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The string partition function in Hull's doubled formalism

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Abstract

T-duality is one of the essential elements of string theory. Recently, Hull has developed a formalism where the dimension of the target space is doubled so as to make T-duality manifest. This is then supplemented with a constraint equation that allows the connection to the usual string sigma model. This Letter analyses the partition function of the doubled formalism by interpreting the constraint equation as that of a chiral scalar and then using holomorphic factorisation techniques to determine the partition function. We find there is quantum equivalence to the ordinary string once the topological interaction term is included.
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1. Introduction

One of the crucial differences between string theory and other theories of quantum gravity is T-duality. The quantum equivalence between different target spaces in which the string propagates is a fascinating consequence of string dynamics. Recently, Hull has developed a formalism where this symmetry is manifest by doubling the dimension of the target space so as to include all duality related geometries in a single target space. The theory is then supplemented with constraints so as to allow a reduction (through gauging) of the theory back to usual string string theory. The formalism has the advantage of providing a geometric manifestation of T-duality but more importantly perhaps it may be crucial in formulating string theory on non-trivial T-folds. The formalism we will use was developed in [1,2] though based on earlier work [3]. We should mention there have been a variety of Letters which have studied T-folds [4] as possible string vacua; it is hoped the doubled formalism will be useful to issues arising in T-fold compactifications even though this Letter will study the formalism on a non-fibred circle.

The goal of this Letter is to study the quantum theory of the doubled formalism. This has already been examined from the point of view of canonical quantisation with a detailed constraint analysis by Hackett-Jones and Moutsopoulos [5] also using results of Hellerman and Walcher [6].

In this Letter we will take the following viewpoint. The constraints are to be viewed as a those determining chiral scalars. The goal is to show that these chiral bosons combine in the right way for us to interpret the partition function for one chiral boson and its dual as that of an ordinary non-chiral boson. (These chiral bosons cannot be interpreted quite in the usual way since they have novel periodicity conditions, as will be seen below.)

The quantum theory of chiral bosons is an extraordinarily subtle business. A host of publications contain the problems and methods of dealing with them [7] but perhaps a special mention should go to [8] where the details of how to calculate the partition function are explained in detail. We will follow many of the techniques explained there (see the Appendices of [8]). The main subtlety arises in the treatment of the sum over instantons, i.e. the sum over the cohomological part of the bosons.

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The method is as follows. To calculate the partition function of a chiral scalar one must first calculate the partition function of the non-chiral scalar and then carry out a variety of resummations and manipulations so as to write it as a product of a holomorphic function and an anti-holomorphic function. One then identifies the holomorphic part as the chiral scalar partition function. Essentially, one is factorising the usual partition function into contributions from the chiral and anti-chiral pieces. (We have skirted over the details of picking spin structures and importantly for our calculation the factorisation is more problematic when the boson is not at the free-Fermion radius.)

In what follows we calculate the partition function of scalars on the doubled torus. Then we carry out a number of non-trivial resummations so as to factorise the partition function into pieces that may be interpreted as the contribution from the chiral modes obeying Hull's constraint. These pieces are then recombined to give the usual string partition function.

2. Factorising the partition function

We begin with the doubled string action for a single toroidally compactified boson, so that we have the boson X, which is associated to a circle of radius R, and its dual $\tilde{X}$ which is associated to a circle of radius R^{-1}. The action is given by

$$S = \frac{\pi}{2} R^2\, dX \wedge *dX + \frac{\pi}{2} R^{-2}\, d\tilde{X} \wedge *d\tilde{X}. \tag{1}$$

The constraint in this geometry (see Appendix A) can be written in a simple form in terms of the linear combinations

$$P = RX + R^{-1}\tilde{X}, \qquad Q = RX - R^{-1}\tilde{X}. \tag{2}$$

Note that the normalisation is a factor of 1/2 times the usual one. This was the normalisation needed by Hull in [2] when gauging the current associated with the constraint to show equivalence with the standard sigma model. Also, here $R = 1$ corresponds to the T-duality self-dual radius, whereas in much of the literature on holomorphic factorisation $R = 1$ is the free-Fermion radius (which would be $R = 1/\sqrt{2}$ in our conventions). A topological term given by

$$L_{\text{top}} = \pi\, dX \wedge d\tilde{X} \tag{3}$$

was also required so as to have invariance under large gauge transformation when gauging the current. We also find the need to include it here and in fact it is crucial in what follows for demonstrating quantum equivalence.

