and $t \ge 0$.
(e) For each $x \in C$, the mapping $t \mapsto T(t)x$ of $[0, \infty)$ into C is strongly continuous.

From (a), we may assume that $L(s + t) \le L(s)L(t)$ for all $s, t \ge 0$. Thus (c) is equivalent to $\sup_{0 \le t \le t_0} L(t) < \infty$ for every $t_0 > 0$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$; that is, $F(\mathcal{S}) = \bigcap_{t \ge 0} F(T(t))$.

Obviously, a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings is a generalization of a one-parameter nonexpansive semigroup. Moreover, the following example shows that the family of all the one-parameter nonexpansive semigroups on C is a proper subclass of the family of mappings satisfying the conditions above.

Example 6.1. Let $p_0 : [0, 2\pi] \to [0, 2\pi]$ satisfy the following conditions.

(i) p_0 is a strictly increasing and belongs to $C^1([0, 2\pi])$.
(ii) $p_0(0) = 0$ and $p_0(2\pi) = 2\pi$.
(iii) $p_0'(0) = p_0'(2\pi)$.
(iv) $0 < \inf_{\theta \in [0, 2\pi]} p_0'(\theta) < \sup_{\theta \in [0, 2\pi]} p_0'(\theta) < \infty$.

For example, a function $p_0 : [0, 2\pi] \to [0, 2\pi]$ defined by

$$p_0(\theta) = \frac{-\theta^3 + 3\pi\theta^2 + 3\theta}{2\pi^2 + 3}$$

for $\theta \in [0, 2\pi]$ satisfies these conditions.

Let $p : [0, \infty) \to \mathbb{R}$ be an extension of p_0 satisfying that $p(\theta + 2\pi n) = p_0(\theta) + 2\pi n$ for every $\theta \in [0, 2\pi)$ and $n \in \mathbb{N}$. Then it follows that p is strictly increasing and $p \in C^1([0, \infty))$. Define a family of mappings $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ on $H = \mathbb{R}^2$ as follows. For $t \ge 0$, using polar coordinates, we define a mapping $T(t) : H \to H$ by

$$T(t)(r, \theta) = (r, q_t(\theta))$$

and $T(t)(0) = 0$, where $q_t(\theta) = p(p^{-1}(\theta) + t)$. Since q_t satisfies that $q_t(\theta + 2\pi n) = q_t(\theta) + 2\pi n$ for every $\theta \in [0, 2\pi)$ and $n \in \mathbb{N}$, we see that $T(t)$ is well defined for every $\theta \ge 0$ by identifying $(r, \theta + 2\pi n) = (r, \theta)$ in H for every $n \in \mathbb{N}$.

For $(r,\theta) \in H$ and $t,s \geq 0$, we have $T(t)T(s)(r,\theta) = T(t)(r, q_s(\theta)) = (r, q_t(q_s(\theta)))$ and

$$\begin{aligned} q_t(q_s(\theta)) &= q_t(p(p^{-1}(\theta)+s)) \\ &= p(p^{-1}(p(p^{-1}(\theta)+s))+t) \\ &= p(p^{-1}(\theta)+s+t) \\ &= q_{t+s}(\theta). \end{aligned}$$

Thus we have $T(t)T(s) = T(t+s)$ for $t,s \geq 0$.

It is easy to see that $F(T(t)) = H$ if $t = 2\pi n$ for some $n \in \mathbb{N}$ and $F(T(t)) = \{0\}$ otherwise. We have

$$\|T(t)x - 0\|^2 = \|T(t)x\|^2 = \|x\|^2 \leq \|x-0\|^2 + \|(I-T(t))x\|^2$$

for $t \geq 0$ and $x \in H$. On the other hand, since the function $t \mapsto q_t(\theta)$ is continuous for fixed $\theta \geq 0$, we have that $t \mapsto T(t)x$ is strongly continuous.

