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A note on projective modules over real affine algebras

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1. Introduction

Let A be an affine domain over $\mathbb{R}$ of dimension d. Let $f \in A$ be an element not belonging to any real maximal ideal of A and let P be a projective A-module of rank $\geqslant d-1$. Let $(a, p) \in A_f \oplus P_f$ be a unimodular element and $Q = A_f \oplus P_f/(a, p)A_f$. If P is free, then a result of Ojanguren and Parimala [6, Theorem] shows that Q is extended from A. A consequence of this result is that, if $d = 3$, then all projective modules over A_f are free, where $A = \mathbb{R}[X_1, X_2, X_3]$ (see [6] for motivation). In this paper, we prove the following result (3.10) which is a generalization of the above result of Ojanguren and Parimala.

Theorem. *Let A be an affine algebra over $\mathbb{R}$ of dimension d. Let $f \in A$ be an element not belonging to any real maximal ideal of A. Let P be a projective A-module of rank $\geqslant d-1$. Let $(a, p) \in A_f \oplus P_f$ be a unimodular element. Then the projective A_f-module $Q = A_f \oplus P_f/(a, p)A_f$ is extended from A.*

2. Preliminaries

In this paper, all the rings are assumed to be commutative Noetherian and all the modules are finitely generated.

Let B be a ring and let P be a projective B-module. Recall that $p \in P$ is called a unimodular element if there exists an $\psi \in P^* = \mathrm{Hom}_B(P, B)$ such that $\psi(p) = 1$. We

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denote by $\mathrm{Um}(P)$, the set of all unimodular elements of P. We write $O(p)$ for the ideal of B generated by $\psi(p)$, for all $\psi \in P^*$. Note that, if $p \in P$ is a unimodular element, then $O(p) = B$.

Let $E_n(B)$ denote the subgroup of $\mathrm{SL}_n(B)$ generated by all the elementary matrices $E_{ij}(z)$, where $E_{ij}(z) \in \mathrm{SL}_n(B)$ is such that its diagonal elements are 1, $i \neq j$, (i, j)th entry is z and the rest of the entries are 0, where $z \in B$.

We begin by stating a classical result of Serre [7].

Theorem 2.1. *Let A be a ring of dimension d. Then, any projective A-module P of rank $> d$ has a unimodular element. In particular, if* $\dim A = 1$, *then any projective A-module of trivial determinant is free.*

Let B be a ring and let P be a projective B-module. Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p of P as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$.

If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and, hence, $1 + \varphi_p$ is a unipotent automorphism of P.

By a "transvection," we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. We denote by $E(P)$, the subgroup of $\mathrm{Aut}(P)$ generated by all transvections of P. Note that, $E(P)$ is a normal subgroup of $\mathrm{Aut}(P)$.

An existence of a transvection of P pre-supposes that P has a unimodular element. Now, let $P = B \oplus Q$, $q \in Q$, $\alpha \in Q^*$. Then $\Delta_q(b, q') = (b, q' + bq)$ and $\Gamma_\alpha(b, q') = (b + \alpha(q'), q')$ are transvections of P. Conversely, any transvection Θ of P gives rise to a decomposition $P = B \oplus Q$ in such a way that $\Theta = \Delta_q$ or $\Theta = \Gamma_\alpha$.

Now, we state a classical result of Bass [1].

Theorem 2.2. *Let A be a ring of dimension d and let P be a projective A-module of rank $> d$. Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$.

The following result is due to Bhatwadekar and Roy [4, Proposition 4.1] and is about lifting an automorphism of a projective module.

Proposition 2.3. *Let A be a ring and let J be an ideal of A. Let P be a projective A-module of rank n. Then, any transvection $\widetilde{\Theta}$ of P/JP (i.e., $\widetilde{\Theta} \in E(P/JP)$) can be lifted to a (unipotent) automorphism Θ of P. In particular, if P/JP is free (of rank n), then any element $\overline{\Psi}$ of $E((A/J)^n)$ can be lifted to $\Psi \in$* $\mathrm{Aut}(P)$. *If in addition, the natural map* $\mathrm{Um}(P) \to \mathrm{Um}(P/JP)$ *is surjective, then the natural map $E(P) \to E(P/JP)$ is surjective.*

Now, we recall some preliminary facts about symplectic modules. Let A be a ring. A bilinear map $\langle\, ,\rangle : A^n \times A^n \to A$ is called *alternating* if $\langle v, v\rangle = 0$, $\forall v \in A^n$. Let us fix a basis of A^n, say $e_1, \ldots, e_n$. Let $\langle e_i, e_j\rangle = a_{ij} \in A$. Then $\alpha = (a_{ij}) \in M_n(A)$ is such that $\alpha + \alpha^t = 0$. Thus, giving an alternating bilinear form $\langle\, ,\rangle$ on A^n is equivalent to giving a $n \times n$ matrix α such that $\alpha + \alpha^t = 0$. Conversely, if $2 \in A^*$ (the set of units of A), then an $n \times n$ matrix $\alpha = (a_{ij})$ such that $\alpha + \alpha^t = 0$ gives rise to a bilinear alternating map $\langle\, ,\rangle : A^n \times A^n \to A$ given by $\langle e_i, e_j\rangle = a_{ij}$.

