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Invariant functions on symplectic representations

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To Ernest B. Vinberg on the occasion of his 70th birthday

Abstract

We study invariants on symplectic representations of a connected reductive group. Our main result is that the invariant moment map is equidimensional. We deduce that the categorical quotient is a fibration over an affine space with rational generic fibers. Of particular interest are representations where these fibers are points. We show that they are cofree. Our main tool is a symplectic version of the local structure theorem.

Keywords: Invariant theory; Symplectic representation; Moment map

1. Introduction

Let G be a connected reductive group over $\mathbb{C}$. Our goal is to study invariants of a *symplectic representation*, i.e., a finite-dimensional G-representations V which is equipped with a non-degenerate symplectic form ω. Our main tool will be the $\mathfrak{g}^*$-valued covariant

$$m : V \to \mathfrak{g}^* : v \mapsto \ell_v \quad \text{where } \ell_v(\xi) := \frac{1}{2}\omega(\xi v, v) \tag{1.1}$$

called the *moment map* (here, $\mathfrak{g}^*$ is the coadjoint representation of G). The key idea is to construct (very special) invariants on V by pulling back the (well-known) invariants on $\mathfrak{g}^*$.

This construction can be described more geometrically as follows. It is known (Chevalley) that G-invariants on $\mathfrak{g}^* \cong \mathfrak{g}$ are in bijection with W_G-invariants on $\mathfrak{t}^*$ (where $\mathfrak{t}^*$ is the dual of a

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Cartan subspace and W_G is the Weyl group). Thus, we can define the composed morphism

$$m/\!\!/G : V \xrightarrow{m} \mathfrak{g}^* \to \mathfrak{t}^*/W_G \tag{1.2}$$

called the *invariant moment map*. Then every function on $\mathfrak{t}^*/W_G$ gives rise to an invariant function on V.

Let $R_0 \subseteq \mathbb{C}[V]^G$ be the ring of invariants obtained this way. It turns out to be more natural to consider the (slightly) larger ring R of invariants which are algebraic over R_0. Then our main results are:

- R is a polynomial ring. More precisely, there is a subspace $\mathfrak{a}^* \subseteq \mathfrak{t}^*$ and a reflection group W_V acting on $\mathfrak{a}^*$ such that

$$R = \mathbb{C}[\mathfrak{a}^*]^{W_V}. \tag{1.3}$$

  The dimension of $\mathfrak{a}^*$ is called the *symplectic rank* of V. The group W_V is a subtle invariant of V called its *little Weyl group*.
- Both the ring of all functions $\mathbb{C}[V]$ and the ring of all invariants $\mathbb{C}[V]^G$ are free R-modules. Geometrically, this means: let

$$\begin{array}{ccc} V & \xrightarrow{\ \pi\ } & V/\!\!/G \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi/\!\!/G} & \\ & \mathfrak{a}^*/W_V & \end{array} \tag{1.4}$$

  be the commutative triangle induced by the inclusions $\mathbb{C}[\mathfrak{a}^*]^{W_V} = R \subseteq k[V]^G \subseteq \mathbb{C}[V]$. Then both ψ and $\psi/\!\!/G$ are faithfully flat. In particular, all fibers have the same dimension.
- The fibers of $\psi/\!\!/G$ are called the *symplectic reductions* of V. They form a flat family of $2c$-dimensional Poisson varieties. Here c is the *symplectic complexity* of V. We show that a generic symplectic reduction contains a dense open subset which is isomorphic to an open subset of $\mathbb{C}^{2c}$ with its standard symplectic structure.
- The generic fibers of $\pi : V \to V/\!\!/G$ are fiber products of the form $G \times^L F$ where L is a Levi subgroup and

$$F = A \times \mathbb{C}^{2m_1} \times \cdots \times \mathbb{C}^{2m_s}. \tag{1.5}$$

  Here A is a torus and L acts on F via a surjective homomorphism

$$L \twoheadrightarrow A \times Sp_{2m_1}(\mathbb{C}) \times \cdots \times Sp_{2m_s}(\mathbb{C}). \tag{1.6}$$

  Of particular interest is the case when $R = \mathbb{C}[V]^G$, i.e., when the moment map furnishes "almost all" invariants. These representations are called *multiplicity free*. We prove:
- The following are equivalent (see Section 9 for unexplained terminology):
  - V is multiplicity free.
  - All invariants Poisson-commute with each other.
  - The algebra of invariants $\mathcal{W}(V)^G$ inside the Weyl algebra $\mathcal{W}(V)$ is commutative.
  - The generic G-orbits are coisotropic.

- Every multiplicity free symplectic representation is cofree, i.e., $\mathbb{C}[V]$ is a free module over $\mathbb{C}[V]^G$ (this condition implies that $\mathbb{C}[V]^G$ is a polynomial ring).

The cofreeness property of multiplicity free representations is very restrictive. We used it in [Kn4] to classify all multiplicity free symplectic representations.

If U is any finite-dimensional representation of G then $V = U \oplus U^*$ carries a canonical symplectic structure. In fact, V can be considered as the cotangent bundle of U. Therefore, the present paper can be seen as an extension of my theory of invariants on cotangent bundles, started in [Kn1]. This also explains some terminology: $V = U \oplus U^*$ is multiplicity free in the symplectic sense if and only if $\mathbb{C}[U]$ is multiplicity free in the usual sense.

The main tool for studying the geometry of a cotangent bundle was the local structure theorem of Brion–Luna–Vust [BLV]. In this paper we prove a symplectic analog of the structure theorem. More precisely, we construct a Levi subgroup $M \subseteq G$ and an M-stable subspace $S \subseteq V$ such that

- The restriction of ω to S is non-degenerate.
- There is a dense open subset $S^0 \subseteq S$ and an embedding $q : S^0 \hookrightarrow V$ such that the diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\;q\;} & V \\ {\scriptstyle m_S/\!\!/M}\downarrow & & \downarrow{\scriptstyle m_V/\!\!/G} \\ \mathfrak{t}^*/W_M & \longrightarrow & \mathfrak{t}^*/W_G \end{array} \tag{1.7}$$

  commutes. One key point is that q is not the natural inclusion of S^0 in V. In general, it will not be even linear. Another key point is that S^0 is not just any open subset: it is the complement of an explicitly given hyperplane. This will imply that S^0 meets the zero-fiber of $m_S/\!\!/M$. This is vital for proving the equidimensionality of $m_V/\!\!/G$ by induction on $\dim V$.
- The map

$$G \times^M S^0 \to V : [g, s] \mapsto gq(s) \tag{1.8}$$

  is étale on an open dense subset (we are actually proving something much more precise). This statement links the generic structure of V with the generic structure of S. It implies, in particular, that S^0 and V have essentially the same image in $\mathfrak{t}^*/W_G$ (see diagram (1.7)).
- The morphism $S^0/\!\!/M \to V/\!\!/G$ induced by q is a Poisson morphism. This will be used to study the generic symplectic reductions.

Notation. (1) We are working in the category of complex algebraic varieties even though $\mathbb{C}$ could be replaced by any algebraically closed field of characteristic zero. The ring of regular functions on a variety X is denoted by $\mathbb{C}[X]$ while $\mathbb{C}(X)$ is its field of rational functions. The dual of a vector space V will be denoted by V^*. In contrast, the one-dimensional torus is $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

(2) If H is any algebraic group and X an affine H-variety we denote the categorical quotient by $X/\!\!/H$, i.e., $X/\!\!/H = \operatorname{Spec} \mathbb{C}[X]^H$.

(3) The Lie algebra of any group is denoted by the corresponding fraktur letter. If X is an H-variety then every $\xi \in \mathfrak{h}$ induces a vector field ξ_* on X. Its value at $x \in X$ is denoted by ξx.

(4) In the whole paper, G will denote a connected reductive group. We sometimes use tacitly that $\mathfrak{g}^* \cong \mathfrak{g}$ as G-varieties. We choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. The

root system of $\mathfrak{g}$ is denoted by Δ, the positive roots by Δ^+. In any root subspace $\mathfrak{g}_\alpha$ we choose a generator ξ_α such that $[\xi_\alpha, \xi_{-\alpha}] = \alpha^\vee$.

2. The moment map

In this section we review properties of the moment map. For a good reference see [GS].

An *affine Poisson variety* is an affine variety X equipped with a bilinear map

$$\mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}[X] : (f, g) \mapsto \{f, g\}, \tag{2.1}$$

called the *Poisson product*, satisfying:

(1) the Poisson product is a Lie algebra structure on $\mathbb{C}[X]$ and
(2) for each $f \in \mathbb{C}[X]$, the map $g \mapsto \{f, g\}$ is a derivation of $\mathbb{C}[X]$.

Because of the second condition, the concept of a Poisson variety is local in nature. In particular, every open subset of X (in the Zariski- or in the étale sense) is again a Poisson variety.

A morphism $\pi : X \to Y$ between two Poisson varieties is a *Poisson morphism* if

$$\{f, g\}_Y = \{f \circ \pi, g \circ \pi\}_X, \quad f, g \in \mathbb{C}[Y]. \tag{2.2}$$

There are two main examples of Poisson varieties:

(1) *Dual Lie algebras:* Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $X = \mathfrak{g}^*$. Since $\mathbb{C}[X] = S^*\mathfrak{g}$ we have $\mathfrak{g} \subseteq \mathbb{C}[X]$. One can show that the Lie bracket on $\mathfrak{g}$ extends uniquely to a Poisson product on $\mathbb{C}[X]$.

(2) *Symplectic varieties:* Let X be a smooth variety X equipped with a non-degenerate closed 2-form ω. Then the Poisson structure is constructed as follows. First, every function $f \in \mathbb{C}[X]$ gives rise to a 1-form df. Since ω is non-degenerate, it identifies the tangent bundle of X with its cotangent bundle. Thus, df corresponds to a vector field H_f, called the *Hamiltonian vector field* attached to f. The relationship of f and H_f is expressed in the formula

$$\omega(\xi, H_f) = \xi(f) = df(\xi) \tag{2.3}$$

where ξ is any tangent vector. Then

$$\{f, g\} := \omega(H_f, H_g) = -df(H_g) = -H_g(f) = dg(H_f) = H_f(g) \tag{2.4}$$

defines a Poisson product. Observe the formula $H_{\{f,g\}} = [H_f, H_g]$. Therefore, the map

$$\mathbb{C}[X] \to \Gamma(T_X) : f \mapsto H_f \tag{2.5}$$

is a Lie algebra homomorphism (with kernel $\mathbb{C}$).

Now assume that the algebraic group G acts on the symplectic variety X in such a way that ω is G-stable. Then every $\xi \in \mathfrak{g}$ induces a vector field ξ_* on X. Moreover, the map $\xi \mapsto -\xi_*$ is a Lie algebra homomorphism.[1] Thus we obtain the diagram

$$\begin{array}{ccc} & & \mathfrak{g} \\ & \overset{m^*}{\swarrow} & \downarrow{\scriptstyle -\xi_*} \\ \mathbb{C}[X] & \xrightarrow[H_f]{} & \Gamma(T_X). \end{array} \tag{2.6}$$

The symplectic G-variety is called *Hamiltonian* if it is equipped with a Lie algebra homomorphism $m^*: \mathfrak{g} \to \mathbb{C}[X]$ such that diagram (2.6) commutes. More geometrically, we consider the morphism

$$m : X \to \mathfrak{g}^* : x \mapsto \bigl[\xi \mapsto m^*(\xi)(x)\bigr] \tag{2.7}$$

called the *moment map* of X. It is also characterized by the equations

$$\omega(\xi x, \xi' x) = \bigl\langle m(x), [\xi, \xi']\bigr\rangle \quad \text{for all } \xi, \xi' \in \mathfrak{g}, \tag{2.8}$$

$$\omega(\xi x, \eta) = \bigl\langle D_x m(\eta), \xi\bigr\rangle \quad \text{for all } \xi \in \mathfrak{g},\ \eta \in T_x X. \tag{2.9}$$

Here, (2.8) expresses the fact that m^* is a Lie algebra homomorphism while (2.9) says that (2.6) commutes. These two equations also imply that m is $\mathfrak{g}$-equivariant. Since G is assumed to be connected we conclude that m is even G-equivariant. Given the G-action and the symplectic structure of X, the moment map is unique up to translation by an element of $(\mathfrak{g}^*)^G$. In fact, if m' is another moment map then (2.9) implies that $m - m'$ has everywhere a zero derivative. Observe the immediate consequence of (2.9):

$$\ker D_x m = (\mathfrak{g}x)^\perp \quad \text{and} \quad \operatorname{Im} D_x m = \mathfrak{g}_x^\perp \tag{2.10}$$

Equation (2.8) implies that m is a Poisson morphism. Moreover, a function $f \in \mathbb{C}[X]$ is G-invariant if and only if

$$\bigl\{m^*(\xi), f\bigr\} = H_{m^*(\xi)}(f) = -\xi_*(f) = 0 \tag{2.11}$$

for all $\xi \in \mathfrak{g}$. Thus, $\mathbb{C}[X]^G$ is the Poisson centralizer of $m^*\mathbb{C}[\mathfrak{g}^*]$ in $\mathbb{C}[X]$. This implies, in particular, that $m^*\mathbb{C}[\mathfrak{g}^*]^G$ is in the Poisson center of $\mathbb{C}[X]^G$. The geometric counterpart to this is the *invariant moment map*

$$m /\!\!/ G : X \to \mathfrak{g}^* /\!\!/ G \cong \mathfrak{t}^*/W_G \tag{2.12}$$

which is the composition of the moment map $X \to \mathfrak{g}^*$ with the categorical quotient map $\mathfrak{g}^* \to \mathfrak{t}^*/W_G$. Not only is this a Poisson map (with the trivial Poisson bracket on $\mathfrak{t}^*/W_G$) but also all fibers are Poisson varieties.