For fixed $t \geq 0$, let us show that $T(t)$ is Lipschitz continuous. Using orthogonal coordinates with variables (u,v), we obtain

$$\begin{aligned} \frac{\partial T(t)}{\partial u} &= \begin{pmatrix} \cos\theta\cos q_t(\theta) + q_t'(\theta)\sin\theta\sin q_t(\theta) \\ \cos\theta\sin q_t(\theta) - q_t'(\theta)\sin\theta\cos q_t(\theta) \end{pmatrix}, \\ \frac{\partial T(t)}{\partial v} &= \begin{pmatrix} \sin\theta\cos q_t(\theta) - q_t'(\theta)\cos\theta\sin q_t(\theta) \\ \sin\theta\sin q_t(\theta) + q_t'(\theta)\cos\theta\cos q_t(\theta) \end{pmatrix}. \end{aligned}$$

Since q_t' is bounded, so are $\partial T(t)/\partial u$ and $\partial T(t)/\partial v$ for $x,y \in H$. By the mean value theorem, we obtain that $T(t)$ is Lipschitz continuous.

We also have that $\{T(t) : 0 \leq t < \infty\}$ is not a nonexpansive semigroup. Indeed, let $t_0 \in [0,\infty)$ and $\theta_0 \in [0,2\pi)$ satisfy that $0 < p'(\theta_0) < p'(\theta_0+t_0)$, and let $\theta_1 = p(\theta_0)$. Then we have

$$q_{t_0}'(\theta_1) = p'(p^{-1}(\theta_1)+t_0)(p^{-1})'(\theta_1) = \frac{p'(p^{-1}(\theta_1)+t_0)}{p'(p^{-1}(\theta_1))} = \frac{p'(\theta_0+t_0)}{p'(\theta_0)} > 1.$$

Thus we can find d_0 satisfying $1 < d_0 < q_{t_0}'(\theta_1)$. Letting $\theta_2 \in (\theta_1,\infty)$ be sufficiently close to θ_1, we get

$$\frac{q_{t_0}(\theta_2) - q_{t_0}(\theta_1)}{\theta_2 - \theta_1} > d_0 \quad \text{and} \quad \frac{q_{t_0}(\theta_2) - q_{t_0}(\theta_1)}{\|T(t_0)(1,\theta_2) - T(t_0)(1,\theta_1)\|} < d_0.$$

Then we have

$$\begin{aligned} \|T(t_0)(1,\theta_2) - T(t_0)(1,\theta_1)\| &> \frac{1}{d_0}(q_{t_0}(\theta_2) - q_{t_0}(\theta_1)) \\ &> \theta_2 - \theta_1 \\ &> \|(1,\theta_2) - (1,\theta_1)\|. \end{aligned}$$

Therefore $T(t_0)$ is not nonexpansive.

We generate a sequence of mappings satisfying the assumptions in the main results by using a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings. For this semigroup, we need the following lemma, given by Suzuki [22], concerning real numbers.

Lemma 6.2 (*Suzuki [22]*). *Let $\{t_n\}$ and τ be a real sequence and a real number, respectively, satisfying $\liminf_{n\to\infty} t_n \leq \tau \leq \limsup_{n\to\infty} t_n$. Suppose that either $\limsup_{n\to\infty}(t_{n+1}-t_n) \leq 0$ or $\liminf_{n\to\infty}(t_{n+1}-t_n) \geq 0$ holds. Then, τ is a cluster point of $\{t_n\}$. Moreover, for $\epsilon > 0$ and $k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $\left|t_j - \tau\right| < \epsilon$ for $j \in \mathbb{N}$ with $m_0 \leq j \leq m_0 + k$.*

By this lemma and the method of [22, Theorem 4], we get the following result for one-parameter Lipschitz quasi-pseudocontractive semigroups.

Lemma 6.3. *Let C be a nonempty closed convex subset of H, and let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings on C with Lipschitz constants $\{L(t)\} \subset [1,\infty)$ such that $F = F(\mathcal{S}) \neq \emptyset$. Let $\{t_n\} \subset [0,\infty)$ be such that $0 = \liminf_{n\to\infty} t_n < s_0 = \limsup_{n\to\infty} t_n \leq \infty$ and $\lim_{n\to\infty}(t_{n+1}-t_n) = 0$. Define a sequence $\{T_n\}$ of mappings on C by*

$$T_n x = T(t_n)(\alpha_n x + (1-\alpha_n)T(t_n)x)$$

for all $n \in \mathbb{N}$ and $x \in C$, where $\sqrt{1+L(t_n)^2}/(\sqrt{1+L(t_n)^2}+1) \leq \alpha_n \leq 1$ for all $n \in \mathbb{N}$. Then, the following hold.

