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Sufficient conditions for semicomplete multipartite digraphs to be Hamiltonian

Yubao Guo[a], Meike Tewes[b], Lutz Volkmann[b,*], Anders Yeo[c]

[a] *Lehrstuhl C für Mathematik, RWTH Aachen, 52056 Aachen, Germany*
[b] *Lehrstuhl II für Mathematik, RWTH Aachen, 52056 Aachen, Germany*
[c] *Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark*

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Abstract

A semicomplete multipartite digraph is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. Very recently, Yeo (J. Graph Theory 24 (1997) 175–185), proved that every regular semicomplete multipartite digraph is Hamiltonian. With this, Yeo confirmed a conjecture of Zhang (Ann. Discrete Math. 41 (1989) 499–514). In the first part of this paper, a generalization of regularity is considered. We extend Yeo's result to semicomplete multipartite digraphs that satisfy this generalized condition apart from exactly two exceptions. In the second part, we introduce the so-called semi-partition complete digraphs and show that this family is Hamiltonian or cycle complementary, when, clearly, the cardinality of each partite set is less than or equal to half the order.

Keywords: Semicomplete multipartite digraphs; Multipartite tournaments; Hamiltonian cycles; Regular digraphs

1. Introduction

A semicomplete multipartite digraph (or semicomplete k-partite digraph) is obtained by replacing each edge of a complete multipartite graph by an arc or by a pair of two mutually opposite arcs. Obviously, a tournament is a semicomplete multipartite digraph without cycles of length 2 where each partite set contains exactly one vertex.

* Corresponding author.
E-mail address: volkm@math2.rwth-aachen.de (L. Volkmann)

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In this paper we give new sufficient conditions for semicomplete multipartite digraphs to be Hamiltonian. To do this we consider semicomplete multipartite digraphs D which have a spanning subdigraph that consists of disjoint cycles, the so-called 1-regular factors.

In 1997, Yeo [9] proved a result about semicomplete multipartite digraphs which have a 1-regular factor. Using this result, he gave affirmative answers to conjectures by Zhang [10], by Bang-Jensen et al. [1] and by Guo and Volkmann [3]. Reusing Yeo's structure result we will extend the first one, that every regular semicomplete multipartite digraph is Hamiltonian. In addition, we will introduce a special family of semicomplete multipartite digraphs, the so-called semi-partition complete digraphs, where the cardinality of the in- and out-neighborhood of every vertex in each partite set is at least half the cardinality of the partite set in consideration. We give sufficient conditions for semi-partition complete digraphs to be Hamiltonian. Further completions of Yeo's theorem that a regular semicomplete multipartite digraph is Hamiltonian can be found in articles of Volkmann [7] and Yeo [8]. Very recently, Bang-Jensen et al. [2] showed that the decision problem 'Hamiltonian cycle in semicomplete multipartite digraphs' is solvable in polynomial time.

2. Terminology and notation

In this paper all digraphs are finite without loops and multiple arcs. A digraph D is determined by its set of vertices and its set of arcs, denoted by $V(D)$ and $E(D)$, respectively. If $xy \in E(D)$ for $x, y \in V(D)$, we write $x \to y$. Let S_1 and S_2 be two disjoint subsets of $V(D)$. Then $d(S_1, S_2)$ denotes the number of arcs leading from S_1 to S_2. If $x \to y$ for every $x \in S_1$ and $y \in S_2$, we write $S_1 \to S_2$. If there is no arc leading from S_2 to S_1, we denote this by $S_1 \Rightarrow S_2$. In this case, there might be non-adjacent vertices $s_1 \in S_1$ and $s_2 \in S_2$. For $S \subseteq V(D)$ and for any vertex $x \in V(D)$ we define $d^+(x,S)$ $(d^-(x,S))$ to be the number of out-neighbors (in-neighbors) of x in S. If $S = V(D)$, we also write $d^+(x,D) = d^+(x)$ $(d^-(x,D) = d^-(x))$. Furthermore, $N^+(S)$ $(N^-(S))$ is the set of out-neighbors (in-neighbors) of S.

All cycles mentioned here are oriented cycles. If x is a vertex on a cycle, then x^+ (x^-) denotes the successor (predecessor) of x on C. A subdigraph F of a digraph D which contains all the vertices of D and which consists of a set of vertex-disjoint cycles is called a *1-regular factor*. F is said to be *minimal* if there is no 1-regular factor F' of D consisting of less cycles than F.

