Journal of Combinatorial Theory, Series B **78**, 185–197 (2000)
doi:10.1006/jctb.1999.1940, available online at http://www.idealibrary.com on IDEAL®

Maximum Balanced Flow in a Network

Haggai Ilani

Department of Mathematics and Computer Science, BG University of the Negev, Beer Sheva 84105, Israel

and

Michael Lomonosov

Department of Mathematics and Computer Science, BG University of the Negev, Beer Sheva 84105, Israel; and Labo. LEIBNIZ, IMAG, 46 Avenue Felix Viallet, 38031 Grenoble Cedex 1, France

Received May 27, 1997

We pose a new network flow problem and solve it by reducing to the b-matching problem. The result has application to integer multiflow optimization. © 2000 Academic Press

1. INTRODUCTION

In this paper we formulate a version of the maximum network flow problem, and solve it by reducing to the **b**-matching problem in a graph. This reduction provides both a max-min theorem, by translating the Tutte–Berge formula [13, 14, 1], and a strongly polynomial solution, by applying a **b**-matching algorithm (see [5, p. 187]). We show in the following paper [7] that this result provides a solution to a class of integer multiflow optimization problems briefly described in Remark 4 below.

1.1. *Main Result*

Throughout, *graph* means an undirected multigraph without loops, and we deal only with Eulerian graphs.

Let G be a graph with the vertex-set $V \cup \{s\}$, where s is considered as the *sink*. By a *flow* in such graph we mean a collection of edge-disjoint paths from V to s. The *degree* $d_{\mathscr{F}}(v)$ of a flow $\mathscr{F}$ in a vertex $v \in V$ is the number of paths of $\mathscr{F}$ having an end in v. A flow whose degrees vanish outside a subset $T \subseteq V$ is referred to as (T, s)-flow, and the vertices of T are called *sources*. A flow is *Eulerian* if its degrees are all even. Given a set U

0095-8956/00 $35.00

of disjoint pairs of vertices of V (shortly, a partial pairing of V), a flow $\mathscr{F}$ is called *balanced* if the equality $d_{\mathscr{F}}(v') = d_{\mathscr{F}}(v'')$ holds for each pair $(v', v'') \in U$.

Consider the following problem.

Problem 0 (Maximum Balanced Eulerian Flow). Given an Eulerian graph G with the vertex-set $V \cup \{s\}$, and a subset $T \subseteq V$ partitioned into pairs, find a maximum balanced Eulerian (T, s)-flow.

The example of a triangle with the verticies t', t'', and s, and U consisting of (t', t'') shows that the requirement of the flow being Eulerian is restrictive even for Eulerian graphs.

Notations. For $X \subseteq V \cup \{s\}$, we denote by $E(X)$ the set of edges of G spanned by X (i.e., having both ends in X); the edge-set of the subgraph $G - s$ is denoted by E.

For sets X and Y of vertices, the number of edges between $X \setminus Y$ and $Y \setminus X$ is denoted by $d(X, Y)$; we denote by $d(v)$ the degree of a vertex v, and by $d(X)$ the number of edges with exactly one end in a set X of vertices. Similarly, given a function m on the edges, we denote by $d_m(v)$ the sum of its values, over the edges incident to a vertex v.

When a function (or "vector") f defined on some set is extended to an *additive* function of subsets, values of this set-function will be written as $f[X]$. According to this rule, for example,

$$d[X] := \sum_{v \in X} d(v) = d(X) + 2\,|E(X)|. \tag{1}$$

DEFINITION. Given a partial pairing U of V, let us call a pair (X, Y) of sets of vertices *sandwich* if $X \cap Y = \emptyset$, $s \in X$, and any pair of U having a member in X has the other one in Y.

MAIN THEOREM. *Let G be an Eulerian graph with the vertex-set $V \cup \{s\}$, U be a partial pairing of V, and T denote the union of the pairs. Then*

$$\max |\mathscr{F}| = \min(d(X) + d(Y) - 2\omega), \tag{2}$$

the maximum over the balanced Eulerian (T, s)-flows, and the minimum over the sandwiches (X, Y); here ω is the number of odd components of $G + U - (X \cup Y)$.

The *parity* of a component with the vertex-set C is defined as the parity of the integer $\frac{1}{2}(d(C \cup Y) - d(Y))$.

