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# On quaternion-free 2-groups

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**Abstract**

Two theorems are proved, the first of them showing that a modular quaternion-free finite 2-group has a characteristic abelian subgroup with metacyclic factor, the second classifying nonmodular finite quaternion-free 2-groups.

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## 1. Introduction

A group is called *modular* if its subgroup lattice is. Let $P$ be a finite 2-group. According to Iwasawa's [1] classification of modular groups, $P$ is modular if and only if it is $D_8$-free if and only if it is either both $D_8$-free and $Q_8$-free or if $P = Q \times E$ with $E$ elementary abelian and $Q \cong Q_8$. Furthermore, $P$ is called *powerful* if $P' \leqslant \mho_2(P)$. In [2] it is shown that every subgroup of $P$ is powerful if and only $P$ is both modular and $Q_8$-free. In the present paper, two theorems will be proved, the first of which establishing a new characterization of the modular $Q_8$-free finite 2-groups, the second classifying nonmodular $Q_8$-free finite 2-groups.

Following notation introduced by King [3], we will call a metacyclic 2-group $H$ *ordinary metacyclic* if $H$ centralizes $C/\mho_2(C)$ for any cyclic normal subgroup $C$ of $H$.

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**Theorem 1.** *Let $P$ be a finite 2-group that is $Q_8$-free as well as $D_8$-free. Then $P$ has a maximal abelian subgroup $A$ such that $A$ is characteristic in $P$, $[P, O^2(\mathrm{Aut}(P))] \leqslant A$ and $P/A$ is ordinary metacyclic.*

**Theorem 2.** *Let $G$ be a finite 2-group that is $Q_8$-free but is not $D_8$-free. Then $G$ is one of the following*:

(a) *$G$ is the semidirect product $N \cdot \langle x\rangle$ where $N$ is a maximal abelian normal subgroup of $G$ with $\exp(N) > 2$ and, if $t$ is the involution in $\langle x\rangle$, then every element of $N$ is inverted by $t$.*
(b) *$G = N\langle x\rangle$ where $N$ is an elementary abelian normal subgroup of $G$ and $[x, N] \nleqslant \langle x\rangle$.*
(c) *$G = N\langle x, t\rangle$ where $N$ is an elementary abelian normal subgroup of $G$ and $t$ is an involution with $[N, t] = 1$. If $o(xN) = 2^k$, then $G/N \cong \mathrm{Mod}_{2^{k+1}}$ and $x^{2^k} \neq 1$; furthermore, $[x^{2^{k-1}}, N] = 1$ and $\langle t, x^{2^{k-1}}\rangle \cong D_8$.*

Let us first check that the groups listed in Theorem 2 are, indeed, $Q_8$-free without being $D_8$-free. We will use the notation introduced in Theorem 2. If $G$ is one of the groups in (a), then $G$ is not $D_8$-free because of $\exp(N) > 2$. Suppose there are $U \lhd V \leqslant G$ with $V/U \cong Q_8$. Then $|V : U(N \cap V)| = 2$, as $V \nleqslant UN$ and $VN/N \cong V/(N \cap V)$, and therefore $V/(N \cap V)$ is cyclic. The set $tN$ is comprised of involutions each of which inverts every element of $N$, so, unless $U \leqslant N$, $\Phi(V \cap N) \leqslant [U, V \cap N] \leqslant U$. As $(N \cap V)U/U \cong C_4$, the latter holds. However, this implies that there is $\hat{t} \in tN \cap V$ such that $V/U \cong (N \cap V, \hat{t})/U \cong D_8$, a contradiction.

Next, take $G$ to be contained in class (b). If $U \lhd V \leqslant G$, then $(N \cap V)U/U$ is an elementary abelian subgroup of $V/U$ with cyclic factor group; $Q_8$ has no such subgroups. Let $n \in N$ with $[n, x, x] \in \langle x\rangle$ and $[n, x] \notin \langle x\rangle$. Then $\langle x^2\rangle \lhd \langle n, x\rangle$ and $\langle n, x\rangle/\langle x^2\rangle \cong D_8$.

The groups in (a) and (b) having been shown to be $Q_8$-free, we know that if $G$ is in class (c) and $U \lhd V \leqslant G$, $V/U \cong Q_8$, then $V \neq U(V \cap N\langle x\rangle)$. Letting $U(V \cap N\langle x\rangle) = H$, we thus get $|V : H| = 2$. Let $H = \langle U, z\rangle$ and let $U = \langle H, v\rangle$. If $\langle v\rangle N > \langle z\rangle N$, then, as $v^2z^2 \in U$, $z^{2k+1}m \in U$ for some $m \in N$ and $k \in \mathbb{N}$. This yields $z \in U$, a contradiction. Accordingly, $v = tz^a m$ for some $m \in N$ and $a \in \mathbb{N}$. But then, $tm \in V \setminus H$ and $tm$ is an involution. Not being $D_8$-free is part of the very definition of the groups in (c).

## 2. Proof of Theorem 1

**Lemma 1.** *Let $P$ be a finite 2-group. Then the following are equivalent*:

(a) *$P$ is powerful and $Q_8$-free.*

(b) *$P$ is modular and $Q_8$-free.*
(c) *If $x, y \in P$, the group $\langle x, y\rangle$ is ordinary metacyclic with* $[x, y] \in \langle x^4, y^4\rangle$.

**Proof.** We will prove that (a) implies (b) by induction on $|P|$. Let $2^n = \exp(P)$ and let $x \in P$ with $o(x) = 2^n$. We may presume that $n \geqslant 3$, $P$ being abelian otherwise. As $P$ is powerful, $\mho_{n-2}(P) = \{y^{2^{n-2}} \mid y \in P\}$ and $\mho_{n-2}(P) \leqslant Z(P)$ (see [2, Proposition 4.1.7 and Corollary 4.1.2]). Let $x^{2^{n-2}} = y$ and $z = x^{2^{n-1}}$; then $y \in Z(P)$. If $P$ is $Q_8$-free, then, via induction, $P/\langle z\rangle$ is $D_8$-free. Suppose there are subgroups $U$ and $V$ of $P$ with $U \lhd V$ and $V/U \cong D_8$. From

$$\langle V, z\rangle/\langle z\rangle/\langle U, z\rangle/\langle z\rangle \cong \langle V, z\rangle/\langle U, z\rangle \ncong D_8,$$

we derive $z \in V\backslash U$ and $Z(U/V) = \langle z\rangle V/V$. As $U \cap \langle y\rangle = 1$ and $|Z(V/U)| = 2$, $y \notin V$; so $\langle V, y\rangle/U \cong C_4 * D_8$, but $C_4 * D_8 \cong C_4 * Q_8$. Analogously, if $P$ is supposed to be $D_8$-free and $V/U \cong Q_8$, then $z \in V\backslash U$ and $V/(U \times \langle z\rangle)$ is elementary abelian of order 4; again, $y \notin V$ and $\langle V, y\rangle/U \cong C_4 * Q_8 \cong C_4 * D_8$.

Next suppose $P$ to satisfy (b); i.e. to be $Q_8$-free as well as $D_8$-free. If $P' \nsubseteq \mho_2(P)$, let $\widehat{P} = P/V$ where $\mho_2(P)[P', P] \leqslant V \lessdot \mho_2(P)P'$. Note that $\exp(\widehat{P}) = 4$ and $|\widehat{P}'| = 2$. Let $x, y \in \widehat{P}$ with $[x, y] \neq 1$; since $x$ and $y$ cannot both be involutions, we may suppose $o(x) = 4$. Let $V = \langle x, y\rangle$. Now $V/\langle x^2, y^2\rangle \ncong D_8$; so either $o(y) = 4$ and $[x, y] = x^2y^2$, or we may suppose that $\widehat{P}' = \langle x^2\rangle$. In the first case, $V/\langle x^2\rangle \cong D_8$, yet in the second case either $V/\langle y^2\rangle \cong D_8$, or $V \cong Q_8$. Thus $P$ is powerful; in other words, (a) and (b) are equivalent.

As (a) and (b) are equivalent, all subgroups of a group $P$ satisfying (a) are powerful, and [2, Theorem 4.3.1] says that $P$ is modular, but not Hamiltonian.

Clearly, (c) implies (a). Conversely, assume $P$ to satisfy (a), let $x, y \in P$ and let $H = \langle x, y\rangle$. By [2, Proposition 4.3.0], $H$ is ordinary metacyclic. Furthermore, [2, Proposition 4.1.9], says that $\mho_2(H) = \langle x^4, y^4\rangle$. □

**Notation.** A finite 2-group $P$ that is $Q_8$-free as well as $D_8$-free will be called $(*)$ for short.

**Lemma 2.** *Let $P$ be $(*)$. Then*

(a) $\operatorname{rank}(P) = \operatorname{rank}(\Omega_1(P))$.
(b) *If $x, y \in P$ and $i \in \mathbb{N}$, then* $(xy)^{2^i} \in x^{2^i} y^{2^i} \langle x^{2^{i+1}}, y^{2^{i+1}}\rangle$.
(c) *Let* $\exp(P) = 2^k$ *and let* $t \in \operatorname{Inv}(P)\backslash Z(P)$. *Then for* $v \in P$, $v^t = v^{1+2^{k-1}}$; *in particular,* $|\Omega_1(P) : \Omega_1(Z(P))| \leqslant 2$ *and* $C_P(t) = \Omega_{k-1}(P)$.

**Proof.** According to [2, Theorem 4.1.12], $\operatorname{rank}(P) \geqslant \operatorname{rank}(\Omega_1(P))$; to prove "$\leqslant$" we use induction on $|P|$. Let $\exp(P) = 2^k$; we may suppose that $k > 2$, $P$ being abelian otherwise. Furthermore, $\Omega_1(P)$ is elementary abelian, $P$ being $D_8$-free. Let $x \in P$ with $o(x) = 2^k$ and let $y = x^{2^{k-2}}$; then $y \in Z(P)$ (see [2, Corollary 4.1.2]) and $\operatorname{rank}(\Omega_1(P)) = \operatorname{rank}(\Omega_1(P/\langle y^2\rangle))$. This proves (a).