A *semicomplete k-partite digraph* (*semicomplete multipartite digraph*) D is obtained by replacing each edge of a complete k-partite graph by an arc or by a pair of two mutually opposite arcs. If $k = 2$, then we call D a *semicomplete bipartite digraph*. Finally, a *multipartite tournament* is a semicomplete multipartite digraph without cycles of length 2.

3. Preliminaries

In this section we list some results which are used later in this paper. The first one deals with a generalization of regularity for general digraphs which was introduced by Ore [6]. We need the following definition.

Definition 3.1. Let D be a digraph and let m be a positive integer. The function $F(D,m)$ is defined as

$$F(D,m)=\sum_{x\in V(D),d^+(x)>m}(d^+(x)-m)+\sum_{x\in V(D),d^-(x)<m}(m-d^-(x)).$$

Remark 3.2. For a digraph D of order n and a positive integer m, consider the function

$$F'(D,m)=\sum_{x\in V(D),d^+(x)<m}(m-d^+(x))+\sum_{x\in V(D),d^-(x)>m}(d^-(x)-m).$$

Note that

$$\sum_{x\in V(D)}d^+(x)=n\cdot m+\sum_{x\in V(D),d^+(x)>m}(d^+(x)-m)-\sum_{x\in V(D),d^+(x)<m}(m-d^+(x)).$$

Since an analogous equation holds for the sum of in-degrees and since $\sum_{x\in V(D)} d^+(x)= \sum_{x\in V(D)} d^-(x)$, it is easy to see that $F'(D,m)=F(D,m)$ for every digraph D and every integer m.

Theorem 3.3 (Ore [6]). *Let D be a digraph. If $F(D,m)\leqslant m-1$ for some positive integer m, then D contains a 1-regular factor.*

For every $m\geqslant 1$ there are families of examples verifying the sharpness of this condition for semicomplete multipartite digraphs, i.e. there exist semicomplete multipartite digraphs where $F(D,m)=m$ and that contain no 1-regular factor. In fact, let D_k denote the semicomplete 3-partite digraph of order $n=3k+2$ with the partite sets V_1, V_2, and V_3, where $k\geqslant 1$ is an integer, $|V_1|=|V_2|=k+1$, and $|V_3|=k$. Let $u\in V_1$, and let $V_1\to(V_2\cup V_3)\to(V_1-u)$ and $V_2\to V_3\to V_2$. Then D_k has no 1-regular factor, since $d^-(u)=0$. Furthermore, it is easy to see that $F(D_k,2k+1)=0+(2k+1-d^-(u))=2k+1$. Analogously, examples can be constructed for even values of m, where $|V_i|=m/2$ for $i=1,2,3$.

The following condition for semicomplete bipartite digraphs to be Hamiltonian is due to Gutin [4] and Häggkvist and Manoussakis [5]. It will be applied in Section 5.

Theorem 3.4 (Gutin [4], Häggkvist and Manoussakis [5]). *A semicomplete bipartite digraph D is Hamiltonian if and only if D is strongly connected and has a 1-regular factor.*

Finally, we state a shortened version of Yeo's [9] result about the structure of semicomplete multipartite digraphs with a 1-regular factor, and the condition for Hamiltonian semicomplete multipartite digraphs that we will generalize in the next section.

Theorem 3.5 (Yeo [9]). *Let D be a semicomplete multipartite digraph containing a 1-regular factor. Then there exists a 1-regular factor F in D whose cycles can be labeled, $C_1, C_2, \dots, C_t$, such that the following holds*:

There exists a partite set V^* in D such that whenever $x_j \to x_1$ where $x_j \in V(C_j)$, $x_1 \in V(C_1)$ and $1 < j \leqslant t$, we have $\{x_j^+, x_1^-\} \subseteq V^*$. Analogously, there is a partite set V^{**} in D (possibly $V^* = V^{**}$) such that whenever $x_t \to x_j$ where $x_t \in V(C_t)$, $x_j \in V(C_j)$ and $1 \leqslant j < t$ we have $\{x_t^+, x_j^-\} \subseteq V^{**}$.

In particular, if F is minimal, then it has the properties described above.