Our problem actually involves only flow degrees. In order to completely eliminate the flows, let us call an integer vector $\mathbf{x} = (x(v) : v \in V)$ *feasible* if there is a flow in G with the degrees $2x(v)$, $v \in V$. By the Gale theorem [4] (see also [2]), $\mathbf{x}$ is feasible iff it satisfies

$$2x[A] \leqslant \lambda(A) := \min\{d(X) \colon A \subseteq X \subseteq V\}, \qquad A \subseteq V. \tag{3}$$

A feasible vector $\mathbf{x}$ *spans* a subset $A \subseteq V$ if $2x[A] = \lambda(A)$, and is called *base* if it spans V. Problem 0 can now be stated in the following equivalent form.

Problem 1. Maximize

$$\beta(\mathbf{x}) := \sum_{(t', t'') \in U} \min\{x(t'), x(t'')\} \qquad \left(= \tfrac{1}{4} |\mathscr{F}|\right) \tag{4}$$

over the feasible vectors $\mathbf{x}$.

It is well known that any network flow is degree-majorated by a maximum flow, and one easily checks that the same is true for Eulerian flows in Eulerian networks. Therefore there always exists a base solving Problem 1. Our Main Theorem is equivalent to the relation

$$\max \beta(\mathbf{x}) = \min \frac{d(X) + d(Y) - 2\omega}{4}, \tag{5}$$

with X, Y, and ω as above, and the maximum taken over the bases.

1.2. *Remarks*

Here we briefly discuss the place of Problem 0 in the field of network flows. The remarks below reveal that Problem 0 majorates some of its apparent extensions; we also mention certain cases of integer multiflow optimization majorated by Problem 0.

Remark 1. It seems natural to permit unpaired sources too, that is to consider the source-set T as consisting of a set K and a partial pairing U of $V \backslash K$. This is equivalent to maximizing the function

$$\beta_1(\mathbf{x}) := x[K] + 2 \sum_{(t', t'') \in U} \min\{x(t'), x(t'')\} \tag{6}$$

over the feasible vectors $\mathbf{x}$. This version is reducible to Problem 0, by taking two disjoint copies of G, merging their sinks into one, and introducing the pairing of the unified source-set, consisting of the respective copies of U and the pairs (v_1, v_2) matching the copies of $v \in K$.

Remark 2. Problem 0 contains its plain generalization, when neither G is assumed, nor the flow is required to be Eulerian. Indeed, let the graph G be arbitrary. Double each of its edges, and denote by G' the Eulerian graph thus obtained. Let $\mathbf{x}$ be a solution of Problem 1 for G', with the same U, and (X, Y) be a sandwich providing the equality in (5). Since $\mathbf{x}$ satisfies the Gale condition $x[A] \leqslant \lambda(A)$, $A \subseteq V$, for the initial G, there is a flow $\mathscr{F}$ in G with the degree vector $\mathbf{x}$. It is easy to check that $\mathscr{F}$ solves the plain version of Problem 1, by comparing it with (X, Y).

In the plain case we have $\max |\mathscr{F}| = \min(d(X) + d(Y) - \omega(X, Y))$, the minimum over the sandwiches (X, Y), where the *parity* of a component with the vertex-set C coincides, by definition, with the parity of $d(C)$.

Remark 3. It might seem tempting to extend the problem, by balancing a flow with respect to an arbitrary graph $S = (T, U)$: it is then natural to call $\mathscr{F}$ balanced with respect to S if there exists an integer nonnegative vector $\alpha = (\alpha(u): u \in U)$ such that $d_{\mathscr{F}}(t) = d_{\alpha}(t)$ for each $t \in T$ incident to U. If no member of T is isolated in S, this problem may be interpreted as maximization of S-flow (see, e.g., [8, 10, 3, 6] and also Remark 4) under the additional condition that all its paths (whose self-intersections cannot now be eliminated) pass through s. The below solution implies that such S-flow problem is tractable for an arbitrary graph S.

It is easy, however, to see that, in contrast to packing S-paths in general, such extension yields nothing new in our case. Indeed, any S-flow maximization problem may be formulated in terms of pairing, by assigning a pair t'_u, t''_u of new sources to each edge $u = (t', t'') \in U$ and connecting them to t' and t'' by a large enough number of edges. For the new source-set we take $T' := \{t'_u, t''_u : u \in U\}$, and put $U' := \{(t'_u, t''_u) : u \in U\}$. There is the obvious one-to-one correspondence between the solutions of both versions of the problem.