[1] The minus sign comes from the fact that ξ acts on functions naturally from the right while a left action is used to define the bracket of vector fields.

In this paper we are mostly concerned with the special case that $X = V$ is a finite-dimensional G-representation equipped with a non-degenerate 2-form $\omega \in \wedge^2 V^*$. Then there is a canonical moment map namely

$$m : V \to \mathfrak{g}^* : v \mapsto \left[\xi \mapsto \frac{1}{2}\omega(\xi v, v)\right]. \tag{2.13}$$

This implies the following useful formula: let $\mathfrak{h} \subseteq \mathfrak{g}$ be a subalgebra and $\mathfrak{h}^\perp \subseteq \mathfrak{g}^*$ its annihilator. Then

$$m^{-1}(\mathfrak{h}^\perp) = \{v \in V \mid \omega(\xi v, v) = 0 \text{ for all } \xi \in \mathfrak{h}\}. \tag{2.14}$$

3. The symplectic local structure theorem

In this section we develop our main technical tool, a symplectic version of the local structure theorem of Brion–Luna–Vust [BLV].

Let V be a symplectic G-representation. The structure theorem will depend on the choice of a highest weight vector $v_0 \in V$. The construction will not work for the defining representation of the symplectic group.

Definition. Let $U \subset V$ be the submodule U generated by v_0. Then U (or v_0) is called *singular* if U is an anisotropic subspace of V (hence itself symplectic) and $G \to Sp(U)$ is surjective. A dominant weight χ of G is *singular* if the corresponding irreducible representation U is singular.

Remarks. (1) If the root system of G has a component of type C_n (for any $n \geqslant 1$, including $\mathsf{C}_1 = \mathsf{A}_1$) then the first fundamental weight of that component is singular. Conversely, all singular dominant weights are of this form.

(2) Clearly, if $U \subseteq V$ is singular then its highest weight is singular. The converse is false since the singularity of U depends on its embedding in V. Let, for example, $G = Sp_{2n}(\mathbb{C})$ and $V = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. Then both summands are singular, but the submodule $\{(v, iv) \mid v \in \mathbb{C}^n\}$ is isotropic, hence non-singular.

3.1. Lemma. *Let U be an irreducible symplectic G-module with highest weight χ. Then $\chi \notin \Delta^+$. Moreover, if one of the following conditions holds then χ is singular*:

(i) $2\chi \in \Delta^+$.
(ii) $2\chi = 2\alpha - \beta$ *with* $\alpha, \beta \in \Delta^+$.
(iii) $2\chi = \alpha + \beta$ *with* $\alpha, \beta \in \Delta^+$.

Proof. We first prove that each of the conditions (i)–(iii) implies that χ is singular.

(i) Assume $2\chi = \alpha \in \Delta^+$. Then $\frac{1}{2}\alpha$ is a weight and $\langle \alpha, \beta^\vee \rangle$ is even for all $\beta \in \Delta^+$. The same holds for the simple root in the W-orbit of α. Thus, α is root in a root system of type C_n, $n \geqslant 1$ and $\chi = \frac{1}{2}\alpha$ is its first fundamental weight.

(ii) Since $\frac{1}{2}\beta$ is a weight, we are as above in a root system of type C_n, $n \geqslant 1$. There, the assertion can be checked directly.

(iii) We claim that if $\chi = \frac{1}{2}(\alpha+\beta)$ is a dominant weight then either 2χ is a root (and we are done by (i)) or the corresponding simple module is not symplectic. First observe, that

$$\langle \chi, \alpha^\vee \rangle = 1 + \frac{1}{2}\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}, \qquad \langle \chi, \beta^\vee \rangle = 1 + \frac{1}{2}\langle \alpha, \beta^\vee \rangle \in \mathbb{Z} \tag{3.1}$$

implies that both $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$ are even integers. This is only possible if both of them are zero. This implies

$$\langle \chi, \alpha^\vee \rangle = 1, \qquad \langle \chi - \alpha, \beta^\vee \rangle = 1. \tag{3.2}$$

Let $L \subseteq G$ be the smallest Levi subgroup having α and β as roots. Its semisimple rank is at most two. Let $U' \subseteq U$ be the L-module generated by a highest weight vector of U. Then (3.2) implies that U' contains also the lowest weight vector of U. Therefore, the restriction of the invariant symplectic form of U to U' is non-zero. We conclude that U' is an irreducible symplectic L-module. Thus we are reduced to $\operatorname{rk} G = 2$ where the claim is easily checked case-by-case.

Finally, if χ were a root since then (iii) implies that χ were singular. Thus $\chi \notin \Delta^+$. □

For the rest of this section we assume that v_0 is non-singular. Let P be the stabilizer of the line $\mathbb{C}v_0$. Denote its unipotent radical by P_u and its Levi subgroup by M. Let $P^- = MP_u^-$ be the opposite parabolic subgroup. Let $v_0^- \in V$ be a lowest weight vector with $\omega(v_0^-, v_0) = 1$. Its weight is necessarily $-\chi$. We are going to need the following observation:

3.2. Lemma. *The subspace $\mathbb{C}v_0 \oplus \mathfrak{p}_u^- v_0 \subseteq V$ is isotropic.*

Proof. Suppose there are root vectors $\xi_{-\alpha}, \xi_{-\beta} \in \mathfrak{p}_u^-$ with $\omega(\xi_{-\alpha}v_0, \xi_{-\beta}v_0) \neq 0$. Then $(\chi-\alpha)+(\chi-\beta) = 0$ which implies that χ is a singular weight (Lemma 3.1). Since v_0 is non-singular, even $\langle Gv_0 \rangle$ would be isotropic. The same argument shows $\omega(\mathfrak{p}_u^- v_0, v_0) = 0$. □

Now consider the following subspace of V:

$$S := (\mathfrak{p}_u^- v_0)^\perp \cap (\mathfrak{p}_u v_0^-)^\perp = \{v \in V \mid \omega(\mathfrak{p}_u^- v_0, v) = 0,\ \omega(\mathfrak{p}_u v_0^-, v) = 0\}. \tag{3.3}$$

It is obviously a representation under M. We will see later (proof of Lemma 3.4) that the restriction of ω to S is non-degenerate. Thus S it is a symplectic M-representation. The goal of this section is to describe a reduction procedure from (V, G) to (S, M).

The natural inclusion of S into V is not compatible with invariant moment maps. To achieve compatibility, we alter it in a non-linear fashion. More precisely, we look at the subset

$$\Sigma := (\mathfrak{p}_u^- v_0)^\perp \cap m^{-1}(\mathfrak{p}_u^\perp) = \{v \in V \mid \omega(\mathfrak{p}_u^- v_0, v) = 0,\ \omega(\mathfrak{p}_u v, v) = 0\} \tag{3.4}$$

whose definition is very similar to that of S.

Definition. For any subset $Z \subseteq V$ we put $Z^0 := \{v \in Z \mid \omega(v, v_0) \neq 0\}$.

Now we have:

3.3. Lemma. *There is a decomposition* $V = \mathfrak{p}_u^- v_0 \oplus (\mathfrak{p}_u v_0^-)^\perp$ *and the projection to the second summand induces an isomorphism* $p : \Sigma^0 \xrightarrow{\sim} S^0$.

Proof. We claim that ω induces a perfect pairing between $\mathfrak{p}_u^- v_0$ and $\mathfrak{p}_u v_0^-$. Indeed, $\omega(\xi_{-\alpha} v_0, \xi_\beta v_0^-) = 0$ for $\alpha \neq \beta$ while $\omega(\xi_{-\alpha} v_0, \xi_\alpha v_0^-) = -\omega([\xi_\alpha, \xi_{-\alpha}] v_0, v_0^-) = \langle \chi | \alpha^\vee \rangle \neq 0$. The claim implies $\mathfrak{p}_u^- v_0 \cap (\mathfrak{p}_u v_0^-)^\perp = 0$, hence $V = \mathfrak{p}_u^- v_0 \oplus (\mathfrak{p}_u v_0^-)^\perp$ for dimension reasons.

Now $\mathfrak{p}_u^- v_0 \subseteq (\mathfrak{p}_u^- v_0)^\perp$ implies that p maps $(\mathfrak{p}_u^- v_0)^\perp$ onto S. In particular, $p(\Sigma) \subseteq S$. Moreover, $\omega(\mathfrak{p}_u^- v_0, v_0) = 0$ implies $p(\Sigma^0) \subseteq S^0$. We are going to construct the inverse map. More precisely, we construct a morphism $\varphi : S^0 \to \mathfrak{p}_u^-$ such that the inverse map is given by $s \mapsto s + \varphi(s) v_0$.

Let $s \in S^0$, $\xi_- \in \mathfrak{p}_u^-$ and put $v = s + \xi_- v_0$. Then $v \in \Sigma$ means $\omega(\xi_+ v, v) = 0$ for all $\xi_+ \in \mathfrak{p}_u$. We have

$$0 = \omega(\xi_+ v, v) = \omega(\xi_+ s, s) + \omega(\xi_+ s, \xi_- v_0) + \omega(\xi_+ \xi_- v_0, s) + \omega(\xi_+ \xi_- v_0, \xi_- v_0). \quad (3.5)$$

The last summand vanishes by Lemma 3.1(ii). Because of $\omega(\xi_+ s, \xi_- v_0) = \omega(\xi_+ \xi_- v_0, s)$, Eq. (3.5) becomes

$$\omega(\xi_+ \xi_- v_0, s) = -\frac{1}{2} \omega(\xi_+ s, s) \quad \text{for all } \xi_+ \in \mathfrak{p}_u. \quad (3.6)$$

Clearly, it suffices to show that for any $s \in S^0$ these equations have a unique solution. First observe that (3.6) is a square system of inhomogeneous linear equations for ξ_-. Thus, it suffices to show that the matrix

$$\big[\omega(\xi_\alpha \xi_{-\beta} v_0, s)\big]_{\alpha, \beta \in \Delta_u} \quad (3.7)$$

is invertible (where Δ_u is the set of roots α with $\xi_\alpha \in \mathfrak{p}_u$). We claim that it is even triangular with non-zero diagonal entries. Let $U := \langle G v_0 \rangle$ and $U^- := \langle G v_0^- \rangle$. These are two (not necessarily distinct) simple G-modules and one is the dual of the other. We have $U^- \cap U^\perp = 0$, and therefore $V = U^- \oplus U^\perp$. Since v_0 and $\xi_\alpha \xi_{-\beta} v_0$ are elements of U, we may replace s by its component in U^-. Then s has a weight decomposition $s = \sum_\eta s_\eta$ with $\eta \geqslant -\chi$ (meaning $\eta + \chi$ being a sum of positive roots). Assume $\omega(\xi_\alpha \xi_{-\beta} v_0, s) \neq 0$. Then $\eta = -\chi + \beta - \alpha \geqslant -\chi$, hence $\beta \geqslant \alpha$. This shows that the matrix is triangular. For the diagonal terms with $\beta = \alpha$ we get

$$\omega(\xi_\alpha \xi_{-\alpha} v_0, s) = \omega\big([\xi_\alpha, \xi_{-\alpha}] v_0, s\big) = \langle \chi | \alpha^\vee \rangle \omega(v_0, s) \neq 0. \qquad \square \quad (3.8)$$

The sought-after embedding of S^0 into V^0 is the composition

$$q : S^0 \xrightarrow[p^{-1}]{\sim} \Sigma^0 \hookrightarrow V^0. \quad (3.9)$$

3.4. Lemma. *Both subsets* S *and* Σ^0 *inherit the structure of a Hamiltonian* M*-variety from* V*. Moreover, the map* $q : S^0 \xrightarrow{\sim} \Sigma^0$ *is an isomorphism of Hamiltonian* M*-varieties. In particular,* *the following diagram commutes*:

$$\begin{array}{ccc} S^0 & \xrightarrow{q} & V \\ {\scriptstyle m_S}\big\downarrow & & \big\downarrow{\scriptstyle m_V} \\ \mathfrak{l}^* & \xleftarrow[\text{res}]{} & \mathfrak{g}^*. \end{array} \tag{3.10}$$

Proof. First, we have $S \cap S^\perp \subseteq (\mathfrak{p}_u v_0^-)^\perp \cap \mathfrak{p}_u^- v_0 = 0$ (Lemma 3.3). This shows that S is anisotropic and therefore a symplectic subspace of V. The set Σ^0 is smooth since it is, via p, isomorphic to an open subset of S.