Theorem 3.6 (Yeo [9]). *Every regular semicomplete multipartite digraph is Hamiltonian.*

4. An extension of Theorem 3.6

In this section, we extend Theorem 3.6 to the more general class of semicomplete multipartite digraphs that satisfy the Ore condition in Theorem 3.3 and that are not isomorphic to two special digraphs.

Definition 4.1. For a positive integer $m \geqslant 2$, let G'_m be a semicomplete 3-partite digraph with the partite sets $V_1 = \{x\}$, $V_2 = \{y_1, y_2, \dots, y_{m-1}\}$, and $V_3 = \{z_1, z_2, \dots, z_m\}$ such that $V_2 \to x \to V_3 \to V_2 \to V_3 \setminus \{z_1\}$ and $z_1 \to x$.

The digraph G''_m is obtained from G'_m by reversing all arcs.

We observe that $F(G'_m, m) \leqslant m - 1$, but G'_m is not Hamiltonian, as a Hamiltonian cycle would contain the arc xz_1 and every second vertex on the cycle would belong to the partite set V_3. Since x has no negative neighbor in $V_3 \setminus \{z_1\}$, this is not possible. Clearly, G''_m has the same properties.

Theorem 4.2. *Let D be a semicomplete multipartite digraph such that $F(D, m) \leqslant m - 1$ for some positive integer m. If D is not isomorphic to G'_m or G''_m, then D is Hamiltonian.*

Proof. By Theorem 3.3, there is a 1-regular factor $F = C_1 \cup C_2 \cup \cdots \cup C_t$ in D, which we may assume is minimal, and therefore has the properties described in Theorem 3.5. Since we are done when $t = 1$, assume that $t \geqslant 2$. We show that this implies that D is isomorphic to G'_m or G''_m. Let $V_1, V_2, \dots, V_k$ be the partite sets of D and let V^* denote the fixed partite set mentioned in Theorem 3.5. Define the following three sets of vertices: $A=V(C_1)$, $B=V(D)\backslash A$, and $S=\{s\in A \mid s\Rightarrow B\}$. By Theorem 3.5, there are no consecutive vertices x,x^- on C_1 such that both have a negative neighbor in B since otherwise it follows that $\{x^-,x^{--}\}\subseteq V^*$, a contradiction. Therefore, $|S|\geqslant |A|/2$.

We prove the following claim before considering the three cases $|A|=2$, $3\leqslant |A|\leqslant m-1$ and $|A|\geqslant m$.

Claim (∗).

$$m-1\geqslant |S|\left(|B|-\max_{1\leqslant i\leqslant k}\{|V_i\cap B|\}\right).$$

Proof. Let $x\in A\backslash S$ and $y\in B$ such that $y\to x$. By Theorem 3.5, x^- and y^+ belong to the same partite set and hence, x and y^+ are adjacent. Since, again by Theorem 3.5, y^+ and y^{++} would belong to the same partite set, if $y^+\to x$, we conclude that $x\to y^+$. Hence, $d(A\backslash S,B)-d(B,A\backslash S)\geqslant 0$. Furthermore, a vertex $s\in S\cap V_i$ dominates $|B|-|V_i\cap B|$ vertices in B. Therefore, $d(S,B)-d(B,S)\geqslant |S|(|B|-\max_{1\leqslant i\leqslant k}\{|V_i\cap B|\})$. The following computations now imply Claim (∗):

$$\begin{aligned}
m-1 &\geqslant F(D,m)=\sum_{x\in V(D),d^+(x)>m}(d^+(x)-m)+\sum_{x\in V(D),d^-(x)<m}(m-d^-(x))\\
&\geqslant \sum_{x\in A,d^+(x)>m}(d^+(x)-m)+\sum_{x\in A,d^-(x)<m}(m-d^-(x))\\
&\geqslant \sum_{x\in A}(d^+(x)-m)+\sum_{x\in A}(m-d^-(x))\\
&=\sum_{x\in A}(d^+(x)-m+m-d^-(x))\\
&=\sum_{x\in A}(d(x,A\backslash\{x\})-d(A\backslash\{x\},x))+\sum_{x\in A}(d(x,B)-d(B,x))\\
&=0+d(A,B)-d(B,A)\\
&=d(A\backslash S,B)-d(B,A\backslash S)+d(S,B)-d(B,S)\\
&\geqslant 0+|S|\left(|B|-\max_{1\leqslant i\leqslant k}\{|V_i\cap B|\}\right).
\end{aligned}$$

Case 1: $|A|=2$.