It makes sense to try various schemes S if some of them can be tractable while the pairing does not. In the case of balanced flow, however, the pairing is both universal and tractable.

Remark 4. In the paper [7] we show that Problem 1 majorates two important integer multiflow optimization problems which we only briefly describe here. Let, again, G be an Eulerian graph with a distinguished subset T of vertices called *terminals*. A T-path in G is a path whose ends are distinct terminals, and a *multiflow* in the network (G, T) is a collection of edge-disjoint T-paths.

For a proper subset A of T, by $(A, T\backslash A)$-*flow* we mean a collection of edge-disjoint paths having one end in A and the other in $T\backslash A$. A multiflow in (G, T) *locks* A if it contains a maximum $(A, T\backslash A)$-flow. If now $\mathscr{H}$ is a hypergraph with the vertex-set T, we say that a multiflow *locks* $\mathscr{H}$ if it locks each $A \in \mathscr{H}$. It is known [8, 10] that a hypergraph is lockable in any Eulerian network (G, T) iff it contains no 3-cross. [Subsets $A, B \subset T$ are called *crossing* if the four atoms, $A\backslash B$, $B\backslash A$, $A \cap B$ and $T\backslash(A \cup B)$, are non-empty; subsets A, B and C form a 3-*cross* if any two of them are crossing, like the subsets $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ of $T = \{1, 2, 3, 4\}$.]

Problem A (Minimum Locking). Given an Eulerian network (G, T) and a 3-cross free hypergraph $\mathscr{H}$, by how few edge-disjoint T-paths can $\mathscr{H}$ be locked?

Further, consider a graph S on T, without loops and isolated vertices, and let us call it *scheme*. An *S-path* is a path whose ends are terminals adjacent in S.

Problem B (Packing S-Paths). What is the maximal number of edge-disjoint S-paths in an Eulerian network (G, T)?

In [7] it is proved that (1) Problem 0 majorates A; and (2) Problem A majorates B for any scheme S whose maximal stable sets form no 3-cross, or, equivalently, whose complement is the line graph of a triangle-free multigraph.

Remark 5. It may be worth noticing that Problem 1 and its reduction to the $\mathbf{b}$-matching maximization can be accurately expressed in terms of two polymatroids on the set V. Let them be denoted by $\mathbf{P}$ and $\mathbf{Q}$: independent in $\mathbf{P}$ are integer nonnegative vectors satisfying the conditions (3), and the independent vectors of $\mathbf{Q}$ are the degree-vectors of $\mathbf{b}$-matchings in the bipartite graph obtained by inserting a 2-valent vertex into each edge of $G-s$ (see Subsection 2.2 for details). Problem 1 may be considered as a polymatroid version of the Matroid Parity problem [11] in $\mathbf{P}$. Theorem 2.3 states that $\mathbf{P}$ and $\mathbf{Q}$ are dual with respect to the function $\frac{1}{2}\mathbf{d}$; from this point of view, the reduction of Problem 1 to $\mathbf{b}$-matching maximization in the graph $\tilde{G}+U$ is similar to the construction of Lawler *et al.* [9].

2. REDUCTION TO MATCHINGS

Throughout, an Eulerian graph G with the vertex-set $V \cup \{s\}$ is fixed, and E denotes the edge-set of the subgraph $G-s$. In the sequel we deal in parallel with flows in G and matchings in some other graph; to avoid confusion, we speak of the latter one in terms of *nodes* and *links*, retaining *vertex* and *edge* to the initial G.

2.1. *Matchings*

Consider a graph with the node-set N and the set L of links, and let $\mathbf{b} \in \mathbb{Z}_+^N$. A function m: $L \to \mathbb{Z}_+$ is called $\mathbf{b}$-*matching* if $d_m(v) \leqslant b(v)$ for each node v. The *size* $\|m\|$ of a $\mathbf{b}$-matching m is the sum of its values; a $\mathbf{b}$-matching of the maximal size is called *maximum*. We say that m *spans* a subgraph (N', L') if its restriction onto L' is a maximum $\mathbf{b}$-matching in the subgraph.