An alternating form $\langle\,,\rangle$ on A^n is called *non-degenerate* if the corresponding $n \times n$ matrix α is invertible. A *symplectic* A-module of rank n is a pair $(A^n, \langle\,,\rangle)$, where $\langle\,,\rangle : A^n \times A^n \to A$ is a non-degenerate alternating bilinear form. If $(A^n, \langle\,,\rangle)$ is a symplectic A-module, then, it is easy to see that n is even.

Two symplectic modules $(A^n, \langle\,,\rangle_1)$ and $(A^n, \langle\,,\rangle_2)$ are said to be isomorphic if there exists an isomorphism $\tau : A^n \xrightarrow{\sim} A^n$ such that $\langle v_1, v_2\rangle_1 = \langle \tau(v_1), \tau(v_2)\rangle_2$, $\forall v_1, v_2 \in A^n$.

To make the notation simple, we will always denote a non-degenerate alternating bilinear form by $\langle\,,\rangle$.

If $(A^n, \langle\,,\rangle)$ and $(A^m, \langle\,,\rangle)$ are two symplectic modules, then non-degenerate alternating bilinear forms on A^n and A^m will give rise (in a canonical manner) to a non-degenerate alternating bilinear form on $A^n \oplus A^m = A^{n+m}$ and we denote the symplectic module thus obtained by $(A^n \perp A^m, \langle\,,\rangle)$. There is a unique (up to scalar multiplication by elements of A^*) non-degenerate alternating bilinear form $\langle\,,\rangle$ on A^2, namely $\langle (a,b),(c,d)\rangle = ad - bc$.

An *isometry* of the symplectic module $(A^n, \langle\,,\rangle)$ is an automorphism of $(A^n, \langle\,,\rangle)$. We denote by $Sp_n(A, \langle\,,\rangle)$ the group of isometries of $(A^n, \langle\,,\rangle)$. It is easy to see that $Sp_n(A, \langle\,,\rangle)$ is a subgroup of $\mathrm{SL}_n(A)$ and it coincides with $\mathrm{SL}_n(A)$ when $n = 2$. Therefore, $\mathrm{SL}_2(A)$ can be identified with a subgroup of $Sp(A^2 \perp A^n, \langle\,,\rangle)$.

Let $(A^n, \langle\,,\rangle)$ be a symplectic A-module and let $u, v \in A^n$ be such that $\langle u, v\rangle = 0$. Let $a \in A$ and let $\tau_{(a,u,v)} : A^n \to A^n$ be a map defined by

$$\tau_{(a,u,v)}(w) = w + \langle w, v\rangle u + \langle w, u\rangle v + a\langle w, u\rangle u, \quad \text{for } w \in A^n.$$

Then $\tau_{(a,u,v)} \in Sp_n(A, \langle\,,\rangle)$. Moreover, it is easy to see that

$$\tau^{-1}_{(a,u,v)} = \tau_{(-a,-u,v)} = \tau_{(-a,u,-v)} \quad \text{and} \quad \alpha\tau_{(a,u,v)}\alpha^{-1} = \tau_{(a,\alpha(u),\alpha(v))}$$

for an element $\alpha \in Sp_n(A, \langle\,,\rangle)$.

An isometry $\tau_{(a,u,v)}$ is called a *symplectic transvection* if either u or v is a unimodular element in A^n. We denote by $ESp_n(A, \langle\,,\rangle)$ the subgroup of $Sp_n(A, \langle\,,\rangle)$ generated by symplectic transvections. It follows from the above discussion that $ESp_n(A, \langle\,,\rangle)$ is a normal subgroup of $Sp_n(A, \langle\,,\rangle)$.

The following result is due to Bhatwadekar [3, Corollary 3.3] and is about lifting an automorphism of a projective module. It is a generalization of a result of Suslin [10, Lemma 2.1].

Proposition 2.4. *Let B be a two dimensional ring and let I be an ideal of B such that $\dim(B/I) \leqslant 1$. Let P be a projective B-module of (constant) rank 2 such that P/IP is free. Then, any element of $\mathrm{SL}_2(B/I) \cap ESp_4(B/I)$ can be lifted to an element of $\mathrm{SL}(P)$.*

Let A be a commutative ring and let I be an ideal of A. For $n \geqslant 3$, let $E^1_n(A, I)$ denote the subgroup of $E_n(A)$ generated by elementary matrices $E_{1i}(a)$ and $E_{j1}(x)$, where $2 \leqslant i, j \leqslant n$, $a \in A$, $x \in I$.

Let $\mathrm{GL}_n(A, I)$ denote the kernel of the canonical map $\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/I)$. For $n \geqslant 3$, we denote $E^1_n(A, I) \cap \mathrm{GL}_n(A, I)$ by $E_n(A, I)$.

Let P be a finitely generated projective A-module of (constant) rank d. Let t be a non-zero divisor of A such that P_t is free. Then it is easy to see that there exits a free submodule $F = A^d$ of P and a positive integer l such that, if $s = t^l$, then $sP \subset F$. Therefore, $sF^* \subset P^* \subset F^*$.