The constraints are

$$\partial_{\bar{z}} P = 0, \qquad \partial_z Q = 0, \tag{4}$$

that is P is holomorphic and Q is anti-holomorphic. We can rewrite the action in terms of P and Q, as follows

$$S = \frac{\pi}{4}\, dP \wedge *dP + \frac{\pi}{4}\, dQ \wedge *dQ. \tag{5}$$

The standard way of obtaining a partition function for a chiral boson is to factorise the partition function for an ordinary boson into holomorphic and anti-holomorphic parts and keep only the factor with the correct holomorphic dependence [8]. Here we would like to do this for P and Q. However, crucially P and Q do not have standard periodicity properties, and are linked through (2). This is why we cannot directly identify these fields as left and right movers on a circle; they do not have the right periodicity conditions.

We will now examine the *instanton* sector. This means we need to examine contributions to the partition function which, from the field theory point of view, will be in the cohomological sector. When writing the action (1), dX should really be replaced by L, where $L = dX + \omega$, with $\omega \in H^1(\Sigma, \mathbb{Z})$. We can express the cohomological part in terms of the standard cohomology basis, which for a toroidal worldsheet consists of just one α cycle and one β cycle.

$$L = dX + n\alpha + m\beta, \qquad \tilde{L} = d\tilde{X} + \tilde{n}\alpha + \tilde{m}\beta, \tag{6}$$

where $n, \tilde{n}, m, \tilde{m} \in \mathbb{Z}$, which in turn means that we should replace dP and dQ by

$$M = dP + \left(Rn + R^{-1}\tilde{n}\right)\alpha + \left(Rm + R^{-1}\tilde{m}\right)\beta, \qquad N = dQ + \left(Rn - R^{-1}\tilde{n}\right)\alpha + \left(Rm - R^{-1}\tilde{m}\right)\beta. \tag{7}$$

The classical or *instanton* sector of the partition function is the sum over all field configurations of $\exp[-S]$ and can be written as

$$\begin{aligned} Z = \sum_{m,n,\tilde{m},\tilde{n}} &\exp\left[-\left(Rm + R^{-1}\tilde{n}\right)^2 \frac{\pi|\tau|^2}{4\tau_2} + \left(Rn + R^{-1}\tilde{n}\right)\left(Rm + R^{-1}\tilde{m}\right)\frac{\pi\tau_1}{2\tau_2} - \left(Rm + R^{-1}\tilde{m}\right)^2 \frac{\pi}{4\tau_2}\right] \\ &\times \exp\left[-\left(Rm - R^{-1}\tilde{n}\right)^2 \frac{\pi|\tau|^2}{4\tau_2} + \left(Rn - R^{-1}\tilde{n}\right)\left(Rm - R^{-1}\tilde{m}\right)\frac{\pi\tau_1}{2\tau_2} - \left(Rm - R^{-1}\tilde{m}\right)^2 \frac{\pi}{4\tau_2}\right], \end{aligned} \tag{8}$$

where the first factor corresponds to P and the second to Q. To holomorphically factorise the partition function we must Poisson resum, but first we separate the sums so that the P and the Q parts of the partition function sum over different independent variables.

The contribution from the topological term within the sum is

$$\exp\big[i\pi(n\tilde{m}-m\tilde{n})\big]. \tag{9}$$

This will only contribute a sign change to the terms in the partition sum, however, this is crucial in showing the equivalence to the standard formulation.

To be able to separate the sums we assume $R^2=\frac{p}{q}$, with p,q coprime integers, and let $k=pq$. Then we have

$$Rn\pm R^{-1}\tilde{n}=\sqrt{k}\left(\frac{n}{q}\pm\frac{\tilde{n}}{p}\right). \tag{10}$$

Making the substitutions $n=cq+q\gamma_q$ and $\tilde{n}=\tilde{c}p+p\gamma_p$ (where $c,\tilde{c}\in\mathbb{Z}$ and $\gamma_q\in\{0,\frac{1}{q},\dots,\frac{q-1}{q}\}$) we can further say

$$\sqrt{k}\left(\frac{n}{q}\pm\frac{\tilde{n}}{p}\right)=\sqrt{k}(c\pm\tilde{c}+\gamma_q\pm\gamma_p). \tag{11}$$

