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Factorization of *J*-Expansive Meromorphic Operator-valued Functions

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The factorization theorems are a generalization for J-biexpansive meromorphic *operator-valued* functions on an *infinite-dimensional* Hilbert space of the theorems on decomposition of J-expansive matrix functions on a finite-dimensional Hilbert space due to A. V. Efimov and V. P. Potapov [*Uspekhi Mat. Nauk* **28** (1973), 65–130; *Trudy Moskov. Mat. Obšč.* **4** (1955), 125–236]. They also generalize theorems on factorization of J-expansive meromorphic operator functions due to Ju. P. Ginzburg [*Izv. Vysš. Učebn. Zaved. Matematika* **32** (1963), 45–53]. Within the framework of generalized network theory, the results can be applied to the J-biexpansive real operators that characterize a Hilbert port. Application of the extraction procedure to a given real operator leads to its splitting into a product of real factors, corresponding to Hilbert ports of a simpler structure. This can be interpreted as an extension of the classical method of synthesis of passive n-ports by factor decomposition.

1.

The theory of electrical circuits has acquired an increasingly complex mathematical structure since the forties. It has attracted the interest of researchers not only due to the practical applications, but also for its theoretical aspects. The basic principles of modern network theory can be found in many books (see, for instance, [1–3]). An important branch of this theory, the synthesis of linear passive n-ports, is intimately related to the concept of a J-expansive matrix: scattering, chain and transfer matrices are J-expansive matrix functions in the right half plane, $\operatorname{Re} p > 0$. This paper studies factorizations of J-expansive operator functions as an extension of the synthesis of n-ports by product decomposition of their characteristic matrices developed by Efimov and Potapov [4], Belevitch [5], Youla [6] and others.

In his profound work, Potapov [7] obtains expressions for the factorization of a J-expansive matrix function $S(p)$ into a product of the form $S(p) = A(p).\,B(p)$, with both $A(p)$ and $B(p)$ J-expansive in the right half

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plane. The matrix $A(p)$ is holomorphic in $\mathrm{Re}\,p > 0$ together with its inverse $A^{-1}(p)$, and $B(p)$ is a Blaschke product formed with the poles of $S(p)$ and $S^{-1}(p)$ in $\mathrm{Re}\,p > 0$ (Potapov considers J-contractive, i.e., $-J$-expansive matrix functions in the unit circle $|z| < 1$). Matrices associated with electrical n-ports (and with physical systems in general) have an additional property which is reality; for such matrices this condition can be expressed by the identity $S(\bar{p}) = \overline{S(p)}$ in $\mathrm{Re}\,p > 0$, where $\bar{p}$ is the complex conjugate of p and $\bar{S}$ is the matrix whose elements are the complex conjugates of the elements of S. Factorization in terms of real J-expansive matrix functions with special emphasis on scattering, chain and transfer matrices, introducing a simplification of Potapov's normalization condition for the convergence of the Blaschke products, has been obtained by González Domínguez [8].

A natural extension of the concept of an n-port, with interesting applications to waveguides, is that of the Hilbert port, which can be described by an operator acting on "signals" that take values on a Hilbert space $\mathcal{H}$ where a conjugation is defined (see, for example, Zemanian [9]). For a passive Hilbert port, the characteristic operators are J-expansive in $\mathrm{Re}\,p > 0$. The factorization theorems contained in the present paper, which generalize results due to Ginzburg [10] and Kovalishina and Potapov [11], can be applied to obtain the decomposition of a real operator in products of "elementary" real operators, thus broadening the concept of synthesis of passive networks by factorization to include Hilbert ports.

2.

Let $\mathcal{H}$ be a Hilbert space, E_+ a projector in $\mathcal{H}$, $E_- = I - E_+$. We define the operator $J = E_+ - E_-$. A linear bounded operator U in $\mathcal{H}$ is *J-unitary* iff $U^*JU = J$, $UJU^* = J$, where U^* denotes the adjoint of U. A linear bounded operator Y is *J-expansive* iff $Y^*JY \geq J$. It is *J-biexpansive* if both Y and Y^* are J-expansive.

