Canonical Lévy process and Malliavin calculus

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Received 17 August 2005; received in revised form 12 June 2006; accepted 20 June 2006
Available online 13 July 2006

Abstract

A suitable canonical Lévy process is constructed in order to study a Malliavin calculus based on a chaotic representation property of Lévy processes proved by Itô using multiple two-parameter integrals. In this setup, the two-parameter derivative $D_{t,x}$ is studied, depending on whether $x = 0$ or $x \neq 0$; in the first case, we prove a chain rule; in the second case, a formula by trajectories.

Keywords: Lévy processes; Malliavin calculus; Skorohod integral

1. Introduction

In general, a Lévy process does not have the property of chaotic representation of the Brownian motion and the Poisson process, then the usual methodology to define a Malliavin derivative $D_t$ through the chaos expansion of random variables (see, for example, Nualart [14]) does not work. However, it is possible to develop a Malliavin calculus for processes with jumps following different procedures. The first studies were undertaken by Bismut [4] and Bichteler et al. [3], but those works were oriented towards the existence and smoothness of the density of the solution of a stochastic differential equation with jumps, and their results need some regularity conditions on the Lévy measure, and exclude, for example, the Poisson process and the compound Poisson process with a discrete distribution of jumps. A different approach is to use a chaotic decomposition of $L^2(\Omega)$ proved by Itô [9] in terms of multiple two-parameter integrals with respect to a certain random measure: this gives a Fock space structure and a two-parameter annihilation...
operator (Malliavin derivative) $D_{t,x}$ can be defined, as can a creation operator (Skorohod integral), see Bentham et al. [2], Øksendal and Proske [19], Di Nunno et al. [7,8], Løkka [13] (all these authors use a different random measure from Itô [9]); see also Lee and Shih [10,11] for a pure white noise approach (with the Itô random measure), and Privault [22] in a context of quantum probability. A third approach is to start with a kind of chaotic property for a Lévy process (under certain integrability conditions of the Lévy measure) proved by Nualart and Schoutens [15] in terms of the so-called Teugels martingales; we followed this approach in Leon et al. [12], and defined derivatives $D_{t}^{(n)}$ in the direction of the $n$ martingale (see also Davis and Johanson [5]).

On the other hand, in order for the Malliavin calculus to be genuinely useful, there is the need for practical rules to compute the derivatives. For example, in the case of Brownian motion, through the identification of the Malliavin derivative with a weak derivative on the canonical space, a chain rule is proven (see Nualart [14]), and for the case of the Poisson process, it is proven that the Malliavin derivative coincides with a difference operator on the canonical space (Nualart and Vives [16]). In Leon et al. [12], in the context of derivatives with respect to the Teugels martingales $D_{t}^{(n)}$, we were only able to give these alternative definitions for the derivative and to develop formulae for a particular class of Lévy process: specifically, for a jump-diffusion process having only a finite number of jump sizes. With the aim of extending these properties to a general Lévy process, we carried out the following: first, we returned to the Itô representation and to the two-parameter derivative $D_{t,x}$, and secondly, from the experience with the Brownian motion and Poisson process, we attempted to work on the canonical space of a Lévy process. However, the usual canonical space of a Lévy process (the set of cadlag functions with the $\sigma$-field of the cylinders and the measure given by the Kolmogorov Theorem) does not suffice and we therefore need a more suitable construction. Following the ideas of Neveu [18] for the Poisson process, we built what is (to the best of our knowledge) a new canonical space for a general Lévy process, which we think is interesting simply for its own sake. We then show that $D_{t,0}F$ is essentially a Brownian derivative and prove a chain rule. We also prove that $D_{t,x}F$, for $x \neq 0$, is a derivative with respect to the jump part of $X$, which can be computed in a pathway sense as an increment quotient

$$
D_{t,x} F(\omega) = \frac{1}{x} (F(\omega_{t,x}) - F(\omega)),
$$

where $\omega_{t,x}$ means, roughly speaking, that we have added a jump of size $x$ at time $t$ in the trajectory $\omega$. This formula is not new: it appears in Lee and Shih [11], and without dividing by $x$ also in Løkka [13] and Picard [21]. Our contribution is to study this formula in the setup of the canonical process, where it is easy to interpret and use.

The paper is organized as follows: Section 2 recalls the chaotic representation proved by Itô [9] and the definition of Malliavin Derivative on the corresponding Fock space. Section 3 is devoted to studying the derivative $D_{t,0}$ using the classical Brownian Malliavin derivative of a Hilbert-valued random variable. In Section 4, we build the canonical Lévy process. In Section 5, we study the derivative $D_{t,x}$, $x \neq 0$ and prove the increment quotient formula. Section 6 deals with the Skorohod integral and its relationship with the integral with respect to a random measure.

2. Chaotic representation for Lévy processes and Malliavin derivatives

2.1. Itô multiple integrals

Let $X = \{X_{t} \geq 0\}$ be a Lévy process (that means, $X$ has stationary and independent increments, is continuous in probability and $X_{0} = 0$), cadlag, in a complete probability space
\((\Omega, \mathcal{F}, \mathbb{P})\), with Lévy triplet \((\gamma, \sigma^2, \nu)\) where \(\gamma \in \mathbb{R}, \sigma \geq 0\) and \(\nu\) is a Lévy measure on \(\mathbb{R}\), with \(\nu(\{0\}) = 0\). For all these concepts we refer to Sato [23]. The process \(X\) admits a Lévy–Itô representation

\[
X_t = \gamma t + \sigma W_t + \int_{(0,t] \times \{x:|x|>1\}} x \, dN(s,x) + \lim_{\varepsilon \downarrow 0} \int_{(0,t] \times \{x:|x| \leq 1\}} x \, d\tilde{N}(s,x),
\]

(1)

where \([W_t, \, t \geq 0]\) is a standard Brownian motion,

\[
N(B) = \#\{t : (t, \Delta X_t) \in B\}, \quad B \in \mathcal{B}((0, \infty) \times \mathbb{R}_0),
\]

is the jump measure of the process, where \(\mathbb{R}_0 = \mathbb{R} - \{0\}\), \(\Delta X_t = X_t - X_{t-}\), \#\(A\) denotes the cardinal of a set \(A\), and

\[
d\tilde{N}(t, x) = dN(t, x) - dt \, d\nu(x)
\]

is the compensated jump measure, and the convergence in (1) is a.s., uniform in \(t\) on every bounded interval.

Following Itô [9], \(X\) can be extended to an independent random measure \(M\) on \((\mathbb{R}_+ \times \mathbb{R}, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}))\): First, consider the measure \(d\mu(t, x) = \sigma^2 \, dt \, d\delta_0(x) + x^2 \, dt \, d\nu(x)\), where \(\delta_0\) is the Dirac measure at point 0. That is, for \(E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\),

\[
\mu(E) = \sigma^2 \int_{E(0)} dt + \int_{E'} x^2 \, dt \, d\nu(x),
\]

where \(E(0) = \{t \in \mathbb{R}_+ : (t, 0) \in E\}\) and \(E' = E - \{(t, 0) \in E\}\); this measure is continuous (Itô [9, page 256]). Now, for \(E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\) with \(\mu(E) < \infty\), define

\[
M(E) = \sigma \int_{E(0)} dW_t + \lim_{n \to \infty} \int_{\{(t, x) \in E: 1/n < |x| < n\}} x \, d\tilde{N}(t, x),
\]

(2)

(convergence in \(L^2(\Omega)\)) which is a centered independent random measure such that for \(E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\) with \(\mu(E_1) < \infty\) and \(\mu(E_2) < \infty\)

\[
\mathbb{E}[M(E_1)M(E_2)] = \mu(E_1 \cap E_2).
\]

Write

\[
L^2_n = L^2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})^n, \mu^\otimes n).
\]

For \(f \in L^2_n\), Itô defines a multiple stochastic integral \(I_n(f)\) through the same steps as in the Wiener case: For

\[
f = 1_{E_1 \times \cdots \times E_n},
\]

where \(E_1, \ldots, E_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})\), pairwise disjoints, with \(\mu(E_1) < \infty, \ldots, \mu(E_n) < \infty\), let

\[
I_n(f) = M(E_1) \cdots M(E_n).
\]

Therefore, \(I_n\) is extended to \(L^2_n\) by linearity and continuity. This integral has the usual properties (Itô [9, Theorem 1]):

(1) \(I_n(f) = I_n(\tilde{f})\), where \(\tilde{f}\) is the symmetrization of \(f\):
It is worth observing that, by construction, 
\[ E_1^F \]

where the convergence of the series is in 
\[ L^F \]

and the representation of a functional 
\[ F \]

known that this filtration is right continuous. Put 
\[ f \]

is unique if we take every 
\[ f \in I(f) \]

and zero over all diagonal sets.

Let \( \{\mathcal{F}_t^X, t \geq 0\} \) be the natural filtration of \( X \) completed with the null sets of \( \mathcal{F} \); it is well known that this filtration is right continuous. Put \( \mathcal{F}_t^X = \bigvee_{s \leq t} \mathcal{F}_s^X \) and write \( L^2(\Omega) = L^2(\Omega, \mathcal{F}_\infty^X, \mathbb{P}) \). Itô [9, Theorem 2], proves a chaotic decomposition property:

\[ L^2(\Omega) = \bigoplus_{n=0}^{\infty} I_n(L^2_n), \]

and the representation of a functional \( F \in L^2(\Omega) \)

\[ F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L^2_n, \]

is unique if we take every \( f_n \) symmetric. We will always assume the symmetry in the kernels \( f_n \) of such decompositions.