(i) *$\|T_n x - z\|^2 \leq \|x - z\|^2 + \alpha_n \|(I-T_n)x\|^2$ for all $n \in \mathbb{N}$, $x \in C$, and $z \in F(T(t_n))$.*
(ii) *If $(L-1)/L < \liminf_{n\to\infty} \alpha_n$, where $L = \sup_{0\leq t\leq s_0} L(t)$, then $\bigcap_{n=1}^{\infty} F(T_n) = F$ and, for a sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply that $z \in F$.*

Proof. First let us show (i). As in the proof of [23, Theorem 3.1], we have

$$\|T_n x - z\|^2 \le \|x - z\|^2 + \beta_n \|(I - T(t_n))x\|^2 + \alpha_n \|(I - T_n)x\|^2$$

for every $n \in \mathbb{N}$, $x \in C$, and $z \in F(T(t_n))$, where $\beta_n = (1-\alpha_n)(L(t_n)^2\alpha_n^2 - 2(1+L(t_n)^2)\alpha_n + (1+L(t_n)^2))$ for $n \in \mathbb{N}$. Since $\sqrt{1+L(t_n)^2}/(\sqrt{1+L(t_n)^2}+1) \le \alpha_n \le 1$, we have $\beta_n \le 0$, and hence $\|T_n x - z\|^2 \le \|x-z\|^2 + \alpha_n \|(I-T_n)x\|^2$.

For (ii), let $\{z_n\}$ be a sequence in C and $z \in C$ such that $z_n \to z$ and $T_n z_n \to z$. Let $0 < t_0 < s_0$ and $0 \le \tau \le t_0$. By Lemma 6.2, there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \ge \tau$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} t_{n_i} = \tau$; see also [22, Theorem 4]. It follows that

$$\begin{aligned}\left\|T(t_{n_i})z_{n_i} - z_{n_i}\right\| &\le \left\|T_{n_i}z_{n_i} - z_{n_i}\right\| + \left\|T_{n_i}z_{n_i} - T(t_{n_i})z_{n_i}\right\| \\ &\le \left\|T_{n_i}z_{n_i} - z_{n_i}\right\| + L(t_{n_i})(1-\alpha_{n_i})\left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\| \\ &\le \left\|T_{n_i}z_{n_i} - z_{n_i}\right\| + L(1-\alpha_{n_i})\left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\|,\end{aligned}$$

which implies that $(1 - L(1-\alpha_{n_i}))\left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\| \le \left\|T_{n_i}z_{n_i} - z_{n_i}\right\|$ for all $i \in \mathbb{N}$. Since $(L-1)/L < \liminf_{n\to\infty} \alpha_n$, we get $\lim_{i\to\infty}\left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\| = 0$. We also have

$$\begin{aligned}\left\|z_{n_i} - T(\tau)z_{n_i}\right\| &\le \left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\| + \left\|T(t_{n_i})z_{n_i} - T(t_{n_i})z\right\| + \left\|T(t_{n_i})z - T(\tau)z\right\| + \left\|T(\tau)z - T(\tau)z_{n_i}\right\| \\ &\le \left\|z_{n_i} - T(t_{n_i})z_{n_i}\right\| + 2L\left\|z_{n_i} - z\right\| + \left\|T(t_{n_i})z - T(\tau)z\right\|,\end{aligned}$$

and hence $z = T(\tau)z$ for every $\tau \in [0, t_0]$. Let $t > 0$. Then, letting m be a maximum integer not exceeding t/t_0, we have $t = mt_0 + t - mt_0$ and $T(t - mt_0)z = T(mt_0)z = z$. Thus we get

$$T(t)z = T(mt_0 + t - mt_0)z = T(mt_0)T(t - mt_0)z = z.$$

Therefore, $z \in F$. From this argument, we get $\bigcap_{n=1}^{\infty} F(T_n) = F$. □

Using this result, we get the following theorems for a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings.