The symbol $\mathcal{H}^2$ denotes the product Hilbert space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$. Given $x = (x_1 x_2)'$ and $y = (y_1 y_2)' \in \mathcal{H}^2$, their scalar product in $\mathcal{H}^2$ is $(x,y)_2 = (x_1,y_1) + (x_2,y_2)$, where $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$. By $(xy)'$ we denote the vector $\binom{x}{y}$.

Let S_J be the class of operators $S(p)$ holomorphic in the open right half plane ($\mathrm{Re}\,p > 0$), except for a set of isolated points, that are equal to a J-biexpansive operator at each point of holomorphism. We are interested in the class M_J of operators $S(p) \in S_J$ such that $S^{-1}(p) \in S_{-J}$. This class M_J is the class of J-biexpansive operators *meromorphic* in the right half plane.

We shall extend the definition of $S(p)$ to the left half plane taking

$$S(p) = JS^{*-1}(-\bar{p})J, \qquad \mathrm{Re}\,p < 0.$$

This is a natural extension for, in the case when $S(p)$ is holomorphic and J-unitary in some segment of the imaginary axis, it coincides with the analytic continuation of $S(p)$ into the left half plane. This symmetry principle, for *operator-valued* functions that take J-unitary values on an arc of the unit circle, has been established by Ginzburg [10, Theorem 1.6].

The operators of the class S_J have the following property.

THEOREM 1. *If the operator $S(p) \in S_J$ has poles at the points p_1, p_2 $(p_1 \neq -\bar{p}_2)$ $(\mathrm{Re}\, p_j \neq 0, j = 1,2)$ and its Laurent expansion in the neighborhood of these points is $S(p) = (p-p_j)^{-n_j} c_j + (p-p_j)^{-n_j+1} d_j + \cdots$, then, given an arbitrary set of vectors $g_1, g_2, h_1 \in \mathcal{H}$, the following inequality is valid.*

$$\begin{aligned}\left((p_1+\bar{p}_1)^{-1} c_1 J c_1^* g_1, g_1\right) &+ \left((p_2+\bar{p}_2)^{-1} c_2 J c_2^* g_2, g_2\right) \\ &+ 2\,\mathrm{Re}\left((p_1+\bar{p}_2)^{-1} c_1 J c_2^* g_2, g_1\right) + 2\,\mathrm{Re}\left((p_1-p)^{-1} c_1 h_1, g_1\right) \\ &+ 2\,\mathrm{Re}\left((p_2-p)^{-1} c_2 h_1, g_2\right) + (W(p) h_1, h_1) \geq 0,\end{aligned} \tag{2.1}$$

where $W(p) = (S^*(p) J S(p) - J)/(p+\bar{p})$.

The proof of the theorem follows from the matrix inequality for J-biexpansive operators contained in [12, p. 12].

Let us define the following operators in $\mathcal{H}^2$, in terms of 2×2 matrices of operators in $\mathcal{H}$.

$$Z = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \qquad D = \begin{bmatrix} (p_1+\bar{p}_1)^{-1} J & (p_1+\bar{p}_2)^{-1} J \\ (p_2+\bar{p}_1)^{-1} J & (p_2+\bar{p}_2)^{-1} J \end{bmatrix},$$

$$A = ZDZ^*, \qquad L(p) = \begin{bmatrix} (p_1-p)^{-1} I & 0 \\ 0 & (p_2-p)^{-1} I \end{bmatrix},$$

$$\hat{W}(p) = \begin{bmatrix} W(p)/2 & 0 \\ 0 & W(p)/2 \end{bmatrix}.$$

Taking these definitions into account, the inequality of Theorem 1 can be expressed in terms of $g = (g_1 g_2)'$ and $h = (h_1 h_1)'$, as follows

$$(Ag, g)_2 \left(\hat{W}(p) h, h\right)_2 \geq \left|(L^*(p) Z^* g, h)_2\right|^2. \tag{2.2}$$

We shall prove the following

Lemma 1. *If the operator $S(p) \in S_J$ has poles at the points p_1, p_2 ($p_1 \neq -\bar{p}_2$) and given f, $u \in \mathcal{H}^2$, then*

$$\gamma(u,u)_2(Af,f)_2 \geq |(Z^*f,u)_2|^2,$$

where $\gamma > 0$ is independent of f and u.