Then we let $h=c+\tilde{c}$ and $l=c-\tilde{c}$. We have rewritten the sum over n and $\tilde{n}$ as a sum over $c,\tilde{c},\gamma_q$ and γ_p, and then rewritten the c and $\tilde{c}$ sums as a sum over h and $l\in\mathbb{Z}$, but $h-l=2\tilde{c}$ so we must restrict to even values of $h-l$ by inserting a factor of

$$\sum_{\phi\in\{0,\frac{1}{2}\}}\frac{1}{2}\exp\big[2\pi i\phi(h-l)\big] \tag{12}$$

in the partition function. Repeating the process to split the $m,\tilde{m}$ sum, the partition function becomes

$$\begin{aligned}Z=\sum_{\phi,\theta,\gamma_q,\gamma_p,\gamma_q',\gamma_p'}\sum_{h,i,j,l}\frac{1}{4}\exp\bigg[&-\frac{k\pi}{4}\bigg\{(h+\gamma_q+\gamma_p)^2\frac{|\tau|^2}{\tau_2}-2(h+\gamma_q+\gamma_p)\big(i+\gamma_q'+\gamma_p'\big)\frac{\tau_1}{\tau_2}+\big(i+\gamma_q'+\gamma_p'\big)^2\frac{1}{\tau_2}\bigg\}\\&-\frac{k\pi}{4}\bigg\{(l+\gamma_q-\gamma_p)^2\frac{|\tau|^2}{\tau_2}-2(l+\gamma_q-\gamma_p)\big(j+\gamma_q'-\gamma_p'\big)\frac{\tau_1}{\tau_2}+\big(j+\gamma_q'-\gamma_p'\big)^2\frac{1}{\tau_2}\bigg\}\\&+2\pi i\big\{\phi(h-l)+\theta(i-j)\big\}+\frac{i\pi k}{2}\big((l+\gamma_q-\gamma_p)\big(i+\gamma_q'+\gamma_p'\big)-(h+\gamma_q+\gamma_p)\big(j+\gamma_q'-\gamma_p'\big)\big)\bigg].\end{aligned} \tag{13}$$

Using the notation $\gamma_\pm=\gamma_q\pm\gamma_p$ we can rewrite the partition function again as

$$\begin{aligned}Z=\sum_{\phi,\theta,\gamma_q,\gamma_p,\gamma_q',\gamma_p'}\sum_{h,i,j,l}\bigg(&\frac{1}{2}\exp\bigg[-\frac{k\pi}{4}\bigg\{(h+\gamma_+)^2\frac{|\tau|^2}{\tau_2}-2(h+\gamma_+)\big(i+\gamma_+'\big)\frac{\tau_1}{\tau_2}+\big(i+\gamma_+'\big)^2\frac{1}{\tau_2}\bigg\}+2\pi i\{\phi h+\theta i\}\bigg]\\&\times\frac{1}{2}\exp\bigg[-\frac{k\pi}{4}\bigg\{(l+\gamma_-)^2\frac{|\tau|^2}{\tau_2}-2(l+\gamma_-)\big(j+\gamma_-'\big)\frac{\tau_1}{\tau_2}+\big(j+\gamma_-'\big)^2\frac{1}{\tau_2}\bigg\}-2\pi i\{\phi l+\theta j\}\bigg]\\&\times\exp\bigg[\frac{i\pi k}{2}\big((l+\gamma_-)\big(i+\gamma_+'\big)-(h+\gamma_+)\big(j+\gamma_-'\big)\big)\bigg]\bigg),\end{aligned} \tag{14}$$

where we have split the terms in the partition sum into three factors, the piece coming from the P kinetic term (which depends on h and i), the piece coming from the Q kinetic term (which depends on l and j), and the topological piece, a cross term depending on all four integer indices.

We now wish to perform Poisson resummation on i and j. Let us focus on the i resummation, replacing it with a sum over r. Here we rewrite the P part of the partition function and the first term of the topological piece:

$$\begin{aligned}Z_P=&\sum_{\phi,\theta,\gamma_q,\gamma_p,\gamma_q',\gamma_p'}\sum_{h,l,r}\frac{1}{2}\sqrt{\frac{4\tau_2}{k}}\exp\bigg[-\frac{k\pi}{4}\bigg\{(h+\gamma_+)^2\frac{|\tau|^2}{\tau_2}-2\gamma_+'(h+\gamma_+)\frac{\tau_1}{\tau_2}+\big(\gamma_+'\big)^2\frac{1}{\tau_2}\bigg\}\\&\quad+2\pi i\phi h+\frac{i\pi k}{2}(l+\gamma_-)\gamma_+'-\frac{4\pi\tau_2}{k}\bigg(r-\theta+ik\frac{(h+\gamma_+)}{4}\frac{\tau_1}{\tau_2}-\frac{ik\gamma_+'}{4\tau_2}-\frac{k}{4}(l+\gamma_-)\bigg)^2\bigg]\\=&\sum_{\phi,\theta,\gamma_q,\gamma_p,\gamma_q',\gamma_p'}\sum_{h,l,r}\sqrt{\frac{\tau_2}{k}}\exp\bigg[\tau_2\bigg(-\frac{k\pi}{4}(h+\gamma_+)^2-4\pi k\bigg(\frac{r-\theta}{k}-\frac{1}{4}(l+\gamma_-)\bigg)^2\bigg)\\&\quad-2\pi i\tau_1(h+\gamma_+)\bigg(r-\theta-\frac{k}{4}(l+\gamma_-)\bigg)+2\pi i\phi h+2\pi i(r-\theta)\gamma_+'\bigg].\end{aligned} \tag{15}$$