From this point, it is possible to apply all the machinery of the annihilation operators (Malliavin derivatives) and creation operators (Skorohod integrals) on Fock spaces, as exposed in Nualart and Vives [16,17].

2.2. Derivative operators

Let \( F \in L^2(\Omega) \), with chaotic representation \( F = \sum_{n=0}^{\infty} I_n(f_n), (f_n \text{ symmetric}) \) such that

\[ \sum_{n=1}^{\infty} n^n \|f_n\|^2_{L^2_n} < \infty. \]

The Malliavin derivative of such an \( F \) is the stochastic process

\[ DF : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \]

defined by

\[ D_z F = \sum_{n=1}^{\infty} n I_n(f_n(z, \cdot)), \quad z \in \mathbb{R}_+ \times \mathbb{R}, \]

where the convergence of the series is in \( L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P}) \). Denote by \( \text{Dom} \ D \) the set of functionals \( F \in L^2(\Omega) \) that satisfies (3), which is a Hilbert space with the escalar product

\[ \langle F, G \rangle = \mathbb{E}[FG] + \mathbb{E} \left[ \int_{\mathbb{R}_+ \times \mathbb{R}} D_z F \, D_z G \, d\mu(z) \right], \]

and \( D \) is a closed operator from \( \text{Dom} \ D \) to \( L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P}) \).
Given the form of the measure $\mu$, for $f : (\mathbb{R}_+ \times \mathbb{R})^n \to \mathbb{R}$ measurable, positive or $\mu^{\otimes n}$ integrable, we have
\[
\int_{(\mathbb{R}_+ \times \mathbb{R})^n} f \, d\mu^{\otimes n} = \int_{\mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f((t, 0), (z_1, \ldots, z_{n-1})) \, dt \, d\mu^{\otimes(n-1)}(z_1, \ldots, z_{n-1})
+ \int_{\mathbb{R}_+ \times \mathbb{R}_0 \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f(z_1, z_2, \ldots, z_n) \, d\mu^{\otimes(n)}(z_1, z_2, \ldots, z_n).
\]
As a consequence, when $\sigma > 0$ and $\nu \neq 0$, it is natural to consider two more spaces: Let Dom $D^0_\sigma$ (if $\sigma > 0$) be the set of $F \in L^2(\Omega)$ with decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$ such that
\[
\sum_{n=1}^{\infty} n! \int_{\mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f^2((t, 0), z_1, \ldots, z_{n-1}) \, dt \, d\mu^{\otimes(n-1)}(z_1, \ldots, z_{n-1}) < \infty.
\]
For $F \in \text{Dom } D^0_\sigma$ we can define the square integrable stochastic process
\[
D_{t,0} F = \sum_{n=1}^{\infty} n I_{n-1}(f_n((t, 0), \cdot)),
\]
convergence in $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$. Analogously, if $\nu \neq 0$, let Dom $D^J$ be the set of $F \in L^2(\Omega)$ such that
\[
\sum_{n=1}^{\infty} n! \int_{\mathbb{R}_+ \times \mathbb{R}_0 \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f_n^2 \, d\mu^{\otimes n} < \infty,
\]
and for $F \in \text{Dom } D^J$, define
\[
D_z F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(z, \cdot)),
\]
convergence in $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, x^2 \, dt \, dv(x) \otimes \mathbb{P})$.

It is clear that when both $\sigma > 0$ and $\nu \neq 0$, then Dom $D = \text{Dom } D^0_\sigma \cap \text{Dom } D^J$.

**Remark 2.2.** Note that when there is no jump part, then $X_t = \gamma t + \sigma W_t$. Therefore, the derivative $D_{t,0} F$ coincides with the classical Malliavin derivative (see Nualart [14]), which we will write as $D_t^W$, except in a factor $\sigma$, due to the fact that in $D_{t,x} F$ we are differentiating with respect to $\sigma W_t$; that is,
\[
D_{t,0} F = \frac{1}{\sigma} D_t^W F.
\]

### 3. Derivatives $D_{t,0}$

The derivative $D_{t,0}$ is, essentially, a derivative with respect to the Brownian part of $X$, and we will see that in many situations the usual rules of classical Malliavin Calculus apply. To this end, we begin studying the filtration associated to the Brownian part and the jump part of $X$. Let

- $\{J_t^W, t \geq 0\}$ the (completed) natural filtration of $\{W_t, t \geq 0\}$.
\[ \{ \mathcal{F}_t^J, t \geq 0 \} \] the (completed) filtration generated by the jumps part of \( X \), that is, by the Lévy process

\[
J_t = \int_{(0,t] \times \{|x| > 1\}} x \, \mathcal{N}(ds, dx) + \lim_{\varepsilon \downarrow 0} \int_{(0,t] \times \{ |x| \leq \varepsilon \}} u \, \tilde{N}(ds, dx).
\]

**Lemma 3.1.** For every \( t \geq 0 \),

\[ \mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J. \]

**Proof.** This lemma has a clear intuitive interpretation: if the entire trajectory \( X_s, 0 \leq s \leq t \) is known then both the continuous part (Brownian part) and the jumps part can be clearly separated and, therefore, be known. The formal proof is as follows: First it is obvious that \( \mathcal{F}_t^X \subset \mathcal{F}_t^W \vee \mathcal{F}_t^J \). For the other inclusion, consider the sequence of \( \mathcal{F}_t^X \) stopping times \( \{ T_k, k \geq 1 \} \) that exhausts the jumps of \( X \): \( \Delta X_{T_k} \neq 0 \), \( \forall k \geq 1 \), and \( X \) only has jumps on these times (see, e.g., Dellacherie et Meyer [6, Théorème B, page XIII] for a construction of this sequence). For \( t > 0 \),

\[
J_t = \int_{(0,t] \times \{|x| > 1\}} x \, d\mathcal{N}(s, x) + \lim_{n} \int_{(0,t] \times \{ |x| \leq \frac{1}{n} \}} x \, d\tilde{\mathcal{N}}(s, x), \text{ a.s.}
\]

Fixed \( n \),

\[
\int_{(0,t] \times \{|x| \leq \frac{1}{n} \}} x \, d\tilde{\mathcal{N}}(s, x) = \sum_{0 < T_k \leq t; \frac{1}{n} < |\Delta X_{T_k}| \leq 1} \Delta X_{T_k} - t \int_{\frac{1}{n} < |x| \leq 1} x \, dv(x),
\]

which is \( \mathcal{F}_t^X \) measurable. Similarly, \( \int_{(0,t] \times \{|x| \geq 1\}} x \, d\mathcal{N}(s, x) \) is \( \mathcal{F}_t^X \) measurable. It follows that, first, \( \mathcal{F}_t^J \subset \mathcal{F}_t^X \), and second, \( \sigma W_t = X_t - \gamma t - J_t \) is \( \mathcal{F}_t^X \) adapted. \( \square \)

Let \( (\Omega, \mathcal{F}, \mathbb{P}, \{ \bar{W}_t, t \geq 0 \}) \) be the canonical Brownian process; that is, \( \Omega_W = C(\mathbb{R}_+) \) is the space of continuous functions on \( \mathbb{R}_+ \), null at the origin, with the topology of the uniform convergence on the compacts, \( \mathcal{F}_W \) the Borel \( \sigma \)-algebra and \( \mathbb{P}_W \) the probability that makes the projections \( \bar{W}_t : \Omega_W \rightarrow \mathbb{R} \) a Brownian motion. On the other hand, let \( (\Omega_J, \mathcal{F}_J, \mathbb{P}_J, \{ J_t, t \in \mathbb{R}_+ \}) \) be the canonical pure jump Lévy process of Section 4 (in this subsection we do not require the structure of this space). Finally, consider \( (\Omega_W \times \Omega_J, \mathcal{F}_W \otimes \mathcal{F}_J, \mathbb{P}_W \otimes \mathbb{P}_J) \), and write

\[
W_t(\omega, \omega') = \bar{W}_t(\omega) \quad \text{and} \quad J_t(\omega, \omega') = \bar{J}_t(\omega'),
\]

and

\[
X_t = \gamma t + \sigma W_t + J_t.
\]

Since there is the isometry \( L^2(\Omega_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J)) \), we can apply the theory of Malliavin derivative of a Hilbert space valued random variable.

### 3.1. Malliavin derivative of Hilbert-valued random variables

In this subsection, we review the definition of the Malliavin derivative of Hilbert-valued random variables following Nualart [14], and we prove some new properties that we require.

Consider a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( W = \{ W_t, t \in \mathbb{R}_+ \} \) a standard Brownian motion, and assume that \( \mathcal{F} \) is generated by \( W \). Denote by \( D^W \) the Malliavin derivative operator.
and let $\text{Dom } D^W$ be its domain, that is, the completion of the set $S^W$ of smooth random variables.