Lemma 2.5. *Let A, P, F, s be as above. If $p \in F$, then $\Delta_p \in E(A \oplus F) \cap E(A \oplus P)$ and if $\alpha \in F^*$, then $\Gamma_{s\alpha} \in E(A \oplus F) \cap E(A \oplus P)$. Hence, if $d \geqslant 2$ and, if we identify $E_{d+1}(A)$ with $E(A \oplus A^d)$, then $E^1_{d+1}(A, As)$ can be regarded as a subgroup of $E(A \oplus P)$.*

We denote by $\mathrm{Um}_n(A, I)$, the set of I-unimodular rows of length n over A (i.e., unimodular rows of the type $[a_1, \ldots, a_n]$, $1 - a_1 \in I$ and $a_i \in I$, $2 \leqslant i \leqslant n$).

For $n \geqslant 3$, $MSE_n(A, I)$ will denote the orbit set $\mathrm{Um}_n(A, I)/E_n(A, I)$. We write $MSE_n(A)$ for $MSE_n(A, A)$.

Let A be a commutative ring and let I be an ideal of A. Let $B = \mathbb{Z} \oplus I$ (with the obvious ring structure on B). Then, for $n \geqslant 3$, the canonical ring homomorphism $B \to A$ gives rise to a map $E_n(B, I) \to E_n(A, I)$, a surjective map $\mathrm{Um}_n(B, I) \twoheadrightarrow \mathrm{Um}_n(A, I)$ and, hence, a surjective map $MSE_n(B, I) \twoheadrightarrow MSE_n(A, I)$.

The following theorem is due to W. van der Kallen [5, Theorem 3.21] and is very crucial for our result.

Theorem 2.6 (Excision). *Let $n \geqslant 3$. Let A be a commutative ring and let I be an ideal of A. Then, the canonical maps $MSE_n(\mathbb{Z} \oplus I, I) \to MSE_n(A, I)$ and $MSE_n(\mathbb{Z} \oplus I, I) \to MSE_n(\mathbb{Z} \oplus I)$ are bijective.*

The following result is due to Vaserstein [12, Theorem].

Theorem 2.7. *Let B be a commutative ring and let $[b_1, \ldots, b_n] \in \mathrm{Um}_n(B)$, $n \geqslant 3$. Let d be a positive integer. Then*

$$[b_1^d, b_2, \ldots, b_n] = [b_1, b_2^d, \ldots, b_n] \quad \big(\mathrm{mod}\ E_n(B)\big).$$

The following corollary is a consequence of Theorems 2.6 and 2.7.

Corollary 2.8. *Let A be a ring and I an ideal of A. Let $[a_1, \ldots, a_n]$ be an element of $\mathrm{Um}_n(A, I)$, $n \geqslant 3$. Let d be a positive integer. Then*

$$[a_1^d, a_2, \ldots, a_n] = [a_1, a_2^d, \ldots, a_n] \quad \big(\mathrm{mod}\ E_n(A, I)\big).$$

The following result of Suslin [9, Lemma 2] is also used in the proof of our result (3.9).

Proposition 2.9. *Let A be a commutative ring and let P be a finitely generated projective A-module of rank d. Let $(c, p_1) \in A \oplus P$ be a unimodular element. Suppose that P/cP is a free A/Ac-module of rank d and that $\overline{p}_1 \in P/cP$ can be extended to a basis of P/cP. Then, there exists an A-automorphism Φ of $A \oplus P$ such that $\Phi(c^d, p_1) = (1, 0)$.*

The following result is due to Ojanguren and Parimala [6, Proposition 3].

Proposition 2.10. *Let $\mathcal{C} = \operatorname{Spec} C$ be a smooth affine curve over a field k of characteristic zero. Suppose that every residue field of $\mathcal{C}$ at a closed point has cohomological dimension $\leqslant 1$. Then, $SK_1(C)$ is divisible.*

The proof of [11, Proposition 1.7] yields the following result.

Proposition 2.11. *Let $\mathcal{C} = \operatorname{Spec} C$ be a curve as in* (2.10). *Then, the natural homomorphism $K_1Sp(C) \to SK_1(C)$ is an isomorphism.*

3. Main Theorem

Given an affine algebra A over $\mathbb{R}$ and a subset $I \subset A$, we denote by $Z(I)$, the closed subset of $X = \operatorname{Spec} A$ defined by I and by $Z_{\mathbb{R}}(I)$, the set $Z(I) \cap X(\mathbb{R})$, where $X(\mathbb{R})$ is the set of all real maximal ideals $\mathfrak{m}$ of A (i.e., $A/\mathfrak{m} \xrightarrow{\sim} \mathbb{R}$). We denote by $\operatorname{Sing} X$, the set of all the prime ideals $\mathfrak{p}$ of A such that $A_{\mathfrak{p}}$ is not a regular ring.

The following lemma is proved in [6, Lemma 2].

Lemma 3.1. *Let A be a reduced affine algebra over $\mathbb{R}$ of dimension d and let $X = \operatorname{Spec} A$. Let $u = (a_1, \ldots, a_n)$ be a unimodular row in A^n. Suppose $a_1 > 0$ on $X(\mathbb{R})$. Then, there exists $b_2, \ldots, b_n \in A$ such that $a_1 + b_2a_2 + \cdots + b_na_n > 0$ on $X(\mathbb{R})$ and $Z(a_1 + b_2a_2 + \cdots + b_na_n)$ is smooth on $X \setminus \operatorname{Sing} X$ of dimension $\leqslant d - 1$.*

Now, we state the Łojasiewicz's inequality [2, Proposition 2.6.2].