The appearance of the squares with τ_2 and the cross term with τ_1 is standard and Poisson resumming to replace j with s in the Q part of the partition function we can rewrite the whole partition function as

$$Z = \sum_{\phi,\theta,\gamma_q,\gamma_p,\gamma'_q,\gamma'_p,h,l,r,s} \left(\sqrt{\frac{\tau_2}{2k}} \exp\left[i\pi k\tau \frac{p_L^2}{2} - i\pi k\bar{\tau}\frac{p_R^2}{2} + 2\pi i\big(\phi h + (r-\theta)\gamma'_+\big)\right] \times \sqrt{\frac{\tau_2}{2k}} \exp\left[i\pi k\tau \frac{q_L^2}{2} - i\pi k\bar{\tau}\frac{q_R^2}{2} + 2\pi i\big(-\phi l + (s+\theta)\gamma'_-\big)\right]\right), \tag{16}$$

where

$$p_L = \frac{1}{2}(h+\gamma_+) - 2\left(\frac{r-\theta}{k} - \frac{1}{4}(l+\gamma_-)\right), \qquad p_R = \frac{1}{2}(h+\gamma_+) + 2\left(\frac{r-\theta}{k} - \frac{1}{4}(l+\gamma_-)\right),$$
$$q_L = \frac{1}{2}(l+\gamma_-) - 2\left(\frac{s+\theta}{k} + \frac{1}{4}(h+\gamma_+)\right), \qquad q_R = \frac{1}{2}(l+\gamma_-) + 2\left(\frac{s+\theta}{k} + \frac{1}{4}(h+\gamma_+)\right). \tag{17}$$

We can now see the clear holomorphic and anti-holomorphic parts of the partition function for both P and Q, as well as additional pieces which restrict the sum over "momenta", p_L, p_R, q_L, q_R. However, because the sums are linked we cannot remove the extra pieces. So we rewrite the sums again, reconstructing them back to a sum over just four integers.

The h, l terms in the momenta can easily be recombined just by undoing the substitutions we made earlier to write

$$h+\gamma_+ = \frac{n}{q} + \frac{\tilde{n}}{p}, \qquad l+\gamma_- = \frac{n}{q} - \frac{\tilde{n}}{p}, \tag{18}$$

where we have replaced the sums over h, l, γ_+, γ_- and ϕ with a sum over integers n and $\tilde{n}$, we also remove one of the factors of $\frac{1}{2}$ we inserted outside the partition function.

The other sum has been Poisson resummed so the recombination is more complicated. We use the identity

$$\sum_{k=0}^{n-1}\left(\exp\left(\frac{2\pi i k}{n}\right)\right)^j = \sum_{\gamma_n}\exp(2\pi i\gamma_n j) = \begin{cases} n, & \text{if } j \equiv 0 \bmod n, \\ 0, & \text{otherwise.}\end{cases} \tag{19}$$

With that in mind we see the only occurrence of γ'_q and γ'_p is in the factor

$$\sum_{\gamma'_q,\gamma'_p}\exp\left[2\pi i\left(\frac{(p_L-p_R)}{2}\gamma'_+ + \frac{(q_L-q_R)}{2}\gamma'_-\right)\right] = \sum_{\gamma'_q,\gamma'_p}\exp\left[2\pi i(r+s)\gamma'_q + 2\pi i(r-s-2\theta)\gamma'_p\right], \tag{20}$$

which has the effect of enforcing

$$r+s \equiv 0 \bmod q, \qquad r-s-2\theta \equiv 0 \bmod p, \tag{21}$$

in the rest of the partition sum. We see that these requirements are fulfilled exactly by putting

$$\frac{r-\theta}{k} = \frac{1}{2}\left(\frac{w}{p} + \frac{\tilde{w}}{q}\right), \qquad \frac{s+\theta}{k} = \frac{1}{2}\left(\frac{w}{p} - \frac{\tilde{w}}{q}\right). \tag{22}$$