Proof. Let us apply the inequality of Theorem 1, taking $h_1 = (p_1 - p)u_1 \in \mathcal{H}$, $g = te^{i\phi}f$, $f = (f_1 f_2)^t \in \mathcal{H}$, $t \in \mathfrak{R}$, $\phi \in \mathfrak{R}$.

We have thus

$$t^2(Af,f) + |p_1 - p|^2(W(p)u_1,u_1) + 2t\,\mathrm{Re}\{e^{i\phi}(c_1u_1,f_1)\} + 2t\,\mathrm{Re}\{e^{i\phi}(p_1 - p)(p_2 - p)^{-1}(c_2u_1,f_2)\} \geq 0.$$

Let p_0 and $\tilde{p}_0$ be points of holomorphism of $S(p)$ such that $p_1 - p_0 = \kappa(p_2 - p_1)$, $p_1 - \tilde{p}_0 = -\kappa(p_2 - p_1)$, where κ is a constant and $0 < \kappa < 1$. It can readily be seen that

$$t^2(Af,f) + \left(\left[|p_1 - p_0|^2W(p_0) + |p_1 - \tilde{p}_0|^2W(\tilde{p}_0)\right]u_1,u_1\right) - 2t|(c_1u_1,f_1)| \geq 0.$$

Therefore,

$$t^2(Af,f) + \kappa^2|p_2 - p_1|^2([W(p_0) + W(\tilde{p}_0)]u_1,u_1) - 2t|(c_1u_1,f_1)| \geq 0.$$

The same arguments show that a similar inequality holds for $h_2 = (p_2 - p)u_2$, that is,

$$t^2(Af,f) + \kappa^2|p_2 - p_1|^2([W(p_0) + W(\tilde{p}_0)]u_2,u_2) - 2t|(c_2u_2,f_2)| \geq 0.$$

Finally, we conclude that

$$4|p_2 - p_1|^2([\hat{W}(p_0) + \hat{W}(\tilde{p}_0)]u,u)_2(Af,f)_2 \geq |(Zu,f)_2|^2. \quad (2.3)$$

From this last inequality the thesis follows.

3.

The following lemma is of major importance towards our objective of factorization of operators.

Lemma 2. *Let A and Z be the bounded linear operators defined previously. Given the operator equation $AQ = Z$, the following properties hold:*

(I) *the operator* $Q = A^{[-1]}Z$ (*where* $A^{[-1]}$ *denotes the inverse of the operator defined by the hermitian operator* A *on its range*) *is defined on a subspace* $\mathcal{L}$ *dense in* $\mathcal{H}^2$;

(II) *the operator* Z^*Q *is bounded on* $\mathcal{L}$.

Proof. Let $T = ZD$, where Z and D are the operators defined previously. It follows that $A = ZDZ^* = TZ^* = ZT^*$. The operator Q is defined on the subspace $\mathcal{L} = \ker Z + \operatorname{rge} T^*$. We shall show that this subspace is dense in $\mathcal{H}^2$. If there exists a vector $x \perp \mathcal{L}$, then $x \in \ker T$ and $x \in \overline{\operatorname{rge} Z^*}$ (the closure of the range of Z^*). Therefore, there exists a sequence $\{x_n\}$ of vectors belonging to the range of Z^* that converges to x, that is $x_n \to x$ and $x_n = Z^*y_n$. Let us write inequality (2.2) taking $g = y_n$ and $h = (h_1\, h_1)'$. We have, then,