Fix $s_1 \in S^0$ and put $v_1 := q(s_1) \in \Sigma^0$. Then we get a map of tangent spaces $Dq : T_{s_1} S^0 \to T_{v_1} \Sigma^0$. Let $s, \bar{s} \in T_{s_1} S^0 = S$ be tangent vectors and consider their images $v = Dq(s)$, $\bar{v} = Dq(\bar{s})$ in $T_{v_1} \Sigma^0$. Now recall that

$$q(s) = s + \varphi(s) v_0 \tag{3.11}$$

where $\varphi : S^0 \to \mathfrak{p}_u^-$ is some morphism. Thus, there are $\xi_-, \bar{\xi}_- \in \mathfrak{p}_u^-$ such that $v = s + \xi_- v_0$, $\bar{v} = \bar{s} + \bar{\xi}_- v_0$. From this we get

$$\omega(v, \bar{v}) = \omega(s, \bar{s}) + \omega(\xi_- v_0, \bar{s}) + \omega(s, \bar{\xi}_- v_0) + \omega(\xi_- v_0, \bar{\xi}_- v_0) = \omega(s, \bar{s}) \tag{3.12}$$

since $S \subseteq (\mathfrak{p}_u^- v_0)^\perp$ and $\mathfrak{p}_u^- v_0$ is isotropic. This shows that $\omega|_{\Sigma^0}$ is non-degenerate and that q is a symplectomorphism.

We also have $v_1 = s_1 + \xi_1^- v_0$ for some $\xi_1^- \in \mathfrak{p}_u^-$. Let $\xi \in \mathfrak{l}$. Then

$$\begin{aligned} 2m_{\Sigma^0}(v_1)(\xi) &= \omega(\xi v_1, v_1) \\ &= \omega(\xi s_1, s_1) + \omega(\xi s_1, \xi_1^- v_0) + \omega(\xi \xi_1^- v_0, s_1) + \omega(\xi \xi_1^- v_0, \xi_1^- v_0) \\ &= \omega(\xi s_1, s_1) = 2m_S(s_1)(\xi) \end{aligned} \tag{3.13}$$

since $\xi s_1 \in S$ and $\xi \xi_1^- v_0 \in \mathfrak{p}_u^- v_0$. This shows the commutativity of (3.10). □

Note that the bottom arrow in (3.10) goes in the "wrong" direction. This is fixed by using the invariant moment maps:

3.5. Theorem. *The following diagram commutes*:

$$\begin{array}{ccc} S^0 & \xrightarrow{q} & V \\ {\scriptstyle m_S /\!\!/ M}\big\downarrow & & \big\downarrow{\scriptstyle m_V /\!\!/ G} \\ \mathfrak{t}^*/W_M & \longrightarrow & \mathfrak{t}^*/W_G. \end{array} \tag{3.14}$$

Proof. We use $\mathfrak{g}^* \cong \mathfrak{g}$. Then $\mathfrak{l}^* \cong \mathfrak{l}$, $\mathfrak{t}^* \cong \mathfrak{t}$, and $\mathfrak{g}^* \to \mathfrak{l}^*$ becomes the orthogonal projection $\mathfrak{g} \to \mathfrak{l}$. The definition of Σ implies $(m_V \circ q)(S^0) = m_V(\Sigma^0) \subseteq \mathfrak{p}_u^\perp = \mathfrak{p}$. In view of Lemma 3.4, the assertion now follows from the commutativity of

$$\begin{array}{ccc} \mathfrak{l} & \twoheadleftarrow & \mathfrak{p} \\ \downarrow & & \downarrow \\ \mathfrak{t}/W_M & \longrightarrow & \mathfrak{t}/W_G. \end{array} \tag{3.15}$$

One can see that this diagram commutes. Observe, e.g., that $\mathfrak{p} \to \mathfrak{l}$ is the categorical quotient by P_u (see e.g. (3.23)). □

The embedding q is, in general, not a Poisson morphism. We just have a statement for P_u-invariants:

3.6. Theorem. *The map $q /\!\!/ P_u : S^0 \to V^0 /\!\!/ P_u$ is a Poisson morphism.*

Proof. In view of Lemma 3.4, we may replace S^0 by Σ^0. For $f \in \mathbb{C}[V^0]$ let f_Σ be its restriction to Σ^0. For all $f, g \in \mathbb{C}[V^0]^{P_u}$ we have to show that $\{f, g\} = \{f_\Sigma, g_\Sigma\}$.

Fix a point $s \in \Sigma^0$. Let $H \in T_s V^0$ be the tangent vector which is dual to dg. It is defined by

$$\omega(H, \eta) = dg(\eta) \quad \text{for all } \eta \in T_s V^0. \tag{3.16}$$

Then we have $\{f, g\}(s) = df(H)$. Similarly, let $H_\Sigma \in T_s \Sigma^0$ be dual to dg_Σ, i.e., with

$$\omega(H_\Sigma, \eta) = dg(\eta) \quad \text{for all } \eta \in T_s \Sigma^0. \tag{3.17}$$

The composition $V \xrightarrow{m} \mathfrak{g}^* \to \mathfrak{p}_u^*$ is the moment map for P_u. Its zero fiber is $Z := m^{-1}(\mathfrak{p}_u^\perp)$, hence $s \in Z$. From (2.10) we get $T_s Z = (\mathfrak{p}_u s)^\perp$. In particular, Z is smooth in s if its isotropy group inside P_u is trivial. Thus, we get from Lemma 3.8 that

$$(\mathfrak{p}_u s)^\perp = T_s Z = \mathfrak{p}_u s \oplus T_s \Sigma^0. \tag{3.18}$$

From $H_\Sigma \in T_s \Sigma^0 \subseteq (\mathfrak{p}_u s)^\perp$ we infer $\omega(H_\Sigma, \eta) = 0$ for all $\eta \in \mathfrak{p}_u s$. On the other hand, the P_u-invariance of g implies $dg(\mathfrak{p}_u s) = 0$. Therefore, (3.17) is valid for all $\eta \in \mathfrak{p}_u s \oplus T_s \Sigma^0 = (\mathfrak{p}_u s)^\perp$. Combined with (3.16) we get $H_\Sigma - H \in (\mathfrak{p}_s u)^{\perp\perp} = \mathfrak{p}_u s$. Hence,

$$\{f_\Sigma, g_\Sigma\}(s) = df(H_\Sigma) = df(H) + df(H_\Sigma - H) = \{f, g\}(s). \tag{3.19}$$

For the last equality, we used the P_u-invariance of f. □

Restricting to G-invariants yields:

3.7. Corollary. *The map $q /\!\!/ G : S^0 /\!\!/ M \to V /\!\!/ G$ is a Poisson morphism.*

Finally, we connect the generic geometry of S with that of V. We do that in two steps.

3.8. Lemma. *There is an isomorphism*

$$P \times^M \Sigma^0 \xrightarrow{\sim} m^{-1}\left(\mathfrak{p}_u^\perp\right)^0. \tag{3.20}$$

Proof. Consider the subspace $N = (\mathfrak{p}_u v_0^-)^\perp \subseteq V$. Then the local structure theorem of Brion–Luna–Vust [BLV, Prop. 1.2], implies that

$$P \times^M N^0 \to V^0 \tag{3.21}$$

is an isomorphism. Intersecting both sides with the P-stable subset $m^{-1}(\mathfrak{p}_u^\perp) = \{v \in V \mid \omega(\mathfrak{p}_u v, v) = 0\}$ yields the assertion. □

Let W_G and W_M be the Weyl groups of G and M, respectively. The morphism $\mathfrak{t}^*/W_M \to \mathfrak{t}^*/W_G$ is smooth in an open subset denoted by $(\mathfrak{t}^*/W_M)_{\text{reg}}$. For every morphism $Z \to \mathfrak{t}^*/W_M$ let Z_{reg} be the preimage of $(\mathfrak{t}^*/W_M)_{\text{reg}}$. For example, $\mathfrak{t}^*_{\text{reg}}$ is the set of points $\chi \in \mathfrak{t}^*$ with $\langle \chi | \alpha^\vee \rangle \neq 0$ for all roots corresponding to $\mathfrak{p}_u$. In particular, it is an affine variety. Its most important property is:

$$\xi \in \mathfrak{t}_{\text{reg}} \quad \Longrightarrow \quad C_G(\xi) \subseteq M \text{ and } C_{\mathfrak{g}}(\xi) \subseteq \mathfrak{l}. \tag{3.22}$$

If Z is a Hamiltonian M-variety with moment map $Z \to \mathfrak{l}^*$ then we can form Z_{reg} with respect to the map $Z \to \mathfrak{l}^* \to \mathfrak{l}^*/\!\!/M = \mathfrak{t}^*/W_M$.

We continue with two well-known lemmas:

3.9. Lemma. *The map*

$$P \times^M \mathfrak{l}_{\text{reg}} \xrightarrow{\sim} \mathfrak{p}_{\text{reg}} \tag{3.23}$$

is an isomorphism.

Proof. We start with a general remark concerning fiber products. Let G be a group, $H \subseteq G$ a subgroup, Y an H-variety, X a G-variety, and $Y \subseteq X$ an H-invariant subvariety. Consider the induced map $\Phi : G \times^H Y \to X$. Then Φ is surjective if and only if

(S) $GY = X$.

Moreover, Φ is injective if and only if

(I) $g \in G$, $y \in Y$, $gy \in Y$ imply $g \in H$.

We use this criterion to show that (3.23) is bijective. Let $\xi \in \mathfrak{p}_{\text{reg}}$ with semisimple part ξ_s. After conjugation with P we may assume $\xi_s \in \mathfrak{l}_{\text{reg}}$. But then $\xi \in C_{\mathfrak{g}}(\xi_s) \subseteq \mathfrak{l}$. This shows (S). Now let $u \in P_u$, $\xi \in \mathfrak{l}_{\text{reg}}$ with $u\xi \in \mathfrak{l}_{\text{reg}}$. Since $u\xi = \xi$ modulo $\mathfrak{p}_u$ we have $u\xi = \xi$. Thus $u \in C_P(\xi_s) \subseteq M$, hence $u = 1$. This shows (I). Thus, (3.23) is a bijective morphism between normal varieties and therefore an isomorphism [Lu, Lemme 1.8]. □

3.10. Lemma. *The following diagram is Cartesian*:

$$\begin{array}{ccc} G \times^P \mathfrak{p}_{\text{reg}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ (\mathfrak{t}/W_M)_{\text{reg}} & \longrightarrow & \mathfrak{t}/W_G. \end{array} \tag{3.24}$$

Proof. Because of (3.23) it suffices to show that the diagram

$$\begin{array}{ccc} G\times^M \mathfrak{l}_{\mathrm{reg}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ (\mathfrak{t}/W_M)_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}/W_G \end{array} \tag{3.25}$$

is Cartesian or, in other words, that

$$G\times^M \mathfrak{l}_{\mathrm{reg}} \to \tilde{\mathfrak{g}} := \mathfrak{g}\times_{\mathfrak{t}/W_G}(\mathfrak{t}/W_M)_{\mathrm{reg}} \tag{3.26}$$

is an isomorphism. We follow the same strategy as for the proof of Lemma 3.9.