Let $\mathcal{H}$ be a real separable Hilbert space. The above notions can be extended to define the derivative of an $\mathcal{H}$-valued random variable (see Nualart [14, page 61]). Specifically, let $S^W, H$ be the set of $\mathcal{H}$-valued smooth random variables of the form

$$F = \sum_{i=1}^{n} G_i H_i,$$

where $G_i \in S^W$ and $H_i \in \mathcal{H}$. Define

$$D^*_t F = \sum_{i=1}^{n} D_t G_i \otimes H_i, \quad (4)$$

and let $\text{Dom } D^*_t$ be the completion of $S^W, H$ with respect to the norm

$$\|F\|_{W, \mathcal{H}} = \left( \mathbb{E}[\|F\|^2_{\mathcal{H}}] + \mathbb{E}\left[ \int_{\mathbb{R}^+} \|D^*_t F\|^2_{\mathcal{H}} \, dt \right] \right)^{1/2}.$$

The following Lemma is straightforward:

**Lemma 3.2.**

$$\text{Dom } D^*_t \simeq \text{Dom } D^t \otimes \mathcal{H}.$$

Consider another probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ such that $L^2(\Omega')$ is separable. Take $\mathcal{H} = L^2(\Omega')$, and let

$$L^2(\Omega; \text{Dom } D^t) = \{F : \Omega' \longrightarrow \text{Dom } D^t \text{ measurable} : \mathbb{E}'[\|F\|^2_W] < \infty\}.$$

Then

$$\text{Dom } D^*_t \simeq \text{Dom } D^t \otimes L^2(\Omega') \simeq L^2(\Omega'; \text{Dom } D^t).$$

As a consequence, we have

**Proposition 3.3.** Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a probability space such that $\mathcal{H} = L^2(\Omega')$ is separable, and consider $F \in L^2(\Omega \times \Omega')$ such that for all $\omega' \in \Omega'$, $\mathbb{P}'$-a.s., $F(\cdot, \omega') \in \text{Dom } D^t$. Then $F \in \text{Dom } D^*_t$ and

$$D^*_t F(\omega, \omega') = D^t F(\cdot, \omega')(\omega), \quad dt \otimes \mathbb{P} \otimes \mathbb{P}' \text{ a.e.}$$

It follows that we can transfer Proposition 1.2.2 of Nualart [14] to this context:

**Corollary 3.4 (Chain Rule).** With the same hypothesis as the above proposition, let $F \in L^2(\Omega \times \Omega')$ be of the form

$$F(\omega, \omega') = f(Z(\omega), Z'(\omega')),$$

where $f(x, y)$ is differentiable in $x$, with bounded partial derivative, and $Z \in \text{Dom } D^t$. Then $F \in \text{Dom } D^*_t$ and

$$D^*_t F = \frac{\partial f}{\partial x}(Z, Z') D_t^* Z.$$
3.2. The derivative $D_t^{W^*} F$ for Lévy processes

We return to the Lévy processes context, taking $\mathcal{H} = L^2(\Omega_J)$. From the chaotic representation property (when $\sigma = 0$) and the fact that the spaces $L^2_n$ are separable, it follows that $L^2(\Omega_J)$ is a separable Hilbert space. Since $L^2(\Omega^*_W \times \Omega_J) \simeq L^2(\Omega_W; L^2(\Omega_J))$ we can compute both $D_{t,0}F$ and $D_t^{W^*} F$. The following proposition gives its relationship:

**Proposition 3.5.** Dom $D^{W^*} \subset$ Dom $D^0$, and for $F \in$ Dom $D^{W^*}$, $D_t^{W^*} F = \sigma D_{t,0} F$.

**Proof.** Step 1. First consider a functional of the form $F_{m,k} = N(B_1) \cdots N(B_m)W(C_1) \cdots W(C_k)$, where $B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ are pairwise disjoints, $0 \notin \overline{B}_i$, and $C_1, \ldots, C_k \in \mathcal{B}(\mathbb{R}_+)$ pairwise disjoints, with $\int_{C_j} dt < \infty$. Itô [9, Proof of Lemma 2], shows that such $F_{m,k}$ can be written as $F_{m,k} = I_0(f_0) + \cdots + I_{m+k}(f_{m+k})$, where $f_j \in L^2$, $j = 1, \ldots, m + k$. This implies $F_{m,k} \in$ Dom $D$. We will check by induction over $m$ that

$$D_t^{W^*} F_{m,k} = \sigma D_{t,0} F_{m,k}. \quad (5)$$

Let $F_{1,k} = N(B)W(C_1) \cdots W(C_k)$.

Then, from definition (4),

$$D_t^{W^*} F_{1,k} = N(B) \sum_{j=1}^k 1_{C_j}(t)W(C_1) \cdots \widehat{W(C_j)} \cdots W(C_k),$$

where $\widehat{W(C_j)}$ means that this element is missing. On the other hand, consider the function

$$f_{1,k}(z_1, \ldots, z_{k+1}) = \frac{1}{\mu_1} \frac{1}{\sigma^k} 1_B(z_1)1_{C_1 \times \{0\}}(z_2) \cdots 1_{C_k \times \{0\}}(z_{k+1}),$$

where $z_j = (t_j, x_j)$, and then,

$$I_{m+1}(f_{1,k}) = (N(B) - \nu(B))W(C_1) \cdots W(C_k) = F_{1,k} - \frac{1}{\sigma^m} \nu(B) I_m(1_{C_1 \times \{0\} \times \cdots \times C_k \times \{0\}}).$$

Hence

$$D_{t,0} F_{1,k} = (k + 1)I_k(\tilde{f}_{1,k}((t, 0), \cdot)) + \frac{1}{\sigma^m} \nu(B) m_{m-1}(1_{C_1 \times \{0\} \times \cdots \times C_k \times \{0\}}((t, 0, \cdot)))$$

$$= \frac{1}{\sigma} \sum_{j=1}^k 1_{C_j}(t)(N(B) - \nu(B))W(C_1) \cdots \widehat{W(C_j)} \cdots W(C_k)$$

$$+ \frac{1}{\sigma} \nu(B) \sum_{j=1}^k 1_{C_j}(t)W(C_1) \cdots \widehat{W(C_j)} \cdots W(C_k) = \frac{1}{\sigma} D_t^{W^*} F_{1,k}.$$
Now, assume that formula (5) is true for $F_{r,k}$ with $r \leq m$, and consider

$$F_{m+1,k} = N(B_1) \cdots N(B_{m+1})W(C_1) \cdots W(C_k).$$

Define the function

$$f_{m+1,k}(z_1, \ldots, z_{m+k+1}) = \frac{1}{x_1} \cdots \frac{1}{x_{m+1}} \left( \frac{1}{\sigma^k} \mathbf{1}_{B_{m+1}}(z_{m+1}) \mathbf{1}_{C_k \times [0]}(z_{m+1}) \right).$$

Then

$$I_{m+1,k}(f_{m+1,k}) = (N(B_1) - \nu(B_1)) \cdots (N(B_{m+1}) - \nu(B_{m+1}))W(C_1) \cdots W(C_k)$$

$$= F_{m+1,k} + V W(C_1) \cdots W(C_k),$$

where

$$V = \prod_{i=1}^{m+1} \left( N(B_i) - \nu(B_i) \right) - \prod_{i=1}^{m+1} N(B_i),$$

that is, $V W(C_1) \cdots W(C_k)$ is a linear combination of different $F_{r,k}$, with $r \leq m$. By the induction hypothesis,

$$D_t F_{m+1,k} = \frac{1}{\sigma} (m + k + 1) I_{m+1,k}(f_{m+1,k}(t, 0, \cdot))$$

$$+ V \frac{1}{\sigma} \sum_{j=1}^{k} \mathbf{1}_{C_j}(t) W(C_1) \cdots \overline{W}(C_j) \cdots W(C_k)$$

$$= \frac{1}{\sigma} (N(B_1) - \nu(B_1)) \cdots (N(B_{m+1}) - \nu(B_{m+1}))$$

$$+ \sum_{j=1}^{k} \mathbf{1}_{C_j}(t) W(C_1) \cdots \overline{W}(C_j) \cdots W(C_k)$$

$$= \frac{1}{\sigma} N(B_1) \cdots N(B_{m+1}) \sum_{j=1}^{k} \mathbf{1}_{C_j}(t) W(C_1) \cdots \overline{W}(C_j) \cdots W(C_k)$$

$$= \frac{1}{\sigma} D_t W^* F_{m+1,k+k}. $$

**Step 2.** Itô [9, Lemma 2], proves that the random variables of the form $N(B_1) \cdots N(B_m)$, with $B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0)$ pairwise disjoints and $0 \notin \overline{B}_i$ (respectively $W(C_1) \cdots W(C_k)$, with $C_1, \ldots, C_k \in \mathcal{B}(\mathbb{R}_+)$ pairwise disjoints and $\int_{C_j} dt < \infty$) constitute a fundamental set in $L^2(\Omega_J)$ (respectively $L^2(\Omega_W)$). Consider an element of the form $F = G H$, where $G$ is a smooth random variable and $H \in L^2(\Omega_J)$. There is a sequence $\{G_n, n \geq 1\}$, where each $G_n$ is a lineal combination of functions $W(C_1) \cdots W(C_k)$ in the conditions given above, such that $G_n$ converges to $G$ in Dom $D^W_t$, and a sequence $\{H_n, n \geq 1\}$ where $H_n$ is a lineal combination of $N(B_1) \cdots N(B_m)$ and such that $H_n$ converges to $H$ is $L^2(\Omega_J)$. It follows that $G_n H_n$ converges
to $F$ in $\text{Dom } D^W$. Then we deduce that $D^W f = \sigma D_{t,0}$ over the set of $L^2(\Omega_1)$-valued smooth functionals, and hence over the whole $\text{Dom } D^W$. \hfill \square

From Corollary 3.4, we get:

**Corollary 3.6 (Chain Rule).** Let $F = f(Z, Z') \in L^2(\Omega_W \times \Omega_J)$ with $Z \in \text{Dom } D^W$ and $Z' \in L^2(\Omega_J)$, and $f(x, y)$ is a continuously differentiable function with bounded partial derivatives in the variable $x$. Then $F \in \text{Dom } D^0$ and

$$D_{t,0} F = \frac{1}{\sigma} \frac{\partial f}{\partial x}(Z, Z') D^W_t Z,$$

where $D^W$ is the Malliavin derivative in $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ and $\text{Dom } D^W$ its domain.