According to Lemma 1(c), (b) holds if $i=1$. Suppose that $(xy)^{2^i} \in x^{2^i}y^{2^i}\langle x^{2^{i+1}}, y^{2^{i+1}}\rangle$. Lemma 1(c) yields that if $u \in \langle x^{2^{i+1}}, y^{2^{i+1}}\rangle$, then $u^2 \in \langle x^{2^{i+2}}y^{2^{i+2}}\rangle = \mho_2(\langle x^{2^i}, y^{2^i}\rangle)$ and that $(x^{2^i}y^{2^i}u)^2 \in x^{2^{i+1}}y^{2^{i+1}}u^2\mho_2(\langle x^{2^i}, y^{2^i}\rangle)$, whence $(xy)^{2^{i+1}} \in x^{2^{i+1}}y^{2^{i+1}}\langle x^{2^{i+2}}y^{2^{i+2}}\rangle$. To prove (c), let $t \in \mathrm{Inv}(P)$ and $v \in P$. According to Lemma 1(c), $\langle v,t\rangle = \langle v\rangle\langle t\rangle$, whence $\langle v\rangle$ is normalized by $t$ and so is every subgroup of $P$. It follows from (b) that if $u, v \in P$ and $o(u) > o(v)$, then $o(uv) = o(u)$; thus $P$ is generated by its elements of order $2^k$. Hence if $t \notin Z(P)$, there is $x \in P$ of order $2^k$ with $x^t = x^{2^{k-1}+1}$. Let $y \in P\backslash\langle x\rangle$, and let $\hat{y}$ be of minimal order, $2^s$ say, in $y\langle x\rangle$. If $o(y) = 2^\ell < 2^k$, then $(xy)^{2^{k-1}} = x^{2^{k-1}}$; moreover, $x^{2^{k-1}} \in Z(P)$; so either $y^t = y$, or $[t, xy] = 1 = [t, y][t, x]$ and $[y, t] = [x, t] = x^{2^{k-1}} = y^{2^{\ell-1}}$. If $\langle \hat{y}\rangle \cap \langle x\rangle \neq 1$, then $\hat{y}^{2^{s-1}} = x^{2^{k-1}}$ and, according to (b), $(\hat{y}x^{2^{k-s}})^{2^{s-1}} = 1$ contrary to the choice of $\hat{y}$. Suppose that $[y,t] \neq 1$. Because of $\ell < k$, $\hat{y} \in y\langle x^2\rangle$; whence $[y,t] = [\hat{y}, t] \notin \langle \hat{y}\rangle$. Thus $[\Omega_{k-1}(P), t] = 1$. If $y \in P\backslash x\Omega_{k-1}(P)$, then, by (b), $y^{2^{k-1}} \neq x^{2^{k-1}}$ and $(xy)^{2^{k-1}} = x^{2^{k-1}}y^{2^{k-1}}$. Both $\langle x\rangle$ and $\langle xy\rangle$ are normalized by $t$, this concludes the proof of (c). □

**Lemma 3.** *Let $P$ be $(*)$ and let $A$ be a maximal abelian normal subgroup of $G$. Let $\exp(A) = 2^k$ and let $|A : \Omega_{k-1}(A)| = 2^n$. Let $B$ be the inverse image in $P$ of $\Omega_1(P/A)$. Then the following hold:*

(a) *If $x \in \Omega_k(B)\backslash A$, then $C_A(x) \geqslant \Omega_{k-1}(A)$.*
(b) *If $n \geqslant 2$, then there is at most one coset $xA$, $x \in B\backslash A$, with $o(x) \leqslant 2^k$. If $x \in \Omega_k(B)\backslash A$, and $a \in A$, then $a^x = a^{2^{k-1}+1}$.*
(c) *If $n \geqslant 3$, then the set $B_1 := \{x \mid x \in B,\ |[x, A]| \leqslant 2\} = \{x \mid x \in B,\ [x, A] \leqslant \langle x\rangle\}$ is a maximal subgroup of $B$.*
(d) *Let $n = 1$ and $A = \langle a, \Omega_{k-1}(A)\rangle$, let $\exp(A/\langle a\rangle) = 2^\ell$. There is at most one coset $A \neq xA \subset B$ with $o(x) \leqslant 2^\ell$; for $b \in A$, $b^x = b^{2^{k-1}+1}$.*

**Proof.** Let $x \in \Omega_k(B)\backslash A$; by Lemma 2(b), $o(x) \leqslant 2^k$. Let $\hat{x}$ be an element of the coset $xA$ of minimal order, $2^s$ say. If $a \in A\backslash\Omega_{k-1}(A)$, then, by Lemma 2(b), $\langle \hat{x}\rangle \cap \langle a\rangle = 1$. The same lemma yields that if $\hat{x}^2 = b^2$ for some $b \in A$, then $(\hat{x}b)^{2^{s-1}} = 1$. Accordingly, $\hat{x}^2 \in \Phi(A)$ if and only if $\hat{x}$ has order 2. We may therefore write $A$ as a direct product $\langle a_1, \ldots, a_n\rangle \times U$ with $o(a_i) = 2^k$ for $1 \leqslant i \leqslant n$, $\exp(U) \leqslant 2^{k-1}$ and $\hat{x}^2 \in U$. The existence of $U$ may be seen as follows: If $A/\langle a_i, \ldots, a_n\rangle$ is cyclic, then $A = \langle a_i, \ldots, a_n, \hat{x}^2\rangle$; if not, there is a cyclic subgroup $\langle z\rangle$ of $A$ of order 2 with $\langle z\rangle \cap \langle a_i, \ldots, a_n, \hat{x}^2\rangle = 1$. Induction may then be applied to obtain that $A = \langle a_i, \ldots, a_n\rangle \cdot U$ with $\langle \hat{x}^2, z\rangle \leqslant U$ and $U \cap \langle a_i, \ldots, a_n\rangle = 1$.

Next apply Lemma 2(c) to obtain that $[a_i, x] \in a_i^{2^{k-1}}\langle \hat{x}^2\rangle$, $1 \leqslant i \leqslant n$; while, as $\exp(A/\langle \hat{x}^2\rangle) = 2^k$, $[\Omega_{k-1}(A), x] \leqslant \langle \hat{x}\rangle$. In particular, $[A, x, x] = 1$; so for $a \in A$, $[a, x^2] = [a, x]^2 = 1$. Note that this implies that $[\Phi(A), x] = 1$.

Let $\tilde{a}_i = a_i^{2k-s}$, $1 \leqslant i \leqslant n$. By Lemma 2(b), $(\hat{x}\tilde{a}_i)^{2^{s-1}} = \hat{x}^{2^{s-1}}\tilde{a}_i^{2^{k-1}}$; so $[x, \Omega_{k-1}(A)] \leqslant \langle \hat{x}^{2^{s-1}} \rangle \cap \langle \hat{x}^{2^{s-1}}\tilde{a}_i^{2^{k-1}} \rangle$, $1 \leqslant i \leqslant n$, implying $[x, \Omega_{k-1}(A)] = 1$, proving (a). If $n > 1$, and $1 \leqslant i \neq j \leqslant n$, then

$$[x, a_i] \in a_i^{2^{k-1}}\langle \hat{x}^{2^{s-1}} \rangle \cap a_i^{2^{k-1}}\langle (\hat{x}^{2^{s-1}} a_j^{2^{k-1}}) \rangle = \{a_i^{2^{k-1}}\},$$

which proves (b). Furthermore, $[A, x] = \mho_{k-1}(A)$; so, unless $n = 1$, $|[A, x]| > 2$.

Next assume $n > 2$ and let $y, y' \in B \backslash A$ with $[A, y] \nleqslant \langle y^2 \rangle$ and $[A, y'] \nleqslant \langle (y')^2 \rangle$. Since $n > 2$, we may assume that $\langle a_1 \rangle \cap \langle y^2, (y')^2 \rangle = 1$. Applying Lemma 2(b), we obtain that $(yy')^2 \in \langle y^2, (y')^2 \rangle$; whence Lemma 2(c) says that $[a_1, yy'] \in \langle (yy')^2 \rangle \geqslant [A, yy']$. If $y \in B$ with $[A, y] \leqslant \langle y^2 \rangle$, then for $a \in A$, $[a, y^2] = 1 = [a, y]^2$, while, if $y \in B$ and $a \in A$, then $[a, y] \in 2^{k-1}\langle y^2 \rangle$. Since $n \geqslant 3$, this yields that $B_1 = \{x \mid x \in B,\ [x, A] \leqslant \langle x \rangle\}$. Conversely, assume that $\{y, y'\} \subset B_1 \backslash A$. According to Lemma 1(d), $(yy')^2 \in \langle y^2, (y')^2 \rangle$ and we are supposing $[A, yy'] \leqslant \langle y^{2^k}, (y')^{2^k} \rangle$. As $n > 2$, we may again take $\langle a_1 \rangle \cap \langle y^2, (y')^2 \rangle = 1$. Accordingly, $[a_1, yy'] \notin a_1^{2^{k-1}}\langle (yy')^2 \rangle$, which entails that $yy' \in B_1$. Thus (c) is fully established.