Lemma 3.2. *Let B be an affine algebra over $\mathbb{R}$ and let $X = \operatorname{Spec} B$. Let $a, b \in B$ be such that a/b is defined on a closed semi-algebraic set $F \subset X(\mathbb{R})$. Then there exists $g \in B$ such that $g > 0$ on $X(\mathbb{R})$ and $|a/b| < g$ on F.*

The following lemma is an easy consequence of (2.1) and (2.2).

Lemma 3.3. *Let B be a ring of dimension n and let Q be a projective B-module of rank n. Let J be an ideal of height $\geqslant 1$. Suppose $(a, q) \in \operatorname{Um}(B \oplus Q)$. Then there exists $\Psi \in \operatorname{Aut}(B \oplus Q)$ such that $(a, q)\Psi = (a_1, \tilde{q})$ with $a_1 = 1 \pmod J$ and $O(\tilde{q}) = B \pmod J$.*

Proof. Let "bar" denotes reduction modulo J. Since $\dim \overline{B} \leqslant n - 1$ and $\overline{Q}$ is a projective $\overline{B}$-module of rank n, by Serre's theorem (2.1), $\overline{Q}$ has a unimodular element. Let $\overline{q_1} \in \overline{Q}$ be a unimodular element, i.e., $O(\overline{q_1}) = \overline{B}$.

Since rank $\overline{Q} > \dim \overline{B}$, by Bass' theorem (2.2), $E(\overline{B} \oplus \overline{Q})$ acts transitively on $\operatorname{Um}(\overline{B} \oplus \overline{Q})$. Hence, there exists $\overline{\Psi} \in E(\overline{B} \oplus \overline{Q})$ such that $(\bar{a}, \bar{q})\overline{\Psi} = (1, \overline{q_1})$.

Applying (2.3), $\overline{\Psi}$ can be lifted to an element $\Psi \in \operatorname{Aut}(B \oplus Q)$. Let $(a, q)\Psi = (a_1, \tilde{q})$. Then, we have $a_1 = 1 \pmod J$ and $O(\tilde{q}) = B \pmod J$. This proves the result. $\square$

Lemma 3.4. *Let B be an affine algebra over $\mathbb{R}$ and let $f \in B$ be an element not belonging to any real maximal ideal of B. Let $K \subset B$ be an ideal and $a \in B$ such that $f^r \in Ba + K$, for some integer r. Then there exists $h \in 1 + Bf$ such that $ah > 0$ on $Z_{\mathbb{R}}(K)$. Moreover, if I is any ideal of B such that $f^l \in I + K$, for some $l \in \mathbb{N}$, then we can choose $h \in 1 + If$.*

Proof. Since $f^r \in Ba + K$, hence, a has no zeros on $Z_{\mathbb{R}}(K)$. Further, it is given that $f^l \in I + K$, hence, $f^{2l} = \lambda + \mu$, for some $\lambda \in I$ and $\mu \in K$. Since $f^{2l} > 0$ on $X(\mathbb{R})$, where $X = \operatorname{Spec} B$, we get that the element $\lambda > 0$ on $Z_{\mathbb{R}}(K)$. Applying Lemma 3.2 for the element $1/af^2\lambda$ (with $F = Z_{\mathbb{R}}(K)$), we get an element $g \in B$ such that $g > 0$ on $X(\mathbb{R})$ and $1/|a|f^2\lambda < g$ on $Z_{\mathbb{R}}(K)$. Thus, it follows that the element $(1 + af^2\lambda g)a > 0$ on $Z_{\mathbb{R}}(K)$. Let us write $h = 1 + af^2\lambda g$. Then $h \in 1 + If$. Further, $ah > 0$ on $Z_{\mathbb{R}}(K)$. This proves the lemma. □

Lemma 3.5. *Let B be an affine algebra over $\mathbb{R}$ and let $X = \operatorname{Spec} B$. Let $f \in B$ be an element such that $f > 0$ on $X(\mathbb{R})$. Let $K \subset B$ be an ideal and let $a_1 \in B$ be such that $a_1 > 0$ on $Z_{\mathbb{R}}(K)$. Then, there exists $c \in K$ such that $a_1 + c > 0$ on $X(\mathbb{R})$. Moreover, if J is any ideal of B such that $f^q - a_1 \in J$, for some $q \in \mathbb{N}$, then we can choose $c \in KJ$.*

Proof. Let W be the closed semi-algebraic subset of $X(\mathbb{R})$ defined by $a_1 \leqslant 0$. Let $f^q - a_1 = \nu \in J$. Since $f > 0$ on $X(\mathbb{R})$, the element $\nu > 0$ on W. On the other hand, we have $Z_{\mathbb{R}}(K) \cap W = \emptyset$, since $a_1 \leqslant 0$ on W and $a_1 > 0$ on $Z_{\mathbb{R}}(K)$. Hence, if $K = (c_1, \ldots, c_n)$, then $c_1^2 + \cdots + c_n^2 > 0$ on W. Therefore, applying (3.2) for the element $a_1/\nu^2(c_1^2 + \cdots + c_n^2)$ (with $F = W$), we get an element $\tilde{c} \in B$ such that $\tilde{c} > 0$ on $X(\mathbb{R})$ and $|a_1|/\nu^2(c_1^2 + \cdots + c_n^2) < \tilde{c}$ on W. Let $c = \tilde{c}\nu^2(c_1^2 + \cdots + c_n^2)$. Then $c \in KJ$. Further, $a_1 + c > 0$ on W. We also have $a_1 + c > 0$ on $X(\mathbb{R})\backslash W$, since $a_1 > 0$ on $X(\mathbb{R})/W$ and $c \geqslant 0$ on $X(\mathbb{R})$. Therefore, we have $a_1 + c > 0$ on the whole of $X(\mathbb{R})$. This proves the result. □