For (S) let $(\xi, W_M\xi')\in\tilde{\mathfrak{g}}$, i.e., $\xi\in\mathfrak{g}$, $\xi'\in\mathfrak{t}_{\mathrm{reg}}$, and ξ,ξ' have the same image in $\mathfrak{t}/W_G$. Then there is $g\in G$ such that $g\xi_s=\xi'$. Hence $g\xi\in C_{\mathfrak{g}}(\xi')=\mathfrak{l}$. Thus, $(\xi, W_M\xi')$ is the image of $[g^{-1}, g\xi]$.

For (I) assume $g\in G$ and $\xi\in\mathfrak{l}_{\mathrm{reg}}$ are elements with $g\xi\in\mathfrak{l}_{\mathrm{reg}}$ and $\xi, g\xi$ have the same image in $\mathfrak{t}/W_M$. Thus, there is $l\in M$ with $lg\xi_s=\xi_s$. Since $C_G(\xi_s)=M$ we conclude $g\in M$.

Finally, $\tilde{\mathfrak{g}}$ is an étale cover of $\mathfrak{g}$, hence smooth. Thus, Luna's Lemma implies that (3.26) is an isomorphism. □

The next statement provides the link between the geometry of S and that of V.

3.11. Theorem. *Let $X := m_V^{-1}(\mathfrak{p}_u^{\perp})_{\mathrm{reg}}$. Then there is a commutative diagram*

$$\begin{array}{ccccc} G\times^M S^0_{\mathrm{reg}} & \xrightarrow{\iota} & G\times^P X & \longrightarrow & V \\ & \searrow^{m_S/\!\!/M} & \downarrow & & \downarrow{\scriptstyle m_V/\!\!/G} \\ & & (\mathfrak{t}^*/W_M)_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}^*/W_G \end{array} \tag{3.27}$$

where the right-hand square is Cartesian and where ι is an open embedding. Moreover, $S^0_{\mathrm{reg}}\neq\emptyset$.

Proof. We use the identifications $\mathfrak{g}^*=\mathfrak{g}$, $\mathfrak{t}^*=\mathfrak{t}$, and $\mathfrak{p}_u^{\perp}=\mathfrak{p}$. Now consider the diagram

$$\begin{array}{ccc} G\times^P m^{-1}(\mathfrak{p}_{\mathrm{reg}}) & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle m} \\ G\times^P \mathfrak{p}_{\mathrm{reg}} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ (\mathfrak{t}/W_M)_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}/W_G. \end{array} \tag{3.28}$$

The top square is clearly Cartesian while the bottom square is Cartesian by Lemma 3.10. This shows that the right square of (3.27) is Cartesian. The commutativity of the left square follows from Theorem 3.5. The morphism ι is induced by q and is an open embedding by Lemma 3.8. Finally, consider $v = v_0 + v_0^- \in S$. Then

$$m_S(v)(\xi) = \frac{1}{2}\omega(\xi v, v) = \begin{cases} 0 & \text{if } \xi \in \mathfrak{p}_u + \mathfrak{p}_u^-, \\ -\chi(\xi) & \text{if } \xi \in \mathfrak{l}. \end{cases} \tag{3.29}$$

This shows $m_S(v) \in S^0_{\text{reg}} \neq \emptyset$. $\square$

4. Terminal representations

Clearly the reduction step of the preceding section does not work if all highest weight vectors are singular. But also a highest weight vector which spans a one-dimensional submodule has to be avoided since in that case $M = G$ and $S^0 = V^0$ and the local structure theorem becomes tautological.

Definition. A highest weight vector of V is called *terminal* if it is either singular or generates a one-dimensional G-module (i.e., $M = G$). The representation V is *terminal* if all of its highest weight vectors are terminal. A dominant weight is called *terminal* if it is singular or a character of $\mathfrak{g}$.

Here is the classification:

4.1. Proposition. *Let V be a terminal symplectic representation. Then (G, V) decomposes as*

$$\begin{aligned} V &= V_0 \oplus V_1 \oplus \cdots \oplus V_s, \\ G &= G_0 \times Sp(V_1) \times \cdots \times Sp(V_s) \end{aligned} \tag{4.1}$$

where V_0 is a direct sum of one-dimensional G_0-modules.

Proof. Let $V = V_0 \oplus \overline{V}$ where V_0 is the sum of all one-dimensional subrepresentations. Let $V_1 \subseteq \overline{V}$ be a simple submodule and let V' be its complement in $\overline{V}$. Then V_1 is anisotropic and $G \to Sp(V_1)$ is surjective. Since the symplectic group is simply connected, there is a unique splitting $G = Sp(V_1) \times G'$. We claim that $Sp(V_1)$ acts trivially on V'. Indeed, otherwise it would contain a simple submodule V_1'. Since also that submodule is singular, there is an isomorphism $\varphi : V_1 \xrightarrow{\sim} V_1$. But then the submodule $\{v + i\varphi(v) \mid v \in V_1\}$ of $V_1 \oplus V_1'$ would be isotropic, hence non-singular. This shows that $(G, \overline{V}) = (Sp(V_1), V_1) \times (G', V')$ and we are done by induction. $\square$

Now we analyze the moment map of terminal representations. First the non-toric part:

4.2. Lemma. *Let $V = \mathbb{C}^{2n}$ be a symplectic vector space and $G = Sp_{2n}(\mathbb{C})$. Then, using the identification $\mathfrak{sp}_{2n}(\mathbb{C})^* \cong \mathfrak{sp}_{2n}(\mathbb{C})$, the moment map maps V to the set of nilpotent matrices of rank $\leqslant 1$. In particular, the invariant moment map $V \to \mathfrak{t}^*/W_G$ is zero.*

Proof. Assume the symplectic structure is given by the skewsymmetric matrix J, i.e., $\omega(u, v) = u^t J v$. Let $A \in \mathfrak{sp}_{2n}(\mathbb{C})$. Then

$$2m(v)(A) = \omega(Av, v) = -v^t J A v = -\operatorname{tr}\bigl(v^t J A v\bigr) = \operatorname{tr}\bigl(\bigl(-vv^t J\bigr)A\bigr). \tag{4.2}$$

Thus $m(v) = -\frac{1}{2} vv^t J$ which has rank one. Moreover, $\operatorname{tr} vv^t J = \omega(v, v) = 0$. Hence $m(v)$ is nilpotent. □

Next, we deal with the toric part:

4.3. Lemma. *Let A be a torus acting with a finite kernel on a symplectic representation V. Let $m : V \to \mathfrak{a}^*$ be the corresponding moment map. Let C be an irreducible component of $m^{-1}(0)$. Then there is a section $\sigma : \mathfrak{a}^* \to V$ with $\sigma(\mathfrak{a}^*) \cap C \neq \emptyset$.*

Proof. There is a direct sum decomposition

$$V = \bigoplus_{i=1}^{n} (\mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i}) \tag{4.3}$$

where $\mathbb{C}_\chi$ denotes the one-dimensional representation of $\mathfrak{a} = \operatorname{Lie} A$ for the character $\chi \in \mathfrak{a}^*$. Let x_i, y_i ($i = 1, \ldots, n$) be the corresponding coordinates. The moment map for A acting on V is $m_V(x_i, y_i) = \sum_{i=1}^n x_i y_i \chi_i$. We will call χ_i *critical* if χ_i is not in the span of $\{\chi_j \mid j \neq i\}$.

We prove the assertion by induction on $\dim V$. First, assume that none of the characters are critical. Then we claim that the zero fiber of m_V is irreducible. To see this, we factor m_V into

$$\bar{m} : V \to \mathbb{C}^n : (x_i, y_i) \mapsto (x_i y_i) \quad \text{and} \quad \pi : \mathbb{C}^n \to \mathfrak{a}^* : (t_i) \mapsto \sum_{i=1}^{n} t_i \chi_i. \tag{4.4}$$

The map π is linear. Let K be its kernel. Then $m_V^{-1}(0) = \bar{m}^{-1}(K)$. Since $\bar{m}$ is visibly faithfully flat, the same holds for $\bar{m}^{-1}(K) \to K$. Therefore, every irreducible component maps dominantly to K, i.e., we just have to check that the generic fiber is irreducible. The non-existence of a critical weight means precisely that every coordinate of a generic point of K is non-zero. Therefore, the fiber is isomorphic to $(\mathbb{C}^\times)^n$, hence irreducible. This proves the claim.

Since the action is locally faithful, the characters χ_i span $\mathfrak{a}^*$, Thus, we may arrange that $\{\chi_1, \ldots, \chi_m\}$ forms a basis of $\mathfrak{a}^*$. Then the set $\mathfrak{a}^0$ defined by the equations

$$x_1 = \cdots = x_m = 1, \qquad x_{m+1} = \cdots = x_n = y_1 = \cdots = y_n = 0 \tag{4.5}$$

is subset of V which maps isomorphically onto $\mathfrak{a}^*$. In particular, it hits the zero fiber C.

Now assume that χ_1, say, is critical. Let $\mathfrak{a}' := \mathbb{C}\chi_1$, $V' := \mathbb{C}_{\chi_1} \oplus \mathbb{C}_{-\chi_1}$ and $\mathfrak{a}'' := \langle \chi_2, \ldots, \chi_n \rangle$, $V'' := \bigoplus_{i=2}^n \mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i}$. Then we have decompositions $V = V' \oplus V''$ and $\mathfrak{a}^* = \mathfrak{a}' \oplus \mathfrak{a}''$, and it suffices to prove the assertion for $(V', \mathfrak{a}')$ and $(V'', \mathfrak{a}'')$ separately. For the latter case we use the induction hypothesis. In the former case, $m^{-1}(0)$ has two components which intersect either the section $\{1\} \times \mathbb{C}$ or $\mathbb{C} \times \{1\}$. □

Now we use the structure of terminal representations and the reduction procedure of Section 3 to give a preliminary description of the generic structure of V. This will be refined further below (Theorem 7.2).

4.4. Lemma. *For every symplectic G-representation V there is a commutative diagram of G-varieties*

$$\begin{array}{ccc} (G \times^L F) \times U & \longrightarrow & V \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle m_V /\!/ G} \\ \mathfrak{t}^*/W_L & \longrightarrow & \mathfrak{t}^*/W_G \end{array} \tag{4.6}$$

which identifies $(G \times^L F) \times U$ with an open subset of the fiber product $V \times_{\mathfrak{t}^*/W_G} \mathfrak{t}^*/W_L$. Here:

- *L is a Levi subgroup of G equipped with a surjective homomorphism*

$$\varphi : L \twoheadrightarrow A \times Sp_{2m_1}(\mathbb{C}) \times \cdots \times Sp_{2m_s}(\mathbb{C}) \tag{4.7}$$

  where A is a torus and s can be 0.
- *$F = A \times \mathbb{C}^{2m_1} \times \cdots \times \mathbb{C}^{2m_s}$ with L-action induced by φ.*
- *U is a non-empty open subset of $\mathfrak{a}^* \times \mathbb{C}^{2c}$.*
- *p is the composition $(G \times^L F) \times U \twoheadrightarrow U \to \mathfrak{a}^* \to \mathfrak{t}^*/W_L$.*

Further properties are $U_{\mathrm{reg}} \neq \emptyset$ and the induced morphism on G-invariants $U \to V /\!/ G$ is a Poisson morphism (where $\mathbb{C}^{2s}$ has the standard symplectic structure).