Finally, assume that $n = 1$ and let $A = \langle a, \Omega_{k-1}(A) \rangle$. Let $x \in B \backslash A$ with $o(x) = 2^s$, $s \leqslant \ell$; there is $b \in \Omega_{k-1}(A)$ of order $2^s$; so $[x, a] \in a^{2^{s-1}}\langle x^{2^{s-1}} \rangle \cap a^{2^{s-1}}\langle x^{2^{s-1}} b^{2^{s-1}} \rangle = \{a^{2^{s-1}}\}$, proving (d). $\square$

According to [4], a non-abelian $Q_8$-free 2-group $G$ has a characteristic maximal subgroup. This implies that $[G, O^2(\mathrm{Aut}(G))]$ is abelian. Let $P$ be $(*)$, $D = [P, O^2(\mathrm{Aut}(P))]$, and $A$ be a maximal characteristic abelian subgroup of $P$ in which $D$ is contained. Let $C = C_P(A)$; then $[C, O^2(\mathrm{Aut}(P))] \leqslant A$. Now $\mathrm{Aut}(P)/O^2(\mathrm{Aut}(P))$ is a 2-group; so either $C = A$, or there is $C_1/A \leqslant C/A$ with $C_1$ char $P$ and $|C_1 : A| = 2$. The group $C_1$ would have to be abelian, contradicting the choice of $A$. Hence $A$ is a maximal abelian subgroup of $P$.

**Notation.** Let $P$ be $(*)$, let $A$ be a characteristic abelian subgroup of $P$ that is maximal subject to the constraint of containing $[P, O^2(\mathrm{Aut}(P))]$. Let $B$ be the inverse image in $P$ of $\Omega_1(P/A)$. It may be worth keeping in mind that, as $B$ is powerful, $B/\mho_1(A)$ is abelian. Let $\exp(A) = 2^k$ and $|A : \Omega_{k-1}(A)| = 2^n$; if $n = 1$ and $a \in A \backslash \Omega_{k-1}(A)$, let $\exp(A/\langle a \rangle) = 2^\ell$.

**Lemma 4.** *Suppose that there are elements $x \in B \backslash A$ and $y \in B \backslash \langle A, x \rangle$ with $[A, z] \leqslant \langle z \rangle$ when $z \in \{x, y, xy\}$. If $n = 1$, then additionally suppose that $o(x) \leqslant 2^k \geqslant o(y)$. Then $|A : C_A(\langle x, y \rangle)| = 2$; furthermore, exactly one of the maximal subgroups of $\langle A, x, y \rangle$ is abelian.*

**Proof.** Let $v \in \{x, y, xy\}$. First of all, $[A, v, v] = 1 = [A, v^2]$; so $|[A, v]| = 2$. By Lemma 3(b), we know that $n = 1$ or $o(x) = o(y) = o(xy) = 2^{k+1}$; while, if $n = 1$, then Lemma 3(a) says that $C_A(x) = C_A(y) = \Omega_{k-1}(A)$. Note that the involution in $\langle xy\rangle$ cannot be contained in $\langle x\rangle \cup \langle y\rangle$; whence $o(x) = o(y)$ and $\langle x\rangle \cap \langle y\rangle = 1$. If $n > 1$, then, as $|[A, (xy)]| = \langle x^{2^k}, y^{2^k}\rangle < [A, x][A, y]$, we have $C_A(y) = C_A(x)$. Accordingly, $v^2 \in C_A(v) = C_A(x)$, in other words $\Phi(\langle x, y\rangle) \leqslant C_A(\langle x, y\rangle)$. Let $o(x) = 2^m$. Then $\langle x, y\rangle' \leqslant \langle x^{2^{m-1}}, y^{2^{m-1}}\rangle$, and therefore one of the groups $\langle x\rangle$, $\langle y\rangle$, $\langle xy\rangle$ must be normal in $\langle x, y\rangle$. We are thus entitled to assume that $[x, y] \in \langle x^{2^{m-1}}\rangle$. Hence either $[x, y] = 1$, or $[x, ay] = 1$; where $a \in A\backslash C_A(x)$. One, but not both, of the groups $\langle x, y, C_A(x)\rangle$ and $\langle x, ay, C_A(x)\rangle$ is therefore abelian; call this group $C$. If $v \in \langle A, x, y\rangle\backslash C$ then $C_C(v) = C_A(x)$; so $C$ is the only abelian maximal subgroup of $\langle A, x, y\rangle$. $\square$

**Lemma 5.** *If $A$ is of maximal order among the set of a maximal abelian normal subgroups of $P$ characteristic in $P$ and containing $[P, O^2(\mathrm{Aut}(P))]$, then $|B : A| \leqslant 4$.*

**Proof.** First consider the case $n \geqslant 3$ and let $B_1$ be defined as in Lemma 3(c); note that $B_1$ char $P$. Whenever $x \in B_1\backslash A$ and $y \in B_1\backslash\langle A, x\rangle$, the set $\{x, y, xy\}$ satisfies the premiss of Lemma 4. In particular, $|A : C_A(B_1)| \leqslant 2$, while $C_A(B_1)$ char $P$; whence $[P, O^2(\mathrm{Aut}(P))] = [A, O^2(\mathrm{Aut}(P))] \leqslant C_A(B_1)$. If $|B_1 : A| \geqslant 4$, then there are $x, y \in B_1$ with $V = \langle x, y, A\rangle$ char $P$ and $|V : A| = 4$. However, Lemma 4 says that exactly one maximal subgroup of $V$ is abelian, so is characteristic in $P$; this contradicts the maximality of $A$. Accordingly, $|B_1 : A| \leqslant 2$ and $|B : A| \leqslant 4$.

Next let $n = 2$ and suppose that $|B : A| = 8$. According to Lemma 3(b), $B = \langle x, y, z, A\rangle$ where $z^2 \in \Omega_{k-1}(A)$ and $A = \langle x^2, y^2, \Omega_{k-1}(A)\rangle$. This entails $\langle x\rangle \cap \langle y\rangle = 1$; taking $z$ to be of minimal order in $zA$, we get $\langle z\rangle \cap \langle x^2, y^2\rangle = 1 = \langle z\rangle \cap \langle x, y\rangle$. Let $o(z) = 2^s$ and let $v \in \{x, y, xy\}$. Since $[\Omega_{k-1}(A), v] \leqslant \langle v^{2^k}\rangle$, we have $|\Omega_{k-1}(A) : C_{\Omega_{k-1}(A)}(\langle x, y\rangle)| \leqslant 2$. From Lemma 3(c), we know that $[z, v^2] = v^{2^k}$; so $[v^4, z] = 1$. Furthermore, $[z^4, v] \in [\Phi(\Omega_{k-1}(A)), v] = 1$. By Lemma 1(d) $\langle z, v\rangle' \leqslant \langle z^4, v^4\rangle \leqslant Z(\langle z, v\rangle)$. Accordingly, $[v^2, z] = [v, z]^2$; so $[v, z] = \hat{v}\hat{z}$ with $\hat{v} \in v^{2^{k-1}}\langle v^{2^k}\rangle$ and $\hat{z} \in \langle z^{2^{s-1}}\rangle$. This yields $[v, z^2] \neq 1$ and $\Omega_{k-1}(A) = \langle z^2, C_{\Omega_{k-1}(A)}(\langle x, y\rangle)\rangle$.

According to Lemma 1(c), the group $\langle x, y\rangle$ is ordinary metacyclic and we may take $\langle x\rangle \lhd \langle x, y\rangle$. Observe that $\langle x^{2^k}, y^{2^k}\rangle \leqslant Z(\langle x, y\rangle)$ and let $\{v, w\} \subset \{x, y, xy\}$. Applying Lemma 1(c) to the group $\langle A, v\rangle/\langle v^2\rangle$, we obtain $[w^2, v] \in \langle w^{2^k}, v^2\rangle$; so $[w^2, v^2] = [w^2, v, v] = 1$. Accordingly, $[A, v] \leqslant \Omega_1(C_A(v))$. This yields $[y^2, x] \in \langle x^{2^k}\rangle$ and $[\Phi(A), \langle x, y\rangle] = 1$; in particular, $[x^4, y] = 1 = [x, y^4]$. As $\langle x, y\rangle$ is powerful, this implies $\mathrm{cl}(\langle x, y\rangle) \leqslant 2$.

We have just seen that $[x, y^2] \in \langle x^{2^k}\rangle = [x, \Omega_{k-1}(A)]$; so $[A, x] = \langle x^{2^k}\rangle$. If $[x, y^2] = x^{2^k}$ then $[x, y] \in x^{2^{k-1}}\langle x^{2^k}\rangle$ and $[x^2, y] = [x^2, z]$; accordingly, $[A, yz] = \langle (yz)^{2^k}\rangle$. Furthermore, $[x^2, xyz] = 1$, while $[y^2, xyz] = x^{2^k} y^{2^k} = (xyz)^{2^k} = [z^2, xyz]$. If $\{v, w\} \subset \{x, yz, xyz\}$ then $[zv, v^2] = v^{2^k}$, while $[w^2, zv] \in w^{2^k}\langle v^{2^k}\rangle$, whence $\langle x, yz, A\rangle = \langle v \mid v \in B,\ |[v, A]| \leqslant 2\rangle$. Moreover, the set

$\{x, yz, xyz\}$ satisfies the premiss of Lemma 4. If $[x, y^2] = 1 = [x, y]^2 = [x^2, y] = 1$, then for $v \in \{x, y, xy\}$, we have $[A, v] = \langle [z^2, v]\rangle = v^{2^k}$; if $w \in \langle x, y, z, A\rangle \backslash \langle x, y, A\rangle$, then $[w, \langle (x^2, y^2\rangle] = [z, \langle x^2, y^2\rangle]$. Hence if $[x, y^2] = 1$, then $\langle v \mid v \in B,\ |[v, A]| \leqslant 2\rangle = \langle x, y, A\rangle$. Furthermore, $\langle x, y\rangle$ satisfies the premiss of Lemma 4.