Let $A=\{x,y\}$ and assume, without loss of generality, that $x\notin V^*$. Since $x=y^-$, Theorem 3.5 implies that $y\Rightarrow B$ and hence, we have $d^+(y)\geqslant |B|/2+1$ and $d^-(y)=1$. Since $F(D,m)\leqslant m-1$, this leads to $d^+(z_1)\leqslant m$ and $d^-(z_2)\geqslant m$ for every $z_1,z_2\in V(D)$, $z_2\neq y$. Again by Theorem 3.5, $d^-(x)\leqslant |B\cap V^*|+1\leqslant |B|/2+1$ and therefore we summarize $m=d^+(y)=|B|/2+1=d^-(x)$. The latter equality implies that y and exactly half the vertices of B belong to V^*. Let $R=B\cap V^*$ and $Q=B\backslash V^*$. From the above, $|R|=|Q|=|B|/2=m-1$. Clearly, $y\Rightarrow Q$. Furthermore, by Theorem 3.5, we obtain that $x\Rightarrow R$. Since $d^-(x)=|B|/2+1$ and $d^+(x)\leqslant m$, we have $Q\Rightarrow x$. Moreover,

$Q \to R$ as $d^-(r) \geqslant m$ for every $r \in R$. Since $d^+(q) \leqslant m$ for every $q \in Q$, this implies that Q is independent. Finally, as $d^-(q) \geqslant m$ for every $q \in Q$, we deduce that $R \to Q$ and therefore D is isomorphic to G'_m.

Case 2: $3 \leqslant |A| \leqslant m - 1$.

Since $m - 1 \geqslant \sum_{s \in S}(m - d^-(s))$ and $d^-(s) \leqslant |A| - 1$ for every $s \in S$, we obtain

$$0 \geqslant 1 - m + \sum_{s \in S}(m - d^-(s)) \geqslant 1 - m + |S|(m - (|A| - 1))$$

$$\geqslant 1 - m + \frac{|A|}{2}(m - (|A| - 1)) = \frac{-(|A| - 2)(|A| - m + 1)}{2}.$$

Since this leads to a contradiction if $3 \leqslant |A| \leqslant m - 2$, we consider the case when $|A| = m - 1$. Clearly, we then have equality everywhere in the above inequation. This implies that $|S| = |A|/2$ and $d^-(s) = |A| - 1$ for every $s \in S$ where the first observation leads to $S \subseteq V^*$. Therefore, $|A| - 1 = d^-(s) = d(A \backslash S, s)$ for every $s \in S$ which implies that $|S| = 1$. But then $|A| = 2$, a contradiction to $|A| \geqslant 3$.

Case 3: $|A| \geqslant m$.

By Claim $(*)$ and since $|S| \geqslant |A|/2$ and $|B| - \max_{1 \leqslant i \leqslant k}\{|V_i \cap B|\} \geqslant |B|/2$, we obtain $m - 1 \geqslant |S||B|/2 \geqslant |A||B|/4 \geqslant m|B|/4$. This is clearly a contradiction when $|B| \geqslant 4$. If $|B| = 3$, then $\max_{1 \leqslant i \leqslant k}\{|V_i \cap B|\} = 1$, and Claim $(*)$ implies that $m - 1 \geqslant |S|(3 - 1) \geqslant |A|$, a contradiction. We may therefore assume that $|B| = 2$. By considering $F'(D, m)$ instead of $F(D, m)$ (see Remark 3.2), we deduce analogously to Case 1 that D is isomorphic to G''_m. □

Theorem 4.2 yields the following corollary.

Corollary 4.3. *Let D be a semicomplete multipartite digraph and let m be a positive integer such that $F(D, m) \leqslant m - 1$. Now the following is true*:

(i) *If D is a multipartite tournament, then D is Hamiltonian.*
(ii) *If D is a semicomplete k-partite digraph with $k \neq 3$, then D is Hamiltonian.*
(iii) *If $|V(D)| \neq 2m$, then D is Hamiltonian.*
(iv) *If $d^+(x), d^-(x) \geqslant 2$ for every $x \in V(D)$, then D is Hamiltonian.*

5. Semi-partition complete digraphs

In this section a new class of semicomplete multipartite digraphs is introduced. We will present sufficient conditions for these digraphs to be Hamiltonian.