We use the following fundamental facts (see, e.g., [11, 12]). The first of them easily follows from the augmenting path theorem for usual bipartite matchings.

CLAIM 2.1. A **b**-matching m in a bipartite graph is not maximum iff there exists an augmenting path, that is an odd path $P=(v_0, v_1, ..., v_{2k+1})$, with the ends v_0 and v_{2k+1} unsaturated by m and $m(v_{2i-1}, v_{2i})>0$, $i=1, ..., k$.

The second is the Tutte–Berge formula for the maximum size of a **b**-matching in a graph.

THEOREM 2.2 (Tutte [13, 14], Berge [1]). *The maximal size of a* **b**-*matching is equal to*

$$\min_{Z \subseteq N} \frac{b[N]+b[Z]-b[I]-\omega(Z)}{2}, \tag{7}$$

where I is the set of isolated vertices of $H-Z$ and $\omega(Z)$ is the number of odd non-trivial components of $H-Z$.

We call a component with the node-set C *non-trivial* if $|C|>1$; its *parity* is the parity of the integer $b[C]$. A set Z achieving the minimum in (7) is called a *Tutte set*.

2.2. *Reduction Theorem*

Given an Eulerian graph G with a sink s, let $\tilde{G}$ denote the bipartite graph with the node-set $N := V \cup E$ in which each $e=(v', v'') \in E$ is linked just to v' and v''. Recall that E is the edge-set of the subgraph $G-s$. Loosely speaking, $\tilde{G}$ is the subgraph $G-s$ subdivided by inserting exactly one two-valent node into each edge. Define a vector $\mathbf{b} \in \mathbb{Z}_+^N$ by assigning

$$b(v) := \tfrac{1}{2}\, d(v) \quad \text{for } v \in V, \qquad \text{and} \qquad b(e) := 1 \quad \text{for } e \in E. \tag{8}$$

(Recall that by $d(v)$ we denote the degree of a vertex v in G.)

From this point, **b** means only the vector defined by (8); therefore we usually omit its indication and use *matching* as an abbreviation for **b**-matching. Clearly, the size of a matching in $\tilde{G}$ cannot exceed $|E|$.

Let us call *normal* any (partial) orientation of G with the zero outdegree in s and the indegree at most $\frac{1}{2}d(v)$ in each vertex $v \in V$.

THEOREM 2.3. *A vector $\mathbf{x} \in \mathbb{Z}_+^V$ is a base if and only if $\tilde{G}$ has a maximum* **b**-*matching $\tilde{m}$ satisfying*

$$\mathbf{x} + \mathbf{d}_{\tilde{m}} = \tfrac{1}{2}\, \mathbf{d}. \tag{9}$$

Here $\mathbf{d}_{\tilde{m}} = (d_{\tilde{m}}(v) \colon v \in V)$, and (9) is a vector relation in $\mathbb{Z}_+^V$.

Proof. Relation between Eulerian flows in G and matchings in $\tilde{G}$ is based on the fact that both are generated by normal orientations of G. There is an obvious correspondence between such orientations and matchings in $\tilde{G}$: given a matching $\tilde{m}$, we direct an edge $e=(v, v') \in E$ towards v' iff $\tilde{m}(e, v')=1$, and an edge incident to s towards s; due to the matching constraints, the obtained orientation is normal. This correspondence is clearly one-to-one because in the reverse way a normal orientation generates a matching in $\tilde{G}$ whose degrees are bounded by 1 in the nodes $e \in E$ and by $\frac{1}{2} d(v)$ in the nodes $v \in V$.

Connection between normal orientations and Eulerian flows is less straightforward, and we first illustrate it by a construction which, in particular, reveals that the maximum size of a matching in $\tilde{G}$ equals $|E|$. Consider any Eulerian orientation of G, and let $C_1, ..., C_k$, $k=\frac{1}{2} d(s)$, be edge-disjoint directed circuits of this orientation, passing through s. Let vertices $v_i \in V(C_i - s)$ (not necessarily distinct) be chosen, and let us reverse in each C_i the direction of the segment sC_iv_i coming out from s. The new orientation of G is normal, and the former circuits C_i form now an Eulerian maximum (V, s)-flow. Since the orientation is total, the generated matching in $\tilde{G}$ has the cardinality $|E|$.

