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Extended finite automata over groups

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Abstract

Some results from Dassow and Mitrana (Internat. J. Comput. Algebra (2000)), Griebach (Theoret. Comput. Sci. 7 (1978) 311) and Ibarra et al. (Theoret. Comput. Sci. 2 (1976) 271) are generalized for finite automata over arbitrary groups. The closure properties of these automata are poorer and the accepting power is smaller when abelian groups are considered. We prove that the addition of any abelian group to a finite automaton is less powerful than the addition of the multiplicative group of rational numbers. Thus, each language accepted by a finite automaton over an abelian group is actually a unordered vector language. Characterizations of the context-free and recursively enumerable languages classes are set up in the case of non-abelian groups. We investigate also deterministic finite automata over groups, especially over abelian groups.

Keywords: Finite automata over groups; Closure properties; Accepting capacity; Interchange lemma; Free groups

1. Introduction

One of the oldest and most investigated device in the automata theory is the finite automaton. Many fundamental properties have been established and many problems are still open.

Unfortunately, the finite automata without any external control have a very limited accepting power. Different directions of research have been considered for overcoming this limitation. The most known extension added to a finite automata is the pushdown memory. In this way, a considerable increasing of the accepting capacity has been achieved: the pushdown automata are able to recognize all context-free languages.

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Another simple and natural extension, related somehow to the pushdown memory, was considered in a series of papers [2,4,5], namely to associate an element of a given group to each configuration, but no information regarding the associated element is allowed. This value is stored in a counter. An input string is accepted if and only if the automaton reaches a designated final state with its counter containing the neutral element of the group.

Thus, new characterizations of unordered vector languages [5] and context-free languages [2] have been reported. These results are, in a certain sense, unexpected since in such an automaton the same choice is available regardless the content of its counter. More precisely, the next action is determined just by the input symbol currently scaned and the state of the machine.

In this paper, we shall consider only acceptors with a one-way input tape read from left to right and a counter able to store elements from a given group. The aforementioned papers deal with finite automata over very-well-defined groups, e.g., the multiplicative group of non-null rational numbers [5], the free group [2] or the additive group of integer vectors. Furthermore, the same idea has been applied to generative grammars [6], where a given number is associated to each production (see also [1]).

The aim of this paper is to provide some general results regardless the associated group. Thus, for any group **K** the family of all languages accepted by finite automata over **K** is a semi-AFL. We shall prove that the addition of any abelian group to a finite automaton is less powerful than the addition of the multiplicative group of rational numbers. An interchange lemma points out the main reason of power decreasing of finite automata over abelian groups. Characterizations of the context-free and recursively enumerable languages classes are set up in the case of non-abelian groups.

As far as the deterministic variants of finite automata over groups are concerned we shall show their considerable lack of accepting power.

2. Preliminaries

We assume the reader to be familiar with the basic concepts in automata and formal language theory and in the group theory. For further details, we refer to Rotman [7] and Rozenberg and Salomaa [8], respectively.

For an alphabet Σ, we denote by Σ^* the free monoid generated by Σ under the operation of concatenation; the empty string is denoted by ε and the semigroup $\Sigma^* - \{\varepsilon\}$ is denoted by Σ^+. The length of $x \in \Sigma^*$ is denoted by $|x|$.

Let $\mathbf{K} = (M, \circ, e)$ be a group under the operation denoted by $\circ$ with the neutral element denoted by e. An extended finite automaton (*EFA*) over the group $\mathbf{K}$ is a construct

$$A = (Z, \Sigma, \mathbf{K}, q_0, F, \delta),$$

where Z, Σ, q_0, F have the same meaning as for a usual finite automaton [8], namely the set of states, the input alphabet, the initial state and the set of final states, respectively, and

$$\delta : Z \times (\Sigma \cup \{\varepsilon\}) \to \mathscr{P}_f(Z \times M).$$

This sort of automaton can be viewed as a finite automaton having a counter in which any element of M can be stored. The relation $(q,m) \in \delta(s,a)$, $q,s \in Z$, $a \in \Sigma \cup \{\varepsilon\}$, $m \in M$ means that the automaton A changes its current state s into q, by reading the symbol a on the input tape, and writes in the register $x \circ m$, where x is the old content of the register. The initial value registered is e.

We shall write

$$(q,aw,m) \models_A (s,w,m \circ r) \text{ iff } (s,r) \in \delta(q,a)$$

for all $s,q \in Z$ $a \in \Sigma \cup \{\varepsilon\}$, $w \in \Sigma^*$, $m,r \in M$. The reflexive and transitive closure of the relation $\models_A$ is denoted by $\models_A^*$. Sometimes, the subscript identifying the automaton will be omitted when it is self-understood. As usual, an automaton A as above is deterministic if the following two conditions hold:

(i) $\delta(q,a)$ contains at most one element for any state q and $a \in \Sigma \cup \{\varepsilon\}$,
(ii) if $\delta(q,\varepsilon) \neq \emptyset$, then $\delta(q,a) = \emptyset$ for all $a \in \Sigma$.

The word $x \in \Sigma^*$ is accepted by the automaton A if and only if there is a final state q such that $(q_0,x,e) \models^* (q,\varepsilon,e)$. In other words, a string is accepted if the automaton completely read the word and reaches a final state when the content of the register is the neutral element of M.

The language accepted by an extended finite automaton A over a group M as above is

$$L(A) = \{x \in \Sigma^* \mid (q_0,x,e) \models_A^* (q,\varepsilon,e) \text{ for some } q \in F\}.$$

We are going to provide some results that will be useful in what follows. The notations $\mathscr{L}(REG)$ and $\mathscr{L}(EFA(\mathbf{K}))$ identify the class of regular languages and the family of languages accepted by extended finite automata over the group $\mathbf{K}$.

Theorem 1. *For any group* $\mathbf{K}$, $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(REG)$ *iff all finitely generated subgroups of* $\mathbf{K}$ *are finite.*

Proof. Let $\mathbf{K}$ be a group such that any finitely generated subgroup of $\mathbf{K}$ is finite. Let $A = (Z,\Sigma,\mathbf{K},z_0,F,\delta)$ be an EFA over $\mathbf{K} = (M,\circ,e)$. We denote by X the finite subset of M,

$$X = \{m \in M \mid (z',m) \in \delta(z,a) \text{ for some } z,z' \in Z, a \in \Sigma \cup \{\varepsilon\}\}.$$

Let $\mathbf{H} = (\langle X\rangle,\circ,e)$ be the subgroup generated by X.