Definition 5.1. Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_k$. If $d^+(x_i, V_j), d^-(x_i, V_j) \geqslant \frac{1}{2}|V_j|$ for every vertex $x_i \in V_i$ and for every $1 \leqslant i, j \leqslant k$, $j \neq i$, then D is called a semi-partition complete digraph.

If the semi-partition complete digraph D contains no cycle of length 2, then D is called a semi-partition complete tournament.

Remark 5.2. Clearly, for semi-partition complete tournaments we have

$$d^+(x_i, V_j) = d^-(x_i, V_j) = \tfrac{1}{2}|V_j|.$$

Since we are still dealing with a selection of cycles covering the whole vertex set of the digraph, we will only consider those semi-partition complete digraphs D, whose partite sets contain at most half the vertices of D each. That is, we consider the case when $\alpha(D) \leqslant n/2$, where n denotes the order of D and $\alpha(D)$ is the cardinality of a maximum independent vertex set in D. First we ensure the existence of a 1-regular factor.

Theorem 5.3. *Let D be a semi-partition complete digraph of order n with $\alpha(D) \leqslant n/2$. Then D contains a* 1-*regular factor.*

Proof. To show this result, we use the fact that a digraph has a 1-regular factor if and only if

$$|S| \leqslant |N^+(S)| \quad \text{for all } S \subseteq V(D), \qquad (*)$$

which was proved by Ore in [6]. Hence, let S be an arbitrary subset of $V(D)$, and let $V_1, V_2, \ldots, V_k$ denote the partite sets of D.

Since $d^+(x,D), d^-(x,D) \geqslant \frac{1}{2}(n - |V_i|)$ for every vertex $x \in V_i$, $1 \leqslant i \leqslant k$, we have $d^+(x,D), d^-(x,D) \geqslant n/4$ for every vertex $x \in V(D)$. Therefore $(*)$ holds for every S satisfying $|S| \leqslant n/4$. Now let $|S| > n/4$. We consider two cases.

Case 1: All the vertices of S belong to the same partite set V_i.

In this case $|S| \leqslant n/2$. Since $|S| > n/4$, S contains more than half of the vertices of V_i and hence $d^-(x,S) \geqslant 1$ for every $x \in V(D) \backslash V_i$. It follows $|N^+(S)| = n - |V_i| \geqslant n/2 \geqslant |S|$, which verifies $(*)$.

Case 2: S contains vertices of at least two distinct partite sets.

Let $x \in S \cap V_i$ and $y \in S \cap V_j$ where $i \neq j$. Since $d^+(x, D \backslash V_i) = d^+(x,D) \geqslant \frac{1}{2}(n - |V_i|)$ and $d^+(y, V_i) \geqslant \frac{1}{2}|V_i|$, we have $|N^+(S)| \geqslant n/2$ and we are done if $|S| \leqslant n/2$.

Hence, let $|S| > n/2$. Then there exists i_0, $1 \leqslant i_0 \leqslant k$, such that $|S \cap V_{i_0}| > \frac{1}{2}|V_{i_0}|$. Analogously to Case 1 we have $N^+(S \cap V_{i_0}) = V(D) \backslash V_{i_0}$.

Subcase 2.1: Suppose that there exists j_0, $1 \leqslant j_0 \leqslant k$, $j_0 \neq i_0$, such that $|S \cap V_{j_0}| > \frac{1}{2}|V_{j_0}|$. Then, analogously, $V_{i_0} \subseteq N^+(S)$. So, altogether we have $N^+(S) = V(D)$ and $(*)$ is true.

Subcase 2.2: If there is no such j_0, then $|S| \leqslant |V_{i_0}| + \frac{1}{2}(n - |V_{i_0}|) = n/2 + \frac{1}{2}|V_{i_0}| \leqslant n - \frac{1}{2}|V_{i_0}|$. On the other hand, by the assumption, there is a vertex $x \in S \backslash V_{i_0}$ with $d^+(x, V_{i_0}) \geqslant \frac{1}{2}|V_{i_0}|$. Hence $|N^+(S)| \geqslant n - |V_{i_0}| + \frac{1}{2}|V_{i_0}| = n - \frac{1}{2}|V_{i_0}|$ and $(*)$ holds.

Since we have considered all possible cases, the theorem is proved. □

By Theorems 5.3 and 3.4, we deduce the following result for the bipartite case.