Let us return to the proof. The below arguments actually show that every Eulerian maximum flow and maximum matching are obtainable in the described way.

(I) *Only if.* Let $\mathbf{x}$ be an arbitrary base; then there is a flow $\mathscr{F}$, $|\mathscr{F}|=d(s)$, having the degrees $2x(v)$, $v \in V$. Let E_0 denote the set of edges of G not used by $\mathscr{F}$; the edge-induced subgraph $G(E_0)$ is clearly Eulerian. Let us direct the paths of $\mathscr{F}$ towards s and choose an arbitrary Eulerian orientation of $G(E_0)$. The orientation of G thus obtained is total and normal; the matching $\tilde{m}$ generated in $\tilde{G}$ by this orientation is, therefore, maximum.

In order to check (9), consider a vertex $v \in V$. The starting edges of the $2x(v)$ paths of $\mathscr{F}$ having the end in v are directed outwards, and exactly half of the other $d(v)-2x(v)$ incident edges are directed towards v; thus, $d_{\tilde{m}}(v)=\frac{1}{2}(d(v)-2x(v))$, as required.

(II) *If.* Let, conversely, $\tilde{m}$ be a maximum matching in $\tilde{G}$. Since its size equals $|E|$, the corresponding orientation of $G-s$ is total. We extend it to a normal orientation of the entire G by directing the edges incident to s towards s. Indeed, the matching constraints imply that the outdegree of each vertex, except s, is not less than the indegree. Therefore G is decomposable into a number of edge-disjoint directed circuits and inclusion-maximal directed paths. Clearly, these paths form an Eulerian maximum (V, s)-flow. So, if by $2x(v)$ we denote the number of these paths starting in a vertex $v \in V$ then the vector $\mathbf{x} := (x(v) \colon v \in V)$ is a base.

We have $2x(v) = \text{outdegree}(v) - \text{indegree}(v) = d(v) - 2d_{\tilde{m}}(v)$, as required. ∎

Let us now append the pairs of U as links to the graph $\tilde{G}$ retaining the constraints vector $\mathbf{b}$ defined by (8). There exists a simple relation between the size of matchings in $\tilde{G} + U$ and the quantity $\beta(\mathbf{x})$ of Problem 1.

For a matching m in the graph $\tilde{G} + U$, let $\tilde{m}$ and m_U denote its restrictions to $\tilde{G}$ and U respectively. Let us confine ourselves to matchings which, first, span $\tilde{G}$ and, second, have a maximal m_U. Such a matching is uniquely defined by choosing for $\tilde{m}$ a maximum matching in $\tilde{G}$ and assigning to each $u = (t', t'') \in U$ the value

$$m(u) := \min\{b(t') - d_{\tilde{m}}(t'), b(t'') - d_{\tilde{m}}(t'')\}. \tag{10}$$

Since $b(v) = \frac{1}{2}d(v)$ for vertices $v \in V$, Theorem 2.3 implies a one-to-one correspondence between the bases $\mathbf{x} \in \mathbb{Z}_+^V$ and the vectors $m_U = (m(u): u \in U)$ of the maximum matchings m spanning $\tilde{G}$, so that

$$m[U] = \sum_{(t', t'') \in U} \min\{x(t'), x(t'')\} = \beta(\mathbf{x}) \tag{11}$$

(cf. (4)). Indeed, given such a matching m, the differences

$$x(v) := b(v) - d_{\tilde{m}}(v), \qquad v \in V, \tag{12}$$

form a base, and conversely, given a base $\mathbf{x}$ the difference $\mathbf{b} - \mathbf{x}$ forms the degree vector of a maximum matching in $\tilde{G}$ whose unique extension onto U is given by (10). By the equality (10), Problem 1 is equivalent to maximizing $m[U]$, or, which is the same, maximizing $m[U] + |E| = m[U] + \|\tilde{m}\| = \|m\|$, over the matchings in $\tilde{G} + U$ spanning $\tilde{G}$.

The requirement that a matching span the subgraph $\tilde{G}$ does not, however, affect its maximum size (but only the balance between $m[U]$ and $\|\tilde{m}\|$), as the following statement shows.

CLAIM 2.4. *There exists a maximum* **b**-*matching in* $\tilde{G} + U$ *which spans* $\tilde{G}$.