We construct the finite automaton with ε-moves $B = (Z \times \langle X\rangle,\Sigma,(z_0,e),F \times \{e\},\varphi)$ with $\varphi((z,m),a) = \{(z',m \circ n) \mid (z',n) \in \delta(z,a)\}$, for all $z \in Z$, $m \in M$, $a \in \Sigma \cup \{\varepsilon\}$. One can easily prove that $(z_0,w,e) \models_A^* (z,\varepsilon,m)$ iff $((z_0,e),w) \models_B^* ((z,m),\varepsilon)$, which imply $L(A) = L(B)$.

It remains to prove that for any infinite group $\mathbf{K}$, finitely generated, there exists an EFA over $\mathbf{K}$ accepting a non-regular language. Let $\mathbf{K} = (\langle X\rangle,\circ,e)$ be such a group with the finite set of generators X. Consider the (deterministic) EFA $A=(\{z\}, Y, \mathbf{K}, z, \{z\}, \delta)$, with $Y=X\cup\{x^{-1}\mid x\in X\}$ and $\delta(z,a)=(z,a)$, for all $a\in Y$. The following facts about $L(A)$ are obvious:

1. For any $m\in\langle X\rangle$, there exists a word $v\in Y^*$ such that $(z,v,e)\models_A^* (z,\varepsilon,m)$.
2. For any $v\in Y^*$, there exists a word $w\in Y^*$ such that $vw\in L(A)$.
3. For any $k\geqslant 0$, the set

$$X_k=\{m\in\langle X\rangle\mid \exists v\in Y^*, |v|\leqslant k\colon (z,v,e)\models_A^* (z,\varepsilon,m)\}$$

is finite.

As a consequence of these facts and of the infiniteness of $\langle X\rangle$, we obtain: For all $k\geqslant 0$, there is a word v_k such that $v_k v\notin L(A)$, for all $v\in Y^*$, $|v|\leqslant k$.

But, one can easily prove that for any regular language $L\subseteq Y^*$, there exists a $k\geqslant 0$ such that for all $vw\in L$, there is $w'\in Y^*$, $|w'|\leqslant k$, with $vw'\in L$. Hence, $L(A)$ cannot be regular. $\square$

A finitely generated abelian group is finite iff all its elements are of finite order. Hence, for an abelian group $\mathbf{K}$, $\mathscr{L}(EFA(\mathbf{K}))=\mathscr{L}(REG)$ iff all elements of $\mathbf{K}$ have finite order. This is not necessarily true for non-abelian groups. We can, however, prove a pumping lemma which is very similar to the pumping lemma for regular languages.

Lemma 1. *Let $\mathbf{K}$ be some group without elements of infinite order. For any language $L\in\mathscr{L}(EFA(\mathbf{K}))$, there is a constant $n\geqslant 1$ such that, for all $x\in L$, $|x|\geqslant n$, there exist a decomposition $x=uvw$ and a natural number $q\geqslant 1$ with $|uv|\leqslant n$, $|v|\geqslant 1$, $uv^{iq+1}w\in L$, for all $i\geqslant 0$.*

Moreover, if $\mathbf{K}$ has the finite exponent p then q can uniformly be chosen as $q=p$.

Proof. Let $A=(Z,\Sigma,\mathbf{K},z_0,F,\delta)$ be an EFA over $\mathbf{K}$. We choose $n=|Z|+1$. Now consider a word $x\in\Sigma^*$ with $|x|\geqslant n$. Similar to the proof of the pumping lemma for regular languages, it can be shown that there is a decomposition $x=uvw$, $|uv|\leqslant n$, $|v|\geqslant 1$, such that

$$(z_0,uvw,e)\models_A^* (z,vw,m_1)\models_A^* (z,w,m_1\circ m_2)\models_A^* (f,\varepsilon,e),\quad z\in Z,\ f\in F.$$

Now choose q such that $m_2^q=e$. Obviously, any word $uv^{iq+1}w$ is accepted by A. $\square$

As a consequence of the aforementioned pumping lemma, we have

Proposition 1. *For any group $\mathbf{K}$, $\mathscr{L}(EFA(\mathbf{K}))$ contains the language $L=\{a^nb^n\mid n\geqslant 1\}$ iff at least one element of $\mathbf{K}$ has an infinite order.*

Proof. Let $\mathbf{K}=(M,\circ,e)$. If M contains an element, say m, of infinite order, then the deterministic $EFA\,A=\{\{z_0,q\},\{a,b\},\mathbf{K},z_0,\{q\},\delta)$ with $\delta(z_0,a)=(z_0,m)$, $\delta(z_0,b)=(q,m^{-1})$, $\delta(q,b)=(q,m^{-1})$, $\delta(q,a)=\emptyset$, accepts L.

If all elements of M have finite order in $\mathbf{K}$, then a simple application of the above pumping lemma yields $L \notin \mathscr{L}(EFA(\mathbf{K}))$. □

For a group $\mathbf{K}$, let $\mathscr{F}(\mathbf{K})$ denote the family of all finitely generated subgroups of $\mathbf{K}$.

Theorem 2. *For any group* $\mathbf{K}$,

$$\mathscr{L}(EFA(\mathbf{K})) = \bigcup_{\mathbf{H} \in \mathscr{F}(\mathbf{K})} \mathscr{L}(EFA(\mathbf{H})).$$

Proof. Let $\mathbf{K} = (M, \circ, e)$ be a group. The inclusion

$$\mathscr{L}(EFA(\mathbf{K})) \supseteq \bigcup_{\mathbf{H} \in \mathscr{F}(\mathbf{K})} \mathscr{L}(EFA(\mathbf{H}))$$

holds, since $\mathscr{L}(EFA(\mathbf{K})) \supseteq \mathscr{L}(EFA(\mathbf{H}))$, for any subgroup $\mathbf{H}$ of $\mathbf{K}$.