We have found out that if $n = 2$ and $|B : A| = 8$, there is a maximal subgroup $A \leqslant U \lessdot B$ with $[A, v] \leqslant \langle v\rangle$ for $v \in U\backslash A$ and $U$ char $P$. According to Lemma 4, $U$ has a unique maximal subgroup $C$ that is abelian; thus the choice of $A$ is contradicted.

Finally, consider the case $n = 1$. Clearly, $|B : \Omega_k(B)| \leqslant 2$. If $x \in \Omega_k(B)\backslash A$, let $A = \langle a, \Omega_{k-1}(A)\rangle$. Recall that by Lemma 3(a), $C_A(x) = \Omega_{k-1}(A)$.

According to Lemma 3(d), $|\Omega_\ell(B)A : A| \leqslant 2$. Let $x, y \in B\backslash A$ and suppose that $\{x, y, xy\} \subset \Omega_k(B)\backslash \Omega_\ell(B)$. If $v \in \{x, y, xy\}$, take $v$ to be of minimal order in $vA$. Using Lemma 2(b), we find that $\langle v\rangle \cap \langle a\rangle = 1$, whence $o(x) = o(y) = o(xy) = 2^{\ell+1}$; note that therefore $\langle x\rangle \cap \langle y\rangle = 1$. We may suppose that $[a, y] \neq a^{2^{k-1}}$; if $[a, y] = a^{2^{k-1}} y^{2^\ell}$, $y$ may be replaced by $y \cdot a^{2^{k-\ell-1}}$ to obtain $[a, y] = y^{2^\ell}$. If $[a, x] = a^{2^{k-1}}$, then $[a, xy] = a^{2^{k-1}} y^{2^\ell} \notin \langle a^{2^{k-1}}, (xy)^{2^\ell}\rangle$, this contradicts Lemma 1(c). Thus $[a, x] \in \{a^{2^{k-1}} x^{2^\ell}, x^{2^\ell}\}$ and we may, possibly upon replacing $x$ by $x \cdot a^{2^{k-\ell-1}}$, assume that $[a, x] = x^{2^\ell}$. The upshot of these considerations is that the set $\{x, y, xy\}$ may be modified to satisfy the premiss of Lemma 4. Hence if $|B : A| > 4$ and there is no element of $P$ of order less than $2^{\ell+1}$ contained in $B\backslash A$, then, since $\Omega_k(B)\,\mathrm{char}\,P$, we might choose $V = \langle A, x, y\rangle\,\mathrm{char}\,P$ and apply Lemma 4 to obtain a contradiction to the choice of $A$.

Suppose that there is $z \in B\backslash A$ with $o(z) \leqslant 2^\ell$. By Lemma 3, we know that $C_A(z) = \Omega_{k-1}(A)$ and that $[a, z] = a^{2^{k-1}}$. Let $x \in \Omega_K(B)\backslash \langle A, z\rangle$; as before, we may assume that $o(x) = 2^\ell$; moreover, $[a, xz] \neq 1$; so $[a, x] \neq a^{2^{k-1}}$ and we may again assume that $[a, x] = x^{2^\ell}$. Accordingly, $[a, xz] = a^{2^{k-1}} x^{2^\ell}$. Furthermore, $[z, x] \in \Omega_{k-1}(A) = C_A(\langle x, z\rangle)$ and $[z^2, x] = 1 = [x^2, z]$. Letting $o(z) = 2^s$, we get $[z, x] \leqslant \langle x^{2^\ell}, z^{2^{s-1}}\rangle$. If $\langle x\rangle$ is not normal in $\langle x, z\rangle$, then, possibly upon replacing $z$ by $zx^{2^{\ell+1-s}}$, we may suppose that $[x, z] = z^{2^{s-1}}$. Yet in this case $[az, x] = a^{2^{k-1}} z^{2^{s-1}} \notin \langle a^{2^{k-1}}, x^{2^\ell}\rangle = \langle (az)^{2^{k-1}}, x^{2^\ell}\rangle$, contradicting Lemma 1(c). Thus $z$ normalizes $\langle a\rangle$ as well as $\langle x\rangle$, and, more specifically, either $[z, x] = 1$, or $[az, x] = 1$. We may choose $U = \langle A, x, z\rangle\,\mathrm{char}\,P$; let $C$ be whichever of the groups $\langle \Omega_{k-1}(A), az, x\rangle$, $\langle \Omega_{k-1}(A), z, x\rangle$ is abelian. Then $C$ is the unique maximal abelian subgroup of $U$, so is characteristic in $P$, too. Thus if $\Omega_\ell(B) > A$, then $\Omega_{\ell+1}(B) \leqslant \langle (\Omega_\ell(B), A\rangle$ and Lemma 4 yields $|B : A| \leqslant 4$. $\square$

Let $A$ be as previously defined. We have just shown that $\Omega_1(P/A)$ is elementary abelian of order at most 4. Now Lemma 2(a) says that $P/A$ is generated by at most 2 elements, and Lemma 1(c) yields that $P/A$ is ordinary metacyclic. This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

**Lemma 6.** *Let $P$ be $(*)$ and let $P/\mho_2(P) = \widehat{P}$. Suppose that $\widehat{P} = \widehat{C} \times \widehat{E}$ with $\widehat{C} \cong C_4$ and $\widehat{E}$ elementary abelian. Then $P = \langle a\rangle \cdot \Omega_1(P)$ and $\Omega_1(P)$ is elementary abelian*; *letting $2^k = o(a)$, we have $[a, \Omega_1(P)] \leqslant \langle a^{2^{k-1}}\rangle$.*

**Proof.** Let $a \in P$ with $\langle a\rangle\mho_2(P)/\mho_2(P) \cong C_4$. Observe that $\Phi(P) = \langle a^2\rangle$, which in particular makes $\langle a\rangle$ a normal subgroup of $P$. As $P$ is $D_8$-free, $\Omega_1(P)$ is elementary abelian. Let $t \in P\backslash\langle a\rangle$; since $t^2 \in \langle a^4\rangle$, $|\langle t, a\rangle/\langle a\rangle| = 2$, and $\langle a, t\rangle$ either is abelian, or is isomorphic with $\mathrm{Mod}_{2^{k+1}}$. Either way, $\langle t, a\rangle$ is the semidirect product of $\langle a\rangle$ with a group of order 2; so $P = \langle a\rangle\Omega_1(P)$ and $|P : C_P(a)| \leqslant 2$. $\square$

In a finite $Q_8$-free 2-group $G$ there are no elements $x$, $o(x) = 2^k > 2$, and $y$, $o(y) = 4$, with $x^y \neq x^{-1}$. The next lemma is an almost immediate, yet very useful, consequence of this simple fact.

**Lemma 7.** *Let $P$ be a finite $Q_8$-free 2-group and let $x$ and $y$ be elements of $P$ of order four with $[x^2, y] = 1 = [y^2, x]$. Then $[x, y] \in \langle x^2y^2\rangle$.*

**Proof.** Let $[y, x] = b$; we may suppose that $b \neq 1$. As $[y, x^2] = 1$, $(yb)^x = y = ybb^x$; likewise, $bb^y = 1$; so $b$ must be an involution with $[b, x] = 1 = [b, y]$. Let $V = \langle by^2, bx^2\rangle$; then $V \leqslant Z(\langle x, y\rangle)$ and, since $\langle x, y\rangle/V \not\cong Q_8$, at least one of $x^2$, $y^2$ is contained in $V$; so $x^2 = by^2$ or $y^2 = bx^2$, i.e. $x^2y^2 = b$. $\square$

**Lemma 8.** *Let $H$ be a finite $Q_8$-free 2-group and suppose $H$ to possess a maximal powerful normal subgroup $A$ which is abelian of exponent at most four. Then one of the following holds*:

(a) $H = A$.
(b) *$H$ is the semidirect product of $A$ and a cyclic group $\langle s\rangle$*; *if $t$ is the involution in $\langle s\rangle$, then $a^t = a^{-1}$ for $a \in A$.*
(c) $|A : \Omega_1(A)| \leqslant 2$.

**Proof.** Suppose there is $x \in H\backslash A$ with $1 \neq x^2 \in \Omega_1(A)$. Let $a \in A\backslash\Omega_1(A)$; then $[a, x]^x = [x, a]$; whence $[a, x] \in C_{\Omega_1(A)}(x)$. In particular, $[a^2, x] = 1 = [a, x^2]$; so Lemma 7 yields $[a, x] \in \langle a^2x^2\rangle$. If $1 \neq b^2 \neq a^2$ and $b \in A$, then either $[b, x] = 1$ and therefore $[ab, x] = a^2x^2 \neq a^2b^2x^2$, or else $[x, b] = x^2b^2$ which implies that $[x, ab] = a^2b^2 = (ab)^2$; neither is possible.

Suppose that $|A : \Omega_1(A)| > 2$, and let $x \in H\backslash A$ with $x^2 \in A$. We have seen that $o(x) \in \{2, 8\}$; first suppose $o(x) = 8$. Let $a \in A\backslash\Omega_1(A)$ and suppose that $a^x = a^{-1}$; then, unless $a^2 = x^4$, $\langle a, x\rangle/\langle x^{-2}a^2\rangle \cong Q_8$. Next let $b \in A$ with $[b, x] \notin \Omega_1(A)$; as $[b, x^2] = 1$, $[b, x]^x = [x, b]$. We have just seen that this entails that $[b, x]^2 = x^4$; so $[b, x] = x^2 z$ with $z \in \Omega_1(A)$. Now $(bx)^2 = b^2x^4z[z, x] \in \Omega_1(A)$, and, since $o(bx) \leqslant 4$, $bx$ is an involution. Replacing $x$ by $x^{-1}$, we obtain that $o(bx^{-1}) = 2$, too; however, $(bx^{-1})^2 = (bx)^2x^4 = x^4$. Thus $[A, x] \leqslant \Omega_1(A)$. Since $[A, x^2] = 1$, we even obtain that $[A, x] \leqslant C_{\Omega_1(A)}(x)$.