Corollary 5.4. *Let D be a strong semi-partition complete digraph of order n that consists of two partite sets. If $\alpha(D) = n/2$, then D is Hamiltonian.*

Furthermore, we can ensure the existence of a Hamiltonian cycle if D contains no cycles of length 2.

Corollary 5.5. *Let D be a semi-partition complete tournament of order n with $\alpha(D) \leqslant n/2$. Then D is Hamiltonian.*

Proof. By Theorem 5.3, D contains a minimal 1-regular factor $F = C_1 \cup C_2 \cup \cdots \cup C_t$. If $t = 1$, there is nothing to prove, hence let $t \geqslant 2$. By Remark 5.2, D is an Eulerian digraph and hence for every subset $S \subseteq V(D)$ we have $d(S, V(D)\backslash S) = d(V(D)\backslash S, S)$. Choose $S = V(C_1)$.

Assume that there exist vertices $x \in V(C_1)$ and $y \in V(C_i)$, $2 \leqslant i \leqslant t$, such that $y \rightarrow x$. It follows by Theorem 3.5 that y^+ and x^- belong to the same partite set and, since $d^+(y^+, V(C_1)) = d^-(x^-, V(D)\backslash V(C_1)) = 0$, we have $x \rightarrow y^+$ and $x^- \rightarrow y$. So whenever there is an arc yx leading from a cycle C_i, $i \neq 1$, to C_1, there are two distinct arcs leading from C_1 to C_i. Since these two arcs are different for any choice of x and y, this contradicts $d(V(C_1), V(D)\backslash V(C_1)) = d(V(D)\backslash V(C_1), V(C_1))$.

If there is no arc leading from another cycle to C_1, we clearly derive the same contradiction. □

If there are cycles of length 2 in a semi-partition complete digraph D, then the existence of a 1-regular factor is not sufficient to guarantee that D contains a Hamiltonian cycle. A slightly weaker result holds.

Theorem 5.6. *Let D be a semi-partition complete digraph of order n with $\alpha(D) \leqslant n/2$. Then D is Hamiltonian or there exist two vertex-disjoint cycles C_1 and C_2 in D such that $V(C_1) \cup V(C_2) = V(D)$ and $|V(C_1) \cap V_j| = |V(C_2) \cap V_j| = |V_j|/2$ for all $1 \leqslant j \leqslant k$, where $V_1, V_2, \ldots, V_k$ are the partite sets of D.*

Proof. By Theorem 5.3, D has a 1-regular factor $F = C_1 \cup C_2 \cup \cdots \cup C_t$, which we may assume is minimal. Then F has the properties described in Theorem 3.5. If $t = 1$, there is nothing to prove. Hence, let $t \geqslant 2$.

Now we assume that there exists i_0, $1 \leqslant i_0 \leqslant k$, such that $|V(C_1) \cap V_{i_0}| > \frac{1}{2}|V_{i_0}|$. Then, in particular, $d^+(u, C_1) \geqslant 1$ for every $u \in V(C_2)\backslash V_{i_0}$. Hence, let $x \in V(C_2)\backslash V_{i_0}$ and $y \in V(C_1) \cap V_{i_0}$ such that $x \rightarrow y$. By Theorem 3.5, $\{x^+, y^-\} \in V^*$ for a partite set V^* of D. Since y and y^- are adjacent, $V^* \neq V_{i_0}$ and hence for $x^+ \in V(C_2)\backslash V_{i_0}$ we have $d^+(x^+, C_1) \geqslant 1$. It follows $x^{++} \in V^*$, a contradiction.

Therefore, $|V(C_1) \cap V_i| \leqslant \frac{1}{2}|V_i|$ for every $1 \leqslant i \leqslant k$. Analogously, we obtain $|V(C_t) \cap V_i| \leqslant \frac{1}{2}|V_i|$ for every $1 \leqslant i \leqslant k$.