On the other hand, let $A = (Z, \Sigma, \mathbf{K}, z_0, F, \delta)$ be an EFA over $\mathbf{K}$. The group $\mathbf{H} = (\langle X \rangle, \circ, e)$, where $X = \{m \in M \mid (q, m) \in \delta(z, a) \text{ for some } q, z \in Z, a \in \Sigma \cup \{\varepsilon\}\}$ is a finitely generated subgroup of $\mathbf{K}$. Obviously, during any computation in the counter of A appear only elements of $\langle X \rangle$. Therefore, the automaton A can be viewed as an automaton over $\mathbf{H}$. More precisely, $A' = (Z, \Sigma, \mathbf{H}, z_0, F, \delta)$ accepts the same language as A does. This proves the second inclusion and thus the theorem. □

3. Closure properties of the families $\mathscr{L}(EFA(\mathbf{K}))$

The closure properties of the languages families accepted by EFA have been investigated for a few groups, e.g. for $\mathbf{Q} = (\mathbb{Q} - \{0\}, \cdot, 1)$, $\mathbf{Z} = (\mathbb{Z}, +, 0)$ (see [6]), or the free group with two generators [2]. The goal of this section is to generalize the results to arbitrary groups. We recall that a family of languages closed under union, intersection with regular languages, homomorphisms and inverse homomorphisms is called a semi-AFL (abstract family of languages) [3].

Theorem 3. *For every group* $\mathbf{K}$, *the family* $\mathscr{L}(EFA(\mathbf{K}))$ *is a semi-AFL closed under concatenation with regular languages.*

Proof. Classical constructions of finite automata for proving the closure of regular languages under union and intersection can be carried over the extended models.

Let $\mathbf{K} = (M, \circ, e)$ be an arbitrary group; we prove now the closure of the class $\mathscr{L}(EFA(\mathbf{K}))$ under substitution with regular sets. To this end, let L be a language over $\Sigma = \{a_1, a_2, \ldots, a_n\}$ recognized by the EFA $A = (Z, \Sigma, \mathbf{K}, z_0, F, \delta)$. Assume that $s : \Sigma^* \to 2^{\Delta^*}$ is a substitution which maps each symbol of Σ into a regular language. More precisely, $s(a_i) = L_i$, $L_i \in \mathscr{L}(REG)$, $1 \leqslant i \leqslant n$. Of course, every language L_i is recognized by a deterministic finite automata $A_i = (Z_i, \Delta, z_0^i, F_i, \delta_i)$.

Without loss of generality, we may suppose the sets Z_i pairwise disjoint. Construct the *EFA* $B=(Z',\Delta,\mathbf{K},z_0,F,\theta)$, where

$$Z' = Z \cup \bigcup_{i=1}^{n} Z_i \times Z$$

and the transition mapping θ is defined as follows:

$$\begin{aligned}
&\theta(z,\varepsilon)=\delta(z,\varepsilon)\cup\{([z_0^i,z],e)\,|\,1\leqslant i\leqslant n\}, \quad z\in Z,\\
&\theta([z,z'],a)=\{([\delta_i(z,a),z'],e)\}, \quad z\in Z_i,\ z'\in Z,\ 1\leqslant i\leqslant n,\\
&\theta([z,z'],\varepsilon)=\{(q,m)\,|\,(q,m)\in\delta(z',a_i)\}, \quad z\in F_i,\ z'\in Z,\ 1\leqslant i\leqslant n.
\end{aligned}$$

Let $x=a_{i_i}a_{i_2}\cdots a_{i_k}\in L(A)$ and $w=w_1w_2\cdots w_k\in s(x)$ with $w_j\in s(a_{i_j})$, $1\leqslant j\leqslant k$. We list below a computation in B for the input w:

$$\begin{aligned}
&(s_0,w_1w_2\cdots w_k,e)\models([z_0^{i_1},s_0],w_1w_2\cdots w_k,e)\models^*([q_1,s_0],w_2w_3\cdots w_k,e)\\
&\quad\models(s_1,w_2w_3\cdots w_k,m_1)\models([z_0^{i_2},s_1],w_2w_3\cdots w_k,m_1)\models^*([q_2,s_1],w_3w_4\\
&\qquad\cdots w_k,m_1)\\
&\quad\models(s_2,w_3w_4\cdots w_k,m_1m_2)\models\cdots\models^*([q_k,s_{k-1}],\varepsilon,m_1m_2\cdots m_{k-1})\models(s_k,\varepsilon,e),
\end{aligned}$$

where

$$\begin{aligned}
&s_0=z_0, \quad s_k\in F,\\
&(s_j,m_j)\in\delta(s_{j-1},a_{i_j}), \quad 1\leqslant j\leqslant k
\end{aligned}$$

and

$$q_j=\delta_{i_j}(z_0^{i_j},w_j)\in F_{i_j}, \quad 1\leqslant j\leqslant k.$$

In conclusion, $s(L(A))\subseteq L(B)$. Also the converse inclusion can be easily checked, hence $s(L(A))=L(B)$ holds.

The closure under homomorphisms is immediate while the closure under inverse homomorphisms follows from Ginsburg and Greibach [3]. Consequently, $\mathscr{L}(EFA(\mathbf{K}))$ is a semi-AFL.

As $\mathscr{L}(EFA(\mathbf{K}))$ is trivially closed under concatenation with symbols, by its closure under substitutions with regular sets one infers its closure under concatenation with regular sets, as well. □

For two groups $\mathbf{K}_1=(M_1,\circ_1,e_1)$ and $\mathbf{K}_2=(M_2,\circ_2,e_2)$, we define the group $\mathbf{K}_1\times\mathbf{K}_2=(M_1\times M_2,\circ,(e_1,e_2))$ with $(m_1,m_2)\circ(n_1,n_2)=(m_1\circ_1 n_1,m_2\circ_2 n_2)$.

Theorem 4. *Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be two groups. The languages $L_1\cap L_2$ and L_1L_2 are in $\mathscr{L}(EFA(\mathbf{K}_1\times\mathbf{K}_2))$ for any two languages $L_i\in\mathscr{L}(EFA(\mathbf{K}_i))$, $i=1,2$.*

Proof. Let $A_i=(Z_i,\Sigma_i,\mathbf{K}_i,z_i^0,F_i,\delta_i)$, $i=1,2$, be two EFA over $\mathbf{K}_i$, respectively. We assume that $Z_1\cap Z_2$ is empty.