Lemma 3.6. *Let B be an affine algebra over $\mathbb{R}$ and let I be an ideal of B. Let $f \in B$ be an element such that $f > 0$ on $X(\mathbb{R})$, where $X = \operatorname{Spec} B$. Let P be a projective B_f-module and let $(a, p) \in \operatorname{Um}(B_f \oplus P)$ such that $a = 1 \pmod{IB_f}$ and $O(p) = B_f \pmod{IB_f}$. Then, there exists $h \in 1 + Bf$ and $\Delta \in \operatorname{Aut}(B_{fh} \oplus (P \otimes B_{fh}))$ such that $(a, p)\Delta = (\tilde{a}, p)$ with $\tilde{a} > 0$ on $X(\mathbb{R}) \cap \operatorname{Spec} B_{fh}$ and $\tilde{a} = 1 \pmod{IB_{fh}}$.*

Proof. Since P is a projective B_f-module, we can find a B-module M such that $P = M_f$. Since $(a, p) \in \operatorname{Um}(B_f \oplus M_f)$, after multiplying by a suitable power of f, we may assume that $(a, p) \in B \oplus M$ such that $a = f^l \pmod{IB}$ and $O(p) \supset f^q B \pmod{IB}$, for some $l, q \in \mathbb{N}$.

We have $(a, p) \in B \oplus M$ and $(a, p)_f \in \operatorname{Um}(B_f \oplus M_f)$. Hence $f^r \in aB + O(p)$, for some $r \in \mathbb{N}$. Write $K = O(p)B$. We also have $f^q \in K + I$. Hence, applying (3.4), there exists $h \in 1 + fI$ such that $a_1 = ha > 0$ on $Z_{\mathbb{R}}(K)$.

Note that we have $a_1 > 0$ on $Z_{\mathbb{R}}(K)$ and $a_1 = f^l \pmod{IB}$ (since $h - 1 \in I$). Hence, applying (3.5), we get $c \in KI$ such that the element $a_2 = a_1 + c > 0$ on $X(\mathbb{R})$. Let $\varphi \in P^*$ be such that $\varphi(p) = c$. Note that we still have $a_2 = f^l \pmod{IB}$. Let $\tilde{a} = a_2/f^l \in B_f$. Then $\tilde{a} = 1 \pmod{IB_f}$.

From the above discussion, it is clear that if $\Gamma_1 = (h, Id)$, $\Gamma_2 = (1/f^l, Id) \in \mathrm{Aut}(B_{fh} \oplus P_{fh})$, then $(a, p)\Gamma_1 = (a_1, p)$, $(a_1, p)\Gamma_\varphi = (a_2, p_1)$ and $(a_2, p)\Gamma_2 = (\tilde{a}, p)$. Further, $\tilde{a} > 0$ on $X(\mathbb{R}) \cap \operatorname{Spec} B_{fh}$ and $\tilde{a} = 1 \pmod{IB_f}$. Take $\Delta = \Gamma_1\Gamma_\varphi\Gamma_2$. Then the result follows. □

The following result is an easy consequence of (3.1).

Lemma 3.7. *Let B be a reduced affine algebra of dimension d over $\mathbb{R}$ and let $X = \operatorname{Spec} B$. Let Q be a projective B-module. Let J be the ideal of B defining the singular locus of B and let $I \subset J$ be an ideal. Let $(\tilde{a}, q) \in \mathrm{Um}(B \oplus Q)$ such that $\tilde{a} > 0$ on $X(\mathbb{R})$ and $\tilde{a} = 1 \pmod I$. Then, there exists $\Phi \in E(B \oplus Q)$ such that $(\tilde{a}, q)\Phi = (b, \tilde{q})$ with $b > 0$ on $X(\mathbb{R})$, $Z(b)$ is smooth on X of dimension $\leqslant d - 1$ and $\tilde{q} \in IQ$.*

Proof. Since $(\tilde{a}, q) \in \mathrm{Um}(B \oplus Q)$, we have $\tilde{a}B + O(q) = B$. Further, $\tilde{a} = 1 \pmod I$. Hence, it is easy to see that if $I = (s_1, \ldots, s_l)$ and $O(q) = (c_1, \ldots, c_n)$, then $(\tilde{a}, s_1^2c_1^2, \ldots, s_1^2c_n^2, s_2^2c_1^2, \ldots, s_l^2c_1^2, \ldots, s_l^2c_n^2)$ is a unimodular row in B^{nl+1}.

Since $\tilde{a} = 1 \pmod I$ and $I \subset J$, hence, $\tilde{a} = 1 \pmod J$. Further, $\tilde{a} > 0$ on $X(\mathbb{R})$. Applying (3.1), we get $h_{ij} \in B$ such that

$$b = \tilde{a} + \sum_{i,j} h_{ij}s_i^2c_j^2 > 0$$

on $X(\mathbb{R})$ and $Z(b)$ is smooth on X of dimension $\leqslant d - 1$.