Let $a \in A\backslash\Omega_1(A)$ with $a^2 \neq x^4$—we are assuming $|A : \Omega_1(A)| > 2$. Applying Lemma 7 to the group $\langle a, x\rangle/\langle x^4\rangle$, we obtain that either $[a, x] \in \langle x^4\rangle$, or $[a, x]\langle x^4\rangle = a^2x^2\langle x^4\rangle$, which is impossible as $[A, x] \leqslant \Omega_1(A)$. Since $A = \langle b \mid b \in A,\ b^2 \notin \langle x^4\rangle\rangle$, we have found that $[A, x] \leqslant x^4$. In particular, the group $\langle A, x\rangle$ is powerful, thus cannot be normal in $H$. If $H/A$ is noncyclic, then, $H$ being quaternion-free, $H/A$ has an elementary abelian subgroup $U/A$ of order four. Let $U = \langle A, t, x\rangle$. We may suppose $[t, H] \subseteq A$, which we have just seen to imply that the coset $tA$ consists entirely of involutions; in other words, every element of $A$ is inverted by $t$. Now $C_H(a) = A$; whence both x and $xt$ may be chosen to be of order 8. We have seen this to entail that $[A, v] = \langle v^4\rangle$ for $v \in \{x, xt\}$. Note that $(xt)^2 = x^2[t, x]^t = x^2[x, t]$.

Furthermore, $[x^2, xt] = x^4$; so $[A, xt] = \langle x^4\rangle = \langle (xt)^4\rangle$. Then $o([x, t]) = 2$ accordingly. Let $a \in C_A(x)\backslash\Omega_1(A)$. Then $[a, xt] = a^2$; so $a^2 = x^4$ and $C_A(x) \leqslant \langle x^2, \Omega_1(A)\rangle$. Now $|A : C_A(x)| = 2$, which leaves only the possibility $|A : \Omega_1(A)| = 4$ and $[\Omega_1(A), x] = 1 = [\Omega_1(A), xt]$. But now $C_A(x) = \langle x^2, \Omega_1(A)\rangle = \langle (xt)^2, \Omega_1(A)\rangle = C_A(xt)$. However, $[A, t] \neq 1$.

If neither (a) nor (c) holds, then, as we have seen, $|\Omega_1(H/A)| = 2$ and there is $t \in H\backslash A$ with the coset $tA$ comprised entirely of involutions; i.e. each element of $A$ inverted by $t$. As $H$ is supposed to be $Q_8$-free, $H/A$ has to be cyclic. □

**Lemma 9.** *Let $H$ and $A$ be as in the previous lemma, with the additional assumption that $|A : \Omega_1(A)| = 2$. Then $\Omega_1(H/\Omega_1(A))$ is elementary abelian and, defining $B$ to be the inverse image in $H$ of $\Omega_1(H/\Omega_1(A))$, we have $[B, \Omega_1(A)] = 1$; furthermore, $B = A \cdot C$, for some elementary abelian subgroup $C$ of $B$ with $C \cap \Omega_1(A) = 1$.*

**Proof.** Let $a \in A\backslash\Omega_1(A)$ and let $x \in H\backslash A$ with $1 \neq x^2 \in \Omega_1(A)$. We may take $a$ to be chosen such that $[a, x] \neq 1$. However, $[a^2, x] = [a, x, x] = 1 = [x, a^2]$; so we may apply Lemma 7 to find $[a, x] = a^2x^2$. Replacing a by $as$, $s \in \Omega_1(A)$, we obtain that either $[as, x] = a^2x^2$ and $[s, x] = 1$, or $[a, x] = [s, x]$. In either case $[A, x] = a^2x^2$ and either $C_A(x) = \Omega_1(A)$, or $A = \Omega_1(A)C_A(x)$ and $x$ acts on $\Omega_1(A)$ as a transvection. Suppose the latter: If $z \in \Omega_1(A)\backslash C_{\Omega_1(A)}(x)$, then $(xz)^2 = x^2[x, z] = a^2$; so $\langle a, xz\rangle/\langle x^2\rangle \cong Q_8$. Thus $[x, \Omega_1(A)] = 1$ and if $1 \neq y^2 \in \Omega_1(A)$, $y \in H$, then $[x, y^2] = 1 = [y, x^2]$; whence, by Lemma 7, $[x, y] \in \langle x^2y^2\rangle \leqslant \Omega_1(A)$. Let $y \in H\backslash\langle A, x\rangle$ with $y^2 \in A$. Then $[A, y] =$

$\langle a^2y^2\rangle$, in particular $y^2 \neq x^2$. If $[x, y] = 1$, then $[a, xy] = a^2x^2a^2y^2 = (xy)^2$; whence $[x, y] = x^2y^2$. Thus both $ax$ and $ay$ are involutions, while $[ax, ay] = [a, y][a, x][x, y] = (a^2x^2)(a^2y^2)(x^2y^2) = 1$. Let $C_1 = \langle y \mid y \in H\backslash A,\ 1 \neq y^2 \in \Omega_1(A)\rangle$ and let $C = \langle ay \mid y \in H\backslash A,\ 1 \neq y^2 \in \Omega_1(H)\rangle$. We already know that $C$ is elementary abelian; furthermore, $AC_1 = AC$.

If $B > AC_1$, then there must be $t \in H$ such that $tA$ consists solely of involutions; in particular, $[t, \Omega_1(A)] = 1$. We will prove that either $B = \langle A, t\rangle$, or $t \in AC_1$. So let $c \in C$; if $(ac)^2 = 1$, then $[a, c] = a^2 = [a, t]$ and $tc \in C_H(A) = A$. Thus in the present case $o(ac) = 4$ for every $c \in C$; in particular, $C\Omega_1(A) = \Omega_1(AC_1)$ which implies that $[C, t] \leqslant C\Omega_1(A)$. If $[C, t] \neq 1$, then there would be $c \in C\backslash A$ with $1 \neq [t, c] \in \Omega_1(A)$; whence $tc \in C_1$. Thus $[\Omega_1(C_1), t] = 1$ and, accordingly, if $c \in C$, then $(ac)^t = a^{-1}c$. Choosing $c \notin A$, we obtain that $(tac)^2 = [c, at] = [c, a] \neq 1$; whence $tac \in C_1$ and $ac \in C_1$, therefore $t \in C_1$. $\square$

**Lemma 10.** *Let $H$ be as before and suppose $A$ to be elementary abelian. Then one of the following is true*:

(a) *There is $z \in H\backslash A$ with $\langle z^2, [z, H]\rangle \leqslant A$ such that whenever $v \in H\backslash A$ and $1 \neq v^2 \in C_A(z)$, then $[A, v] = \langle v^2\rangle$ and $C_A(z) = C_A(v)$.*

(b) *$H/A$ is cyclic.*

**Proof.** We will conduct some preliminary investigations, listed as (1)–(3).

Let $x \in H\backslash A$, $y \in H\backslash\langle x, A\rangle$ and assume that $\langle x^2, y^2\rangle \leqslant A$. If $v \in \{x, y\}$, the group $\langle v, A\rangle$ cannot be elementary abelian, and we will assume that $x^2 \neq 1 \neq y^2$ throughout.

(1) First suppose that $[x, y] = 1$ and that $[x, yt] = 1 = [xs, y]$ whenever $s, t \in A$ with $(yt)^2 \neq 1 \neq (xs)^2$ and $[(yt)^2, x] = 1 = [(xs)^2, y]$. There is $t \in A$ with $[y, t] \neq 1 = [y, t, x]$. As $[y, t, x] = [x, t, y] = [(xt)^2, y] = 1 \neq [xt, y]$, our present assumption yields $[x, t] = x^2 = [x, yt] \neq [x, (yt)^2] = 1$. Hence $[y, t] = y^2$. Accordingly, $[x, \hat{t}\,] = x^2$ whenever $\hat{t} \in tC_A(y)$ and if $\hat{t} \in tC_A(x)$, then $[\hat{t}, y] = y^2$. This implies that $C_A(y) = C_A(x)$, and if $t \in A\backslash C_A(x)$, then $[t, y] \in C_A(x)$. We have found this to imply $[A, y] = \langle y^2\rangle$ and $[A, x] = \langle x^2\rangle$.

(2) Next assume that $[x, y^2] = 1 = [y, x^2] \neq [x, y]$. Then Lemma 7 says that $[x, y] = x^2y^2$. In particular, $[x, y] \in A$ and $o(xy) = 2$.

Suppose that $C_A(x) \neq C_A(y)$, and take notation to be chosen such that $C_A(x)\backslash C_A(y) \neq \emptyset$. Let $t \in C_A(x)\backslash C_A(y)$. Then $(xyt)^2 = [y, t] \neq 1$ and $[y, t] \in C_A(x) \cap [A, y] \leqslant C_A(\langle x, y\rangle)$. According to Lemma 7, $[xyt, y] \in \langle y^2(xyt)^2\rangle$. If $[xyt, y] = 1$, then $[y, t] = [y, x]$; if not, we obtain that $[xyt, y] = [x, y][t, y] = y^2(xyt)^2 = y^2[t, y]$. As $[x, y] = x^2y^2$, the second implies $x^2 = 1$, yet if $[y, t] = [y, x]$, then $(yt)^2 = y^2[y, t] = x^2$, so $[x, yt] = 1 = [x, y]$ by Lemma 7. Thus $C_A(x) = C_A(y)$. Let $v \in A$: We have seen that $[A, x][A, y] \leqslant C_A(\langle x, y\rangle)$; so $[x, (yv)^2] = [x, y^2[y, v]] = [y, x^2[x, v]] = [y, (xv)^2] = 1$. First assume that

$[x, yv] = 1$. Then $(xv)^2 = x^2[x, y] = y^2$, and Lemma 7 yields that $[y, xv] = 1$; i.e. $[y, v] = x^2y^2 = [x, v]$. Thus unless either $[v, y] = y^2$ or $[v, y] = x^2y^2 = [v, x]$, we obtain, again with the help of Lemma 7, that $[x, yv] = [x, y][x, v] = x^2y^2[x, v] = x^2(yv)^2 = x^2y^2[y, v]$; interchanging the roles of $x$ and $y$ in the preceding argument, we obtain that if $[x, v] \neq [y, v]$, then $[x, v] = x^2$ and $[y, v] = y^2$. Since $A$ is a maximal normal abelian subgroup of $H$, this is possible only if $|A : C_A(x)| = 2$ and $[A, x] = \langle x^2\rangle$ and $[A, x] = \langle y^2\rangle$. In other words, $x$ and $y$ act on $A$ as transvections to the same hyperplane, and both the cosets $xA$ and $yA$ contain involutions, while $o(xy) = 2$.