We have $L(B) = L_1 \cap L_2$ and $L(C) = L_1 L_2$, where $B = (Z_1 \times Z_2, \Sigma_1 \cap \Sigma_2, \mathbf{K}_1 \times \mathbf{K}_2, (z_1^0, z_2^0), F_1 \times F_2, \delta)$ with

$$\begin{aligned}\delta((z_1,z_2),a) = \{((z_1',z_2'),(m_1,m_2)) \,|\, (z_1',m_1) \in \delta_1(z_1,a), (z_2',m_2) \\ \in \delta_2(z_2,a)\}, \quad z_1 \in Z_1, z_2 \in Z_2,\ a \in \Sigma_1 \cap \Sigma_2,\\ \delta((z_1,z_2),\varepsilon) = \{((z_1',z_2),(m_1,e_2)) \,|\, (z_1',m_1) \in \delta_1(z_1,\varepsilon)\} \\ \cup \{((z_1,z_2'),(e_1,m_2)) \,|\, (z_2',m_2) \in \delta_2(z_2,\varepsilon)\}, \quad z_1 \in Z_1,\ z_2 \in Z_2\end{aligned}$$

and $C = (Z_1 \cup Z_2, \Sigma_1 \cup \Sigma_2, \mathbf{K}_1 \times \mathbf{K}_2, z_1^0, F_2, \delta)$ with

$$\delta(z,a) = \{(z',(m,e_2)) \,|\, (z',m) \in \delta_1(z,a)\}, \quad z \in Z_1,\ a \in \Sigma_1,$$

$$\delta(z,\varepsilon) = \begin{cases} \{(z',(m,e_2)) \,|\, (z',m) \in \delta_1(z,\varepsilon)\}, & z \in Z_1 \setminus F_1,\\ \{(z',(m,e_2)) \,|\, (z',m) \in \delta_1(z,\varepsilon)\} \cup \{(z_2^0,(e_1,e_2))\}, & z \in F_1,\end{cases}$$

$$\delta(z,a) = \{(z',(e_1,m)) \,|\, (z',m) \in \delta_2(z,a)\}, \quad z \in Z_2,\ a \in \Sigma_2 \cup \{\varepsilon\}$$

which concludes the proof. □

4. EFA over abelian groups

Valence grammars and EFA have initially been introduced for the groups $\mathbf{Z}_k = (\mathbb{Z}^k, +, 0)$, $k \geqslant 1$ and $\mathbf{Q} = (\mathbb{Q} - \{0\}, \cdot, 1)$. In what follows, we shall show that the accepting capacity of EFA does not increase if we consider arbitrary abelian groups instead of $\mathbf{Q}$. Thus, every language accepted by an EFA over an abelian group is a (unordered) vector language [1]. The reason of this fact is the following fundamental result in the group theory.

Theorem 5. *A finitely generated abelian group is the direct product of a finite number of cyclic groups.*

As a consequence, a finitely generated abelian group is either finite or isomorphic to a group $\mathbf{Z}_k \times \mathbf{H}$, where k is a positive integer and H is a finite abelian group.

Theorem 6. *For a group* $\mathbf{K}$ *and a finite group* $\mathbf{H}$,

$$\mathscr{L}(EFA(\mathbf{K} \times \mathbf{H})) = \mathscr{L}(EFA(\mathbf{K})).$$

Proof. Let $\mathbf{K}$ and $\mathbf{H}$ be given by $\mathbf{K} = (M_1, \circ_1, e_1)$ and $\mathbf{H} = (M_2, \circ_2, e_2)$, and let $A = (Z, \Sigma, \mathbf{K} \times \mathbf{H}, z_0, F, \delta)$ be an *EFA* over $\mathbf{K} \times \mathbf{H}$. We construct the *EFA* over $\mathbf{K}$, $A' = (Z', \Sigma, \mathbf{K}, z_0', F', \delta')$ with $Z' = Z \times M_2$, $z_0' = (z_0, e_2)$, $F' = F \times \{e_2\}$ and

$$\begin{aligned}\delta'((z,n_2),a) = \{((z', n_2 \circ_2 m_2), m_1) \,|\, (z',(m_1,m_2)) \in \delta(z,a)\},\\ z,z' \in Z, a \in \Sigma \cup \{\varepsilon\}, \quad m_1 \in M_1,\ m_2,n_2 \in M_2.\end{aligned}$$

By induction on the number of steps, one can show that $((z_0,e_2),w,e_1) \models_{A'}^* ((z,m_2), w',m_1)$ iff $(z_0,w,(e_1,e_2)) \models_A^* (z,w',(m_1,m_2))$, and hence $L(A) = L(A')$. □

We are now ready to prove the main result of this section.

Theorem 7. *For an abelian group* $\mathbf{K}$, *one of the following relations hold*:

$$\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(REG),$$

$$\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(EFA(\mathbf{Z}_k)) \quad \textit{for some } k,$$

$$\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(EFA(\mathbf{Q})).$$

Proof. As it was shown in Theorem 2, $\mathscr{L}(EFA(\mathbf{K})) = \bigcup_{\mathbf{H} \in \mathscr{F}(\mathbf{K})} \mathscr{L}(EFA(\mathbf{H}))$. Every $\mathbf{H} \in \mathscr{F}(\mathbf{K})$ is either finite or isomorphic to a group $\mathbf{Z}_k \times \mathbf{H}'$ where $k \geqslant 1$ and $\mathbf{H}'$ is a finite group. Hence for all $\mathbf{H} \in \mathscr{F}(\mathbf{K})$, either $\mathscr{L}(EFA(\mathbf{H})) = \mathscr{L}(REG)$, or $\mathscr{L}(EFA(\mathbf{H})) = \mathscr{L}(EFA(\mathbf{Z}_k \times \mathbf{H}')) = \mathscr{L}(EFA(\mathbf{Z}_k))$, for some $k \geqslant 1$.