Let $\varphi \in Q^*$ be such that $\varphi(q) = \sum_{i,j} h_{ij}s_i^2c_j^2$. Let $\Delta_1 = \Gamma_\varphi \in E(B \oplus Q)$. Then $(\tilde{a}, q)\Delta_1 = (b, q)$. Note that $b = 1 \pmod I$. Therefore, there exists $\Delta_2 \in E(B \oplus Q)$ such that $(b, q)\Delta_2 = (b, \tilde{q})$, where $\tilde{q} \in IQ$. Write $\Phi = \Delta_1\Delta_2$. Then $(\tilde{a}, q)\Phi = (b, \tilde{q})$ has the required properties. This proves the lemma. □

The following result is a generalization of [6, Proposition 1].

Lemma 3.8. *Let A be a reduced affine algebra of dimension d over $\mathbb{R}$ and let $X = \operatorname{Spec} A$. Let J be an ideal of A of height $\geqslant 1$. Let $f \in A$ be an element not belonging to any real maximal ideal of A. Let P be a projective A_f-module of rank $d - 1$ and let $(a, p) \in \mathrm{Um}(A_f \oplus P)$. Then there exists $h \in 1 + fA$ and $\Delta \in \mathrm{Aut}(A_{fh} \oplus P_{fh})$ such that if $(a, p)\Delta = (b, \tilde{p})$, then*

(1) $b > 0$ *on* $X(\mathbb{R}) \cap \operatorname{Spec} A_{fh}$,
(2) $Z(b)$ *smooth on* $\operatorname{Spec} A_{fh}$ *of dimension* $\leqslant d - 1$, *and*
(3) $(b, \tilde{p}) = (1, 0) \pmod{JA_{fh}}$.

Proof. By replacing f by f^2, we may assume that $f > 0$ on $X(\mathbb{R})$. Let J_1 be the ideal of A defining the singular locus of $\operatorname{Spec} A$. Since A is reduced and char $\mathbb{R} = 0$, J_1 is an ideal of height $\geqslant 1$. Let $I = JJ_1$. Then $\operatorname{ht} I \geqslant 1$.

Write $A_1 = A_{f(1+fA)}$. Then $\dim A_1 = d - 1$. Recall, rank $P = d - 1$ and $(a, p) \in \mathrm{Um}(A_1 \oplus (P \otimes A_1))$. Applying (3.3) with $B = A_1$, $Q = P \otimes A_1$, and $J = IA_1$, there exists $\Psi \in \mathrm{Aut}(A_1 \oplus (P \otimes A_1))$ such that $(a, p)\Psi = (a_1, p_1)$, where $a_1 = 1 \pmod{IA_1}$ and $O(p_1) = A_1 \pmod{IA_1}$.

It is easy to see that there exists $h_1 \in 1 + fA$ such that, if we write $B = A_{h_1}$, then $\Psi \in \mathrm{Aut}(B_f \oplus (P \otimes B_f))$ and $(a, p)\Psi = (a_1, p_1)$, where $a_1 = 1 \pmod{IB_f}$ and $O(p_1) = B_f \pmod{IB_f}$. Applying (3.6), there exists an element $h' \in 1 + fB$ and $\Gamma \in \mathrm{Aut}(B_{fh'} \oplus (P \otimes B_{fh'}))$ such that $(a_1, p_1)\Gamma = (\tilde{a}, p_1)$ with $\tilde{a} > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, B_{fh'}$ and $\tilde{a} = 1 \pmod{IB_f}$

Recall that $h_1 \in 1 + fA$, $B = A_{h_1}$, and $h' \in 1 + fB$. Let s be an integer such that $h = h_1^s h' \in A$. Then $B_h = A_h$ and $h \in 1 + fA$. Therefore, there exists $\Gamma \in \mathrm{Aut}(A_{fh} \oplus (P \otimes A_{fh}))$ such that $(a_1, p_1)\Gamma = (\tilde{a}, p_1)$ with $\tilde{a} > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, A_{fh}$ and $\tilde{a} = 1 \pmod{IA_{fh}}$.

Write $C = A_{fh}$. Then, we have $(\tilde{a}, p_1) \in \mathrm{Um}(C \oplus (P \otimes C))$ such that $\tilde{a} > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, C$ and $\tilde{a} = 1 \pmod{IC}$, where $I \subset J_1$ (recall that J_1 is an ideal of A defining the singular locus of $\mathrm{Spec}\, A$). Applying (3.7), we get $\Phi \in \mathrm{Aut}(C \oplus (P \otimes C))$ such that $(\tilde{a}, p_1)\Phi = (b, \tilde{p})$ with $b > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, C$, $Z(b)$ is smooth on $\mathrm{Spec}\, C$ of dimension $\leqslant d - 1$ and $\tilde{p} \in I(P \otimes C)$.