(3) Thirdly, assume that $[x, y] \in A$ and that $[y^2, x] = 1 \neq [y, x^2]$. Let $V = \langle [y, x^2]\rangle$ and consider the group $\langle x, y\rangle/V$. As $V \leqslant C_A(x) \cap [A, y]$, $[V, x] = [V, y] = 1$; as $[x^2, y] = [y, x, x]$ neither of the elements $x^2$ and $[x, y]$ is contained in $V$. According to Lemma 7, either $y^2 \in V$, or else $[x, y] \in x^2y^2V$. If the second held, then $(xy)^2 \in V$; in particular, $[y^2, xy] = 1 = [(xy)^2, y]$. As $V = \langle [xy, x^2]\rangle$, we could take $V = \langle (xy)^2\rangle$; whence either (1) would apply to the pair $(y, xy)$, or there would be $\hat{y} \in yA$ and $\widehat{xy} \in (xy)A$ such that $\hat{y}$ and $\widehat{xy}$ could replace $x$ and $y$ in (2). Either way, we would obtain that $[y, A] = \langle y^2\rangle$, $[xy, A] = V = \langle (xy)^2\rangle$ and $C_A(y) = C_A(xy) = C_A(x)$. This, however, is possible only if $[x^2, y] = 1$. Thus $[x^2, y] = y^2$.

We have seen that $y^2 \in V = \langle [x^2, y]\rangle$; the order of $y$ was supposed to be 4; whence $V = \langle y^2\rangle$. Next assume there is $t \in A$ with $[(yt)^2, x] \neq 1$. Let $W = \langle [x^2, y], [y, t, x]\rangle$; then $W \leqslant C_A(\langle x, y\rangle)$; in particular, none of the elements $[x, y]$, $x^2$, $(yt)^2$ can be contained in $W$. Observe that $V \leqslant W$, thus $y^2 \in W$. If $[x, yt] = [x, t][x, y] \in W$, then $1 = [x, y, y][x, t, y] = [x, t, y] = [y, t, x] = [(yt)^2, x]$. Applying Lemma 7 to the group $\langle x, yt\rangle/W$, we therefore obtain that $x^2(yt)^2W = x^2[y, t]W = [x, yt]W = [x, y][x, t]W$. Since $[W, x] = 1$, this implies that $[y, t, x] = [x, t, y] = [y, x, x] = [y, x^2]$; in particular, $W = V$. Furthermore, $[(xt)^2, y] = 1$; so unless $[t, x] = x^2$, either (1) is satisfied by the pair $(xt, y)$, or there are elements $\hat{x}$ of $xA$ and $\hat{y}$ of $yA$ such that $\hat{x}$ and $\hat{y}$ can replace $x$ and $y$ in (2). Either way, we would obtain that $C_A(x) = C_A(y)$; so $[x^2, y] = 1$. We have found that if $[(yt)^2, x] \neq 1$, $t \in A$, then $[x, t] = x^2$. Thus if $B = \{s \mid s \in A,\ [s, y] \in C_A(x)\}$, then $A\backslash B$ is covered by only one coset of $C_A(x)$ in $A$. This is possible only if $[A, x] = \langle x^2\rangle$ and $B = C_A(x) \lessdot A$.

But now $[A, x] = \langle x^2\rangle \leqslant C_A(y)$, contradicting the initial assumption.

We have seen that $[(yt)^2, x] = 1$ for $t \in A$. We have also seen this to imply $(yt)^2 \in \langle [x^2, yt]\rangle$; whence either $[t, y] = y^2$, or $(yt)^2 = y^2[y, t] = [x^2, yt] = [x^2, y] = y^2$. Hence $[A, y] = \langle y^2\rangle$; i.e. $y$ acts on $A$ as a transvection and there are involutions contained in $yA$.

Aided by (1)–(3) we are now ready to prove the lemma. Let $u \in H\backslash A$ with $\langle u^2, [u, H]\rangle \leqslant A$ and assume $u^2 \neq 1$ (this is feasible because of $\exp(\langle w, A\rangle) = 4$). First, assume that there is $v \in C_H(u^2)\backslash\langle u, A\rangle$ with $v^2 \in C_A(u)$; then either $u$ and $v$ can take the roles of $x$ and $y$ in (1), or there are $\hat{u} \in uA$ and $\hat{v} \in vA$ with $\hat{u}^2 \neq 1 \neq \hat{v}^2$ that satisfy the conditions imposed on $x$ and $y$ in (2). Either way, we obtain that $C_A(u) = C_A(v)$, while $[A, v] = \langle v^2\rangle$ and $[A, u] = \langle u^2\rangle$; in particular, $[A, u] = \langle u^2\rangle \leqslant Z(H)$; so whenever $w \in H\backslash A$ and $w^2 \in C_A(u)$, then applying (1) or (2) we get $C_A(u) = C_A(w)$ and $[A, w] = \langle w^2\rangle$. In other words, we may take $z = u$ to satisfy assertion (a) of this lemma.

Suppose there is $v \in H\backslash\langle u, A\rangle$ with $[v^2, u] = 1$, but that no such $v$ centralizes $u^2$. Applying (3) (with $u = x$ and $v = y$) we obtain that if $1 \neq v^2 \in C_A(u)$ and $v \notin \langle A, u\rangle$, then $[A, v] = \langle v^2\rangle$. In particular, $v^2 = [v, u^2] = [v, u, u]$ and $[A, u, v] = [A, v, u] = 1$, which implies that $u^2 \notin [A, u]$; in other words, the coset $uA$ does not contain involutions. Let $\mathcal{M} = \{v \mid v \in H\backslash\langle u, A\rangle,\ 1 \neq v^2 \in [A, u] \leqslant C_A(v)\}$; our present assumption has been seen to entail that $\mathcal{M} \neq \emptyset$. If $i \in \mathrm{Inv}(H)\backslash A$, then certainly $1 \neq [i, A] \cap C_A(u)$; so, setting $v = it$ with $[i, t, u] = 1 \neq [i, t]$, we obtain that $v \in \mathcal{M}$. If $v, v' \in \mathcal{M}$ with $vA \neq v'A$, then clearly $[v^2, v'] = 1 = [v'^2, v]$; so either the pair $(v, v')$ can take the place of $(x, y)$ in (1), or there are elements $\hat{v}$ of $vA\backslash \mathrm{Inv}(H)$ and $\hat{v}'$ of $v'A\backslash \mathrm{Inv}(H)$ such that $\hat{v}$ and $\hat{v}'$ can replace $x$ and $y$ in (2). Either way, $C_A(v) = C_A(v')$, while $\langle v^2\rangle = [A, v]$ and $\langle v'^2\rangle = [A, v']$. According to Lemma 1, either $[v, v'] = 1$ and $[A, vv'] = \langle v^2(v')^2\rangle = \langle (vv')^2\rangle$, i.e. $vv' \in \mathcal{M}$, or $[v, v'] = v^2(v')^2$ whence $(vv')$ is an involution. We have seen that in that case there is $t \in A$ with $vv't \in \mathcal{M}$. Let $B = \langle \mathcal{M}\rangle$; we have just seen that, whenever $b \in B$, then either $b \in A$ or $bA$ does contain some element of $\mathcal{M}$; furthermore, if $v, v' \in \mathcal{M}$, then $v$ and $v'$ centralize one and the same hyperplane in $A$, while for $w \in \{v, v'\}\langle w^2\rangle = [w, A]$. Thus $B/A$ is an elementary abelian normal subgroup of $H/A$—in fact, $B \lhd H$—and, as $uA \cap \mathcal{M} = \emptyset$, $u \notin BA$. Now let $z \in H\backslash A$ with $zA \in B/A \cap Z(H/A)$ and $z \in \mathcal{M}$. Then $[z, A] = \langle z^2\rangle \leqslant Z(G)$ and, employing (1) or (2) in the now familiar way, we obtain that $z$ satisfies (a).

We still have to consider the possibility that if $v^2 \in A$, $v \in H\backslash\langle u, A\rangle$, then $[v^2, u] \neq 1$. If we could choose $v$ with $[v^2, u] = 1$, then (3)—$u$ now taking the part of $y$ and $v$ that of $x$—would yield that $[A, u] = \langle u^2\rangle$; moreover, $\Omega_1(H/C_A(u)) = \langle A, u\rangle/C_A(u) = \Omega_1(H)/C_A(u)$ and, setting $u = z$, we see that (a) is rather trivially satisfied.