We have replaced the sums over r, s, θ, γ'_p and γ'_q by a sum over integers $w, \tilde{w} \in \mathbb{Z}$. Note that importantly it is the term with q in the denominator which changes sign between the two combinations. Also, due to (19) we get a factor of $k = pq$ outside the exponential, which cancels the factor of $1/k$ obtained from Poisson resummation. We can now rewrite (17) as

$$p_L = \frac{n}{q} - \left(\frac{w}{p} + \frac{\tilde{w}}{q}\right), \qquad p_R = \frac{\tilde{n}}{p} + \left(\frac{w}{p} + \frac{\tilde{w}}{q}\right), \qquad q_L = -\frac{\tilde{n}}{p} - \left(\frac{w}{p} - \frac{\tilde{w}}{q}\right), \qquad q_R = \frac{n}{q} + \left(\frac{w}{p} - \frac{\tilde{w}}{q}\right). \tag{23}$$

The doubled partition function is now in the simple form

$$Z_d = \sum_{p_L,p_R}\sqrt{2\tau_2}\exp\left[i\pi k\tau\frac{p_L^2}{4} - i\pi k\bar{\tau}\frac{p_R^2}{4}\right]\sum_{q_L,q_R}\sqrt{2\tau_2}\exp\left[i\pi k\tau\frac{q_L^2}{4} - i\pi k\bar{\tau}\frac{q_R^2}{4}\right]. \tag{24}$$

Now we can make the following final substitution

$$u = n - \tilde{w}, \qquad v = -w, \qquad \tilde{u} = \tilde{w}, \qquad \tilde{v} = \tilde{n} + w, \tag{25}$$

leading to

$$p_L = \frac{u}{q} + \frac{v}{p}, \qquad p_R = \frac{\tilde{u}}{q} + \frac{\tilde{v}}{p}, \qquad q_L = \frac{\tilde{u}}{q} - \frac{\tilde{v}}{p}, \qquad q_R = \frac{u}{q} - \frac{v}{p}. \tag{26}$$

It is the shift in the momenta caused by the topological term that allows us to rewrite n and $\tilde{w}$ in terms of *independent* summation variables u, $\tilde{u}$, v and $\tilde{v}$.

The pieces of the appropriate holomorphicity that we wish to keep are now summed over the same indices, and are not linked to the pieces which we wish to remove. We may therefore remove the anti-holomorphic part of the partition function coming from P (the p_R piece) and the holomorphic part of the partition function coming from Q (the q_L piece). This leaves us with

$$Z_f = \sum_{p_L, p_R} \sqrt{2\tau_2} \exp\left[i\pi\tau \frac{p_L^2}{4} - i\pi\bar{\tau}\frac{q_R^2}{4}\right], \tag{27}$$

where

$$p_L = uR + \frac{v}{R}, \qquad q_R = uR - \frac{v}{R}. \tag{28}$$

Alternatively we can see that the final form of the instanton part of the partition function, Z_f, is the holomorphic square root of the doubled contribution to the partition function:

$$Z_d = Z_f \times \bar{Z}_f. \tag{29}$$

Z_f is the instanton part of the partition function for a standard boson with radius R (or R^{-1} by relabelling the indices), up to a factor outside the exponential. We will find that when we consider the rest of the partition function in the doubled formalism in the next section, the inverse of this factor will occur, giving an identical total partition function.

The approach taken here has been to treat the bosons in the doubled formalism as chiral bosons when trying to quantise. There is a key difference however, for chiral bosons one must pick a spin structure [7] and we have not done so here. If one were to do so then there would not be enough degrees of freedom to reconstruct the usual non-chiral boson. Thus when one is holomorphically factorising here, one is effectively keeping a sum of chiral bosons with all spin structures. This prescription is an essential part of the quantum prescription of the theory.

3. The oscillators

So far we have only included the sum over solutions to the classical equations of motion, to complete the quantum path integral we must include the fluctuations around these classical solutions [10]. For a boson X with action

$$S = -\frac{\pi R^2}{2} dX \wedge *dX, \tag{30}$$

we must perform the Gaussian integral

$$\int \mathcal{D}X e^{-\int \frac{\pi R^2}{2} X \Box X}, \tag{31}$$

where $\Box$ is the Laplacian. The $\mathcal{D}X$ integration is split into the zero-mode piece and the integral over $\mathcal{D}X'$, orthogonal to the zero-mode. As X has period 1 in our conventions, the zero-mode contribution is only a factor of 1. To normalise the measure we insert a factor of

$$\left(\int dx e^{-\frac{\pi R^2}{2}\int x^2}\right)^{-1} = \left(\frac{\pi}{\frac{\pi R^2}{2}\int 1}\right)^{-1/2} = \frac{R}{\sqrt{2}}, \tag{32}$$