Proof. We construct L, F, and U by induction on $\dim V$. Assume first, that V is terminal. Then it has a decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s \tag{4.8}$$

as in Proposition 4.1. By Lemma 4.2, the invariant moment map of V factors through V_0. Since V_0 is a symplectic representation, there is a G-stable decomposition $V_0 = \bigoplus_{i=1}^n (\mathbb{C}_{\chi_i} \oplus \mathbb{C}_{-\chi_i})$ as in (4.3). Then $V_0 = U_0 \oplus U_0^*$ with $U_0 = \bigoplus_{i=1}^n \mathbb{C}_{\chi_i}$. Consider the torus $C := (\mathbb{C}^\times)^n$. Then the map

$$C \times \mathfrak{c}^* = \left(\mathbb{C}^\times\right)^n \times \left(\mathbb{C}^n\right)^* \to U_0 \oplus U_0^* : (u_i, v_i) \mapsto \left(u_i, u_i^{-1} v_i\right) \tag{4.9}$$

realizes the cotangent bundle T^*C with its natural symplectic structure as a dense open subset of $U_0 \oplus U_0^*$. The group G acts on V_0 via a homomorphism $G \to C$. Let $A \subseteq C$ be its image. Since A is connected it has a complement $\overline{A}$ in C, i.e., we get a splitting $C = A \times \overline{A}$. Using (4.9), we get a G-equivariant open embedding

$$T^*A \times T^*\overline{A} = T^*C \hookrightarrow V_0 \tag{4.10}$$

which is compatible with the symplectic structure and where G acts trivially on $\overline{A}$. Using (4.9) for $\overline{A}$ instead of C we see that $T^*\overline{A}$ is isomorphic to an open subset of $\mathbb{C}^{2c}$, $c = \dim \overline{A}$, with its standard symplectic structure. Thus, we have proved the existence of the diagram (4.6) with $L = G$, $F = A \times V_1 \times \cdots \times V_s$, and $U = \mathfrak{a}^* \times T^*\overline{A}$.

If V is not terminal then from Theorem 3.11 we get a diagram

$$\begin{array}{ccc} G \times^M S^1 & \longrightarrow & V \\ \downarrow & & \downarrow \\ (\mathfrak{t}^*/W_M)_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}^*/W_G \end{array} \tag{4.11}$$

(with $S^1 := S^0_{\mathrm{reg}}$) which identifies the upper left corner with an open subset of the fiber product. Since V is not terminal we may assume $S \neq V$. Thus, the induction hypothesis furnishes a diagram

$$\begin{array}{ccc} (M \times^L F) \times U & \xrightarrow{\alpha} & S \\ \downarrow & & \downarrow \\ \mathfrak{t}^*/W_L & \longrightarrow & \mathfrak{t}^*/W_M. \end{array} \tag{4.12}$$

Since S_1 is an *affine* open subset of S, its preimage under α is so as well. This implies that we may shrink U such that, in diagram (4.12), we may replace S by S^1. Now replace α by its fiber product $G \times^M \alpha$ and compose the ensuing diagram with diagram (4.11) yielding diagram (4.6) with required properties.

By induction, we know that the image $\overline{U}$ of U in $\mathfrak{t}^*/W_M$ intersects the locus over which $\mathfrak{t}^*/W_L \to \mathfrak{t}^*/W_M$ is unramified. From the fact that $S^1 \neq \emptyset$ we get that $\overline{U}$ contains points in which $\mathfrak{t}^*/W_M \to \mathfrak{t}^*/W_G$ is unramified. Since $\overline{U}$ is irreducible it contains points where both conditions hold showing $U_{\mathrm{reg}} \neq \emptyset$. Finally, the statement about the Poisson structure follows from Corollary 3.7. $\square$

In the proofs of Theorem 5.1 and Lemma 5.3 we need to deal with one-dimensional submodules directly. For that we use the following reduction:

4.5. Lemma. *Assume $v_0 \in V$ generates a one-dimensional submodule with character χ. Let $v_0^- \in V$ be a G-eigenvector with $\omega(v_0^-, v_0) = 1$. Put $\overline{V} := (\mathbb{C}v_0 \oplus \mathbb{C}v_0^-)^\perp$ and $\bar{\mathfrak{g}} := \ker\chi \subseteq \mathfrak{g}$. Let $\bar{m} : \overline{V} \to \bar{\mathfrak{g}}^*$ be the moment map for $\bar{\mathfrak{g}}$. Then there is a G-equivariant isomorphism Φ such that the following diagram commutes*:

$$\begin{array}{ccc} \overline{V} \times \mathbb{C} \times \mathbb{C}^\times & \xrightarrow{\Phi} & V^0 \\ {\scriptstyle m_0}\downarrow & & \downarrow{\scriptstyle m} \\ \bar{\mathfrak{g}}^* \oplus \mathbb{C}\chi & \xrightarrow{\sim} & \mathfrak{g}^*. \end{array} \tag{4.13}$$

Here m_0 is the map $(v, t, y) \mapsto (\bar{m}(v), ty)$. Moreover, $\mathfrak{g}$ acts trivially on $\mathbb{C}$ and with the character $-\chi$ on $\mathbb{C}^\times$.

Proof. If $\chi = 0$ then we define $\Phi(v, t, y) = v + tv_0 + yv_0^-$. So assume $\chi \neq 0$ from now on. Choose ξ in the center of $\mathfrak{g}$ with $\chi(\xi) = 1$. This yields splittings $\mathfrak{g} = \bar{\mathfrak{g}} \oplus \mathbb{C}\xi$ and $\mathfrak{g}^* = \bar{\mathfrak{g}}^* \oplus \mathbb{C}\chi$. We define a G-invariant function on $\overline{V}$ by

$$f(v) = \langle m(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v). \tag{4.14}$$

Then $m(v + xv_0 + yv_0^-) = \bar{m}(v) + (f(v) + xy)\chi$ and

$$\Phi(v, t, y) := v + \frac{t - f(v)}{y} v_0 + yv_0^- \tag{4.15}$$

has the required property. □

5. Equidimensionality

Next, we use the results of Section 3 to investigate the geometry of the invariant moment map. In abuse of notation (W_G does not, in general, act on $\mathfrak{a}^*$) we denote the image of $\mathfrak{a}^*$ in $\mathfrak{t}^*/W_G$ by $\mathfrak{a}^*/W_G$.

5.1. Theorem. *Let V be a symplectic representation and let $\mathfrak{a}^* \subseteq \mathfrak{t}^*$ be as in Lemma* 4.4. *Then*:

- (i) *The image of V in $\mathfrak{t}^*/W_G$ is $\mathfrak{a}^*/W_G$.*
- (ii) *There is a morphism $\sigma : \mathfrak{a}^* \to V$ such that*

$$\begin{array}{ccc} \mathfrak{a}^* & \xrightarrow{\sigma} & V \\ & \searrow & \downarrow{\scriptstyle m/\!\!/G} \\ & & \mathfrak{a}^*/W_G \end{array} \tag{5.1}$$

  is commutative.
- (iii) *Let C be an irreducible component of $(m/\!\!/G)^{-1}(0)$. Then one can choose σ such that $\sigma(\mathfrak{a}^*) \cap C \neq \emptyset$.*

Before we prove the theorem, we mention the following consequence:

5.2. Corollary. *The invariant moment map $V \to \mathfrak{a}^*/W_G$ is equidimensional.*

Proof. By general properties of fiber dimensions, every fiber has dimension $\geqslant \dim V - \dim \mathfrak{a}^*$. We have to show the opposite inequality. By homogeneity and semicontinuity, the fiber of maximal dimension is the zero fiber. Let C be an irreducible component and let σ be as in Theorem 5.1(ii) and (iii). Then

$$0 = \dim \sigma(\mathfrak{a}^*) \cap C \geqslant \dim \mathfrak{a}^* + \dim C - \dim V \tag{5.2}$$

shows indeed $\dim C \leqslant \dim V - \dim \mathfrak{a}^*$. □

Proof of Theorem 5.1. Lemma 4.4 implies that the image of V in $\mathfrak{t}^*/W_G$ is contained in $\mathfrak{a}^*/W_G$. Therefore, part (i) will follows from part (ii). To show (ii) and (iii) we use induction on $\dim V$.

If V is terminal then the assertions are covered by Lemmas 4.2 and 4.3. Otherwise, the component C contains a non-terminal highest weight vector v_1. We defer the proof to Lemma 5.3 since it is quite technical. Then $v_0^- := w_0 v_1$ (with $w_0 \in W_G$ the longest element) is a lowest weight vector contained in C and there is a highest weight vector v_0 (possibly not in C) with $\omega(v_0^-, v_0) = 1$. Now we use the notation of Section 3 and consider the commutative diagram (3.14). Since $v_0^- \in C$, the preimage C_0 of C in S^0 is not empty. Then we have

$$\dim \mathfrak{a}^* \geqslant \operatorname{codim}_V C \geqslant \operatorname{codim}_S C_0 \geqslant \operatorname{codim}_S m_S^{-1}(0) \geqslant \dim \mathfrak{a}^*. \tag{5.3}$$

The last inequality holds because, by induction, $S \to \mathfrak{a}^*/W_M$ is equidimensional. Thus, we have equality throughout. This implies in particular that every component of C_0 is a component of $m_S^{-1}(0)$. Let C_1 be one of them. From Lemma 4.5 applied to (S, M, v_0) we obtain a diagram

$$\begin{array}{ccc} \bar{S} \times \mathbb{C} \times \mathbb{C}^\times & \xrightarrow{\sim} & S^0 \\ {\scriptstyle m_0}\downarrow & & \downarrow{\scriptstyle m} \\ \bar{\mathfrak{l}}^* \oplus \mathbb{C}\chi & \xrightarrow{\sim} & \mathfrak{l}^*. \end{array} \tag{5.4}$$

In particular, $C_1 \cong \bar{C} \times \mathbb{C}^\times$ where $\bar{C}$ is a component of $\bar{m}^{-1}(0) \subseteq \bar{S}$. By induction, there is a section $\bar{\sigma} : \bar{\mathfrak{a}}^* \to \bar{S}$ meeting $\bar{C}$. Thus we get a section $\mathfrak{a}^* = \bar{\mathfrak{a}}^* \oplus \mathbb{C}\chi \to S^0$ defined by

$$\alpha + t\chi \mapsto \Phi\big(\bar{\sigma}(\alpha), t, 1\big) \tag{5.5}$$

which meets C_1 and therefore C_0. Finally, the composition σ with the inclusion $S^0 \to V$ has the required properties. □

The following lemma was used in the preceding proof.

5.3. Lemma. *Let C be an irreducible component of $(m/\!/G)^{-1}(0)$ and assume V is not terminal. Then C contains a non-terminal highest weight vector.*

Proof. Let $U \subseteq V$ be the submodule generated by C. Assume first, that U has a non-terminal highest weight. We claim that there is a 1-parameter subgroup $\rho : \mathbf{G}_m \to T \subseteq G$ with the following properties:

$$\langle \alpha | \rho \rangle > 0 \quad \text{for all positive roots } \alpha. \tag{5.6}$$

$$\langle \chi_1 | \rho \rangle < \langle \chi_2 | \rho \rangle \quad \text{for all } \chi_1 \text{ terminal, } \chi_2 \text{ non-terminal highest weights of } V. \tag{5.7}$$

$$\langle \chi_1 | \rho \rangle \neq \langle \chi_2 | \rho \rangle \quad \text{for all weights } \chi_1 \neq \chi_2 \text{ of } V. \tag{5.8}$$

An explicit calculation for the group Sp_{2n} shows that (5.6) and (5.7) hold for ρ equal to the sum of the fundamental coweights. Now a slight perturbation of ρ yields additionally (5.8).

Let Γ be the set of weights of U. Choose $v \in C$ such that its χ-component v_χ is non-zero for every $\chi \in \Gamma$. By the last condition (5.8) on ρ, the maximum N of $\{\langle\chi|\rho\rangle \mid \chi \in \Gamma\}$ is attained for a unique weight χ_0. Because of (5.6), this weight is a highest weight. Moreover, since U contains a non-terminal highest weight, the weight χ_0 is non-terminal (condition (5.7)). Thus, the limit

$$v_0 := \lim_{t\to\infty} t^{-N}\rho(t)v \tag{5.9}$$

exists. It is a non-terminal highest weight vector (condition (5.8)), and is contained in C since C, as a component of $(m/\!\!/G)^{-1}(0)$, is a cone.