$$(ZDZ^*y_n, y_n)_2(\hat{W}(p)h, h)_2 \geq |(L^*(p)Z^*y_n, h)_2|^2.$$

Taking limits for $n \to \infty$,

$$(Dx, x)_2(\hat{W}(p)h, h)_2 \geq |(x, L(p)h)_2|^2. \tag{3.1}$$

Since $x \in \ker T$, then $Tx = ZDx = 0$. Therefore $Dx \in \ker Z$. Taking into account that $x \in \overline{\operatorname{rge} Z^*}$, we conclude that $x \perp Dx$. Relationship (3.1) implies that $(x, L(p)h)_2 = 0$. Putting $x = (x_1\, x_2)'$ we have

$$(p_1 - p)^{-1}(x_1, h_1) + (p_2 - p)^{-1}(x_2, h_1) = 0.$$

Since $(p_1 - p)^{-1}$ and $(p_2 - p)^{-1}$ are linearly independent and noting that h_1 is arbitrary we obtain that $x_1 = x_2 = 0$, that is $x = 0$. Thus, $\mathcal{L}$ is dense in $\mathcal{H}^2$.

Using Lemma 1 with $f = Qu$, $u \in \mathcal{L}$ and taking into account that $AQu = Zu$ so that $(Zu, Qu)_2 \geq 0$, we obtain the following inequality.

$$(Zu, Qu)_2^2 \leq \gamma(u, u)_2(Zu, Qu)_2.$$

Therefore

$$0 \leq (Zu, Qu)_2 / (u, u)_2 \leq \gamma. \tag{3.2}$$

This proves that the operator Z^*Q is bounded on $\mathcal{L}$.

The boundedness of Z^*Q on a dense subspace permits the extension of the operator to all $\mathcal{H}^2$. We shall denote this operator with the symbol K. For every $u \in \mathcal{L}$, $Ku = Z^*Qu$. Moreover, inequality (3.2) implies that $K \geq 0$.

4.

Let us define the operators in $\mathcal{H}^2$, $G = (G)_{ij} = J$, $B = (B)_{ij} = p\delta_{ij}I$. These operators satisfy the following identities.

$$ZGZ^* = AB^* + BA, \qquad ZB = BZ.$$

Then, for every $u \in \mathcal{L}$ we have

$$\begin{aligned} ZGZ^*Qu &= AB^*Qu + BAQu \\ &= AB^*Qu + BZu \\ &= AB^*Qu + ZBu. \end{aligned}$$

Therefore,

$$Z(GZ^*Q - B)u = AB^*Qu.$$

Since the operator Q^* is defined on the range of A, the following identities hold,

$$Q^*Z(GZ^*Q - B)u = Q^*AB^*Qu = Z^*B^*Qu = B^*Z^*Qu. \tag{4.1}$$

The fact that K is the extension of the bounded operator Z^*Q to $\mathcal{H}^2$ allows us to conclude that, from (4.1), we have

$$K(GK - B)u = B^*Ku, \qquad u \in \mathcal{L}.$$

The operators K, G and B are bounded, therefore this identity is valid for all $x \in \mathcal{H}^2$. That is,

$$KGK = KB + B^*K. \tag{4.2}$$

It is readily seen that

$$ZDK = Z. \tag{4.3}$$

The operator K acting on $\mathcal{H}^2$ can be written in matrix form as

$$K = (K)_{ij}.$$

From (4.3) it follows that, for $j = 1,2$,

$$c_j = c_j\left\{\left(p_j + \bar{p}_1\right)^{-1}J(K_{11} + K_{12}) + \left(p_j + \bar{p}_2\right)^{-1}J(K_{21} + K_{22})\right\}. \tag{4.4}$$

Since $K \geq 0$, we obtain the following relationships

$$(KL(p)x, L(p)x)_2 = \sum_{i,j=1}^{2} \left((\bar{p} - \bar{p}_i)^{-1}(p - p_j)^{-1}K_{ij}x_j, x_i\right) \geq 0,$$

where $x = (x_1 x_2)^t$. Taking (4.2) into account, we have

$$(K_{i1} + K_{i2})J(K_{1j} + K_{2j}) = K_{ij}(\bar{p}_i + p_j), \qquad (i,j = 1,2).$$