where we have used the fact that with our conventions $\int 1$ over the torus is 1. This means

$$Z_{\text{osc}} = \frac{R}{\sqrt{2}} \frac{1}{\det' \Box}. \tag{33}$$

We evaluate the determinant of $\Box = -4\tau_2 \partial\bar{\partial}$ as a regularised product of eigenvalues, where the $'$ indicates this does not include zero-modes. We use a basis of eigenfunctions

$$\psi_{nm} = \exp\left[\frac{2\pi i}{2i\tau_2}\big(n(z - \bar{z}) + m(\tau\bar{z} - \bar{\tau}z)\big)\right], \tag{34}$$

which is single valued under $z \to z + 1$ and $z \to z + \tau$, where $z = \sigma_1 + \tau\sigma_0$ and τ is the complex structure of the toroidal worldsheet. The regularised determinant is then the product of eigenvalues

$$\det' \Box = \prod_{\{m,n\} \neq \{0,0\}} \frac{4\pi^2}{\tau_2}(n - \tau m)(n - \bar{\tau}m). \tag{35}$$

This can be evaluated using ζ-function regularisation (see for example [9]) as

$$\det{}'\Box = \tau_2\eta^2(\tau)\bar{\eta}^2(\bar{\tau}), \tag{36}$$

where $\eta(\tau)$ is the Dedekind η-function,

$$\eta(\tau) = e^{i\pi\tau/12}\prod_{n>1}\left(1-e^{2\pi in\tau}\right). \tag{37}$$

We now have the oscillator part of the partition function, given by

$$Z_{\mathrm{osc}} = \frac{R}{\sqrt{2\tau_2}|\eta|^2}. \tag{38}$$

The contribution due to the $\tilde{X}$ functional integral is an identical factor with R replaced by $1/R$, so the square root of the contribution of the doubled oscillators, which we expect to be, and is, the same as the constrained contribution, is given by

$$Z_{\mathrm{osc}} = \frac{1}{\sqrt{2\tau_2}|\eta|^2}. \tag{39}$$

To factorise the classical part of the partition function we worked in terms of P and Q and used holomorphic factorisation, and we can check that we get the same answer if we do that here. The substitution (2) introduces a Jacobian factor of $1/2$. Once the substitution is made we do the path integral for the two bosons, P and Q, just like the path integral for X and $\tilde{X}$, except for a factor of $1/2$ in the action and the more complex target space boundary conditions that P and Q inherit as a result of their definition (2) in terms of X and $\tilde{X}$. As the eigenfunctions of $\Box$, (34), do not depend on these boundary conditions (unlike the instanton pieces) the determinants for P and Q are the same as those for X and $\tilde{X}$. However, the zero-mode integral does depend on the boundary conditions: although P and Q can take any value, the periodicity condition means we should only integrate over one fundamental region, we choose the one inherited from the fundamental region for X and $\tilde{X}$, where X and $\tilde{X}$ are allowed to range from 0 to 1. The volume of this region is given by an integral over the values P can take, of the range of values Q can take for that value of P, that is

$$\int_{P=0}^{R^{-1}} 2P\,dP + \int_{P=R^{-1}}^{R} 2R^{-1}\,dP + \int_{P=R}^{R+R^{-1}} 2\left(\left(R+R^{-1}\right)-P\right)dP = 2, \tag{40}$$

cancelling the factor from the Jacobian.

The normalisation factor (32) remains the same as the additional factor of $1/2$ on the action is cancelled by the Jacobian which should also be included in the normalisation integral (32) (or rather the root of the Jacobian as there is one Jacobian to be split between both the P and the Q normalisation integrals). The P oscillator contribution is then

$$Z_{\mathrm{osc};P} = \frac{1}{\sqrt{2\tau_2}|\eta|^2}. \tag{41}$$

The Q contribution is identical, and again one can take the τ dependent holomorphic square root of the P factor and the $\bar{\tau}$ dependent anti-holomorphic square root of the Q factor and multiply them together to again get (41). Taking this together with (27) we obtain that the partition function for a boson of radius R in the doubled formalism:

$$Z = \sum_{p_L,p_R}\frac{1}{|\eta|^2}\exp\left[i\pi\tau\frac{p_L^2}{4} - i\pi\bar{\tau}\frac{q_R^2}{4}\right] \tag{42}$$

with

$$p_L = mR + \frac{n}{R}, \qquad q_R = mR - \frac{n}{R}. \tag{43}$$

This is exactly what one gets for the same boson using the undoubled formalism, as we will now calculate with our conventions.