Now we are reduced to the case where all highest weights of U are terminal. If U contains a one-dimensional submodule then Lemma 4.5 reduces the statement to $\overline{V}$ and we are done by induction. Thus U contains only irreducible submodules U_1 such that the image of G in $GL(U_1)$ is the full symplectic group. Assume that there is such a U_1 and it is isotropic. Then there is a second isotropic submodule U_2 which is G-isomorphic to U_1 and such that $\overline{U} := U_1 \oplus U_2$ is anisotropic. The moment map m is invariant under the $\mathbf{G}_m$-action which is $(tu_1, t^{-1}u_2)$ on $\overline{U}$ and trivial on $\overline{U}^\perp$. Hence also C is $\mathbf{G}_m$-stable. Now choose $v \in C$ which has a non-zero component in U_1. Then

$$v_1 := \lim_{t\to\infty} t^{-1}(t \cdot v) \tag{5.10}$$

exists, is non-zero, and lies in $C \cap U_1$. Thus there is $g \in G$ such that gv_1 is a non-singular highest weight vector in C.

Now we are reduced to the case that $U = U_1 \oplus \cdots \oplus U_s$ where each U_i is anisotropic and such that $G \to Sp(U_1) \times \cdots \times Sp(U_s)$ is surjective. Let $Q := U^\perp$. Then $V = U \oplus Q$. Clearly, we may assume that the action of G on V is (locally) effective. Assume $Sp(U_i) \subseteq G$ acts trivially on Q. Then the moment map m_V factors through V/U_i and we are done by induction. Thus, we may assume that even the action of G on Q is locally effective. From $G \subseteq Sp(Q)$ we infer $\operatorname{rk} G \leqslant \frac{1}{2}\dim Q$. On the other hand, C has codimension $\leqslant \dim \mathfrak{t}^* = \operatorname{rk} G$ in V. From $C \subseteq U$ we get

$$\dim Q = \operatorname{codim}_V U \leqslant \operatorname{codim}_V C \leqslant \operatorname{rk} G \leqslant \frac{1}{2}\dim Q. \tag{5.11}$$

This implies that $V = U$ is terminal. □

6. The little Weyl group

Next, we analyze the equidimensional morphism $V \to \mathfrak{a}^*/W_G$. Let $N(\mathfrak{a}^*)$ and $C(\mathfrak{a}^*)$ be the normalizer and the centralizer of $\mathfrak{a}^*$ in W_G, respectively. Then the quotient $N(\mathfrak{a}^*)/C(\mathfrak{a}^*)$ acts faithfully on $\mathfrak{a}^*$.

6.1. Theorem. *Let $\mathfrak{a}^* \subseteq \mathfrak{t}^*$ be as in Lemma* 4.4. *Then there is subgroup W_V of $N(\mathfrak{a}^*)/C(\mathfrak{a}^*)$ and a morphism $\psi : V \to \mathfrak{a}^*/W_V$ such that the generic fibers of ψ are irreducible and such that the* *following diagram commutes*:

$$\begin{array}{ccc} V & & \\ {\scriptstyle\psi}\downarrow & \searrow^{m/\!\!/G} & \\ \mathfrak{a}^*/W_V & \longrightarrow & \mathfrak{a}^*/W_G. \end{array} \tag{6.1}$$

Moreover, the pair $(\mathfrak{a}^*, W_V)$ *is, up to conjugation by* W_G*, uniquely determined by this property.*

Proof. First we claim that, up to isomorphism, there is a unique diagram

$$\begin{array}{ccc} V & & \\ {\scriptstyle\psi}\downarrow & \searrow^{m/\!\!/G} & \\ Z & \xrightarrow{\beta} & \mathfrak{a}^*/W_G \end{array} \tag{6.2}$$

such that Z is connected and normal, β is finite, and the generic fibers of ψ are connected. The morphism $V \to \mathfrak{a}^*/W_G$ is surjective. Moreover, since β is finite, ψ has at least to be dominant. Thus we get inclusions

$$\mathbb{C}[\mathfrak{a}^*/W_G] \subseteq \mathbb{C}[Z] \subseteq \mathbb{C}[V]. \tag{6.3}$$

The requirement that the generic fibers of ψ are connected means that $\mathbb{C}(Z)$ is algebraically closed in $\mathbb{C}(V)$. This shows that $\mathbb{C}(Z)$ is the algebraic closure of $\mathbb{C}(\mathfrak{a}^*/W_G)$ in $\mathbb{C}(V)$. Since Z is normal and finite over $\mathfrak{a}^*/W_G$ we see that $\mathbb{C}[Z]$ is the integral closure of $\mathbb{C}[\mathfrak{a}^*/W_G]$ in $\mathbb{C}(Z)$. Combined we get that $\mathbb{C}[Z]$ is the integral closure of $\mathbb{C}[\mathfrak{a}^*/W_G]$ in $\mathbb{C}(V)$ which proves uniqueness. For existence, we take $Z = \operatorname{Spec} R$ where R is the $\mathbb{C}(V)$. This integral closure is contained in $\mathbb{C}[V]$ since V is normal. Thus we get the required factorization (6.2).

It rests to identify Z with $\mathfrak{a}^*/W_V$. For that consider the morphism $\sigma : \mathfrak{a}^* \to V$ from Theorem 5.1(ii). Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{a}^* & & \\ \downarrow & \searrow & \\ Z & \longrightarrow & \mathfrak{a}^*/W_G \end{array} \tag{6.4}$$

and therefore $\mathbb{C}[\mathfrak{a}^*/W_G] \subseteq \mathbb{C}[Z] \subseteq \mathbb{C}[\mathfrak{a}^*]$. Let $\varGamma := N(\mathfrak{a}^*)/C(\mathfrak{a}^*)$. Then the morphism $\mathfrak{a}^* \twoheadrightarrow \mathfrak{a}^*/W_G$ factors through $\mathfrak{a}^*/\varGamma$ and we claim that

$$\mathfrak{a}^*/\varGamma \twoheadrightarrow \mathfrak{a}^*/W_G \tag{6.5}$$

is the normalization morphism. In fact, since it is clearly finite and surjective and since $\mathfrak{a}^*/\varGamma$ is normal, we have to show that (6.5) is birational. Suppose $w \in W_G$ with $w \notin N(\mathfrak{a}^*)$. Then $w\mathfrak{a}^* \neq \mathfrak{a}^*$ and therefore $w\xi \notin \mathfrak{a}^*$ for ξ in an open dense subset of $\mathfrak{a}^*$. Since W_G is finite, there is a dense open subset U of $\mathfrak{a}^*$ such that $W_G\xi \cap \mathfrak{a}^* = \varGamma\xi$ for all $\xi \in U$. Thus (6.5) is injective on U, proving the claim.

From the claim and the fact that Z is normal, we get

$$\mathbb{C}[\mathfrak{a}^*]^\Gamma \subseteq \mathbb{C}[Z] \subseteq \mathbb{C}[\mathfrak{a}^*]. \tag{6.6}$$

For the field of fraction we obtain

$$\mathbb{C}(\mathfrak{a}^*)^\Gamma \subseteq \mathbb{C}(Z) \subseteq \mathbb{C}(\mathfrak{a}^*). \tag{6.7}$$

From Galois theory we get a unique subgroup W_V of Γ with $\mathbb{C}(Z) = \mathbb{C}(\mathfrak{a}^*)^{W_V}$. The normality of Z implies

$$\mathbb{C}[Z] = \mathbb{C}(Z) \cap \mathbb{C}[\mathfrak{a}^*] = \mathbb{C}(\mathfrak{a}^*)^{W_V} \cap \mathbb{C}[\mathfrak{a}^*] = \mathbb{C}[\mathfrak{a}^*]^{W_V} = \mathbb{C}[\mathfrak{a}^*/W_V]. \tag{6.8}$$

This implies $Z = \mathfrak{a}^*/W_V$.

For the uniqueness statement observe that the preimage of $m/\!\!/G(V)$ in $\mathfrak{t}^*$ is the union of the subspaces $w\mathfrak{a}^*$, $w \in W_G$. Thus, $\mathfrak{a}^*$ is unique up to conjugation by W_G. Now assume also $\mathfrak{a}^*/W_V'$ fits into the diagram (6.1). Since Z is unique, there would be a $\mathfrak{a}^*/\Gamma$-isomorphism $\bar{n} : \mathfrak{a}^*/W_V \to \mathfrak{a}^*/W_V'$. Since $\mathfrak{a}^*$ is Galois over $\mathfrak{a}^*/\Gamma$, the isomorphism $\bar{n}$ extends to an automorphism $n \in \Gamma$ of $\mathfrak{a}^*$. This shows $W_V' = nW_Vn^{-1}$. □

Remark. The subspace $\mathfrak{a}^*$ determines the Levi subgroup L of Lemma 4.4. In fact, it follows from $U_{\mathrm{reg}} \neq \emptyset$ in Lemma 4.4 that L is the pointwise centralizer of $\mathfrak{a}^*$ (and not larger than that). Thus, the roots of L are those α with $\langle \mathfrak{a}^*, \alpha^\vee \rangle \neq 0$. On the other hand, there is no canonical parabolic P with Levi subgroup L. More precisely, after repeated application of the construction in Section 3 one winds up with a parabolic subgroup with Levi part L but that subgroup would depend on the choice of highest weights. The most trivial example is $G = GL(n)$ with $n \geqslant 3$ and $V = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. This representation has two highest weight vectors and the corresponding parabolic subgroups are not conjugated. Nevertheless, their Levi parts are.

6.2. Theorem. *The subgroup W_V of $GL(\mathfrak{a}^*)$ is generated by reflections. In particular, $\mathfrak{a}^*/W_V$ is smooth.*

Proof. It follows from Corollary 5.2 and the finiteness of $\mathfrak{a}^*/W_V \to \mathfrak{a}^*/W_G$ that also $\psi : V \to \mathfrak{a}^*/W_V$ is equidimensional. From the proof of Theorem 6.1 we know that $\mathbb{C}[\mathfrak{a}^*/W_V]$ is integrally closed in $\mathbb{C}[V]$. Thus, the assertion follows from the following lemma, essentially due to Panyushev [Pa]. □

6.3. Lemma. *Let X and Y be vector spaces, $\Gamma \subseteq GL(Y)$ a finite subgroup and $\psi : X \to Y/\Gamma$ a dominant equidimensional morphism. Assume that $\mathbb{C}[Y/\Gamma]$ is integrally closed in $\mathbb{C}[X]$. Then Γ is a generated by reflections and Y/Γ is smooth.*

Proof. Let $\Gamma_r \subseteq \Gamma$ be the normal subgroup generated by reflections. Then, by the Shepherd–Todd–Chevalley theorem [Bo2, Ch. V §5 nº 5.5] the quotient Y/Γ_r is again an affine space. Moreover, Γ/Γ_r acts linearly on it and does not contain any reflections. Thus we may replace Y and Γ by Y/Γ_r and Γ/Γ_r, respectively. Thereby we achieve that Γ does not contain any reflections anymore and we have to show $\Gamma = 1$.

Let $\overline{X} := X \times_{Y/\Gamma} Y$. Since $Y \to Y/\Gamma$ is surjective, also $\overline{X} \to X$ is surjective. Thus, we can find an irreducible component $\tilde{X}$ of $\overline{X}$ such that $\tilde{X} \to X$ is dominant. Thus, we get the following diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\pi}} & X \\ {\scriptstyle\tilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y & \xrightarrow[\pi]{} & Y/\Gamma. \end{array} \tag{6.9}$$

Since $\tilde{\pi}$ is finite, it is in fact surjective. Because Γ does not contain reflections, the morphism π is étale outside a subset $Y_s \subseteq Y$ of codimension $\geqslant 2$. Then $\tilde{\pi}$ is étale outside $X_s := \tilde{\psi}^{-1}(Y_s)$. Since ψ is equidimensional and dominant the same holds for $\tilde{\psi}$. This implies that the codimension of $\tilde{X}_s$ is $\tilde{X}$ is at least 2. Since X is smooth, the Zariski–Nagata theorem (stating that the ramification locus is pure of codimension one, see [SGA2, Exp. X, Th. 3.4]) implies that $\tilde{\pi}$ is (everywhere) étale and therefore a covering. The affine space X does not possess non-trivial coverings. We conclude that $\tilde{\pi}$ is an isomorphism. From this, we get a diagonal morphism $X \to Y$ and therefore $\mathbb{C}[Y]^\Gamma \subseteq \mathbb{C}[Y] \subseteq \mathbb{C}[X]$. Since $\mathbb{C}[Y/\Gamma]$ is integrally closed in $\mathbb{C}[X]$ we conclude $\Gamma = 1$. □

Remark. One can replace the Zariski–Nagata theorem by the following (well-known) topological argument: let $X_s := \psi^{-1}(Y_s/\Gamma)$. Then $\tilde{X}_r := \tilde{X} \setminus \tilde{X}_s \to X_r := X \setminus X_s$ is a finite étale morphism. The real codimension of X_s in X is at least 4 and X is 1-connected. Therefore, X_r is still 1-connected. Because $\tilde{X}_r$ is at least connected we conclude that $\tilde{X}_r \to X_r$ is an isomorphism. Hence, $\tilde{\pi}$ is birational and therefore an isomorphism.