### 4. Canonical pure jump Lévy processes

In this section we build a canonical pure jump Lévy process and prove some measurability properties that are needed to study the Malliavin derivative $D_{t,x}$, $x \neq 0$.

The usual canonical Lévy processes are defined on the space of measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ or on the space of càdlàg functions, in both cases with the $\sigma$-field generated by the cylinders and the Kolmogorov extension theorem (see Sato [23, pages 36 and 121]). However, for our purposes, we need a different construction. The main reason is that in the Poisson case the derivative operator coincides with a difference (translation) operator on a convenient canonical space that consists, roughly speaking, in the set of finite sequences of jump times of the Poisson process in a fixed finite time interval. This canonical space for the Poisson process was introduced by Neveu [18] and used in the context of Malliavin calculus by Nualart and Vives [16]. Here we propose an extension to Lévy processes of the space of Neveu. Later, in the next section, we will see that the Malliavin derivative with respect to a pure jump Lévy process coincides with an increment quotient operator in that space. When the Lévy process reduces to a Poisson process our space and derivative operator are the same as that defined by Neveu [18] and Nualart and Vives [16] respectively.

The construction of the canonical space is undertaken in three steps: first, for a compound Poisson process indexed in $[0, T]$, second, the extension for a compound Poisson process indexed in $\mathbb{R}_+$, and third, the extension to a general pure jump Lévy process. Specifically, the program is the following:

**1st step.** Fix $T > 0$ and consider a compound Poisson process of the form

$$Y_t = \sum_{j=1}^{N_t} Z_j, \quad t \in [0, T],$$

where $\{N_t, t \in [0, T]\}$ is a Poisson process of parameter $\lambda > 0$, and $\{Z_n, n \geq 1\}$ is a sequence of i.i.d. random variables with law $Q$, with support $S \in \mathcal{B}(\mathbb{R}_0)$, that is, $\mathbb{P}(Z_j \in S) = Q(S) = 1$.

Any trajectory of $Y$ is totally described by a finite sequence $((t_1, x_1), \ldots, (t_n, x_n))$ where $t_1, \ldots, t_n \in [0, T]$ are the jump instants, and $x_1, \ldots, x_n \in \mathbb{R}_0$ are the size of the jumps. Then, let $\Omega_T$ be the set of all finite sequences of the type just described. On $\Omega_T$ we define a $\sigma$-field $\mathcal{F}_T$ and a probability $\mathbb{P}_T$ such that the application

$$X_t(\omega) = \sum_{j=1}^n x_j 1_{[0, t]_J}(t_j), \quad \text{if } \omega = ((t_1, x_1), \ldots, (t_n, x_n))$$

is a compound Poisson process with the same Lévy measure as $Y$. 
2nd step. For each \( m \geq 1 \), consider the canonical space \((\Omega_m, \mathcal{F}_m, \mathbb{P}_m)\) of the first step. We prove that \((\Omega_m, \mathcal{F}_m, \mathbb{P}_m)\), \( m \geq 1 \), is a projective system and the projective limit works as a canonical space for a compound Poisson process on the whole \( \mathbb{R}_+ \).

3rd step. Let \( \nu \) be an arbitrary Lévy measure. We define a partition

\[
\mathbb{R}_0 = \bigcup_{k \geq 1} S_k,
\]

where \( S_k \in \mathcal{B}(\mathbb{R}_0) \) are pairwise disjoints, with \( 0 < \nu(S_k) < \infty \). For each \( k \), we consider a canonical compound Poisson process of intensity parameter \( \lambda_k = \nu(S_k) \) and jumps’ law \( Q_k = \nu 1_{S_k}/\nu(S_k) \). The final canonical space is the cartesian product of all these spaces.

4.1. Canonical compound Poisson process: Finite time interval

Fix a time \( T > 0 \). Let \( \lambda > 0 \) and let \( Q \) be a probability on \( \mathbb{R} \), supported by \( S \in \mathcal{B}(\mathbb{R}_0) \). We build a compound process of intensity of jumps \( \lambda \) and size of jumps given by \( Q \). Write

\[
\Omega_T = \bigcup_{n \geq 0} ([0, T] \times S)^n,
\]

where \((0, T] \times S)^0 = \{\alpha\}, \alpha\) being a distinguished element that represents the empty sequence. Let

\[
\mathcal{F}_T = \left\{ B \subset \Omega_T : B = \bigcup_{n \geq 0} B_n, B_n \in \mathcal{B}([0, T] \times S)^n \right\} = \bigvee_{n \geq 0} \mathcal{B}(([0, T] \times S)^n)
\]

and define the probability \( \mathbb{P}_T \) on \((\Omega_T, \mathcal{F}_T)\) in the following way: for \( B = \bigcup_n B_n, B_n \in \mathcal{B}([0, T] \times S)^n \) write

\[
\mathbb{P}_T(B) = e^{-\lambda T} \sum_{n=0}^{\infty} \frac{\lambda^n (dt \otimes Q)^{\otimes n}(B_n)}{n!},
\]

where \((dt \otimes Q)^0 = \delta_\alpha\).

Finally, define \( \{X_t, t \in [0, T]\} \) by

\[
X_t(\omega) = \begin{cases} 
\sum_{j=1}^{n} x_j 1_{[0,t]}(t_j), & \text{if } \omega = ((t_1, x_1), \ldots, (t_n, x_n)), \\
0, & \text{if } \omega = \alpha. 
\end{cases}
\]

Roughly speaking, for \( \omega = ((t_1, x_1), \ldots, (t_n, x_n)) \) the trajectory \( X_\omega(\omega) \) on \([0, T]\) has jumps at (and only at) the points \( t_1, \ldots, t_n \) of sizes \( x_1, \ldots, x_n \) respectively (note that \( t_1, \ldots, t_n \) does not need to be increasing); and when \( \omega = \alpha \), then \( X_\omega(\omega) \) does not jump at all on \([0, T]\).

**Theorem 4.1.** With the above definitions, \( \{X_t, t \in [0, T]\} \) is a compound Poisson process of Lévy measure \( \lambda. Q \).

**Proof.** First, \( X_t \) is measurable: Fixed \( n \geq 0 \) and \( t \in [0, T] \), the application

\[
T_{t,n} : ([0, T] \times S)^n \longrightarrow \mathbb{R}
\]

\[
((t_1, x_1), \ldots, (t_n, x_n)) \rightarrow \sum_{j=1}^{n} x_j 1_{[0,t]}(t_j)
\]
is measurable. Then, for \( B \in \mathcal{B} (\mathbb{R}) \)
\[
X^{-1}_t (B) \cap [0, T]^n^n
= \{(z_1, \ldots, z_n) \in ([0, T] \times S)^n : T_{t, n}(z_1, \ldots, z_n) \in B\} \in \mathcal{B} (([0, T] \times S)^n).
\]

Fix \( 0 = s_0 < s_1 < \cdots < s_r \leq T \). We compute the characteristic function of \( X_{s_1}, X_{s_2} - X_{s_1}, \ldots, X_{s_r} - X_{s_{r-1}} \). Take \( y_1, \ldots, y_r \in \mathbb{R} \).

\[
\mathbb{E}_T \left[ \exp \left\{ i \sum_{k=1}^r y_k (X_{s_k} - X_{s_{k-1}}) \right\} \right]
= \int_{\Omega} \exp \left\{ i \sum_{k=1}^r y_k (X_{s_k} (\omega) - X_{s_{k-1}} (\omega)) \right\} d\mathbb{P}_T (\omega)
= e^{-\lambda T} \frac{\lambda^n}{n!} \int_{[0, T] \times S} \cdots \int_{[0, T] \times S} \exp \left\{ i \sum_{k=1}^r y_k \sum_{j=1}^n x_j 1_{(s_{k-1}, s_k)} (t) \right\}
\times dt_1 dQ (x_1) \cdots dt_n dQ (x_n)
= e^{-\lambda T} \frac{\lambda^n}{n!} \left( \int_{[0, T] \times S} \exp \left\{ i \sum_{k=1}^r y_k 1_{(s_{k-1}, s_k)} (t) \right\} dt dQ (x) \right)^n,
\]
and

\[
\int_{[0, T] \times S} \exp \left\{ i \sum_{k=1}^r y_k 1_{(s_{k-1}, s_k)} (t) \right\} dt dQ (x)
= \sum_{k=1}^r \int_{s_{k-1}}^{s_k} \int_S \exp (iy_k x) dQ (x) dt + \int_{s_k}^T \int_S dQ (x) dt
= \sum_{k=1}^r (s_k - s_{k-1}) \varphi_Q (y_k) + T - s_r = \sum_{k=1}^r (s_k - s_{k-1}) (\varphi_Q (y_k) - 1) + T,
\]
where \( \varphi_Q \) is the characteristic function of the probability \( Q \). It follows that

\[
\mathbb{E}_T \left[ \exp \left\{ i \sum_{k=1}^r y_k (X_{s_k} - X_{s_{k-1}}) \right\} \right] = \prod_{k=1}^r \exp \{ \lambda (s_k - s_{k-1}) (\varphi_Q (y_k) - 1) \}.
\]