Proof. Let m be a maximum matching in $\tilde{G} + U$ having the greatest possible value of $\|\tilde{m}\|$. If m does not span $\tilde{G}$ then, by Claim 2.1, the graph $\tilde{G}$ has an augmenting path for the matching $\tilde{m}$, say P. Since the sets V and E form a bipartition of $\tilde{G}$, P has exactly one end in V, say v. This node is unsaturated by $\tilde{m}$ (by the definition of augmenting path) and saturated by m, for otherwise the augmentation of $\tilde{m}$ along P would augment m too. This means that v belongs to a pair $u = (v, v') \in U$ having $m(u) > 0$.

Let us augment $\tilde{m}$ along P and decrease $m(u)$, both by 1. The new maximum matching, m_1, has $\|\tilde{m}_1\| > \|\tilde{m}\|$, contradiction. ∎

Thus, the following intermediate result is established.

THEOREM 2.5 (Reduction Theorem). *Let the source-set T consist of a set U of disjont pairs, and μ denote the maximum size of* **b**-*matching in $\tilde{G}+U$. Then*

(i) *the maximum size of a balanced Eulerian (T, s)-flow equals* $4(\mu-|E|)$; *and*

(ii) *if m is a maximum* **b**-*matching in $\tilde{G}+U$ spanning $\tilde{G}$ then any decomposition of the corresponding normal orientation of G into directed circuits and maximal directed paths contains a maximum balanced Eulerian (T, s)-flow.*

3. PROOF OF MAIN THEOREM

Here again N is the node-set of the graph $\tilde{G}+U$, that is $N=V\cup E$. Recall that given a subset $Z\subseteq N$, we denote by I the set of isolated nodes and by $\omega(Z)$ the number of odd non-trivial components of the subgraph $\tilde{G}+U-Z$. We will use here the notation

$$f(Z):=2(b[N]+b[Z]-b[I]-2\,|E|-\omega(Z)).$$

For a set of nodes A, let A_V and A_E denote the intersections $A\cap V$ and $A\cap E$ respectively. By the definition (8) of the vector **b**,

$$b[A]=\tfrac{1}{2}\,d(A_V)+|E(A_V)|+|A_E|. \tag{13}$$

By the assertion (i) of Theorem 2.5 and the Tutte–Berge formula (see Theorem 2.2), the inequality

$$|\mathscr{F}|\leqslant f(Z) \tag{14}$$

holds for every Eulerian balanced (T, s)-flow $\mathscr{F}$ and any set of nodes Z, and there exist $\mathscr{F}$ (a maximum balanced (T, s)-flow) and Z (a Tutte set) providing the equality. We are to express $f(Z)$ in terms of the graph G and the pairing U.

(I) Consider relation between sets of nodes in $\tilde{G}+U$ and sandwiches in $G+U$.

STATEMENT (I.1). *For an arbitrary set Z of nodes of $\tilde{G}+U$, the sets $X=I_V\cup\{s\}$ and $Y=Z_V$ form a sandwich (X, Y) in G.*

Indeed, a vertex $v \in X \backslash \{s\}$ is an isolated node of $\tilde{G} + U - Z$, so that all its neighbours belong to Z. In particular, if v participates in a pair $(v, v') \in U$ then its mate v' should belong to $Z_V = Y$.

A given sandwich (X, Y) may, however, be generated in the above way by various sets of nodes. Each such set has $Z_V = Y$, and its E-part is characterised by the following condition: for any vertex $v \in V$, the set of incident edges of $G - s$ is contained by Z_E iff $v \in X$. We canonize the inclusion-minimal sets of this form, by adopting the following

DEFINITION. A set Z of nodes will be called *regular* if Z_E coincides with the set of edges of $G - s$ incident to I_V.

Thus, any sandwich in $G + U$ is generated by a unique regular set of nodes. Moreover, we have the following property

STATEMENT (I.2). *If Z and Z' generate the same sandwich and $Z'_E \subset Z_E$ then $f(Z') \leqslant f(Z)$.*

Proof. Suppose Z is not regular, so that there is an edge $e = (a, b) \in Z_E$ with $\{a, b\} \cap I = \varnothing$. It suffices to prove the inequality for the set $Z' := Z \backslash \{e\}$. Indeed, let I' denote the set of isolated nodes of $\tilde{G} + U - Z'$. Since $b[Z'] = b[Z] - 1$, we are only to check that $b[I'] + \omega(Z') \geqslant b[I] + \omega(Z) - 1$. Consider the possible locations of the ends of e.