If all finitely generated subgroups of $\mathbf{K}$ are finite, then $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(REG)$ holds, due to Theorem 1.

Otherwise, let $N(\mathbf{K})$ be the set of all k such that $\mathscr{L}(EFA(\mathbf{H})) = \mathscr{L}(EFA(\mathbf{Z}_k))$, for some $\mathbf{H} \in \mathscr{F}(\mathbf{K})$. If $N(\mathbf{K})$ is finite then $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(EFA(\mathbf{Z}_k))$, where $k = \max(N(\mathbf{K}))$. If $N(\mathbf{K})$ is infinite then $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(EFA(\mathbf{Q}))$. $\square$

It is known that languages as $L_1 = \{a^n b^n \mid n \geqslant 1\}^*$ or $L_2 = \{wcw^R \mid w \in \{a,b\}^+\}$ are not in $\mathscr{L}(EFA(\mathbf{Q}))$. Therefore, $L_1, L_2 \notin \mathscr{L}(EFA(\mathbf{K}))$, for any abelian group $\mathbf{K}$. In [2] it was conjectured that the commutativity of the multiplication of rational numbers is responsible for this fact. We shall formally prove this conjecture by help of the following “interchange lemma”.

Lemma 2. *Let* $\mathbf{K} = (M, \circ, e)$ *be some abelian group, and let* L *be a language in* $\mathscr{L}(EFA(\mathbf{K}))$. *There is a constant* k *such that, for any* $x \in L$, $|x| \geqslant k$, *and any decomposition* $x = v_1 w_1 v_2 w_2 \dots v_k w_k v_{k+1}$, $|w_i| \geqslant 1$, *exist two integers* $1 \leqslant r < s \leqslant k$ *such that the word* $x' = v_1 w_1' v_2 w_2' \dots v_k w_k' v_{k+1}$ *with* $w_r' = w_s$, $w_s' = w_r$, $w_i' = w_i$, *for* $i \notin \{r, s\}$, *is in* L.

Proof. Let $A = (Z, \Sigma, \mathbf{K}, z_0, F, \delta)$ be an EFA over $\mathbf{K} = (M, \circ, e)$. We choose $k = |Z|^2 + 1$. For a word $x \in L(A)$, $|x| \geqslant k$, let be given a decomposition $x = v_1 w_1 v_2 w_2 \dots v_k w_k v_{k+1}$, $|w_i| \geqslant 1$. There are states $y_i, z_i \in Z$, $1 \leqslant i \leqslant k$, $q \in F$, with

$$(z_{i-1}, v_i, e) \models^* (y_i, \varepsilon, m_i), \quad 1 \leqslant i \leqslant k,$$

$$(y_i, w_i, e) \models^* (z_i, \varepsilon, n_i), \quad 1 \leqslant i \leqslant k,$$

$$(z_k, v_{k+1}, e) \models^* (q, \varepsilon, m_{k+1})$$

and $m_1 \circ n_1 \circ m_2 \circ n_2 \circ \cdots \circ m_k \circ n_k \circ m_{k+1} = e$.

By the pigeon-hole principle, there are two numbers $1 \leqslant r < s \leqslant k$ with $(y_r, z_r) = (y_s, z_s)$.

Now consider $x' = v_1 w_1' v_2 w_2' \ldots v_k w_k' v_{k+1}$ with $w_r' = w_s$, $w_s' = w_r$, $w_i' = w_i$, for $i \notin \{r, s\}$. For the words v_i, $1 \leqslant i \leqslant k+1$, and w_i', $1 \leqslant i \leqslant k+1$ the relations

$$\begin{aligned}(z_{i-1}, v_i, e) &\models^* (y_i, \varepsilon, m_i), \quad 1 \leqslant i \leqslant k,\\ (y_i, w_i', e) &\models^* (z_i, \varepsilon, n_i'), \quad 1 \leqslant i \leqslant k,\\ (z_k, v_{k+1}, e) &\models^* (q, \varepsilon, m_{k+1})\end{aligned}$$

hold with $n_r' = n_s$, $n_s' = n_r$, $n_i' = n_i$, for $i \notin \{r, s\}$. By the commutativity of $\mathbf{K}$ it follows that

$$\begin{aligned}&m_1 \circ n_1' \circ m_2 \circ n_2' \circ \cdots \circ m_k \circ n_k' \circ m_{k+1}\\ &\quad = m_1 \circ n_1 \circ m_2 \circ n_2 \circ \cdots \circ m_k \circ n_k \circ m_{k+1}\\ &\quad = e,\end{aligned}$$

implying that x' is accepted by A. □

Proposition 2. *For any abelian group* $\mathbf{K}$, *the languages* $L_1 = \{a^n b^n \mid n \geqslant 1\}^*$ *and* $L_2 = \{wcw^R \mid w \in \{a, b\}^+\}$ *are not in* $\mathscr{L}(EFA(\mathbf{K}))$.

Proof. Assume that $L_1 \in \mathscr{L}(EFA(\mathbf{K}))$, for some $\mathbf{K}$, and let k be the constant from the interchange lemma. Now consider the word $x = aba^2b^2 \ldots a^k b^k$ and the decomposition $v_i = a^i, w_i = b^i$, $1 \leqslant i \leqslant k$, $v_{k+1} = \varepsilon$. There are $1 \leqslant r < s \leqslant k$ such that $x' = aw_1' a^2 w_2' \ldots a^k w_k'$ with $w_r' = b^s$, $w_s' = b^r$, $w_i' = b^i$, for $i \notin \{r, s\}$, is in L_1, contradiction.

A similar reasoning for the relation $L_2 \notin \mathscr{L}(EFA(\mathbf{K}))$ is left to the reader. □

Theorem 8. *For an abelian group* $\mathbf{K}$, *either* $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(REG)$, *or* $\mathscr{L}(EFA(\mathbf{K}))$ *is closed neither under Kleene operation* * *nor under substitutions.*

Proof. Let $\mathbf{K} = (M, \circ, e)$. We distinguish two cases: either the order of every element of M is finite or there is an element of infinite order.