Let $\Delta = \Psi\Gamma\Phi$. Then $\Delta \in \mathrm{Aut}(A_{fh} \oplus P_{fh})$ is such that $(a, p)\Delta = (b, \tilde{p})$ with $b > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, A_{fh}$ and $Z(b)$ is smooth on $\mathrm{Spec}\, A_{fh}$ of dimension $\leqslant d - 1$. Moreover, $(b, \tilde{p}) = (1, 0) \pmod{JA_{fh}}$. This proves the lemma. □

Proposition 3.9. *Let A be an affine algebra over $\mathbb{R}$ of dimension d. Let $f \in A$ be an element not belonging to any real maximal ideal of A and let $A' = A_{f(1+Af)}$. Then, every projective A'-module P of rank $d - 1$ is cancellative.*

Proof. By replacing f by f^2, we may assume that $f > 0$ on $X(\mathbb{R})$, where $X = \mathrm{Spec}\, A$. It is enough to show that, if $(a, p) \in \mathrm{Um}(A' \oplus P)$, then, there exists $\Lambda \in \mathrm{Aut}(A' \oplus P)$ such that $(a, p)\Lambda = (1, 0)$. Without loss of generality, we may assume that A is reduced.

We can choose $g \in 1 + Af$ such that P is a projective A_{fg}-module of rank $d - 1$. Write $\widetilde{A} = A_{fg}$. Let $t \in \widetilde{A}$ be a non-zero divisor such that P_t is a free $\widetilde{A}_t$-module of rank $d - 1$. Let $F = \widetilde{A}^{d-1}$ be a free submodule of P such that $F_t = P_t$ and let $s = t^l$ be such that $sP \subset F$. Let $(e_1, \ldots, e_{d-1})$ denote the standard basis of $\widetilde{A}^{d-1}$.

Let J be the ideal of A defining the singular locus of A. Since A is reduced and char $\mathbb{R} = 0$, J is an ideal of height $\geqslant 1$. Let $I = sJ$. Then $\mathrm{ht}\, I = 1$. Applying (3.8) for $\widetilde{B} = A_g$, there exists $h' \in 1 + f\widetilde{B}$ and $\Gamma \in \mathrm{Aut}(\widetilde{B}_{fh'} \oplus P_{fh'})$ such that if $(a, p)\Gamma = (a_1, p_1)$, then

(1) $a_1 > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, \widetilde{B}_{fh'}$,
(2) $\mathrm{Spec}(\widetilde{B}_{fh'}/a_1\widetilde{B}_{fh'})$ is smooth of dimension $\leqslant d - 1$, and
(3) $(a_1, p_1) = (1, 0) \pmod{s\widetilde{B}_{fh'}}$.

We can choose a suitable positive integer r such that $h = g^r h' \in A$ (in fact, $h \in 1 + Af$ and $\widetilde{B}_{fh} = A_{fh}$) and $\Gamma \in \mathrm{Aut}(A_{fh} \oplus P_{fh})$ such that $(a, p)\Gamma = (a_1, p_1)$, satisfies the above properties $(1) - (3)$, i.e.,

(1) $a_1 > 0$ on $X(\mathbb{R}) \cap \mathrm{Spec}\, A_{fh}$,
(2) $\mathrm{Spec}(A_{fh}/a_1A_{fh})$ is smooth of dimension $\leqslant d - 1$, and
(3) $(a_1, p_1) = (1, 0) \pmod{sA_{fh}}$.

Since $p_1 \in sP_{fh}$ and we have $sP \subset F$, hence, we can write $p_1 = b_1e_1 + \cdots + b_{d-1}e_{d-1}$, for some $b_i \in A_{fh}$. Let $B = \mathbb{R}(f) \otimes_{\mathbb{R}[f]} A_{fh}$. Then B is an affine algebra over $\mathbb{R}(f)$ of dimension $d-1$. We write $\widetilde{P} = P \otimes B$.

Let "bar" denotes reduction modulo the ideal a_1B. Since $a_1B + sB = B$ and $F_s = P_s$, it follows that the inclusion $F \subset P$ gives rise to the equality $\overline{F} = \overline{P}$. In particular, $\overline{P}$ is free of rank $d-1$ with a basis $(\bar{e}_1, \ldots, \bar{e}_{d-1})$ and $\overline{p}_1$ is a unimodular element of $\overline{P}$.

If $d = 3$, then $C = B/a_1B$ is smooth of dimension 1. Note that, every maximal ideal $\mathfrak{m}$ of C is the image in $\operatorname{Spec} C$ of a prime ideal $\mathfrak{p}$ of A_{fh} of height 2 containing a_1. Since a_1 does not belong to any real maximal ideal of A_{fh}, by [8], the residue field $\mathbb{R}(\mathfrak{p}) = k(\mathfrak{m})$ has cohomological dimension $\leqslant 1$. By (2.11), $SK_1(C)$ is divisible and the natural map $K_1Sp(C) \to SK_1(C)$ is an isomorphism. Hence, there exists $\Gamma' \in \mathrm{SL}_2(C) \cap ESp_4(C)$ and $c_1, c_2 \in B$ such that, if $q = c_1^2e_1 + c_2e_2 \in F$, then $\Gamma'(\overline{p}_1) = \bar{q}$. By (2.4), Γ' has a lift $\Gamma_1 \in SL(\widetilde{P})$. Recall that $\widetilde{P} = P \otimes B$.