Finally, assume that there is $v \in H\backslash\langle u, A\rangle$ with $1 \neq v^2 \in A$, but that $[v^2, u] \neq 1 \neq [u, v^2]$ for any $v$ so chosen (note that this implies that $\Omega_1(H) \leqslant \langle u, A\rangle$). Let $v^2 \in A$ such that $[v^2, u] \neq 1 \neq [u, v^2]$ and let $W = \langle [v^2, u], [u^2, v]\rangle$. Since $[u, v] \in A$, $W \leqslant C_A(u) \cap C_A(v)$; in particular, none of the elements $u^2$, $v^2$, $[u, v]$ is contained in $W$. Thus Lemma 1 yields that $[u, v]W = u^2v^2W$, so $(uv)^2 \in W$. Moreover, $[u, v^2] = [u, v, v] = [u^2, v]$, so $|W| = 2$. Since $\langle [v^2, u]\rangle = \langle [v^2, uv]\rangle = W$, we may take $W = \langle (uv)^2\rangle$; as $[u, (uv)^2] = 1$, this is a contradiction.

We have seen that if $H$ does not possess an element $z$ to satisfy (a), then $uA$ is the only involution in $H/A$. This says that $H/A$ is either cyclic or generalized quaternion; so $H/A$ is cyclic, $H$ being $Q_8$-free. □

**Lemma 11.** *Let $P$ be a finite $Q_8$-free 2-group. Suppose $P$ to possess a powerful subgroup $N$ with $|P:N|=2$ and assume that, if $P=\langle N,t\rangle$, then every element of $N/\mho_2(N)$ is inverted by $t$. Then $N$ is abelian and $t$ inverts every element of $N$.*

**Proof.** We will proceed by induction on $|P|$. Let $2^n=\exp(N)$; if $n\in\{1,2\}$, then there is nothing to prove; so we may suppose that $n>2$. Let $z\in C_{\Omega_1(Z(P))}(t)\backslash\{1\}$. Via induction, $P/\langle z\rangle$ is abelian with every element of $P/\langle z\rangle$ inverted by $t$, while $t^2\in\langle z\rangle$. Suppose there are $x,y\in P$ with $[x,y]=z$. Lemma 1(d) yields that every element of $N$ of order no more than 4 normalizes every subgroup of $N$, and that $\Omega_2(N)$ is abelian. Furthermore, $\langle x,y\rangle$ is ordinary metacyclic. If $\langle x,y\rangle$ is non-abelian, we may therefore presume $z=x^{2^{k-1}}$ where $o(x)=2^k$ and $k\geqslant 3$. Now $t^2\in\langle x^{2^{k-1}}\rangle$ and $x^t\in x^{-1}\langle x^{2^{k-1}}\rangle$, which, since $P$ must not have subgroups that are generalized quaternion or semidihedral, yields $\langle x,t\rangle\cong D_{2^{k+1}}$ and $t^2=1$. Suppose that there is $v\in\langle x,y^2\rangle$ with $v^t=v^{-1}z$. Then $(vt)^2=z$, which makes $\langle vt,x\rangle$ generalized quaternion; thus every element of $C_{\langle x,y\rangle}(x)$ is inverted by $t$. By Lemma 2(b), we are allowed to assume that $\langle x\rangle\cap\langle y\rangle=1$. However, $\langle y^2\rangle\lhd\langle x,y,t\rangle$ and $\langle ty,x\rangle/\langle y^2\rangle\cong SD_{2^{k+1}}$. □

**Lemma 12.** *Let $P$ be a finite $Q_8$-free 2-group and assume $\Omega_1(P)$ to be elementary abelian. Let $N$ be a maximal powerful normal subgroup of $P$ with $\Omega_1(P)\subseteq N$. Then $P=N$ or $P/\Omega_1(P)$ is cyclic.*

**Proof.** As before, let $\widehat{P}=P/\mho_2(N)$. According to Lemma 8, either $P/N$ is cyclic or $|\widehat{N}:\Omega_1(\widehat{N})|\leqslant 2$. If $P>N$ and $|\widehat{N}:\Omega_1(\widehat{N})|>2$, then Lemma 2(b) says that $P$ has a subgroup $M=\langle N,t\rangle$ with $\hat{y}^{\hat{t}}=\hat{y}^{-1}$, $\hat{y}\in\widehat{N}$. In this case, the previous lemma applies to $M$; in particular, $M\backslash N$ would have to contain involutions.

We have just seen that, unless $P=N$, $|\widehat{N}:\Omega_1(\widehat{N})|\leqslant 2$. If $|\widehat{N}:\Omega_1(\widehat{N})|=2$, then, by Lemma 6, there is $a\in P$, $o(a)>2$, such that $N=\langle a\rangle\Omega_1(P)$ and $[\Omega_1(P),a]\leqslant\Omega_1(\langle a\rangle)$. Furthermore, Lemma 9 says that if $t\in\widehat{P}\backslash\widehat{N}$ and $t^2\in\Omega_1(\widehat{N})$, then $\hat{t}\widehat{N}$ contains an involution in $\widehat{P}$. Accordingly, suppose there is $s\in P\backslash N$ with $s^2\in\mho_2(N)=\langle a^4\rangle$ and note that $\langle a^2\rangle=\Phi(N)\lhd P$. Let $B=\langle a^2,s\rangle$; then $|B:\langle a^2\rangle|=2$, and, since $P$ is $Q_8$-free, $B$ is either abelian, dihedral or modular. In each of these three cases there is an involution contained in $B\backslash\langle a^2\rangle$, hence in $P\backslash N$. We have found that $P/\langle a^4,\Omega_1(P)\rangle$ has only got one involution, thus is cyclic, $P$ being $Q_8$-free. Accordingly, $P=\Omega_1(P)\langle y\rangle$ where $a$ may be taken to be a power of $y$.

Thirdly, suppose $\widehat{N}$ to be elementary abelian; then of course $\mho_2(N)=1$; in other words, $N=\Omega_1(P)$ and $\Omega_1(P)$ is a maximal abelian normal subgroup of $P$. Applying Lemma 10, we immediately obtain that $P/N$ is cyclic. □

**Notation.** Let $G$ be a finite $Q_8$-free 2-group and let $N$ be a maximal powerful normal subgroup of $G$. Let $\widehat{G}=G/\mho_2(N)$. Then $\widehat{N}$ is a maximal powerful normal subgroup of $\widehat{G}$ which is abelian of exponent at most 4. Thus Lemmas 8 to 10 pertain to $\widehat{G}$; in particular, if $|\widehat{N} : \Omega_1(\widehat{N})| > 2$, then Lemmas 8 and 11 yield that $G$ is one of the groups specified in Theorem 2(a).

**Lemma 13.** *Suppose that* $|\widehat{N} : \Omega_1(\widehat{N})| = 2$ *and let* $2^k = \exp(N)$. *Let* $\widehat{M}/\Omega_1(\widehat{N}) = \Omega_1(\widehat{G}/\Omega_1(\widehat{N}))$ *and let* $\hat{a} \in \widehat{N}$ *be of order* 4. *If* $k \geqslant 3$ *or else* $\hat{a}^{\hat{c}} \neq \hat{a}^{-1}$, *for any involution* $\hat{c} \in \widehat{M}$, *then* $G$ *satisfies Theorem* 2.

**Proof.** According to Lemma 9, $\widehat{M} = \widehat{N} \cdot \widehat{C}$ where $\widehat{C}$ is elementary abelian and $[\widehat{C}, \Omega_1(\widehat{N})] = 1$. Lemma 6 further says that $N = \langle a\rangle \cdot E$ where $o(a) = 2^k$ and $E \leqslant G$ is elementary abelian with $[E, a] \leqslant \langle a^{2^{k-1}}\rangle$. Let $\widehat{M} = \widehat{N}\widehat{D}$ where $\widehat{D}$ is elementary abelian and $\widehat{D} \cap \Omega_1(\widehat{N}) = \langle \hat{a}^2\rangle$.

(a) Let $c \in D\backslash A$ be such that $\hat{a}^{\hat{c}} \neq \hat{a}^{-1}$. Let $V = \langle a, c\rangle\Omega_1(N)/\Omega_1(N)$. Since $a^c \in a\langle a^4, \Omega_1(N)\rangle$, $V$ is either abelian or modular; whence unless $o(a) = 8$ and $\langle a^2, c\rangle = D_8$, the group $\langle a^2, c\rangle$ is either abelian or modular, too. In each of these cases we have $[c, a^4] = 1$; furthermore, there are involutions contained in $c\langle a^2\rangle$; in particular, $D = \langle a^2\rangle\Omega_1(D)$. Let $c \in \mathrm{Inv}(D)$: Setting $a^c = abv$ with $b \in \langle a^4\rangle$ and $v \in \Omega_1(N)\backslash\langle a\rangle$, we obtain that $a^{c^2} = a = ab^2[v, c]$ and $[v, c] \in \langle a^4\rangle \cap \Omega_1(N) = \langle a^{2^{k-1}}\rangle$; so $o(b) \leqslant 4$. If $[a^{2^{k-2}}, c] = 1$ and $[u, c] \neq 1$, $u \in \Omega_1(N)$, then $[u, c] = a^{2^{k-1}}$ and $\langle a^{2^{k-2}}u, a^{2^{k-2}}c\rangle \cong Q_8$. If $k > 3$, then $[a^{2^{k-2}}, D] = 1$; so $[\Omega_1(N), D] = 1$ and $[a, D] \leqslant \Omega_1(N)$. If $k = 3$ and $c \in \mathrm{Inv}(D)$ with $a^{2c} = a^{-2}$, then $a^c = av$, $v \in \Omega_1(N)$, $[a, v] = a^4$ and $[v, c] = 1$. If there was $u \in \Omega_1(N)$ with $[u, c] = a^4$, then $\langle a^2, uc\rangle \cong Q_8$; hence $[\Omega_1(N), c] = 1$ in this case, too.