Now we obtain that ψ is not only equidimensional but even flat:

6.4. Corollary. *The morphisms $\psi : V \to \mathfrak{a}^*/W_V$ and $\psi /\!\!/ G : V /\!\!/ G \to \mathfrak{a}^*/W_V$ are faithfully flat.*

Proof. Since ψ is equidimensional the same holds for $\psi /\!\!/ G$. Since the source of ψ and $\psi /\!\!/ G$ is Cohen–Macaulay (this is clear for V and follows from the Hochster–Roberts theorem for $V /\!\!/ G$) and the target is smooth we conclude that both morphisms are flat [EGA, §15.4.2]. This shows in particular that $\psi(V)$ is open. On the other hand, this is a homogeneous subset of $\mathfrak{a}^*/W_V$ which contains 0. Thus, $\psi(V) = \mathfrak{a}^*/W_V$. □

In algebraic terms, we get:

6.5. Corollary. *The rings $\mathbb{C}[V]$ and $\mathbb{C}[V]^G$ are free as $\mathbb{C}[\mathfrak{a}^*/W_V]$-modules.*

Proof. For positively graded algebras, freeness and flatness are the same [Bo1, Ch. 2, §11, no. 4, Prop. 7]. □

7. The generic structure

In this section, we refine Lemma 4.4 to give a very precise description of the generic structure of a symplectic representation. We start with a rather general remark. Recall that an open subset

U of an affine G-variety X is called *saturated* if it is the preimage of an open subset of $X/\!\!/G$. This is equivalent to saying that every closed orbit of U is closed in X.

7.1. Proposition. *Let V be a selfdual representation of G. Then*:

(i) *For every G-stable divisor $D \subset V$ there is a non-zero invariant function $f \in \mathbb{C}[V]^G$ vanishing on D.*
(ii) *Every invariant rational function is the quotient of two regular invariants.*
(iii) *Let F be a general fiber of $\pi : V \to V/\!\!/G$. Then F contains an open orbit F_0 with* $\operatorname{codim}_F F \setminus F_0 \geqslant 2$.
(iv) *Let $U \subseteq V$ be a G-stable non-empty open subset of V. Assume moreover that U is affine. Then U contains a saturated non-empty open subset.*

Proof. (i) The divisor D is the zero set of a function $f \in \mathbb{C}[V]$. Moreover, f is unique up to a scalar which implies that $gf = \chi(g)f$ for all $g \in G$ and a character χ of G. Since V is selfdual also $\mathbb{C}[V]_{\leqslant d} = \bigoplus_{i=0}^{d} S^i V^*$ is selfdual. Thus there is a semiinvariant $f^* \in \mathbb{C}[V]$ with $gf^* = \chi(g)^{-1} f^*$ for all $g \in G$. Hence, $F = ff^*$ is an invariant vanishing on D.

(ii) It is well known that every invariant rational function f is the quotient of two regular semiinvariants $f = p/q$ where p and q transform with the same character χ. As above, there is a semiinvariant q^* with character χ^{-1}. Thus, $f = pq^*/qq^*$ is a ratio of two invariants.

(iii) Suppose the assertion is false. Then F contains a G-stable divisor. Using, e.g., the slice theorem this implies that V contains a G-stable divisor D which meets the general fiber of $V \to V/\!\!/G$. But then no non-zero invariant would vanish on D.

(iv) Since U is affine, its complement D in V is a G-stable divisor. Let $Z := \overline{\pi(D)}$. Then $Z \neq V/\!\!/G$ by (i). Hence $\pi^{-1}(V/\!\!/G - Z)$ has the required properties. □

For symplectic representations we can say much more:

7.2. Theorem. *Let V be a symplectic G-representation. Then there is a commutative diagram*

$$\begin{array}{ccccc} (G \times^L F) \times U & \longrightarrow & V_0 & \hookrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \pi_V \\ U & \xrightarrow{\alpha} & U_0 & \hookrightarrow & V/\!\!/G \\ \downarrow & & \downarrow & \swarrow {\scriptstyle \psi/\!\!/G} & \\ \mathfrak{a}^* & \longrightarrow & \mathfrak{a}^*/W_V & & \end{array} \tag{7.1}$$

where all squares are Cartesian and where α is finite and étale. Here, L, F, $\mathfrak{a}^*$, and U are as in Lemma 4.4. *Moreover, U_0 is an open dense subset of $V/\!\!/G$ and V_0 is its preimage in V.*

Proof. First, I claim that $\tilde{V} := V \times_{\mathfrak{a}^*/W_V} \mathfrak{a}^*$ is irreducible. In fact, since $\mathfrak{a}^*/W_V$ is an affine space, $\tilde{V}$ is a complete intersection. Hence all irreducible components of $\tilde{V}$ have the same dimension as V and they all map dominantly to V. It follows that W_V acts transitively on the set of irreducible components and therefore each component maps dominantly to $\mathfrak{a}^*$, as well. But the generic fibers of $\tilde{V} \to \mathfrak{a}^*$ are the same as the ones of $V \to \mathfrak{a}^*/W_V$ and therefore irreducible which shows that $\tilde{V}$ is irreducible.

Let $X := G \times^L F$. Then Lemma 4.4 says that $X \times U$ is an open subset of $\tilde{V}$. Clearly, we may shrink U to be affine. Then the complement $\tilde{D}$ of $X \times U$ in $\tilde{V}$ is a divisor. Since $\tilde{V} \to V$ is finite, the image D of $\tilde{D}$ in V is a divisor as well. From Proposition 7.1(iv) we get a non-empty affine saturated open subset $V_0 \subseteq V$ with $V_0 \cap D = \emptyset$. The preimage of V_0 in $\tilde{V}$ is affine and is contained in $X \times U$. Hence its complement in $X \times U$ is a divisor and therefore of the form $X \times U_0$ with $U_0 \subseteq U$ open. Replace U by U_0. Then the image of $X \times U$ in V is V_0 and the morphism $X \times U \to U$ is finite. Put $U_0 := V_0 /\!\!/ G$ and let $\alpha : U \to U_0$ the morphism on G-quotients. This finishes the construction of diagram (7.1).

The upper right square is Cartesian since V_0 is saturated. Let $\tilde{V}_0 = V_0 \times_{\mathfrak{a}^*/W_V} \mathfrak{a}^*$. As an open subset of $\tilde{V}$ it is irreducible, as well. Moreover, the morphism $X \times U \to \tilde{V}_0$ is both finite and an open embedding. We conclude that it is an isomorphism. Thus, the big left square is Cartesian. Going over to G-invariants we conclude that the lower left square is Cartesian. This implies, by general nonsense, that the upper left square is Cartesian.

Finally, by shrinking, if necessary, U_0 we may assume that the image of U_0 in $\mathfrak{a}^*/W_V$ lies in the part over which the map $\mathfrak{a}^* \to \mathfrak{a}^*/W_V$ is unramified. This makes α étale. □

7.3. Corollary. *In the étale topology there is a non-empty open subset of $V /\!\!/ G$ over which the quotient morphism $V \to V /\!\!/ G$ is a trivial fiber bundle with fiber $G \times^L F$.*

Remark. It is follows from Luna's slice theorem that every quotient map is generically a fiber bundle in the étale topology but in general the trivializing étale map cannot be controlled. Therefore, it might be surprising that for symplectic representations a Galois covering whose group is a subquotient of the Weyl group suffices.

7.4. Corollary. *Let H the principal isotropy group of V (i.e., the isotropy group of a generic closed orbit). Then there is a Levi subgroup L with $(L, L) \subseteq H \subseteq L$ and $L/H = A$.*

In the next statement, we denote the isotropy group of a non-zero vector in $\mathbb{C}^{2n}$ inside $Sp_{2n}(\mathbb{C})$ by $Sp_{2n-1}(\mathbb{C})$.

7.5. Corollary. *Let H be the generic isotropy group of V. Then*

$$H \cong H_0 \times Sp_{2m_1-1}(\mathbb{C}) \times \cdots \times Sp_{2m_s-1}(\mathbb{C}) \tag{7.2}$$

with H_0 reductive.

Remark. A non-reductive generic isotropy group is a quite exceptional phenomenon. Clearly $G = Sp_{2m}(\mathbb{C})$ acting on $V = \mathbb{C}^{2m}$, $m \geqslant 1$ is an example. The same holds more generally for $G = Sp_{2m}(\mathbb{C}) \times SO_{2n-1}(\mathbb{C})$ acting on $V = \mathbb{C}^{2m} \otimes \mathbb{C}^{2n-1}$ with $2m > 2n - 1 \geqslant 1$. More examples can be found in [Kn4].

8. Symplectic reductions

Definition. Let V be a symplectic vector space. A fiber of $\psi /\!\!/ G : V /\!\!/ G \to \mathfrak{a}^*/W_V$ is called a *symplectic reduction* of V.

8.1. Theorem. *The symplectic reductions of V form a flat family parametrized by an affine space. Moreover, for a generic symplectic reduction there exists a birational Poisson morphism to* $\mathbb{C}^{2c}$ *with its standard symplectic structure.*

Proof. The first part is just a reformulation of parts of Theorem 6.2 and Corollary 6.4. The second part follows from Theorem 7.2. □

Definition. We call $\operatorname{rk}_s V := \dim \mathfrak{a}^*$ the (*symplectic*) *rank* and $c_s(V) := \frac{1}{2}(\dim V/\!\!/G - \dim \mathfrak{a}^*)$ the (*symplectic*) *complexity* of V.

Thus the symplectic reductions are of dimension $2c_s(V)$ and they are parametrized by an affine space of dimension $\operatorname{rk}_s V$.

There already exists a notion of rank and complexity for an arbitrary G-variety X: the complexity $c(X)$ is the transcendence degree of $k(X)^B$ over k and the rank $\operatorname{rk} X$ is the transcendence degree of $k(X)^U$ minus the complexity of X. This previous notion is related to our present definition:

8.2. Proposition. *For a finite-dimensional* (*non-symplectic*) *representation X of G consider the symplectic representation* $V = X \oplus X^*$. *Then* $\operatorname{rk}_s V = \operatorname{rk} X$ *and* $c_s(V) = c(X)$.

Proof. Observe that $V = X \oplus X^*$ is just the cotangent bundle T_X^* of X. Then we recognize the assertion as a special case of [Kn1, Thm. 7.1]. □

8.3. Theorem. *If we regard* $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ *as a subalgebra of* $\mathbb{C}[V]^G$ *then it is precisely the Poisson center. In particular, the Poisson center of* $\mathbb{C}[V]^G$ *is a polynomial ring over which it is a free module.*

Proof. Let $f \in \mathbb{C}[\mathfrak{a}^*/W_V]$. Then f is finite over $\mathbb{C}[\mathfrak{a}^*/W_G]$, i.e., satisfies a monic equation $p(f) = 0$ with coefficients in $\mathbb{C}[\mathfrak{a}^*/W_G]$. Assume that p is of minimal degree. Since $\mathbb{C}[\mathfrak{a}^*/W_G]$ is in the Poisson center of $\mathbb{C}[V]^G$ (see Section 2) we infer for any $g \in \mathbb{C}[V]^G$:

$$0 = \bigl\{p(f), g\bigr\} = p'(f)\{f, g\}. \tag{8.1}$$

This shows $\{f, g\} = 0$ since, by minimality $p'(f) \neq 0$. Hence f is in the Poisson center of $\mathbb{C}[V]^G$.