If $d \geqslant 4$, then, since $\overline{P}$ is a free of rank $d-1$, using (2.9) and (2.11), one can deduce from the proof of [10, Theorem 2.4] that there exists $\widetilde{\Gamma} \in E(\overline{P})$ and $c_i \in B$, $1 \leqslant i \leqslant d-1$ such that, if $q = c_1^{d-1}e_1 + c_2e_2 + \cdots + c_{d-1}e_{d-1} \in F$, then $\widetilde{\Gamma}(\overline{p}_1) = \bar{q}$. By (2.3), $\widetilde{\Gamma}$ can be lifted to an element $\Gamma_1 \in SL(\widetilde{P})$. (In particular, the above argument shows that every stably free B/a_1B-module of rank $\geqslant d-2$ is cancellative.)

Therefore, in either case, there exists $q_1 \in \widetilde{P}$ and $\Gamma_1 \in \mathrm{SL}(\widetilde{P})$ such that

$$\Gamma_1(p_1) = q - a_1q_1, \quad \text{where } q = c_1^{d-1}e_1 + c_2e_2 + \cdots + c_{d-1}e_{d-1}.$$

Now, the rest of the argument is similar to [3, Theorem 4.1]. We give the proof for the sake of completeness.

Now, Γ_1 induces an automorphism $\Psi_1 = (Id_B, \Gamma_1)$ of $B \oplus \widetilde{P}$. Let $\widetilde{\Psi} = \Psi_1 \Delta_{q_1}$. Recall that $\Delta_{q_1} \in E(B \oplus \widetilde{P})$ is defined as $\Delta_{q_1}(b, q') = (b, q' + bq_1)$. Therefore, we have $(a_1, p_1)\widetilde{\Psi} = (a_1, q)$. Let us write $\Lambda_1 = \Gamma\widetilde{\Psi} \in \mathrm{Aut}(B \oplus \widetilde{P})$. Then $(a, p)\Lambda_1 = (a_1, q)$.

Recall that $a_1 = 1 \pmod{sB}$. Hence, there exists $x \in B$ such that $sx + a_1 = 1$. Let $\mu_i = sxc_i$. Then $\mu_i - c_i \in a_1B$. Let

$$q_2 = \mu_1^{d-1}e_1 + \sum_2^{d-1} \mu_i e_i \in sF \quad \text{and} \quad q_3 = \sum_1^{d-1} \mu_i e_i.$$

Then $q_2 - q = a_1p_2$, for some $p_2 \in F$. Hence, we have $(a_1, q)\Delta_{p_2} = (a_1, q_2)$. Let $\Lambda_2 = \Lambda_1\Delta_{p_2}$. Then $(a, p)\Lambda_2 = (a_1, q_2)$.

Since $1 - a_1 \in sB$ and $\mu_i \in sB$ for $1 \leqslant i \leqslant d-1$. Hence, the row $[a_1, \mu_1, \ldots, \mu_{d-1}] \in \mathrm{Um}_d(B, Bs)$. Therefore, by (2.6),

$$\left[a_1^{d-1}, \mu_1, \ldots, \mu_{d-1}\right] = \left[a_1, \mu_1^{d-1}, \ldots, \mu_{d-1}\right] \quad \left(\text{mod } E_d(B, Bs)\right).$$

By (2.5), there exists $\Phi \in E(B \oplus \widetilde{P})$ such that $(a_1, q_2)\Phi = (a_1^{d-1}, q_3)$. Write $\widetilde{\Phi} = \Lambda_2\Phi$. Then, we have $(a, p)\widetilde{\Phi} = (a_1^{d-1}, q_3)$.

Since $\widetilde{P}/a_1\widetilde{P}$ is free of rank $d-1$ and every stably free B/a_1B-module of rank $\geqslant d-2$ is cancellative, $\bar{q}_3 \in \widetilde{P}/a_1\widetilde{P}$ can be extended to a basis of $\widetilde{P}/a_1\widetilde{P}$. Therefore, by (2.9), there exists $\Phi_1 \in \mathrm{Aut}(B \oplus \widetilde{P})$ such that $(a_1^{d-1}, q_3)\Phi_1 = (1, 0)$.

Let $\Lambda = \widetilde{\Phi}\Phi_1$. Then $\Lambda \in \mathrm{Aut}(B \oplus \widetilde{P})$ and $(a, p)\Lambda = (1, 0)$. Note that $A' = A_{f(1+Af)} = B \otimes_{R(f)} A'$. Therefore, we get the result. □

As a consequence of above Proposition 3.9, we prove the following result. If $P = A^{d-1}$ in the following Theorem 3.10, then we get [6, Theorem].

Theorem 3.10. *Let A be an affine algebra over $\mathbb{R}$ of dimension d and let $f \in A$ be an element not belonging to any real maximal ideal of A. Let P be a projective A-module of rank $\geqslant d - 1$. Let $(a, p) \in A_f \oplus P_f$ be a unimodular element. Then, the projective A_f-module $Q = A_f \oplus P_f/(a, p)A_f$ is extended from A.*

Proof. Let $A' = A_{f(1+Af)}$. By (3.9), $P \otimes A' \xrightarrow{\sim} Q \otimes A'$. Hence, there exists $g \in 1 + Af$ and an isomorphism $\Psi : P \otimes A_{fg} \xrightarrow{\sim} Q \otimes A_{fg}$. The module Q over A_f and P over A_g together with an isomorphism Ψ yield a projective module over A whose extension to A_f is isomorphic to Q. This proves the result. □

Remark 3.11. Theorem 3.10 is valid for an affine algebra A over any real closed field k. For simplicity, we have taken $k = \mathbb{R}$.

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