Theorem 6.4. *Let C be a nonempty closed convex subset of H, and let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings on C with Lipschitz constants $\{L(t)\} \subset [1,\infty)$ such that $F = F(\mathcal{S}) \ne \emptyset$. Let $\{t_n\} \subset [0,\infty)$ be such that $0 = \liminf_{n\to\infty} t_n < s_0 = \limsup_{n\to\infty} t_n \le \infty$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Let $\{\alpha_n\} \subset \mathbb{R}$ be such that $\sqrt{1+L(t_n)^2}/(\sqrt{1+L(t_n)^2}+1) \le \alpha_n \le 1$ for all $n \in \mathbb{N}$, $(L-1)/L < \liminf_{n\to\infty}\alpha_n$, and $\limsup_{n\to\infty}\alpha_n < 1$, where $L = \sup_{0\le t\le s_0} L(t)$. Let f be a Meir–Keeler contraction of C into itself, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T(t_n)(\alpha_n x_n + (1-\alpha_n)T(t_n)x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n\|x_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

Theorem 6.5. *Let C be a nonempty closed convex subset of H, and let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a one-parameter Lipschitz semigroup of quasi-pseudocontractive mappings on C with Lipschitz constants $\{L(t)\} \subset [1,\infty)$ such that $F = F(\mathcal{S}) \ne \emptyset$. Let $\{t_n\} \subset [0,\infty)$ be such that $0 = \liminf_{n\to\infty} t_n < s_0 = \limsup_{n\to\infty} t_n \le \infty$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Let $\{\alpha_n\} \subset \mathbb{R}$ be such that $\sqrt{1+L(t_n)^2}/(\sqrt{1+L(t_n)^2}+1) \le \alpha_n \le 1$ for all $n \in \mathbb{N}$, $(L-1)/L < \liminf_{n\to\infty}\alpha_n$, and $\limsup_{n\to\infty}\alpha_n < 1$, where $L = \sup_{0\le t\le s_0} L(t)$. Let f be a Meir–Keeler contraction of C into itself, and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = T(t_n)(\alpha_n x_n + (1-\alpha_n)T(t_n)x_n), \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n\|x_n - y_n\|^2\}, \\ Q_n = \begin{cases} C & (n = 1) \\ \{z \in Q_{n-1} : \langle f(x_{n-1}) - x_n, x_n - z\rangle \ge 0\} & (n \ge 2), \end{cases} \\ x_{n+1} = P_{C_n \cap Q_n} f(x_n) \end{cases}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $z_0 \in F$, which satisfies $P_F f(z_0) = z_0$.

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References

[1] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.
[2] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 58 (1992) 486–491.
[3] A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000) 46–55.
[4] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.
[5] T. Suzuki, Moudafi's viscosity approximations with Meir–Keeler contractions, J. Math. Anal. Appl. 325 (2007) 342–352.
[6] Y. Haugazeau, Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes, Thèse, Université de Paris, Paris, France, 1968.
[7] H.H. Bauschke, P.L. Combettes, A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces, Math. Oper. Res. 26 (2001) 248–264.
[8] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003) 372–379.
[9] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces, Taiwanese J. Math. 10 (2006) 339–360.
[10] M.V. Solodov, B.F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, Math. Program. Ser. A 87 (2000) 189–202.
[11] W. Takahashi, Y. Takeuchi, R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008) 276–286.
[12] Y. Kimura, K. Nakajo, W. Takahashi, Strongly convergent iterative schemes for a sequence of nonlinear mappings, J. Nonlinear Convex Anal. 9 (2008) 407–416.
[13] Y. Kimura, W. Takahashi, On a hybrid method for a family of relatively nonexpansive mappings in a Banach space, J. Math. Anal. Appl. 357 (2009) 356–363.
[14] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
[15] W. Takahashi, Convex Analysis and Approximation of Fixed Points, Yokohama Publishers, Yokohama, 2000 (in Japanese).
[16] S. Banach, Sur les opération dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math. 3 (1922) 133–181.
[17] A. Meir, E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969) 326–329.
[18] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Adv. Math. 3 (1969) 510–585.
[19] G. Beer, Topologies on Closed and Closed Convex Sets, Kluwer Academic Publishers Group, Dordrecht, 1993.
[20] M. Tsukada, Convergence of best approximations in a smooth Banach space, J. Approx. Theory 40 (1984) 301–309.
[21] X.L. Qin, Y.J. Cho, S.M. Kang, H.Y. Zhou, Convergence theorems of common fixed points for a family of Lipschitz quasi-pseudocontractions, Nonlinear Anal. 71 (2009) 685–690.
[22] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005) 227–239.
[23] Y. Kimura, K. Nakajo, W. Takahashi, Convexity of the set of fixed points of a quasi-pseudocontractive type Lipschitz mapping and the shrinking projection method, Sci. Math. Jpn. 70 (2009) 213–220.