By assuming that there is j_0, $1 \leqslant j_0 \leqslant k$, such that $|V(C_1) \cap V_{j_0}| < \frac{1}{2}|V_{j_0}|$, we have $|(V(D)\backslash V(C_1)) \cap V_{j_0}| > \frac{1}{2}|V_{j_0}|$ and by using the analogous arguments we derive the same contradiction. Therefore, we obtain $|V(C_1) \cap V_i| = \frac{1}{2}|V_i|$ for all $1 \leqslant i \leqslant k$ and, analogously, $|V(C_t) \cap V_i| = \frac{1}{2}|V_i|$ for all $1 \leqslant i \leqslant k$. Hence $t = 2$ and Theorem 5.6 is proved. □

Theorem 5.6 is best possible with respect to the number of cycles in a minimal 1-regular factor, which is shown by the following example. Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_k$ such that every partite set consists of an even number of vertices. Let $A \subseteq V(D)$ be a subset of the vertex set of D containing half the vertices of each partite set (arbitrarily chosen) and let $B = V(D) \setminus A$. Furthermore, let $x \to y$ for all $x, y \in A$, where x and y belong to distinct partite sets, let $u \to v$ for all $u, v \in B$, where u and v belong to distinct partite sets, and let $x \to u$ for all $x \in A$, $u \in B$, where x and u belong to distinct partite sets. Then D is semi-partition complete and not Hamiltonian.

But obviously the following holds.

Corollary 5.7. *Let D be a semi-partition complete digraph of order n such that $\alpha(D) \leqslant n/2$. If there is a partite set of D containing an odd number of vertices, then D is Hamiltonian.*

Now we consider strong semi-partition complete digraphs with more than two partite sets. If we strengthen the condition on the cardinality of the partite sets, it is possible to ensure the existence of a Hamiltonian cycle.

Theorem 5.8. *If D is a strong semi-partition complete digraph of order n with $\alpha(D) < n/2$, then D is Hamiltonian.*

Proof. Assume that D is not Hamiltonian. By Theorem 5.6, D has a minimal 1-regular factor $F = C_1 \cup C_2$ which has the properties described in Theorem 3.5, where C_1 and C_2 contain exactly half the vertices of every partite set of D. Since D is strong, there are vertices $u_2 \in V(C_2)$ and $u_1 \in V(C_1)$ such that $u_2 \to u_1$. By Theorem 3.5, u_2^+ and u_1^- belong to the same partite set of D, say V^*. Therefore u_1^{--} and u_2^+ are adjacent and it is easy to see that $u_1^{--} \to u_2^+$. By connecting C_1 and C_2 with the arcs u_2u_1 and $u_1^{--}u_2^+$, we obtain a cycle of length $n-1$ in D which does not contain the vertex u_1^-.

Again by Theorem 3.5, $d^-(u_1^-, C_2) = 0$. Since C_1 and C_2 contain half the vertices of every partite set of D, we deduce $x \to u_1^-$ for every $x \in V(C_1) \setminus V^*$. Furthermore, $d^-(x, C_2 \cap V^*) = 0$ for every $x \in V(C_1) \setminus V^*$ and hence $V(C_1) \cap V^* \to V(C_1) \setminus V^*$. In particular, $u_1^- \to x$ for every $x \in V(C_1) \setminus V^*$. Assume that there is a vertex $x \in V(C_1)$ such that $\{x, x^+\} \cap V^* = \emptyset$. Since $x \to u_1^- \to x^+$, the vertex u_1^- can be included in the $(n-1)$-cycle mentioned above, a contradiction to the assumption. Hence $|V(C_1) \cap V^*| = |V(C_1)|/2$.

Analogously, $u_1^- \to u_2^{++}$ and hence there is a cycle of length $n-1$ in D which does not contain u_2^+. Since $d^+(u_2^+, C_1) = 0$, we conclude with the same arguments that $|V(C_2) \cap V^*| = |V(C_2)|/2$ and therefore $|V^*| = n/2$, a contradiction. □

The following example shows that Theorem 5.8 is best possible. Let D be a digraph of order n with the partite sets $V_1, V_2, \ldots, V_k$ $(k \geqslant 3)$ where the cardinality of every partite set is even and $|V_1| = n/2$ and $|V_i| < n/2$ for every $2 \leqslant i \leqslant k$. Let $A \subset V(D)$ such that A contains half the vertices of every partite set and let $B = V(D) \setminus A$. Now let $A \setminus V_1 \rightarrow A \cap V_1 \rightarrow V(D) \setminus V_1 \rightarrow B \cap V_1 \rightarrow B \setminus V_1$. Furthermore, let the arcs connecting vertices of $V(D) \setminus V_1$ be chosen such that D is semi-partition complete and strong (for example, let $V(D) \setminus V_1$ induce a complete $(k-1)$-partite digraph). Then D is not Hamiltonian since in a Hamiltonian cycle, every second vertex would belong to V_1 which is clearly not possible.

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