Let us define the operators

$$P_j = J(K_{1j} + K_{2j}), \qquad (j = 1,2). \tag{4.5}$$

The following identities can be readily obtained.

$$(\bar{p}_i + p_j)^{-1} P_i^* J P_j = (\bar{p}_i + p_j)^{-1}(K_{i1} + K_{i2})J(K_{1j} + K_{2j}) = K_{ij}, \tag{4.6}$$

$$\sum_{i=1}^{2} (\bar{p}_i + p_j)^{-1} P_i^* J P_j = K_{1j} + K_{2j} = JP_j. \tag{4.7}$$

LEMMA 3. *The operator* $\Omega(p) = I + (p - p_1)^{-1}P_1 + (p - p_2)^{-1}P_2$, *where* P_1 *and* P_2 *are defined in* (4.5), *is J-biexpansive in* $\operatorname{Re} p > 0$ *and J-unitary on* $\operatorname{Re} p = 0$.

Proof. Using relationships (4.6) and (4.7), we obtain

$$\Omega^*(p)J\Omega(p) - J = (p + \bar{p}) \sum_{i,j=1}^{2} (\bar{p} - \bar{p}_i)^{-1} K_{ij}(p - p_j)^{-1}$$

$$= (p + \bar{p})(I I)L^*(p)KL(p)(I I)^t. \tag{4.8}$$

Therefore $\Omega^*(p)J\Omega(p) - J$ is ≥ 0 for $\operatorname{Re} p > 0$, $= 0$ for $\operatorname{Re} p = 0$ and ≤ 0 for $\operatorname{Re} p < 0$. When $\operatorname{Re} p_1 > 0$, $\operatorname{Re} p_2 > 0$, the J-biexpansivity of $\Omega(p)$ in $\operatorname{Re} p > 0$ follows directly from Lemma 6.2 of Ref. [13].

Taking into account (4.5) and (4.6) it is easy to show that

$$J\Omega^*(-\bar{p})J\Omega(p) = I. \tag{4.9}$$

The fact that for p large enough (that is $|p| > \rho$) we have

$$\|(p - p_1)^{-1}P_1 + (p - p_2)^{-1}P_2\| < \text{const} < 1,$$

implies that $\Omega^{-1}(p)$ exists for $|p| > \rho$. The identity (4.9) points out that $\Omega^{-1}(p) = J\Omega^*(-\bar{p})J$, which is equivalent to

$$\big(I + (p - p_1)^{-1}P_1 + (p - p_2)^{-1}P_2\big)\big(I - (p + \bar{p}_1)^{-1}JP_1^*J - (p + \bar{p}_2)^{-1}JP_2^*J\big) = I, \qquad |p| > \rho.$$

This implies that

$$P_jJ = \sum_{i=1}^{2} (p_j + \bar{p}_i)^{-1} P_j J P_i^*, \qquad (j = 1,2).$$

After a few simple calculations we obtain

$$\Omega(p)J\Omega^*(p) - J$$
$$= (p+\bar{p}) \sum_{i,j=1}^{2} (\bar{p}-\bar{p}_i)^{-1}(p_j+\bar{p}_i)^{-1} P_j J P_i^* (p-p_j)^{-1}.$$

Thus, $\Omega(p)$ is J-unitary on the imaginary axis.

The following symmetry principle is, therefore, established:

$$\Omega^{-1}(p) = J\Omega^*(-\bar{p})J.$$

Making use of the fact that

$$\Omega^*(p)J\Omega(p) - J \le 0 \qquad \text{in } \operatorname{Re} p < 0,$$

we obtain the following inequalities:

$$J - \Omega^{*-1}(p)J\Omega^{-1}(p) \le 0 \qquad \text{in } \operatorname{Re} p < 0,$$
$$J - \Omega(-\bar{p})J\Omega^*(-\bar{p}) \le 0 \qquad \text{in } \operatorname{Re} p < 0,$$
$$J - \Omega(p)J\Omega^*(p) \le 0 \qquad \text{in } \operatorname{Re} p > 0.$$

This completes the proof of Lemma 3.