Case 1. a, b belong to the same non-trivial component of $\tilde{G} + U - Z$, say C. In $\tilde{G} + U - Z'$, it is transformed into the component $C' := C \cup \{e\}$, so that $\omega(Z') \geqslant \omega(Z) - 1$. Since $I' = I$, the required relation holds.

Case 2. a, b belong to distinct nontrivial components, C_1 and C_2. In $\tilde{G} + U - Z'$, they are unified into the component $C' := C_1 \cup C_2 \cup \{e\}$. Again, we have $\omega(Z') \geqslant \omega(Z) - 1$ (the worst is the case when just one of C_1, C_2 is odd) and $I' = I$.

Case 3. $a \in Z_V$ and b belongs to a component C of $\tilde{G} + U - Z$. Then C is transformed into $C' \cup \{e\}$, so that $\omega(Z') \geqslant \omega(Z) - 1$, while $I' = I$.

Case 4. $a, b \in Z_V$. The graph $\tilde{G} + U - Z'$ has the same non-trivial components (so that $\omega(Z') = \omega(Z)$), and $I' = I \cup \{e\}$. The required relation obviously holds.

The assertion is proved. ∎

Summarizing, let us state the following implicit characterization of maximum Eulerian balanced flows in terms of sandwiches.

STATEMENT (I.3). *The relations* $X = I_V \cup \{s\}$ *and* $Y = Z_V$ *establish a one-to-one correspondence between the sandwiches in* $G + U$ *and the regular sets of nodes, and*

$$\max |\mathscr{F}| = \min f(Z),$$

the minimum over the regular sets $Z \subset N$.

(II) It remains to interpret $f(Z)$; this is done in the statements (II.1)–(II.3) below.

STATEMENT (II.1). *If* Z *is a regular set of nodes and* (X, Y) *is the corresponding sandwich then*

$$f(Z) = d(X) + d(Y) - 2\omega(Z).$$

Proof. First, by (1), the term independent of Z equals

$$2(b[N] - 2\,|E|) = d[V] - 2\,|E| = d(V) = d(s).$$

Note, further, that Z being regular implies the equality $I_E = E(Z_V)$ (cf. (I.2), Case 4 of the proof). Therefore, subtracting the expressions

$$2b[Z] = d(Z_V) + 2\,|E(Z_V)| + 2\,|Z_E| \quad \text{and}$$
$$2b[I] = d(I_V) + 2\,|E(I_V)| + 2\,|I_E|$$

(cf. (13)) we obtain

$$2(b[Z] - b[I]) = d(Z_V) + 2\,|Z_E| - d(I_V) - 2\,|E(I_V)|. \tag{15}$$

By the definition of regularity, Z_E is the set of edges of $G - s$ incident to I_V, so that

$$|Z_E| = |E(I_V)| + d(I_V) - d(I_V, s), \tag{16}$$

because $d(\cdot)$ counts edges of the entire graph G while E is the edge-set of $G - s$. Therefore, we obtain from (15)

$$2(b[Z] - b[I]) = d(I_V) - 2d(I_V, s) + d(Z_V),$$

whence

$$\begin{aligned} f(Z) &= [d(s) + d(I_V) - d(I_V, s)] + d(Z_V) - 2\omega(Z) \\ &= d(I_V \cup \{s\}) + d(Z_V) - 2\omega(Z), \end{aligned}$$

as required. ∎

It remains to find connection between the non-trivial components of $\tilde{G}+U-Z$ and the components of $G+U-(X\cup Y)$, and check preserving the component parity. This will imply the equality $\omega(Z)=\omega$, thus completing the proof of Main Theorem.

When speaking of a connectivity component we always mean its vertex-set (or node-set).