In the first case, consider a finite subset $X = \{m_1, \ldots, m_r\}$ of M and let $\mathbf{K}' = (\langle X \rangle, \circ, e)$ be the subgroup generated by X. By the commutativity of $\circ$ and for each m_i, $1 \leqslant i \leqslant r$, has a finite order it follows that $\langle X \rangle$ is finite, and by Theorem 1, $\mathscr{L}(EFA(\mathbf{K})) = \mathscr{L}(REG)$.

In the second case, the language $L = \{a^n b^n \mid n \geqslant 1\}$ belongs to $\mathscr{L}(EFA(\mathbf{K}))$ (by Proposition 1), but L^* does not, as stated in Proposition 2.

The non-closure under substitutions follows immediately since $\mathscr{L}(EFA(\mathbf{K}))$ contains all regular languages and it is not closed under Kleene closure. □

5. EFA over non-abelian groups

In this section, we restrict our investigation to the free groups, since for any (non-abelian) group $\mathbf{K}$ there is a homomorphism from a free group to $\mathbf{K}$ [7].

In this way, we get a characterization of the context-free languages class in terms of languages accepted by extended finite automata over the free group with just two generators [2]. The free group with n generators is denoted by $\mathbf{F}_n$.

Recall from Dassow and Mitrana [2].

Theorem 9. *The family of context-free languages equals* $\mathscr{L}(EFA(\mathbf{F}_2))$.

It is well known that every recursively enumerable language can be expressed as the homomorphical image of the intersection of two linear languages.

In conclusion, due to the previous theorem as well as to Theorems 3 and 4, we have just proved

Theorem 10. $\mathscr{L}(EFA(\mathbf{F}_2 \times \mathbf{F}_2))$ *equals the family of recursively enumerable languages.*

6. Deterministic EFA

In the case of EFA, the determinism significantly decreases the accepting capacity. Denote by $\mathscr{L}(DEFA(\mathbf{K}))$ the family of languages recognized by deterministic EFA over the group $\mathbf{K}$.

Lemma 3. *For any group* $\mathbf{K}$, *the languages* $L_1 = \{a^n \mid n \geqslant 1\} \cup \{a^n b^n \mid n \geqslant 1\}$, $L_2 = \{a^m b^n \mid m \geqslant n\}$ *and* $L_3 = \{a,b\}^* \setminus \{a^n b^n \mid n \geqslant 1\}$ *are not in* $\mathscr{L}(DEFA(\mathbf{K}))$.

Proof. Let $\mathbf{K} = (M, \circ, e)$, and let $A = (Z, \{a,b\}, \mathbf{K}, z_0, F, \delta)$ be a DEFA over $\mathbf{K}$ such that $L_1 = L(A)$. Since $a^n \in L(A)$, for all $n \geqslant 1$, there are the integers $1 \leqslant r < s \leqslant |F|+1$ such that $(z_0, a^r, e) \models_A^* (q, \varepsilon, e)$ and $(z_0, a^s, e) \models_A^* (q, \varepsilon, e)$, for some $q \in Z$. Since $a^n b^n \in L(A)$, for all $n \geqslant 1$, we have also $(z_0, a^r b^r, e) \models_A^* (q, b^r, e) \models_A^* (q', \varepsilon, e)$, for some $q' \in F$. Hence, $(z_0, a^s b^r, e) \models_A^* (q, b^r, e) \models_A^* (q', \varepsilon, e)$, implying $a^s b^r \in L(A)$, contradiction. In conclusion, $L_1 \neq L(A)$.

By similar arguments it can be shown that L_2 and L_3 are not in $\mathscr{L}(DEFA(\mathbf{K}))$. □

On the other hand, $\mathscr{L}(DEFA(\mathbf{K}))$ generally contains languages with undecidable membership problem.

Theorem 11. *There is a finitely generated group* $\mathbf{K}$ *such that the following question is undecidable. Given a DEFA A over* $\mathbf{K}$ *and a word w over the input alphabet of A, is* $w \in L(A)$?

Proof. For a finitely generated group $\mathbf{K} = (M, \circ, e)$ with the set of generators X, the word problem is the following question. Is a given term $x_1 \circ x_2 \circ \cdots \circ x_n, x_i \in X \cup X^{-1}$, $1 \leqslant i \leqslant n$, $X^{-1} = \{x^{-1} \mid x \in X\}$ equal to the neutral element e? It is a well-known result from group theory that there exist finitely generated groups $\mathbf{K}$ for which the word problem is undecidable.

As one can easily see, the undecidability of the membership problem for A follows directly from the undecidability of the word problem for $\mathbf{K}$. $\square$

Theorem 12. *For every group* $\mathbf{K}$ *having at least one element of infinite order, we have* $L(DEFA(\mathbf{K})) \subset \mathscr{L}(EFA(\mathbf{K}))$, *strict inclusion.*

Proof. Let $\mathbf{K}$ be an arbitrary given group and $A = (Z, \Sigma, \mathbf{K}, z_0, F, \delta)$ be a deterministic *EFA* over $\mathbf{K}$.

If there is an element of M of infinite order, then the language $\{a^n \mid n \geqslant 1\} \cup \{a^n b^n \mid n \geqslant 1\} \in \mathscr{L}(EFA(\mathbf{K}))$, since $\mathscr{L}(EFA(\mathbf{K}))$ is closed under union. In conclusion, $\mathscr{L}(DEFA(\mathbf{K})) \subset \mathscr{L}(FEA(\mathbf{K}))$. $\square$

As regards closure properties of language families defined by DEFA, we first remark

Theorem 13. *For any group* $\mathbf{K}$, $\mathscr{L}(DEFA(\mathbf{K}))$ *is closed under intersection with regular languages and inverse homomorphism.*

Proof. We prove the closure under inverse homomorphism and let the proof for the intersection with regular languages to the reader.