All the properties of the Lévy processes follow from that characteristic function. Note that, by construction, \( X_{\cdot} (\omega) \) is cadlag. \( \square \)

**Remark 4.2.** Symmetric \( \sigma \)-field.

1. Let \( E \) be an arbitrary set, \( n \geq 1 \), and consider \( E^n \). Given a permutation \( \pi \in \mathfrak{S}_n \) and an element \( x = (x_1, \ldots, x_n) \in E^n \), write

\[
\pi (x) = (x_{\pi(1)}, \ldots, x_{\pi(n)}),
\]
and for a set \( C \subset E^n \), let

\[
\pi (C) = \{ \pi (x), \text{ for } x \in C \}.
\]
A set $C$ is called symmetric if $\pi(C) = C$, $\forall \pi \in G_n$. Denote by $\tilde{C}$ the symmetrized set of the set $C$:

$$\tilde{C} = \bigcup_{\pi \in G_n} \pi(C).$$

Obviously, $\tilde{C}$ is symmetric, and $C$ is symmetric if and only if $C = \tilde{C}$. A function $f : E^n \rightarrow \mathbb{R}$, is called symmetric if $f(\pi(x)) = f(x)$, $\forall x \in E^n$, $\forall \pi \in G_n$.

Consider now a measurable space $(E, \mathcal{E})$. The collection $\mathcal{E}^n_{\text{sym}} = \{C \in \mathcal{E}^n : C \text{ is symmetric}\}$ is a $\sigma$-field. An interesting property is that a function $f : E^n \rightarrow \mathbb{R}$ is $\mathcal{E}^n_{\text{sym}}$ measurable if and only if $f$ is $\mathcal{E}^n$ measurable and symmetric.

2. We now return to the case in which $\Omega_T = \bigcup_{n \geq 0} ([0, T] \times S)^n$, and let $\mathcal{F}_{T, \text{sym}}$ the sub-$\sigma$-field of $\mathcal{F}_T$ defined by

$$\mathcal{F}_{T, \text{sym}} = \bigvee_{n \geq 0} B([0, T] \times S)^n_{\text{sym}}.$$

Let $\mathcal{F}^X_T$ be the $\sigma$-field generated by $\{X_t, t \in [0, T]\}$, without completion; we study this more specifically.

**Proposition 4.3.**

$$\mathcal{F}^X_T = \mathcal{F}_{T, \text{sym}}.$$

**Proof.** Consider $B \in B(\mathbb{R})$, and compute with further details $X^{-1}(B)$:

$$X_t^{-1}(B) \cap [0, T]^n = \bigcup_{j=0}^n ((\text{Sum}_j^{-1}(B) \times [0, t]^j) \times (S^{n-j} \times (t, T]^{n-j})), $$

where $\text{Sum}_k$ is the sum function on $S^k$:

$$\text{Sum}_k : S^k \rightarrow \mathbb{R}$$

$$(x_1, \ldots, x_k) \rightarrow \sum_{j=1}^k x_j.$$

Then, $\mathcal{F}^X_T \subseteq \mathcal{F}_{T, \text{sym}}$.

For the other inclusion, we should prove that all symmetric sets in $B([0, T] \times S)^n)$ belong to the $\sigma$-field trace $([0, T] \times S)^n \cap \mathcal{F}^X_T$; however, from the fact that all symmetric sets in $B([0, T] \times S)^n)$ are of the form $A$, for some $A \in B([0, T] \times S)^n)$, and that every rectangle $C_1 \times \cdots \times C_n$ can be decomposed in a disjoint union of rectangles of the form $C_1 \times \cdots \times C_n$, with $C_i \cap C_j = \emptyset$, or $C_i = C_j$, $i \neq j$, by a monotone classes argument, it is enough to work with the sets of the form $(A_1 \times B_1) \times \cdots \times (A_n \times B_n)$, where the $A_i$ are intervals in $[0, T]$, and $B_i$ are Borel sets of $S$, and $A_1 \times B_1, \ldots, A_n \times B_n$ are pairwise disjoints or equal. Then, consider
0 ≤ s_1 < ⋯ < s_{m+1} ≤ T, m ≤ n, and let A_i = [s_j, s_{j+1}] for some j = 1, . . . , m. It follows that

\[(A_1 × B_1) × ⋯ × (A_n × B_n) = ([s_1, s_2] × B_{d_1}) × ⋯ × ([s_m, s_{m+1}] × B_{d_m})^{r_m}\]

where r_j is the number of A_i = [s_j, s_{j+1}], and r_1 + ⋯ + r_m = n, and d_1, . . . , d_m ∈ \{1, . . . , n\}. Then

\[
(A_1 × B_1) × ⋯ × (A_n × B_n) = \{(X_{s_2} - X_{s_1})^{-1}(S_{r_1}^{-1}(B_{d_1})), \ldots, (X_{s_{m+1}} - X_{s_m})^{-1}(S_{r_m}^{-1}(B_{d_m}))\} \cap ([0, T] × S)^n.
\]

that belongs to \(\mathcal{F}^X \cap ([0, T] × S)^n\). □

**Remark 4.4.** In the next subsection we require the fact that the space

\[H = \bigcup_{n ≥ 0} ([0, T] × \mathbb{R})^n\]

is a metric space; specifically, for v, w ∈ H, v ∈ ([0, T] × \mathbb{R})^n, w ∈ ([0, T] × \mathbb{R})^m, we can define the distance

\[d(v, w) = \begin{cases} 1, & \text{if } n \neq m, \text{ or } n = m \text{ and } d_{2n}(v, w) > 1, \\ d_{2n}(v, w), & \text{if } n = m \text{ and } d_{2n}(v, w) ≤ 1, \end{cases}\]

where d_k is the Euclidean distance on \(\mathbb{R}^k\). Then H is a Polish space (metric, separable and complete) and the \(\sigma\)-field \(\bigvee_{n ≥ 0} \mathcal{B}([0, T] × \mathbb{R})^n\) coincides with the Borel \(\sigma\)-field. It follows, in the terminology of Parthasarathy [20], page 133, Definition 2.2 and Theorem 2.2, that \((\Omega_T, \mathcal{F}_T)\) is a separable standard Borel space.

### 4.2. Canonical compound Poisson process: Infinite time interval

In this subsection we extend the construction above to a canonical compound Poisson process (with the same \(λ\) and \(Q\)) over the whole time interval \(\mathbb{R}_+\) through a projective system of probability spaces. With this purpose, for \(m ≥ 1\), let \((\Omega_m, \mathcal{F}_m, \mathbb{P}_m)\) be the canonical space corresponding to the interval \([0, m]\), that from **Remark 4.4** is a separable standard Borel space, and consider the maps

\[\pi_m: \Omega_{m+1} → \Omega_m\]

defined by

\[\pi_m((l_1, x_1), . . . , (l_r, x_r)) = ((l_{i_1}, x_{i_1}), . . . , (l_{i_s}, x_{i_s}))\]

where \(i_1, . . . , i_s\) are the points of \(l_1, . . . , l_r\) such that they belong to \([0, m]\). And

\[\pi_m((l_1, x_1), . . . , (l_r, x_r)) = α\]

if \(l_1, . . . , l_r ∈ (m, m + 1]\). It is straightforward to check that

\[\mathbb{P}_m(B) = \mathbb{P}_{m+1}(\pi_{m+1}^{-1}(B)), \quad ∀ B ∈ \mathcal{F}_m.\]

It follows that we can apply Theorem 3.2 of Parthasaraty [20, page 139], and we deduce that there is a unique probability \(\mathbb{P}\) on \((Ω, \mathcal{F})\), where \(Ω\) is the projective limit of the system \((Ω_m, π_m, m ≥ 1)\), \(\mathcal{F}\) is the \(\sigma\)-field generated by the canonical projections \(π_m: Ω → Ω_m\), and

\[\mathbb{P}_m(B) = \mathbb{P}(π_m^{-1}(B)), \quad ∀ B ∈ \mathcal{F}_m.\]
By construction, the projective limit $\Omega$ is the set of all sequences $(\omega_1, \omega_2, \ldots)$ with $\omega_m \in \Omega_m$ such that $\pi_m(\omega_{m+1}) = \omega_m$; in our setup, this condition is that if $\omega_m = ((t_1, x_1), \ldots, (t_r, x_r))$, then

$$\omega_{m+1} = ((s_1, v_1), \ldots, (t_1, x_1), \ldots, (s_j, v_j), \ldots, (t_r, x_r), \ldots, (s_m, v_m))$$

where $(s_i, v_i)$ are points with $s_i \in (m, m + 1]$. This means that $\Omega$ is equivalent to the set with:

- The empty sequence $\alpha$, corresponding to the element $(\alpha, \alpha, \ldots)$.
- All finite sequences $(((t_1, x_1), \ldots, (t_r, x_r)))$, corresponding to the elements $(\omega_1, \omega_2, \ldots)$ such that $\omega_r = \omega_{r+1} = \ldots$, for some $r$.
- All infinite sequences $((t_1, x_1), (t_2, x_2), \ldots)$ such that for every $T > 0$, there is only a finite number of $t_i \leq T$.