(b) Suppose that $\hat{a}^{\hat{c}} \neq \hat{a}^{-1}$ for any $\hat{c} \in \widehat{C}$. In that case, $\Omega_1(\widehat{M}) = \Omega_1(\widehat{G})$ is elementary abelian; so the preceding lemma yields that $\widehat{G}/\Omega_1(\widehat{G})$ is cyclic (observe that if $\widehat{G}$ is powerful, then so is $G$ and $N/\Omega_1(N)$ is cyclic anyway).

Furthermore, (a) yields $[D, a^4] = 1$ and $[a, D] \leqslant \Omega_1(N)$. Accordingly, $\Phi(D) \leqslant C_N(D)$, implying $\exp(D') = 2$ and therefore $D' \leqslant \langle a^{2^{k-1}}\rangle$. In case $c, c' \in \mathrm{Inv}(C_D(a^{2^{k-2}}))$ we have $[c, c'] = 1$; for otherwise $\langle ca^{2^{k-2}}, c'a^{2^{k-2}}\rangle \cong Q_8$. If $k = 3$ and $c, c' \in \mathrm{Inv}(D)$ with $[a^2, c] = [c, c'] = a^4$, while $[a^2, c'] = 1$, then $\langle cc', a^2\rangle \cong Q_8$.

We have seen the group $\Omega_1(D)\Omega_1(N)$ to be elementary abelian; let $\Omega_1(D) = D_1$. We have further seen that either $k = 3$ and $D_1 = C_{D_1}(a^2)\langle c\rangle$, with $[a, c] \in \Omega_1(N)\backslash C_N(a)$ or $\Omega_1(M) = \Omega_1(G) = D_1\Omega_1(N)$. Moreover, $[a, M] \in \Omega_1(N)$; so if $i \in \mathrm{Inv}(M)$, then $(ai)^4 = (a^2[a, i]^2) = a^4$.

Note that $\langle a^2\rangle = \Phi(N) \lhd G$. Let $H = C_G(a^2/\langle a^8\rangle)$. We have seen that, unless $k = 3$, $H = G$. Furthermore, $\Omega_1(H)$ is elementary abelian, so $H/\Omega_1(H)$ is cyclic. Accordingly, let $H = \Omega_1(H)\langle x\rangle$ with $o(x) = 2^m$; let $o(xN) = 2^\ell$ and let $x^{2^\ell} = y$. By the remark concluding the previous paragraph, $y = a^i d$, $d \in D_1\Omega_1(N)$, and $y^4 \in \langle a^4\rangle$; as $\Omega_1(H)$ is elementary abelian, $\langle y\rangle \lhd N$, and we may therefore take $y = a$. Finally, suppose that $k = 3$ and that $G = \langle H, c\rangle$ with $c \in D_1$ and $a^{2c} = a^{-2}$. Set $\widetilde{G} = G/\Omega_1(H)$. If $\widetilde{G}$ is abelian, then $\langle c, \Omega_1(H)\rangle$ is an elementary abelian normal subgroup of $G$ with cyclic factor group. As $\Omega_1(H) \cap N = \Omega_1(N)$, $\widetilde{G}$ cannot be dihedral, so is abelian or modular. Accordingly, we have seen that $G$ belongs to one of the categories (b) and (c) of Theorem 2.

(c) Finally, assume that there is $t \in D$ with $\hat{a}^{\hat{t}} = \hat{a}^{-1}$. Therefore $a^t \in a^{-1}\langle a^4\rangle$ and, $\langle a, t\rangle$ being neither generalized quaternion nor semidihedral, we get $\langle a, t\rangle \cong D_{2^{k+1}}$. Recall that $[\Omega_1(N), t] \in \langle a^4\rangle \cap \Omega_1(N) = \langle a^{2^{k-1}}\rangle$. If $x \in \Omega_1(N)\backslash\{1\}$, then $\langle a, tx\rangle$ is not allowed to be generalized quaternion or else semidihedral; so $[C_{\Omega_1(N)}(a), t] = 1 = [C_{\Omega_1(N)}(t), a]$. If $s \in \Omega_1(N)$, then $(as)^{2^{k-1}} = a^{2^{k-1}}$, and $\langle as, t\rangle$ must not be semidihedral; so $N$ is abelian and $[t, \Omega_1(N)] = 1$.

Let $U = C_G(a\Omega_1(N)/\Omega_1(N))$; we apply (a) and apply (b) to the group $U$ instead of $G$; taking into account the fact that $N$ is abelian, we obtain that $U = \Omega_1(U) \cdot \langle x\rangle$ where $a$ is a power of $x$; as $k \geqslant 3$, we obtain $x^t \in x^{-1}\Omega_1(U)$.

Let $v \in G\backslash NC$ and suppose that $\hat{v}^2 \in \widehat{C}\Omega_1(\widehat{N})$. Applying Lemma 7 to the group $\langle \hat{a}, \hat{v}\rangle/\langle[\hat{a}, \hat{v}^2]\rangle$, we find that, unless $[\hat{a}, \hat{v}^2] = \hat{a}^2$, $[\hat{a}, \hat{v}] \in \hat{a}^2\hat{v}^2\langle[\hat{a}, \hat{v}^2]\rangle$, which is impossible. Now if $v \in U\backslash NC$, then $v^t = v^{-1}d$ for some $d \in \Omega_1(U)$; accordingly, $(vt)^2 = d$. If $d \notin \Omega_1(N)$, then, as we have just seen, $\hat{a}^{\hat{d}} = \hat{a}^{-1}$, which contradicts the fact that $d \in U$. Yet if $v^t = v^{-1}z$, $z \in \Omega_1(N)\backslash\{1\}$, then $(vt)^2 = z$, so $v \in NC$. We have shown every element of $U\backslash\Omega_1(U)$ to be inverted by $t$; whence $t$ inverts every element of $U$, and $U = N$. Finally, consider the group $G/\Omega_1(N)$; if $s \in G\backslash\langle t, N\rangle$ with $[s^2, a] \in \Omega_1(N)$, then the group $\langle s, a\rangle\Omega_1(N)/\Omega_1(N)$ has to be modular; yet now $\langle st, a\rangle\Omega_1(N)/\Omega_1(N)$ is semidihedral, which is forbidden. Thus $G/\langle t, N\rangle$ is isomorphic with a cyclic subgroup of $\mathrm{Aut}(\langle a\rangle\Omega_1(N)/\Omega_1(N))$, hence $G = \langle N, t\rangle$. In particular, $G$ belongs to the class of groups specified in Theorem 2(a). □

**Proof of Theorem 2.** We need yet to investigate the possibility that all the powerful normal subgroups $U$ of $G$ have $|U : \Omega_1(U)| \leqslant 2$. Take $N$ to be of maximal order among the powerful normal subgroups of $G$. If $N$ is elementary abelian and $G/N$ is not cyclic, then, by Lemma 10, there is $z \in G$ of order 4 with $[z, G] \leqslant N$ and $[z, N] = \langle z^2\rangle$; accordingly, $\langle y \mid y \in \langle z, N\rangle,\ o(y) = 4\rangle = \langle z, C_N(z)\rangle \lhd G$. We may therefore restrict our attention to the case $N = \langle a\rangle \times E$, $E$ elementary abelian, $o(a) = 4$. As before, let $M$ be the inverse image of $\Omega_1(G/\Omega_1(N))$. Lemma 9 says that $M = \Omega_1(N)C$, where $C\Omega_1(N)$ is elementary abelian and $C \cap \Omega_1(N) = 1$. Furthermore, we may assume that there is $t \in C$ with $a^t = a^{-1}$; otherwise we would be on ground already covered in Lemma 13. If $|C| > 2$, then $G$ has a normal subgroup $V = \langle N, t, s\rangle$ with $s \in C\backslash\langle N, t\rangle$. Then $\langle t, s, \Omega_1(N)\rangle$ is the unique elementary abelian maximal subgroup of $V$, hence is normal in $G$, contrary to the choice of $N$. Accordingly, $C = \langle t\rangle$. Let $\widetilde{G} = G/\langle a^2\rangle$; obviously, $\Omega_1(\widetilde{G})$ is elementary abelian, so, applying Lemma 12, we obtain that $\widetilde{G}$ is powerful or $G/\langle N, t\rangle$ is cyclic. Suppose the first: Then $\Omega_2(\widetilde{G})$ is abelian and if $|\Omega_2(\widetilde{G})/\Omega_1(\widetilde{G})| > 2$, then, with some appropriate modification of notation, there are $u, v \in G$ with $[u, v] \in \langle a^2\rangle$, while $u^2 = a$ and $v^2 = t$; yet certainly $[u^2, v^2] = [u, v, u, v] = 1$. We conclude that $G/\langle N, t\rangle$ is cyclic. Let $G = \langle N, t, x\rangle$ and let $o(x\Omega_1(N)) = 2^\ell$. Since $\langle N, t\rangle = \Omega_1(G)$, we may either take $x^{2^{\ell-1}} = a$, or $x^{2^{\ell-1}} = t$. Considering the first of these possibilities, assume that $\ell \geqslant 2$ and let $y = x^{2^{\ell-2}}$. Let $\hat{y} \in y\Omega_1(N)$. If $\hat{y}^t = y^{-1}z$, $z \in \Omega_1(N)\setminus\{1\}$, then $(\hat{y}t)^2 = z$; whence $yt \in M = \langle N, t\rangle$, this is a contradiction. By the maximality of $N$, $\langle y, \Omega_1(N)\rangle$ cannot be abelian; accordingly, the group $G/\Omega_1(N)$ is either abelian (in which case we may replace $N$ by $\langle \Omega_1(N), t\rangle$ and find $G$ to belong into category (b)) or modular (in which case $G$ is in category (c)). If $x^{2^{\ell-1}} = t$, then, since $[a, x] \in \Omega_1(N)$, we may replace $N$ by $\langle \Omega_1(N), at\rangle$ and place $G$ in category (b). $\square$

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