STATEMENT (II.2). *Let Z be a regular set of nodes, and (X, Y) be the corresponding sandwich. A set of nodes C is a non-trivial component of $\tilde{G}+U-Z$ if and only if C_V is a component of $G+U-(X\cup Y)$ and C_E consists of $E(C_V)$ and the edges between C_V and Y.*

Proof. Let C be a non-trivial component of $\tilde{G}+U-Z$. Since a component is connected, any edge $e\in E$ belonging to C has at least one end in C_V. On the other hand, an edge incident to C_V belongs to either C or Z_E. By the regularity of Z, each edge in Z_E is incident to I_V; so, the set of edges $E(C_V)$ and the edges between C_V and Z_V should be in C. This also means that in the graph $G+U$ the set C_V is adjacent only to $X=I_V\cup\{s\}$ and $Y=Z_V$, so that C_V is a union of components of $G+U-(X\cup Y)$.

Conversely, a component of $G+U-(X\cup Y)$, say D, is adjacent only to X and Y. Therefore the set of nodes C consisting of D, $E(D)$ and the edges between D and Y is adjacent in $\tilde{G}+U$ only to Z_V and Z_E, so that C is a union of components of $\tilde{G}+U-Z$. These components are non-trivial because $D\cap X=\varnothing$. ∎

STATEMENT (II.3). *A non-trivial component C of $\tilde{G}+U-Z$ and the corresponding component C_V of $G+U-(X\cup Y)$ have the same parity.*

Indeed, we have by (13)

$$b[C]=\tfrac{1}{2}d(C_V)+|E(C_V)|+|C_E|=\tfrac{1}{2}d(C_V, X)+\tfrac{3}{2}d(C_V, Y) \qquad \bmod 2$$

which is easily seen to coincide modulo 2 with

$$\tfrac{1}{2}(d(C_V, X)-d(C_V, Y))=\tfrac{1}{2}(d(C_V\cup Y)-d(Y)).$$

Thus, $\omega(Z)=\omega$. This completes the proof of Main Theorem.

ACKNOWLEDGMENTS

The authors appreciate stimulating discussions with András Sebő and his valuable comments and are grateful to the referee who pointed out a serious omission in the proof of the Main Theorem.

REFERENCES

1. C. Berge, Sur le couplage maximum d'un graph, *C. R. Acad. Sci. Paris* **247** (1958), 258–259.
2. L. R. Ford, Jr., and D. R. Fulkerson, "Flows in Networks," Princeton Univ. Press, Princeton, NJ, 1962.
3. A. Frank, A. Karzanov, and A. Sebő, On integer multiflow maximization, *SIAM J. Discrete Math*. **10**, No. 1 (1997).
4. D. Gale, A theorem on flows in networks, *Pacific J. Math*. **7** (1957), 1073–1082.
5. A. M. H. Gerards, Matching, *in* "Handbooks in OR and MS" (M. O. Ball *et al*., Eds.), Vol. 7, pp. 135–224, 1995.
6. H. Ilani, E. Korach, and M. Lomonosov, A new max-min theorem on packing paths, Rapport de recherche, Labo. LEIBNIZ, IMAG, RR 970-I-, Fevrier 1997.
7. H. Ilani, E. Korach, and M. Lomonosov, On extremal multiflows, *J. Combin. Theory Ser. B*, in press.
8. A. Karzanov and M. Lomonosov, Systems of flows in undirected networks, *in* "Mathematical Programming" (O. I. Larichev, Ed.), Institute for Systems Studies, Issue 1, Moscow, 1978. [in Russian]
9. E. L. Lawler, Po Tong, and V. V. Vazirani, Solving the weighted parity problem for gammoids by reduction to graphic matching, *in* "Progress in Combinatorial Optimization" (W. R. Pulleyblank, Ed.), pp. 363–374, 1984.
10. M. Lomonosov, Combinatorial approaches to multiflow problems, *Appl. Discrete Math*. **11**, No. 1 (1985), 1–93.
11. L. Lovász and M. D. Plummer, "Matching Theory," Akad. Kiadó, Budapest, 1986.
12. A. Schrijver, Min-max results in combinatorial optimization, *in* "Mathematical Programming, the State of the Art: Bonn, 1982" (A. Bachem, M. Grötschel, and B. Korte, Eds.), pp. 439–500, Springer-Verlag, Berlin, 1983.
13. W. T. Tutte, The factors of graphs, *Canad. J. Math*. **4** (1952), 314–328.
14. W. T. Tutte, A short proof of the factor theorem for finite graphs, *Canad. J. Math*. **6** (1954), 347–352.