On the other hand, the canonical projection

$$\pi_m : \Omega \rightarrow \Omega_m$$

$$(\omega_1, \omega_2, \ldots) \rightarrow \omega_m$$

with the interpretation of $\Omega$ as the set of finite or infinite sequences $(((t_1, x_1), (t_2, x_2), \ldots)$ exposed above, can be interpreted as the map such that $\pi_m((t_1, x_1), (t_2, x_2), \ldots)$ is the finite sequence of points $(t_i, x_i)$ such that $t_i \in [0, m]$. In the sequel, both $\Omega$ and $\pi_k$ should be understood in this sense.

Now define the $\sigma$-field on $\Omega$

$$\mathcal{F}_{\text{sym}} = \bigvee_{n \geq 0} \pi_m^{-1}(\mathcal{F}_m, \text{sym}).$$

The following theorem is a consequence of the properties of $\mathbb{P}$:

**Theorem 4.5.** The process $\{X_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$X_t(\omega) = \begin{cases} 
\sum_j x_j 1_{[0,t]}(t_j), & \text{if } \omega = ((t_1, x_1), (t_2, x_2), \ldots), \\
0, & \text{if } \omega = \alpha,
\end{cases}$$

is a compound Poisson process with Lévy measure $\lambda Q$, and the $\sigma$-field generated by this process, $\mathcal{F}^X$, is $\mathcal{F}_{\text{sym}}$.

### 4.3. Canonical pure jump Lévy process

We build a canonical pure jump Lévy process with Lévy measure $\nu$. Let $\{\varepsilon_k, k \geq 1\}$ be a strictly decreasing sequence of positive real numbers, $\varepsilon_1 = 1$, $\lim_k \varepsilon_k = 0$, and let

$$S_1 = \{x \in \mathbb{R} : 1 < |x|\}, \quad \text{and} \quad S_k = \{x \in \mathbb{R} : \varepsilon_k < |x| \leq \varepsilon_{k-1}\}, \quad k \geq 2.$$ 

Since $\nu$ is a Lévy measure, $\nu(S_k) < \infty$, $\forall k$. To simplify the notation, assume that we have suppressed from the sequence $\{\varepsilon_k\}$ those $\varepsilon_k$ such that $\nu(S_k) = 0$, and corrected the numbering. For each $k \geq 1$ construct the canonical compound Poisson process

$$(\Omega^{(k)}, \mathcal{F}_{\text{sym}}^{(k)}, \mathbb{P}^{(k)}), \quad \{X_t^{(k)}, t \geq 0\}$$
corresponding to the intensity \( \lambda_k = \nu(S_k) \) and jumps given by the probability distribution \( Q_k := \nu 1_{S_k} / \nu(S_k) \) supported by \( S_k \). Now consider the product probability space 
\[
(\Omega, \mathcal{F}, \mathbb{P}) = \bigotimes_{k \geq 1} (\Omega^{(k)}, \mathcal{F}^{(k)}_{\text{sym}}, \mathbb{P}^{(k)}).
\]

For \( \omega = (\omega^{(k)})_{k \geq 1} \in \Omega \), define 
\[
X_t(\omega) = \lim_n \sum_{k=2}^n \left( X_t^{(k)}(\omega^{(k)}) - t \int_{S_k} x \nu(dx) \right) + X_t^{(1)}(\omega^{(1)}).
\]

The existence for almost all \( \omega \) of this limit is proved exactly as the Itô–Lévy representation of a Lévy process (see, for example, Sato [23, Chapter 4]), which gives the convergence a.s., uniform on \( t \in [0, T] \), for any \( T > 0 \), of an equivalent sequence.

**Theorem 4.6.** With the above definitions, \( X = \{X_t, t \geq 0\} \) is a cadlag pure jump process Lévy process with Lévy measure \( \nu \).

**Proof.** Since \( X \) is a limit a.s. of Lévy processes, it has independent stationary increments, and is also clear in the measurability of \( X_t \). Furthermore, from the fact that the limit is uniform on \( t \in [0, T] \), for any \( T > 0 \), \( X \) inherits the cadlag property of the \( X^{(n)} \). Now we compute the characteristic function of \( X_t \).

Let \( y \in \mathbb{R} \). For \( k \geq 2 \), by the computations into the proof of Theorem 4.1
\[
\mathbb{E} \left[ \exp \left\{ iy \sum_{j=2}^k X^{(j)}_t \right\} \right] = \prod_{j=2}^k \mathbb{E}^{(j)}[\exp(iyX^{(j)}_t)]
\]
\[
= \prod_{j=2}^k \exp \left\{ \lambda_j t \left( \int_{S_j} (\exp(iyx) - 1)Q_j(dx) \right) \right\}
\]
\[
= \exp \left\{ t \left( \int_{B_k^c} (\exp(iyx) - 1)\nu(dx) \right) \right\},
\]
where \( B_k = \bigcup_{j=2}^k S_j \). It follows that
\[
\mathbb{E}[\exp(iyX_t)] = \lim_k \exp \left\{ t \left( \int_{B_k} (\exp(iyx) - 1)\nu(dx) - iy \int_{B_k} x \nu(dx) \right) \right. \]
\[
+ \left. \int_{B_k} (\exp(iyx) - 1)\nu(dx) \right) \right\}
\]
\[
= \exp \left\{ t \int_{R_0} (\exp(iyx) - 1 - iyx1_{0 < |x| < 1})\nu(dx) \right\}.
\]

From this characteristic function it is clear that \( X \) is continuous in probability. \( \square \)

Let \( \mathcal{F}^X \) be the \( \sigma \)-field generated by \( \{X_t, t \geq 0\} \), without completion.

**Proposition 4.7.** Denote by \( \mathcal{N} \) the collection of \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Then 
\[
\mathcal{F}^X \vee \mathcal{N} = \mathcal{F} \vee \mathcal{N}.
\]
Proof. By construction, $\mathcal{F}^X \vee \mathcal{N} \subset \mathcal{F} \vee \mathcal{N}$. The proof of the other inclusion is an adaptation of that of Lemma 3.1 to the current setup. Let $p_k$ be the projection from $\Omega$ on $\Omega^{(k)}$, and let $X^{(k)} = X(k) \circ p_k$. Then,

$$
\overline{X}^{(n)}_t 1_{A \cap \Gamma} = X_t 1_{A \cap \Gamma} 1_{|\Delta X_t| \in S_n, s \in [0,t]} \times (\times_m \{(|\Delta X_t| \notin S_m, s \in [0,t])},
$$

and we deduce

$$
\mathcal{F}^X \vee \mathcal{N} = \bigvee_{k=1}^{\infty} \mathcal{F}^X^{(k)} \vee \mathcal{N}.
$$

Now let $\mathcal{T} = \{\emptyset, \Omega\}$ be the trivial $\sigma$-field. Then

$$
\mathcal{F}^{X^{(k)}} = \mathcal{T} \otimes (k-1) \otimes \mathcal{F}^{X^{(k)}} \otimes \mathcal{T} \otimes (\infty),
$$

and the proposition follows.

4.4. Transformation of random variables

In this subsection, we study the following transformation: Given $\omega \in \Omega$ and $z = (t, x) \in \mathbb{R}_+ \times \mathbb{R}_0$, we introduce in $\omega$ a jump of size $x$ at instant $t$, and call the new element $\omega_z = ((t, x), (t_1, x_1), (t_2, x_2), \ldots)$. For a random variable $F$, we define

$$
F_z(\omega) = F(\omega_z).
$$

We will prove the following proposition:

**Proposition 4.8.** Let $F$ be a $\mathcal{F}^X$ random variable. Then

$$
F(\cdot) : \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}
$$

$$(z, \omega) \rightarrow F_z(\omega)
$$

is $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}_0) \otimes \mathcal{F}^X$ measurable and

$$
F = 0, \ \mathbb{P} \text{ a.s.} \implies F_z(\cdot) = 0, \ \ \mathbb{d}t \otimes \nu \otimes \mathbb{P} \ \text{a.e.}
$$

(6)

**Proof.** We only consider the case of a compound Poisson process, since the extension to a general Lévy process is direct. Consider the canonical space of a compound Poisson process of Lévy measure $\lambda Q$, with support $S \in \mathcal{B}(\mathbb{R}_0)$. The application

$$
\theta : \mathbb{R}_+ \times S \times \Omega \rightarrow \Omega
$$

$$(z, \omega) \rightarrow (z, \omega)
$$

is $\mathcal{B}(\mathbb{R}_+ \times S) \otimes \mathcal{F}_{\text{sym}}/\mathcal{F}_{\text{sym}}$- measurable since $\mathcal{F}_{\text{sym}} \subset \mathcal{B}(\mathbb{R}_+ \times S) \otimes \mathcal{F}_{\text{sym}}$. To prove this, it is enough to see that $\forall A_1, \ldots, A_n \in \mathcal{B}([0, m] \times S),

$$
A_1 \otimes \cdots \otimes A_n \in \mathcal{B}([0, m] \times S) \otimes \mathcal{F}_{\text{sym}}.
$$

but this is obvious:

$$
A_1 \otimes \cdots \otimes A_n = \bigcup_{i=1}^{n} (A_i \times (A_{i_1} \times \cdots \times A_{i_{n-1}})),
$$

where in each union, $i_1, \ldots, i_{n-1}$ are the indices different from $i$. 