4. The ordinary boson

In order to aid comparison with the result of the doubled formalism, we describe below the partition function of the ordinary boson at one loop using appropriate conventions so as to compare results. We proceed in the same way as above. The action is

$$S = -\pi R^2 L \wedge *L, \tag{44}$$

with $L = dX + n\alpha + m\beta$, $m, n \in \mathbb{Z}$. We can then write the instanton sum part of the partition function as

$$Z_{\text{inst}} = \sum_{m,n} \exp\left[-\pi R^2\left(n^2\frac{|\tau|^2}{\tau_2} - 2mn\frac{\tau_1}{\tau_2} + \frac{m^2}{\tau_2}\right)\right]. \tag{45}$$

Poisson resummation on m gives

$$\begin{aligned} Z_{\text{inst}} &= \sum_{n,w}\sqrt{\frac{\tau_2}{R^2}}\exp\left[-\pi R^2\frac{n^2|\tau|^2}{\tau_2} - \frac{\tau_2\pi}{R^2}\left(w - \frac{in\tau_1 R^2}{2\tau_2}\right)^2\right] \\ &= \sum_{n,w}\sqrt{\frac{\tau_2}{R^2}}\exp\left[-\pi\tau_2\left(R^2n^2 + \frac{w^2}{R^2}\right) + 2\pi inw\tau_1\right] \\ &= \sum_{n,w}\sqrt{\frac{\tau_2}{R^2}}\exp\left[i\pi\tau\frac{p_L^2}{2} - i\pi\bar{\tau}\frac{p_R^2}{2}\right], \end{aligned} \tag{46}$$

where

$$p_L = Rn + \frac{w}{R}, \qquad p_R = Rn - \frac{w}{R}. \tag{47}$$

Performing Poisson resummation on n, rather than m, leads to a the same result up to a modular transformation taking $\tau \to -1/\tau$.

Evaluation of the oscillator part of the partition function proceeds much as the previous section, leading up to (38). The only difference is that there is no factor of $1/\sqrt{2}$ due to the standard normalisation of the action, giving

$$Z_{\text{osc}} = \frac{R}{\sqrt{\tau_2}|\eta|^2}, \tag{48}$$

leading to the full partition function

$$Z = \sum_{m,n}\frac{1}{|\eta|^2}\exp\left[i\pi\tau\frac{p_L^2}{2} - i\pi\bar{\tau}\frac{p_R^2}{2}\right], \tag{49}$$

where

$$p_L = Rn + \frac{w}{R}, \qquad p_R = Rn - \frac{w}{R}. \tag{50}$$

The partition function is now invariant for $R \to 1/R$, after the relabelling of sums. In general there will be an R-dependent factor outside the exponential which is absorbed into the dilaton shift, but in the case of the torus there is no dilaton shift due to the vanishing Euler character. In our doubled calculation there was also no R-dependence in the partition function, but both the instanton and oscillator pieces were separately independent of R. For higher genus we expect both pieces of the doubled partition function to remain R-independent, but for the ordinary boson the instanton part will give higher powers of R whereas the R-dependence of the oscillator part will remain the same (this contribution effectively comes from the volume of the zero-mode but we have scaled into the target space metric). This R-dependence will give the dilaton shift which is not present in the doubled formalism, as here perturbation theory is in terms of a differently defined, T-duality invariant, dilaton [2].

5. Discussion

We have shown the equivalence of the partition sum in Hull's formulation with that of the ordinary string. Crucial to this was the inclusion of the topological term introduced by Hull in [2]. We have not included all the subtleties of quantising chiral bosons. In particular we have not discussed the possibility, crucial for nontrivial T-folds, of a non-global choice of polarisation. This is left for future work.

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Appendix A. The constraint

In the formalism of Hull, toroidally compactified dimensions (or fibres of torus fibrations) are doubled, which elucidates how T-duality acts on the theory. This, of course, doubles the number of degrees of freedom: to ensure no additional physical degrees of freedom are introduced a constraint is required to maintain the original number. For one flat compactified dimension X (with dual dimension $\tilde{X}$) this constraint is that $dX + d\tilde{X}$ be self-dual, and that $dX - d\tilde{X}$ be anti-self-dual. This can be re-expressed as $X + \tilde{X}$ is left-moving, or, once we have moved to a Euclidean world sheet, that $X + \tilde{X}$ is a holomorphic function of the complex world sheet co-ordinate $z = \sigma_1 + \tau\sigma_0$, where τ is the complex structure of the toroidal worldsheet.

