NON-COMPACTLY GENERATED CATEGORIES

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1. INTRODUCTION

Let $\mathcal{T}$ be a triangulated category. An object $c \in \mathcal{T}$ is called compact if any map from it to a coproduct of objects of $\mathcal{T}$ factors through a finite coproduct. The category $\mathcal{T}$ is called compactly generated if, for every non-zero object $x \in \mathcal{T}$, there is a non-zero map $c \to x$ with $c$ compact. We produce an example of a triangulated category which is not compactly generated.

More precisely, if $\mathcal{T}$ is a triangulated category we know and love, like the category of spectra or the derived category of a commutative noetherian ring, one can produce many more by localising with respect to homology. A year and a half ago, Palmieri asked me if these can ever fail to be compactly generated. Our counterexample shows they can.

We remind the reader that examples of categories which are not compactly generated exist in the literature. Boardman proved in [1] that the dual of the category of spectra contains no compact objects at all. It is easy to see that the category of $H$-acyclic objects for a homology theory $H$ need not be compactly generated. Why is the case of the category of $H$-local objects, which we study here, so very much more interesting? What led Palmieri to specifically ask about it?

The answer is that the category of $H$-local objects is nice in many ways that the others I mentioned are not. It has a tensor product, and this tensor product has an adjoint (a mapping space). Hovey, Palmieri and Strickland extensively studied this nice situation, assuming also the existence of compact objects. In [4] they develop a rich theory. The question Palmieri asked is very natural; do the categories they studied always Bousfield localise to give more of the same?

It should be noted that the counterexample given here is not contrived. It is obtained from the derived category of a commutative, local, noetherian integral domain $R$ of dimension $\geq 2$. We simply localise $D(R)$ with respect to the homology theory given by smashing with $k + K$. Here, $k$ is the residue field, $K$ is the quotient field.

This suggests there should be plenty of other examples. The only real virtue of the particular example we chose is that it is very easily computable.

2. BACKGROUND

Let $\mathcal{T}$ be a compactly generated triangulated category with a smash product. Examples are the homotopy category of spectra, and the derived category of a commutative, noetherian ring $R$. Let $H$ be any object. Then $\mathcal{S}_H$ will be the full subcategory whose objects are $E \in \mathcal{T}$ such that $H_* E = 0$, i.e.

$$\mathcal{S}_H = \{ E \in \mathcal{T} \mid H \wedge E = 0 \}.$$ 

Two objects $H$ and $H'$ in $\mathcal{T}$ are defined to be equivalent if $\mathcal{S}_H = \mathcal{S}_{H'}$. 

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Fix $H$, and let $\mathcal{S} = \mathcal{S}_H$. Following Verdier [6], one can form the quotient category $\mathcal{S}/\mathcal{S}$, and following Bousfield [2], the natural functor

$$j^* : \mathcal{S} \to \mathcal{S}/\mathcal{S}$$

has a right adjoint

$$j_* : \mathcal{S}/\mathcal{S} \to \mathcal{S}.$$ 

Precisely, given any object $x \in \mathcal{S}$, there is the unit of adjunction

$$x \to j_* j^* x$$

which may be completed to a triangle

$$t \to x \to j_* j^* x \to \Sigma t.$$ 

Bousfield’s construction asserts that $t \in \mathcal{S}$, and $j_* j^* x$ is a $\mathcal{S}$-local object; for all $s \in \mathcal{S}$, any map $s \to j_* j^* x$ vanishes. When we want to remind ourselves of the dependence of the above on $H$ in $\mathcal{S} = \mathcal{S}_H$, we will write

$$t \to x \to j_* j^* x \to \Sigma t$$

for

$$t \to x \to j_* j^* x \to \Sigma t.$$ 

If $\mathcal{T}$ is the derived category of a noetherian, commutative ring $R$, then the author’s [5] gives a complete classification of the equivalence classes of $H \in \mathcal{T}$. Let $\text{Spec}(R)$ be the spectrum of $R$, that is the set of all prime ideals. If $p \in \text{Spec}(R)$, that is $p$ is a prime ideal of $R$, let $k(p)$ denote its residue field. Let $X \subseteq \text{Spec}(R)$ be any subset. Define