As a consequence, \( F_\varepsilon(\omega) = F(\theta(z, \omega)) \) is \( B(\mathbb{R}_+ \times S) \otimes \mathcal{F}^X \) measurable.

To prove (6) observe that, fixed \( m \geq 1 \), on \( ([0, m] \times S)^n \) the probability \( \mathbb{P} \) is equivalent to \( (dt \otimes Q)^{\otimes n} \). Then \( F = 0 \), a.s. implies
\[
F(z_1, \ldots, z_n) = 0, \quad (dt \otimes Q)^{\otimes n} \text{ a.e., } \forall n \geq 0.
\]
It follows that
\[
F_\varepsilon(z_1, \ldots, z_n) = F(z, z_1, \ldots, z_n) = 0, \quad (dt \otimes Q)^{\otimes(n+1)} \text{ a.e., } \forall n \geq 0,
\]
which implies (6). \(\square\)

5. Derivatives \( D_{t,x}, x \neq 0 \)

5.1. Pure jump \( \text{Lévy process} \)

In this subsection we will consider that \( \sigma = 0 \) and will work with the canonical jump process. With the notations of the above section, consider a random variable \( F \), and define an increment quotient operator
\[
\Psi_{t,x} F(\omega) = \frac{F_{t,x}(\omega) - F(\omega)}{x}.
\]
Thanks to Proposition 4.8, \( \Psi_{t,x} \) is a measurable operator from \( L^0(\Omega) \) to \( L^0(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega) \). It should be pointed out that this operator appears in Lee and Shih [11, Theorem 5.15], and also, but without dividing by \( x \), in Løkka [13, Proposition 10] and Picard [21, Theorem 3.1].

The following properties are obvious:

**Proposition 5.1.** Let \( F, G \in L^0(\Omega) \). Then
1. \( \Psi(F + G) = \Psi(F) + \Psi(G) \).
2. \( \Psi_{t,x}(FG) = G \Psi_{t,x} F + F \Psi_{t,x} G + x \Psi_{t,x}(F) \Psi_{t,x}(G) \).

Using the same ideas as Nualart and Vives [17], we will see that when \( \Psi F \) is a square integrable process, then \( \Psi F = DF \). First, we prove a couple of lemmas:

**Lemma 5.2.** For every \( f \in L^2_n \),
\[
DI_n(f) = \Psi I_n(f), \quad \mu \otimes \mathbb{P} \text{ a.e.}
\]

**Proof.** Let \( z = (t, x) \in \mathbb{R}_+ \times \mathbb{R}_0 \). Fix \( T > t \), and assume \( x \in S_k \) (see notations in Section 4). Write
\[
\tilde{f}_{T,k} = f 1_{([0,T] \times S_k)^n}.
\]
For every \( \omega \), there is a finite sequence \( (s_1, v_1), \ldots, (s_r, v_r) \in [0, T] \times S_k \) (see notations in Section 4), such that
\[
I_n(\tilde{f}_{T,k})(\omega) = \int_{([0,T] \times S_k)^n} f((t_1, x_1), \ldots, (t_n, x_n)) \prod_{i=1}^n \left( \sum_{j=1}^r v_j \delta_{(s_j, dv_j)} (dt_i, dx_i) - dt_i \nu(dx_i) \right), \tag{7}
\]
where \( ([0, T] \times S_k)^n = \{ (t_1, x_1), \ldots, (t_n, x_n) \in ([0, T] \times S_k)^n : (t_i, x_i) \neq (t_j, x_j) \text{ if } i \neq j \} \).
This is proved first for \( f = 1_{E_1 \times \cdots \times E_n} \), with \( E_1, \ldots, E_n \in \mathcal{B}([0, T] \times A_k) \), pairwise disjoints, \( \mu(E_j) < \infty \), \( i = 1, \ldots, n \), and then extending the result to \( L^2(([0, T] \times S_k)^n, \mathcal{B}([0, T] \times S_k)^\otimes n, \mu \otimes \mathbb{P}) \) using that both sides of (7) are closed operators on that space. Note that, by Remark 2.1, we can assume that \( f(z_1, \ldots, z_n) = 0 \) if \( z_i = z_j \) for some \( i \neq j \).

Now using the formula \( \prod_{i=1}^n a_i - \prod_{i=1}^n b_i = \sum_{i=1}^n a_1 \cdots a_i - 1 (a_i - b_i) b_i+1 \cdots b_n \), we have

\[
\Psi_z I_n(f_{T,k})(\omega) = \frac{1}{x} \int \cdots \int_{([0, T] \times S_k)^n} f((t_1, x_1), \ldots, (t_n, x_n)) \prod_{i=1}^n (x \delta_{(t_i, x)}(dt_k, dx_k)) + \sum_{j=1}^r v_j \delta_{(s_j, v_j)}(dt_k, dx_k) - dt_k v(dx_k) (u d\delta_{(t_i, x)}(t_i, x_i)) \cdot \prod_{k=i+1}^n \left( \sum_{j=1}^r v_j \delta_{(s_j, v_j)}(dt_k, dx_k) - dt_k v(dx_k) \right) = n I_{n-1}(f_{T,k}((t, x), \cdot))(\omega) = D_z I_n(f_{T,k})(\omega).
\]

On the other hand,

\[
D_z I_n(f) = D_z I_n(f_{T,k}) \quad \text{and} \quad \Psi_z I_n(f) = \Psi_z I_n(f_{T,k}).
\]

Also, it is obvious that

\[
D_z I_n(f_{T,j}) = \Psi_z I_n(f_{T,j}) = 0, \quad \forall j \neq k,
\]

and, finally, we get

\[
D_z I_n(f) = \Psi_z I_n(f). \quad \square
\]

**Lemma 5.3.** Let \( F_m, F \in L^2(\Omega) \) such that

\[
\lim_m F_m = F, \quad \text{in } L^2(\Omega),
\]

and

\[
\lim_m \Psi F_m = G, \quad \text{in } L^2(\mathbb{R}^+ \times \mathbb{R}_0 \times \Omega, \mu \otimes \mathbb{P}),
\]

for some \( G \in L^2(\mathbb{R}^+ \times \mathbb{R}_0 \times \Omega, \mu \otimes \mathbb{P}) \). Then

\[
G = \Psi F, \quad \mu \otimes \mathbb{P} \text{ a.e.}
\]

**Proof.** First assume that the Lévy process is a compound Poisson process with Lévy measure \( \nu = \lambda Q \), and consider the restriction to \([0, T]\). On the one hand,

\[
\mathbb{E}[(F_m - F)^2] = e^{-\lambda T} \sum_{n=0}^\infty \frac{\lambda^n}{n!} \int_{([0, T] \times \mathbb{R}_0)^n} (F_m(z_1, \ldots, z_n) - F(z_1, \ldots, z_n))^2 \times (dt \otimes dQ)^\otimes n(z_1, \ldots, z_n),
\]

and this implies that for all \( n \),

\[
\lim_m F_m(z_1, \ldots, z_n) = F(z_1, \ldots, z_n) \quad \text{in } L^2(([0, T] \times \mathbb{R}_0)^n, (dt \otimes Q)^\otimes n).
\]
Moreover, working in a similar way with $\Psi^{m} - G$, we obtain for all $n$,
\[
\lim_{m}(F_{m}((t, x), (t_{1}, x_{1}), \ldots, (t_{n}, x_{n})) - F_{m}((t_{1}, x_{1}), \ldots, (t_{n}, x_{n})) = xG((t, x), (t_{1}, x_{1}), \ldots, (t_{n}, x_{n}))
\]
in $L^{2}([0, T] \times \mathbb{R})^{|\mathbb{N}|}$, $(dr \otimes Q)^{\otimes n+1}$, and we deduce
\[
G_{z}(\omega, \omega') = \Psi_{z}F(\omega), \quad \forall (z, \omega) \in [0, T] \times \mathbb{R}, \quad \mu \otimes \mathbb{P}_{T} \text{ a.e.}
\]
The extension to $\mathbb{R}_{+}$ follows directly from the construction of $\mathbb{P}$.