Let $A = (Z, \Sigma, \mathbf{K}, z_0, F, \delta)$ be a DEFA, and let $h : \Delta^* \to \Sigma^*$ be a homomorphism. We construct the DEFA $C = (Y, \Delta, \mathbf{K}, z_0, F, \varphi)$. The set of states is defined as $Y = Z \cup \{[w_1, z, w_2] \mid z \in Z, w_1 w_2 = h(a) \text{ for all } a \in \Delta\}$. The transition fuction φ is

$$\varphi(z, a) = ([\varepsilon, z, h(a)], e), \quad z \in Z, a \in \Delta,$$

$$\varphi([w_1, z, uw_2], \varepsilon) = ([w_1 u, z', w_2], m) | (z', m) \in \delta(z, u),\ u \in \Sigma \cup \{\varepsilon\},\ uw_2 \in \Sigma^+\},$$

$$\varphi([w, z, \varepsilon], \varepsilon) = \begin{cases} ([w, z', \varepsilon], m) & \text{if } (z', m) \in \delta(z, \varepsilon), \\ (z, e) & \text{if } \delta(z, \varepsilon) \text{ is not defined.} \end{cases}$$

It is easy to see that $(z, a, e) \models_C^* (z', \varepsilon, m)$, for $z \in Z, a \in \Delta \cup \{\varepsilon\}$ iff $(z, h(a), e) \models_A^* (z', \varepsilon, m)$, i.e., a computation of C on the input word $w \in \Delta^*$ simulates a computation of A on the input $h(w) \in \Sigma^*$. $\square$

Theorem 14. *For every group* $\mathbf{K}$ *having at least one element of infinite order, the family* $\mathscr{L}(DEFA(\mathbf{K}))$ *is not closed under complement, concatenation, union and letter-to-letter homomorphisms.*

Proof. The languages $L_1 = \{a^n \mid n \geqslant 1\}$, $L_2 = \{a^n b^n \mid n \geqslant 1\}$, $L_3 = \{c^n \mid n \geqslant 1\} \cup \{a^n b^n \mid n \geqslant 1\}$, are in $\mathscr{L}(DEFA(\mathbf{K}))$. As shown above, the complement of L_2 is not in $\mathscr{L}(DEFA(\mathbf{K}))$. Let $L = \{a^n \mid n \geqslant 1\} \cup \{a^n b^n \mid n \geqslant 1\}$. Obviously, $L = L_1 \cup L_2$ and $L = h(L_3)$ with the homomorphism $h : \{a, b, c\}^* \to \{a, b\}^*$, $h(a) = h(c) = a$, $h(b) = b$. Moreover $L_1 L_2 = \{a^m b^n \mid m \geqslant n\}$ which completes the proof. $\square$

Corollary 1. *For any group* $\mathbf{K}$, *if* $\mathscr{L}(DEFA(\mathbf{K})) \neq \mathscr{L}(REG)$, *then the family* $\mathscr{L}(DEFA(\mathbf{K}))$ *is not closed under concatenation, union, letter-to-letter homomorphisms, Kleene closure.*

7. Deterministic EFA over abelian groups

The next result is a consequence of Theorem 12.

Theorem 15. *For every abelian group* $\mathbf{K}$, *we have either* $\mathscr{L}(DEFA(\mathbf{K})) = \mathscr{L}(REG)$ *or* $\mathscr{L}(DEFA(\mathbf{K})) \subset \mathscr{L}(EFA(\mathbf{K}))$, *strict inclusion.*

Obviously, the statement of Theorem 7 is also valid for the deterministic EFA over abelian groups.

As we have seen, neither $\mathscr{L}(EFA(\mathbf{Q}))$ [6] nor $\mathscr{L}(DEFA(\mathbf{Q}))$ are closed under complement. In the end of this section we will show that the complement of a language from $\mathscr{L}(DEFA(\mathbf{Q}))$ is in $\mathscr{L}(EFA(\mathbf{Q}))$. In order to handle the difficulties owing to the existence of ε-steps, we introduce some notations.

Let $A = (S, \Sigma, \mathbf{Z}_k, s_0, F, \delta)$ be a *DEFA* over $(\mathbf{Z}_k)$, for some $k \geqslant 1$. For all $s \in S$, we define the sets

$$N_s = \{r \in \mathbb{Z}^k \mid (s, \varepsilon, 0) \models^* (q, \varepsilon, r) \text{ for some } q \in F\}.$$

Lemma 4. *Let A be a DEFA as above. For all* $s \in S$, *there is an EFA* $A_s = (Y_s, \Sigma, \mathbf{Z}_k, s, \{q_s\}, \delta_s)$ *such that* $\delta_s(s, a) = \emptyset$, *for all* $a \in \Sigma$, *and* $(s, \varepsilon, 0) \models^* (q_s, \varepsilon, r)$ *iff* $r \in \mathbb{Z}^k \setminus N_s$.

Proof. In what follows let e_i, $1 \leqslant i \leqslant k$, denote the ith unit vector of $\mathbb{Z}^k$. For any $s \in S$, we construct the EFA $A_1(s) = (S \cup \{q\}, T, \mathbf{Z}_k, s, \{q\}, \delta_1)$ with $q \notin S$, $T = \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}$, and the transition relation δ_1 defined as

$$\delta_1(p, a) = \emptyset \quad \text{for } p \in S, a \in T,$$

$$\delta_1(p, \varepsilon) = \delta(p, \varepsilon) \quad \text{for } p \in S \setminus F,$$

$$\delta_1(p, \varepsilon) = \delta(p, \varepsilon) \cup \{(q, 0)\} \quad \text{for } p \in F,$$

$$\delta_1(q, a_i) = \{(q, -e_i) \quad \text{for } 1 \leqslant i \leqslant k,$$

$$\delta_1(q, b_i) = \{(q, e_i) \quad \text{for } 1 \leqslant i \leqslant k,$$

$$\delta_1(q, \varepsilon) = \emptyset.$$

Obviously, the language accepted by $A_1(s)$ is

$$L_s = \{w \in T^* \mid (|w|_{a_1} - |w|_{b_1}, \ldots, |w|_{a_k} - |w|_{b_k}) \in N_s\}.$$

Let $\Psi : T^* \to \mathbb{N}^{2k}$ be the Parikh mapping with $\Psi(w) = (|w|_{a_1}, |w|_{b_1}, \ldots, |w|_{a_k}, |w|_{b_k})$, for all $w \in T^*$. The Parikh set $\Psi(L_s)$ is semilinear, and its complement $\overline{\Psi(L_s)}$ is semilinear, too. Therefore, there is a finite automaton $A_2(s) = (Y_s, T, s, \{q_s\}, \delta'_s)$ such that the Parikh set of $L(A_2(s))$ is $\overline{\Psi(L_s)}$. From $A_2(s)$ we can construct A_s as follows:

$A_s = (Y_s, \Sigma, \mathbf{Z}_k, s, \{q_s\}, \delta_s)$ with

$$\delta_s(p,a) = \emptyset \quad \text{for } p \in Y_s, a \in \Sigma,$$
$$\delta_s(p,\varepsilon) = \{(p', e_i) \mid p' \in \delta'_s(p,a_i), 1 \leqslant i \leqslant k\}$$
$$\cup \{(p', -e_i) \mid p' \in \delta'_s(p,b_i), 1 \leqslant i \leqslant k\} \quad \text{for } p \in Y_s.$$

Obviously, $(s,\varepsilon,0) \models^*_{A_s} (q_s,\varepsilon,r)$ iff $L(A_2(s))$ contains a word w with $r = (|w|_{a_1} - |w|_{b_1}, \ldots, |w|_{a_k} - |w|_{b_k})$, i.e. iff L_s contains no word v with $r = (|v|_{a_1} - |v|_{b_1}, \ldots, |v|_{a_k} - |v|_{b_k})$, hence iff $r \notin N_z$. □

Theorem 16. *For all $L \in \mathscr{L}(DEFA(\mathbf{Q}))$, the complement of L is contained in $\mathscr{L}(EFA(\mathbf{Q}))$.*

Proof. Let $A = (S, \Sigma, \mathbf{Z}_k, s_0, F, \delta)$ be a *DEFA* over $\mathbf{Z}_k$. The set of states can be partitioned into the set R consisting of all states $s \in S$, such that A can perform only ε-steps if s is reached, and its complement $S \setminus R$.

The complement of $L(A)$ consists of two sets; first, the set of all words $w = w_1 w_2 \in \Sigma^+$ with $w_2 \neq \varepsilon$ and $(s_0, w, 0) \models^*_A (p, w_2, r)$ for some $r \in \mathbb{Z}^k$ and some $p \in R$; second, the set of all words $ww_1a \in \Sigma^+$ with $a \in \Sigma$ and $(s_0,w,0) \models^*_A (p,a,r') \models_A (s,\varepsilon,r)$ for some $p,s \in S$, and $r + t \neq 0$, for all $t \in N_s$. The last condition is equivalent to $r + t = 0$, for some $t \in \mathbb{Z}^k \setminus N_s$.

Now let for all $s \in S$, $A_s = (Y_s, \Sigma, \mathbf{Z}_k, s, \{q_s\}, \delta_s)$ be the *DEFA* constructed in the last lemma. Without loss of generality, we may assume that Y_s and $Y_{s'}$ are disjoint for $s \neq s'$ and that Y_s and S are disjoint for all $s \in S$. Now we construct $B = (S', \Sigma, \mathbf{Z}_k, s_0, F', \delta')$ with the set of states $S' = S \cup \bigcup_{s \in S} Y_s \cup \{f\}$, the set of final states $F' = \{q_s \mid s \in S\} \cup \{f\}$, and the following transition mapping:

$$\delta'(s,\varepsilon) = \begin{cases} \delta(s,\varepsilon) & \text{if } s \in S - \{s_0\}, \\ \delta(s,\varepsilon) \cup \{(f,0)\} & \text{if } s = s_0, \\ \delta_x(s,\varepsilon) & \text{if } s \in Y_x, \text{for some } x \in S, \end{cases}$$

$$\delta'(s,a) = \begin{cases} \delta(s,a) \cup \{(y_{s'}, r) \mid (s',r) \in \delta(s,a)\} & \text{if } s \in S - R, \\ \delta(s,a) \cup \{(y_{s'}, r) \mid (s',r) \in \delta(s,a)\} \cup \{(f,0)\} & \text{if } s \in R, \end{cases}$$

$$\delta'(f,a) = \{(f,0)\},$$
$$\delta'(f,\varepsilon) = \{(f, m \cdot e_i)\}, \quad m \in \{-1,0,1\},\ 1 \leqslant i \leqslant k$$

for all $a \in \Sigma$.

It is easy to see that B accepts all words not in $L(A)$. On the other hand, no word from $L(A)$ is accepted by B. □

An important consequence of the last theorem is the decidability of the inclusion problem (and of the equivalence problem, too) for DEFA over $\mathbf{Q}$, which is an interesting contrast to the undecidability of the universe problem for nondeterministic one-turn counter automata, a proper subclass of *EFA* over $(\mathbb{Z}, +, 0)$.

Theorem 17. *Let A and B be DEFA over* $\mathbf{Q}$. *It is decidable whether or not* $L(A) \subseteq L(B)$.

Proof. Given B, one can construct a (nondeterministic) *EFA* $\bar{B}$ over $\mathbf{Q}$ with $L(\bar{B}) = \overline{L(B)}$. Clearly, $L(A) \subseteq L(B)$ iff $L(\bar{B}) \cap L(A)$ is empty. Now an EFA C over $\mathbf{Q}$ can be constructed such that $L(C) = L(\bar{B}) \cap L(A)$. Hence, the inclusion problem for DEFA over $\mathbf{Q}$ is reduced to the emptiness problem for EFA over the same group, which is decidable [1]. □

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References

[1] J. Dassow, Gh. Păun, Regulated Rewriting in Formal Language Theory, Akademie-Verlag, Berlin, 1989.
[2] J. Dassow, V. Mitrana, Internat. J. Comput. Algebra (2000), in press.
[3] S. Ginsburg, S.A. Greibach, Mem. Amer. Math. Soc. 87 (1969) 1.
[4] S.A. Griebach, Theoret. Comput. Sci. 7 (1978) 311.
[5] O.H. Ibarra, S.K. Sahni, C.E. Kim, Theoret. Comput. Sci. 2 (1976) 271.
[6] Gh. Păun, Rev. Roumaine Math. Pures Appl. 25 (6) (1980) 911.
[7] J.J. Rotman, An Introduction to the Theory of Groups, Springer, Berlin, 1995.
[8] G. Rozenberg, A. Salomaa (Eds.), The Handbook of Formal Languages, Springer, Berlin, 1997.