The general form of the constraint (in the absence of sources) is

$$\mathcal{P}^I = L^{IJ}\mathcal{H}_{JK} * \mathcal{P}^K. \tag{A.1}$$

$\mathcal{P}^I$ are the doubled momenta, which can locally be written as $\mathcal{P}^I = d\mathbb{X}^I$, where $\mathbb{X}^I$ are the doubled torus, $I, J, \ldots$ run over the $2n$ co-ordinates of the doubled torus (note that we have not yet picked a polarisation, so there is no distinction over which half of the co-ordinates are fundamental, and which are dual). $\mathcal{H}_{IJ}$ is the positive metric on the doubled torus, and L_{IJ} is an $O(n,n)$ invariant metric and $*$ is the world sheet Hodge dual. We also need to introduce the vielbein, $\mathcal{V}$, such that

$$\mathcal{H}_{IJ} = \mathcal{V}^t{}_I{}^A \delta_{AB} \mathcal{V}^B{}_J, \tag{A.2}$$

where $A, B, \ldots$ are $O(n) \times O(n)$ indices which split into the two factors as $A = (a, a')$. In the $O(n) \times O(n)$ basis L^{AB} can be written

$$L^{AB} = \begin{pmatrix} \mathbb{1}^{ab} & 0 \\ 0 & -\mathbb{1}^{a'b'} \end{pmatrix} \tag{A.3}$$

so in terms of $\mathcal{P}^A = \mathcal{V}^A{}_I \mathcal{P}^I$ the constraint becomes

$$\mathcal{P}^a = *\mathcal{P}^a, \qquad \mathcal{P}^{a'} = - * \mathcal{P}^{a'}. \tag{A.4}$$

As long as we are dealing with a trivial bundle this implies that, in terms of the light cone co-ordinates on the world sheet, $\partial_- X^a = 0$ and $\partial_+ X^{a'} = 0$. Hence the relationship with the chiral boson. After Wick rotation these conditions become $\partial_{\bar{z}} X^a = 0$ and $\partial_z X^{a'} = 0$, allowing us to use a form of holomorphic factorisation.

We now turn to our example where we have one toroidally compactified dimension, X, of constant radius R. We introduce a dual co-ordinate, $\tilde{X}$, of radius R^{-1}. Choosing X as the original co-ordinate means we are working with a specific polarisation, that is we have chosen a basis where the co-ordinates separate into the fundamental and the dual representations of $GL(n,\mathbb{R})$ ($n = 1$ here), labelled by $\tilde{I} = ({}^i, {}_i)$. The projectors

$$\Phi^{\tilde{I}}{}_I = \begin{pmatrix} \Pi^i{}_I \\ \Pi_{iI} \end{pmatrix} \tag{A.5}$$

take us to this basis and

$$X = X^i = \Pi^i{}_I \mathbb{X}^I, \qquad \tilde{X} = X_i = \Pi_{iI}\mathbb{X}^I. \tag{A.6}$$

Also in this basis we have

$$\mathcal{H}_{\tilde{I}\tilde{J}} = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix} \tag{A.7}$$

and

$$L_{\tilde{I}\tilde{J}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.8}$$

We introduce $K^A{}_{\tilde{I}}$ via $K = \mathcal{V}\Phi^t$ which will allow us to relate things in the $O(n,n)$ basis (where the constraint is simplest) to the $GL(n,\mathbb{R})$ basis (where our original boson can be seen). As we know L and $\mathcal{H}$ in both bases we can determine that K is given by[1]

$$K^A{}_{\tilde{I}} = \frac{1}{\sqrt{2}} \begin{pmatrix} R & R^{-1} \\ R & -R^{-1} \end{pmatrix}. \tag{A.9}$$

This means that $\mathcal{P}^a$ appearing in the constraint (A.4) is related to $\mathcal{P}^i = dX$ and $\mathcal{P}_i = d\tilde{X}$ via $\mathcal{P}^a = K^a{}_{\tilde{I}}\mathcal{P}^{\tilde{I}}$, giving

$$\mathcal{P}^a = \frac{1}{\sqrt{2}}\big(R\,dX + R^{-1}\,d\tilde{X}\big), \qquad \mathcal{P}^{a'} = \frac{1}{\sqrt{2}}\big(R\,dX - R^{-1}\,d\tilde{X}\big). \tag{A.10}$$

[1] We have made some sign choices which do not affect the constraint.

We will work in terms of P and Q where $\sqrt{2}\mathcal{P}^a = dP$ and $\sqrt{2}\mathcal{P}^{a'} = dQ$. As a consequence of (A.10) they obey

$$\partial_{\bar{z}} P = 0, \tag{A.11}$$

$$\partial_z Q = 0. \tag{A.12}$$

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