Conversely, let f be in the Poisson center of $\mathbb{C}[V]^G$. Then the restriction of f to a generic symplectic reduction R is in the center of $\mathbb{C}[R]$. According to Theorem 8.1, R has a dense open subset R° which is a symplectic variety. Since f Poisson-commutes with all of $\mathbb{C}[R]$, the same holds for the localization $\mathbb{C}(R)$ and therefore $\mathbb{C}[R^\circ]$. This implies that f is constant on R and therefore $f \in \mathbb{C}[V]^G \cap \mathbb{C}(\mathfrak{a}^*/W_V) = \mathbb{C}[\mathfrak{a}^*/W_V]$. □

We end this section with a permanence property. It is very useful in computing complexity and rank of a symplectic representation. See [Kn4] for examples.

8.4. Theorem. *Let V be a symplectic G-representation and let* (S, M) *be as in Section* 3. *Then* $\operatorname{rk}_s(S, M) = \operatorname{rk}_s(V, G)$ *and* $c_s(S, M) = c_s(V, G)$. *Moreover, there is an* $w \in W_G$ *with* $\mathfrak{a}_V^* = w\mathfrak{a}_S^*$ *and* $L_V = wL_S w^{-1}$.

Proof. We already noted that $\mathfrak{a}^*$ is, up to conjugation by W_G, determined by the image of V in $\mathfrak{t}^*/W_G$. Thus $\mathfrak{a}_V^* = w\mathfrak{a}_S^*$ follows from Theorem 3.11. This immediately implies the statement on ranks. The assertion on L follows from the fact that L is the point-wise stabilizer of $\mathfrak{a}^*$. Finally, the equality of complexities follows, for example from (4.12) and the fact that $\dim U = \mathrm{rk}_s + 2c_s$. □

9. Multiplicity free symplectic representations

Of particular interest is the case of complexity zero.

Definition. A symplectic representation V is called *multiplicity free* if $c_s(V) = 0$.

There are many equivalent characterizations of multiplicity free symplectic representations. Some of them are summarized below. Observe that (v) is a precise version of the assertion "Almost all G-invariants on V are pull-backs of invariants on $\mathfrak{g}^*$ via the moment map." For (vii) recall that a subspace U of a symplectic vector space is *coisotropic* if $U^\perp \subseteq U$. A smooth subvariety is coisotropic if each of its tangent spaces is coisotropic.

9.1. Proposition. *Let V be a symplectic representation. Then the following statements are equivalent*:

(i) *V is multiplicity free.*
(ii) *The morphism $V/\!\!/G \to \mathfrak{a}^*/W_V$ is an isomorphism.*
(iii) *All symplectic reductions are points* (*in the scheme sense*).
(iv) *The morphism $V/\!\!/G \to \mathfrak{t}^*/W_G$ is unramified* (*i.e., injective on tangent spaces*) *on a dense open subset.*
(v) *The morphism $V/\!\!/G \to \mathfrak{t}^*/W_G$ is finite.*
(vi) *The Poisson algebra $\mathbb{C}[V]^G$ is commutative.*
(vii) *The generic G-orbit of V is coisotropic.*

Proof. The morphism $\psi : V/\!\!/G \to \mathfrak{a}^*/W_V$ is faithfully flat of relative dimension $2c_s(V)$. Thus, (i) implies (ii) which implies in turn (iii)–(v). Conversely, each of these conditions implies $c_s(V) = 0$ and therefore (i).

The equivalence of (ii) and (vi) is a special case of Theorem 8.3. Finally, for (vii) consider $Z := \overline{m(V)} \subseteq \mathfrak{g}^*$. Then we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{m} & Z & \hookrightarrow & \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi_{\mathfrak{g}^*}} \\ V/\!\!/G & \xrightarrow{m'} & \mathfrak{a}^*/W_G & \hookrightarrow & \mathfrak{t}^*/W_G \end{array} \tag{9.1}$$

where all vertical arrows are categorical quotients. Choose a generic point $v \in V$ and denote its image in Z, $V/\!\!/G$, and $\mathfrak{a}^*/W_G$ by z, $\bar{v}$, and $\bar{z}$, respectively. Let $V_{\bar{z}}$, $Z_{\bar{z}}$ etc. denote the fiber over $\bar{z}$. Then $V_{\bar{z}} = m^{-1}(Z_{\bar{z}})$. The fibers of π_Z contain only finitely many G-orbits (it inherits this property from $\pi_{\mathfrak{g}^*}$). This implies that Gz is open in $Z_{\bar{z}}$. Thus, the preimage of Gz in $V_{\bar{z}}$ is open as well. This preimage is isomorphic to the fiber product $G \times^{G_z} V_z$ where $V_z = m^{-1}(z)$. The tangent space of V_z in v is $(\mathfrak{g}v)^\perp$ (see (2.10)). Thus $T_v V_{\bar{z}} = \mathfrak{g}v + (\mathfrak{g}v)^\perp$. On the other hand, Gv is dense in $V_{\bar{v}} = \pi_V^{-1}(\bar{v})$ (Proposition 7.1(iii)). This (and the fact that $V_{\bar{z}}$ is smooth in v) shows $T_v V_{\bar{v}} = \mathfrak{g}v$. Thus, we get

$$T_{\bar{v}}(V /\!\!/ G)_{\bar{z}} = \bigl(\mathfrak{g}v + (\mathfrak{g}v)^\perp\bigr)/\mathfrak{g}v. \tag{9.2}$$

This space is zero if and only if $(\mathfrak{g}v)^\perp \subseteq \mathfrak{g}v$, i.e., Gv is coisotropic. This shows the equivalence of (iv) and (vii). □

The notion "multiplicity free" is justified by:

9.2. Proposition. *For a finite-dimensional* (*non-symplectic*) *representation X of G consider the symplectic representation $V = X \oplus X^*$. Then V is multiplicity free* (*as symplectic representation*) *if and only if $\mathbb{C}[X]$ is a multiplicity free G-module.*

Proof. Using Proposition 8.2, this is the main statement of [VK] (see also [Kn3, Thm. 3.1]). □

Associated to a symplectic vector space V is another object, namely the Weyl algebra $\mathcal{W}(V)$. By definition, it is an associative unital algebra which is generated by V with the relations

$$uv - vu = \omega(u, v) \quad \text{for all } u, v \in V. \tag{9.3}$$

If G acts on V then it will also act on $\mathcal{W}(V)$ by way of automorphisms. On the Lie algebra level, this action is inner: there is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathcal{W}(V)$ such that

$$\xi a = \bigl[\rho(\xi), a\bigr] \quad \text{for all } \xi \in \mathfrak{g} \text{ and } a \in \mathcal{W}(V). \tag{9.4}$$

(It suffices to show this for $\mathfrak{g} = \mathfrak{sp}(V)$. In that case see [Ho, Thm. 5]). This means in particular, that the algebra of invariants $\mathcal{W}(V)^G$ can be interpreted as centralizer of $\rho(\mathfrak{g})$.

The map ρ induces an algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{W}(V)$. With $\mathcal{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G$, the center of $\mathcal{U}(\mathfrak{g})$, we get a homomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathcal{W}(V)^G$.

The connection with Poisson algebras is as follows: both algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{W}(V)$ come with natural filtrations by placing the generators in degree one. For a filtered vector space $F_{\leqslant \bullet}$ let $\operatorname{gr} F := \bigoplus_{n \in \mathbb{Z}} F_{\leqslant n}/F_{<n}$ be the associated graded space. Then $\operatorname{gr} \mathcal{U}(\mathfrak{g}) = S^*(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ while $\operatorname{gr} \mathcal{W}(V) = S^*(V) = \mathbb{C}[V^*] = \mathbb{C}[V]$. Moreover, the commutator on a filtered algebra induces a Poisson structure on the associated graded algebra. In our case, we arrive exactly at the usual Poisson structures on $\mathbb{C}[\mathfrak{g}^*]$ and $\mathbb{C}[V]$. Since G is linearly reductive we have $\operatorname{gr} \mathcal{Z}(\mathfrak{g}) = \operatorname{gr} \mathcal{U}(\mathfrak{g})^G = \mathbb{C}[\mathfrak{g}^*]^G = \mathbb{C}[\mathfrak{g}^* /\!\!/ G]$. Similarly, $\operatorname{gr} \mathcal{W}(V)^G = \mathbb{C}[V]^G$. The homomorphism $\mathcal{Z}(\mathfrak{g}) \to \mathcal{W}(V)^G$ gives on the associated graded level a homomorphism $\mathbb{C}[\mathfrak{g}^* /\!\!/ G] \to \mathbb{C}[V]^G$ which corresponds precisely to the morphism $V \to \mathfrak{g}^* /\!\!/ G$ induces by the moment map.

Now we have further characterizations in terms of the Weyl algebra. Here statement (ii) is a precise version of the assertion "Almost all elements of $\mathcal{W}(V)$ commuting with $\rho(\mathfrak{g})$ come from $\mathcal{Z}(\mathfrak{g})$."

9.3. Proposition. *Let V be a symplectic representation. Then the following statements are equivalent*:

(i) *V is multiplicity free.*
(ii) *$\mathcal{W}(V)^G$ is a finitely generated $\mathcal{Z}(\mathfrak{g})$-module.*
(iii) *The algebra $\mathcal{W}(V)^G$ is commutative.*

Proof. (i) $\Rightarrow$ (ii): if V is multiplicity free then $\mathbb{C}[V]^G$ is a finitely generated $\mathbb{C}[\mathfrak{g}^* /\!/ G]$-module (Proposition 9.1(v)). This implies the corresponding statement (ii) on filtered objects.

(ii) $\Rightarrow$ (i): every $x \in \mathcal{W}(V)^G$ satisfies a (monic) equation with coefficients in $\mathcal{Z}(\mathfrak{g})$. Looking at highest degree terms, we see that $\mathbb{C}[V]^G$ algebraic over $\mathbb{C}[\mathfrak{g}^* /\!/ G]$. Thus, Proposition 9.1(iv) is satisfied.

(i) $\Rightarrow$ (iii): we adapt the argument of [Kn2]. Suppose $\mathcal{W}(V)^G$ were not commutative. Let $x \in \mathcal{W}(V)^G$ be not in the center and consider the derivation $\vartheta := \operatorname{ad} x$ of $\mathcal{W}(V)^G$. Then there is a minimal number $m \in \mathbb{Z}$ such that $\deg \vartheta(y) \leqslant \deg y + m$. Thus, ϑ induces a non-zero derivation $\bar{\vartheta}$ of $\mathbb{C}[V]^G$ which is of degree m. Moreover, $\bar{\theta}$ is trivial on $\operatorname{gr} \mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^* /\!/ G]$. By Proposition 9.1(v), that ring is finite in $\mathbb{C}[V]^G$. Hence, $\bar{\vartheta} = 0$.

(iii) $\Rightarrow$ (i): apply Proposition 9.1(vi). □

We finish with a summary of our results specialized to multiplicity free representations:

9.4. Corollary. *Let V be a multiplicity free symplectic representation.*

(i) *The morphism $V \to \mathfrak{a}^*/W_V$ identifies $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ with $\mathbb{C}[V]^G$.*
(ii) *V is a cofree representation, i.e., the invariant ring $\mathbb{C}[V]^G$ is a polynomial ring and $\mathbb{C}[V]$ (or any module of covariants) is a free $\mathbb{C}[V]^G$-module.*
(iii) *The invariant algebra $\mathcal{W}(V)^G$ is a polynomial ring and $\mathcal{W}(V)$ is a free (left or right) $\mathcal{W}(V)^G$-module.*

Proof. Part (i) is a restatement of Proposition 9.1(ii). Part (ii) is just a specialization of Corollary 6.5 to the multiplicity free case. Finally, (iii) follows from Proposition 9.3(iii) and its associated graded version (ii). □

Because the moment map is quadratic, we get:

9.5. Corollary. *Let V be a multiplicity free symplectic representation. Then the degrees of the generators of $\mathbb{C}[V]^G$ are twice the degrees of the generators of $\mathbb{C}[\mathfrak{a}^*]^{W_V}$. In particular, they are all even.*

Remark. This last statement is often a convenient way to compute W_V. See [Kn4] where W_V is listed for all cases.

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