For the general case, for $k \geq 1$, decompose $\Omega = \Omega_{k} \times \widehat{\Omega}_{k}$, where $\widehat{\Omega}_{k} = \otimes_{j \neq k} \Omega_{j}$, and similarly, $\mathbb{P} = \mathbb{P}_{k} \otimes \widehat{\mathbb{P}}_{k}$, where $\widehat{\mathbb{P}}_{k} = \otimes_{j \neq k} \mathbb{P}_{j}$. From the preceding proof (taking if necessary a subsequence of the initial sequence), we get that $\forall \omega' \in \widehat{\Omega}_{k}$, $\widehat{\mathbb{P}}_{k}$ a.s.,
\[
G_{z}(\omega^{(k)}, \omega') = \Psi_{z}F(\omega^{(k)}, \omega'), \quad \forall (z, \omega^{(k)}) \in \mathbb{R}_{+} \times S_{k} \times \Omega_{k} \quad \mu_{0} \otimes \mathbb{P}_{k} \text{ a.e.}
\]
and joining all zero measure the result follows from the Fubini theorem. □

**Proposition 5.4.** Let $F \in L^{2}(\Omega)$. Then $F \in \text{Dom} D$ if and only if $\Psi F \in L^{2}(\mathbb{R}_{+} \times \mathbb{R} \times \Omega)$, and in this case
\[
DF = \Psi F, \quad \mu \otimes \mathbb{P} \text{ a.e.}
\]

**Proof.** Assume $\Psi F \in L^{2}(\mathbb{R}_{+} \times \mathbb{R} \times \Omega)$. Then (see Lemma 1.3.1 by Nualart [14]) we can write
\[
\Psi_{z}F = \sum_{n=0}^{\infty} I_{n}(g_{n}(z, \cdot)),
\]
where $g_{n} \in L^{2}_{n+1}$, symmetric in the last $n$-variables, and the convergence of the series is in $L^{2}(\mathbb{R}_{+} \times \mathbb{R} \times \Omega)$. Now consider the decomposition $F = \sum_{n=1}^{\infty} I_{n}(f_{n})$, $f_{n}$ symmetric. From Lemma 5.2 it is clear that $g_{n}(z, \cdot) = (n + 1)f_{n+1}(z, \cdot)$, a.e., and the proposition follows.

Assume that $F \in \text{Dom} D$, and let the decomposition of $F$ given by $F = \sum_{n=1}^{\infty} I_{n}(f_{n})$. Let
\[
F_{N} = \sum_{n=1}^{N} I_{n}(f_{n}).
\]
From the fact that $DF_{N} = \Psi F_{N}$, $\forall N \geq 1$, we have
\[
\lim_{N} F_{N} = F, \quad \text{in } L^{2}(\Omega), \quad \text{and} \quad \lim_{N} \Psi F_{N} = DF, \quad \text{in } L^{2}(\mathbb{R}_{+} \times \mathbb{R} \times \Omega),
\]
and from Lemma 5.3 we get $\Psi F = DF$. □

**5.2. Derivatives $D_{t, x}, x \neq 0$, for a general Lévy process**

We now return to the general case. With the notations of Section 3, consider the canonical space $(\Omega_{W} \times \Omega_{J}, \mathcal{F}_{W} \otimes \mathcal{F}_{J}, \mathbb{P}_{W} \otimes \mathbb{P}_{J})$. Given $z = (t, x) \in \mathbb{R}_{+} \times \mathbb{R}$, for $\omega = (\omega_{W}^{t}, \omega_{J}^{t}) \in$
\(\Omega_W \times \Omega_J\) define \(\omega_z = (\omega_W, \omega_J^z)\), where \(\omega_J^z\) was defined in Section 5.1, and for a random variable \(F \in L^0(\Omega_W \times \Omega_J)\) let \(F_z(\omega) = F(\omega_z)\). Define the increment quotient operator
\[
\Psi_{t,x} F = \frac{F(\omega_{t,x}) - F(\omega)}{x}.
\]

We can repeat all the reasoning in Section 3 and get the following proposition:

**Proposition 5.5.** Let \(F \in L^2(\Omega_W \times \Omega_J)\) such that
\[
\mathbb{E}\left[ \int_{\mathbb{R}_+ \times \mathbb{R}_0} (\Psi_z F)^2 \mu(\mathrm{d}z) \right] < \infty.
\]
Then \(F \in \text{Dom } D^J\) and
\[
D_z F(\omega) = \Psi_z F(\omega), \quad \mu \otimes \mathbb{P} \text{ a.e. } (z, \omega) \in \mathbb{R}_+ \times \mathbb{R}_0 \times \Omega.
\]

6. The Skorohod integral

Following the scheme of Nualart and Vives [16], we can define a creation operator (Skorohod integral). Let \(f \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P})\). As we commented in the proof for Proposition 5.4, there is a chaotic decomposition
\[
f(z) = \sum_{n=0}^{\infty} I_n(f_n(z, \cdot)), \quad (8)
\]
where \(f \in L^2_{n+1}\) is symmetric in the \(n\) last variables. Now, denote by \(\hat{f}_n\) the symmetrization in all \(n + 1\) variables. If
\[
\sum_{n=0}^{\infty} (n + 1)! \| \hat{f}_n \|_{L^2_{n+1}}^2 < \infty, \quad (9)
\]
define the Skorohod integral of \(f\) by
\[
\delta(f) = \sum_{n=0}^{\infty} I_{n+1}(\hat{f}_n),
\]
convergence in \(L^2(\Omega)\). Denote by \(\text{Dom } \delta\) the set of \(f\) that satisfies (9). For the sake of completeness, we recall the main properties of the operator \(\delta\):

1. **Duality formula:** A process \(f \in L^2([0, T] \times \mathbb{R} \times \Omega, \mu \times \mathbb{P})\) belongs to \(\text{Dom } \delta\) if and only if there is a constant \(C\) such that for all \(F \in \text{Dom } D\),
\[
\mathbb{E} \left| \int_{\mathbb{R}_+ \times \mathbb{R}} f(z) D_z F \, \mathrm{d}\mu(z) \right| \leq C (\mathbb{E}[F^2])^{1/2}.
\]
If \(f \in \text{Dom } \delta\), then \(\delta(f)\) is the element of \(L^2(\Omega)\) characterized by
\[
\mathbb{E}[\delta(f) F] = \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} f(z) D_z F \, \mathrm{d}\mu(z), \quad (10)
\]
for any \(F \in \text{Dom } D\).
2. Covariance of Skorohod integrals: Denote by $\mathbb{L}^{1,2}$ the set of elements $f \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathcal{F}_t^X, \mu \otimes \mathbb{P})$ such that $f(z) \in \text{Dom } D, \forall z \mu\text{ e and that } D.f(\cdot) \in L^2((\mathbb{R}_+ \times \mathbb{R})^2 \times \Omega)$. In terms of the chaotic expression (8) of $f$, both conditions are equivalent to

$$\sum_{n=1}^{\infty} n^2 \|\hat{f}_n\|_{L_{2n+1}^2}^2 < \infty,$$

and, in particular, this implies $\mathbb{L}^{1,2} \subset \text{Dom } \delta$. For $f, g \in \mathbb{L}^{1,2}$,

$$E[\delta(f)\delta(g)] = \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} f(z) g(z) d\mu(z) + \mathbb{E} \int_{(\mathbb{R}_+ \times \mathbb{R})^2} D_z f(z') D_z g(z') d\mu(z) d\mu(z').$$

3. Differentiability of $\delta$. Let $f \in \mathbb{L}^{1,2}$ such that $D_z f \in \text{Dom } \delta, \forall z, \mu\text{ a.e. Then } \delta(f) \in \text{Dom } D$ and

$$D_z \delta(f) = f(z) + \delta(D_z f), \quad \forall z, \mu\text{ a.e.}$$

Fix a finite time $T > 0$ and consider the process $X$ indexed on $[0, T]: \{X_t, t \in [0, T]\}$. The independent random measure $M$ defined in (2) is now restricted to $[0, T] \times \mathbb{R}$, and the multiple integrals are of functions of $L^2(([0, T] \times \mathbb{R})^n, \mathcal{B}(([0, T] \times \mathbb{R})^n), \mu \otimes \mathbb{P})$; in the same way, the Skorohod integral is defined for a process $\{f(z), z \in [0, T] \times \mathbb{R}\}$.

When $\int_{[0,T]} x^2 d\nu(x) < \infty$, then the random measure $M$, with the filtration $\{\mathcal{F}_t^X, t \in [0, T]\}$, induces a martingale-valued measure and it can be defined a stochastic integral $\int_{[0,T]} f(z) dM_z$ of a predictable process $f = \{f(z), z \in [0, T] \times \mathbb{R}\}$ such that $\int_{[0,T]} f^2(z) \mu(dz) < \infty$ a.s. (see Applebaum [1, Chapter 4] for details). In particular, for $f$ and $g$ square integrable predictable processes

$$\mathbb{E} \left[ \int_{[0,T]} f(z) dM_z \cdot \int_{[0,T]} g(z) dM_z \right] = \mathbb{E} \left[ \int_{[0,T]} f(z) g(z) d\mu(z) \right].$$

As in the Brownian case, the Skorohod integral restricted to predictable processes coincides with the integral with respect to the random measure $M$. The next theorem is a version of Privault [22, Proposition 11], Di Nunno et al. [7, Proposition 3.15], and Oksendal and Proske [19, Proposition 3.7]:

**Theorem 6.1.** Assume that $\int_{\mathbb{R}} x^2 d\nu(x) < \infty$. Let $f \in L^2([0, T] \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$ be predictable. Then $f \in \text{Dom } \delta$ and

$$\delta(f) = \int_{[0,T]} f(z) dM_z.$$

**Proof.** The equality is first proved using the duality relationship (10) for an elementary predictable process of the form $f(z) = \beta \mathbb{1}_{(r,s) \times B}(z)$, where $0 \leq r < s \leq T, \beta$ is a bounded $\mathcal{F}_r^X$ random variable, and $B \in \mathcal{B}(\mathbb{R})$ and then extended by linearity and density to a square integrable predictable processes. \□
References