It is interesting to note that, taking into account (4.4) and the definition of P_1 and P_2,

$$c_j\Omega^{-1}(p_j) = c_j - c_j\Big((p_j+\bar{p}_1)^{-1}J(K_{11}+K_{12})$$
$$+ (p_j+\bar{p}_2)^{-1}J(K_{21}+K_{22})\Big) = 0 \qquad (j=1,2). \quad (4.10)$$

Let us write inequality (2.2), taking $g = Qu_n$, $u_n \in \mathfrak{L}$. Since

$$(Ag,g)_2 = (AQu_n, Qu_n)_2 = (Q^*AQu_n, u_n)_2 = (Ku_n, u_n)_2 \ge 0,$$

and

$$(L^*(p)Z^*g, h)_2 = (Z^*Qu_n, L(p)h)_2 = (Ku_n, L(p)h)_2,$$

therefore,

$$(Ku_n, u_n)_2(\hat{W}(p)h, h)_2 \ge |(Ku_n, L(p)h)_2|^2.$$

Let u_n be a sequence in $\mathfrak{L}$ that converges to $L(p)h$. Taking limits in the preceding inequality we conclude that

$$(\hat{W}(p)h, h)_2 = (W(p)h_1, h_1) \ge (L^*(p)KL(p)h, h)_2 \ge 0.$$

Using the definition of $W(p)$ and Eq. (4.8), we have

$$(L^*(p)KL(p)h,h)_2 = (p+\bar{p})^{-1}((\Omega^*(p)J\Omega(p) - J)h_1,h_1),$$

and

$$(p+\bar{p})^{-1}(S^*(p)JS(p) - J) \geq (p+\bar{p})^{-1}(\Omega^*(p)J\Omega(p) - J) \geq 0. \tag{4.11}$$

5.

The preceding results will be used to prove the following

THEOREM 2. *Let the operator $S(p) \in M_J$ have poles at the points p_1,p_2 of the right half plane* ($\mathrm{Re}\,p > 0$) *and a Laurent expansion in the neighborhood of these points of the form*

$$S(p) = (p-p_j)^{-n_j}c_j + (p-p_j)^{-n_j+1}d_j + \cdots, \qquad (j=1,2).$$

If we define the operator $\Omega(p) = I + (p-p_1)^{-1}P_1 + (p-p_2)^{-1}P_2$, where P_1 and P_2 are given in (4.5), *then,*

(I) *$S(p) = S_1(p)\Omega(p)$, with $S_1(p) \in M_J$ and $\Omega(p) \in M_J$; the operator $\Omega(p)$ is J-unitary on the imaginary axis;*

(II) *at the point p_j $(j = 1,2)$ the operator $S_1(p)$ has a pole of order $n_j - 1$;*

(III) *if p_j is a pole of order $m_j > 0$ of the operator $S^{-1}(p)$, the order of the pole of $S_1^{-1}(p)$ at p_j is m_j.*

Proof.

(I) We have already shown that $\Omega(p)$ belongs to the class M_J and that it is J-unitary on the imaginary axis. Let p be a point of holomorphism of $S(p)$ and $S^{-1}(p)$. From (4.11) we obtain

$$\Omega^*(p)S_1^*(p)JS_1(p)\Omega(p) - \Omega^*(p)J\Omega(p) \geq 0, \qquad \mathrm{Re}\,p > 0.$$

Taking into consideration the fact that $\Omega(p)$ has an inverse, we get

$$S_1^*(p)JS_1(p) - J \geq 0, \qquad \mathrm{Re}\,p > 0.$$

On the other hand, since $S^{-1}(p) \in S_{-J}$, we have,

$$S_1(p)JS_1^*(p) - J = S(p)(\Omega^{-1}(p)J\Omega^{-1*}(p) - S^{-1}(p)JS^{-1*}(p))S^*(p) \geq 0, \qquad \mathrm{Re}\,p > 0.$$

Therefore $S_1(p) \in M_J$.

(II) In the neighborhood of p_1 we may write

$$S_1(p) = S(p)\Omega^{-1}(p) = S(p)\big(\Omega^{-1}(p_1) + (\Omega^{-1}(p) - \Omega^{-1}(p_1)\big)$$
$$= \big[(p-p_1)^{-n_1}c_1 + (p-p_1)^{-n_1+1} + \cdots\big]\Big\{\Omega^{-1}(p_1) + (p-p_1)$$
$$\big[(p_1+\bar{p}_1)^{-1}(p+\bar{p}_1)^{-1}JP_1^*J + (p_1+\bar{p}_2)^{-1}(p+\bar{p}_2)^{-1}JP_2^*J\big]\Big\}.$$

From Eq. (4.10), we have $c_1\Omega^{-1}(p_1) = 0$. Therefore $S_1(p)$ has a pole of order $n_1 - 1$ at $p = p_1$. A similar argument proves that $S_1(p)$ has a pole of order $n_2 - 1$ at $p = p_2$.

(III) In the neighborhood of p_j $(j = 1,2)$, let

$$S^{-1}(p) = (p-p_j)^{-m_j}a_j + (p-p_j)^{-m_j+1}b_j + \cdots.$$

In that case, we must have $c_ja_j = 0$.

Now, $S_1^{-1}(p) = \Omega(p)S^{-1}(p)$, so that in the neighborhood of p_j

$$S_1^{-1}(p) = \big(I + (p-p_1)^{-1}P_1 + (p-p_2)^{-1}P_2\big)$$
$$\big((p-p_j)^{-m_j}a_j + (p-p_j)^{-m_j+1}b_j + \cdots\big).$$

We will show that $P_ja_j = 0$.

Given that $c_ja_j = 0$, for all $x_1, x_2 \in \mathcal{H}$, the vector $u = (a_1x_1\, a_2x_2)^t$ belongs to the kernel of Z, so it belongs to $\mathfrak{L}$. This implies that $Ku = Z^*Qu = Z^*A^{[-1]}Zu = 0$, or equivalently, $K_{11}a_1 = K_{21}a_1 = K_{12}a_2 = K_{22}a_2 = 0$. Since $P_j = J(K_{1j} + K_{2j})$, then $P_ja_j = 0$. That is, the operator $S_1^{-1}(p)$ has a pole of order m_j at $p = p_j$. This ends the proof of Theorem 2.

The -J-biexpansivity of $S^{-1}(p)$ accounts for the validity of the following theorem, whose proof we shall not carry out since it is similar to that of Theorem 2.

THEOREM 3. *Let the operator $S(p)$ belong to M_J. If $S^{-1}(p)$ at p_1 and p_2 has poles of order r_1 and r_2 respectively, the following factorization formula holds*

$$S(p) = \hat{\Omega}(p)\cdot\hat{S}_1(p),$$

having the properties:

(I) $\hat{S}_1(p) \in M_J$;

(II) $\hat{\Omega}(p) \in M_J$ *and is J-unitary on the imaginary axis*;

(III) *at the point p_j $(j = 1,2)$ the operator $\hat{S}_1^{-1}(p)$ has a pole of order $r_j - 1$*;

(IV) *if p_j $(j = 1,2)$ is a pole of order $q_j > 0$ of the operator $S(p)$, the order of the pole of $\hat{S}_1(p)$ at p_j is q_j also.*

The proof of the theorems may readily be extended to extract any number of poles simultaneously, by combining the ideas contained in this presentation with the methods described in [12].

6.

If $S(p)$ is a real operator (assuming that there is a conjugation operation defined in $\mathcal{H}$), that is if $S(\bar{p}) = \overline{S(p)}$, where $\bar{S}$ is the conjugate operator of S, and p_0 is a pole of $S(p)$, then $\bar{p}_0$ is also a pole.

It can readily be seen that in this case, if $J = \bar{J}$, then

$$\Omega(p) = I + (p - p_0)^{-1}P + (p - \bar{p}_0)^{-1}\bar{P},$$

and is therefore a real operator.

Thus, the factorization procedure provides automatically a decomposition into real factors, which can be readily applied to operators that characterize Hilbert ports.

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