$$H_X = \prod_{p \in X} k(p).$$

Then every $H \in \mathcal{T}$ is equivalent to some $H_X$ as above, and no two $H_X$’s are equivalent to each other. What is more, the category $\mathcal{S}_H$, can be described as the smallest triangulated subcategory of $\mathcal{T}$ closed under coproducts, containing $k(p)$ for all $p \notin X$. An object $x$ is $H_X$-local if and only if, for every $p \notin X$, $\text{RHom}(k(p), X) = 0$.

Let $\mathcal{T}$ still be the derived category of a noetherian ring $R$. There are two cases in which we understand the Bousfield localisation. First, let us give the old, trivial case. Let $p$ be a prime ideal in $R$. If $X$ is the set of primes contained in $p$, then $H_X$ is equivalent to $R_p$, the localisation of $R$ at $p$. That is,

$$E \otimes H_X = 0 \iff E \otimes R_p = 0.$$ 

Recall that $R_p$ is the ring in which every element outside $p$ is inverted. The map $x \to j_* j^* x$ is just the localisation at $p$. It is

$$x \to x \otimes R_p.$$ 

In particular, the $H_X$ local objects all can be written in the form $x \otimes R_p$.

For the second case in which we understand the localisation, choose some ideal $I \triangleleft R$. Let $H = R/I$, and let $\mathcal{S}_H$ be as above, for the given choice of $H$. Then Greenlees and May [3] identified the adjoint $j_*$ of $j^*$; the unit of adjunction

$$x \to j_* j^* x$$

is just the map from $x$ to its completion. More precisely, it is the map

$$x \to \text{Holim} \ x \otimes \begin{pmatrix} R \\ \mathcal{T}^n \end{pmatrix}$$

where the tensor is, of course, the derived tensor, in the derived category.
3. THE COUNTEREXAMPLE

Let \( R \) be a commutative, noetherian, regular local ring of height 2. Let the maximal ideal be \( m \). Let \( k = k(m) \) be the residue field, and \( K = R_0 \) the fraction field of the integral domain \( R \). We wish to consider the Bousfield localisation with respect to two possible \( H \)'s, namely \( H = k = k(m) \), and \( H = k \oplus K \). The first localisation we understand, by Greenlees–May; after all, \( H = k(m) = R/m \). The second localisation will give us our counterexample.

Define therefore \( \mathcal{S}_H \) to be the full subcategory of all \( E \) with \( E \not\in k \), while \( \mathcal{S}_k \) will be the full subcategory of all \( E \) with \( E \not\in M_k = K \). Clearly, \( \mathcal{S}_H \subseteq \mathcal{S}_k \). For any object \( x \) we have a map

\[
\begin{align*}
x \to x_k
\end{align*}
\]

and a triangle

\[
\begin{align*}
t_H \to x \to x_H \to \Sigma t_H.
\end{align*}
\]

But, \( t_H \in \mathcal{S}_H \subseteq \mathcal{S}_k \), and the composite

\[
\begin{align*}
t_H \to x \to x_k
\end{align*}
\]

is a map from \( t_H \in \mathcal{S}_k \) to a \( \mathcal{S}_k \)-local object \( x_k \), and must vanish. Hence,

\[
\begin{align*}
x \to x_k
\end{align*}
\]

must factor through

\[
\begin{align*}
x \xrightarrow{f} x_H \xrightarrow{g} x_k.
\end{align*}
\]

The third edge of the triangle on \( f \) is \( t_H \in \mathcal{S}_H \). The third edge of the triangle on \( gf \) is \( t_k \in \mathcal{S}_k \).

Since \( \mathcal{S}_H \subseteq \mathcal{S}_k \), both lie in the larger \( \mathcal{S}_k \). Let us consider the triangle

\[
\begin{align*}
t \to x_H \xrightarrow{g} x_k \to \Sigma t.
\end{align*}
\]

By the octahedral axiom there is a triangle

\[
\begin{align*}
t_k \to t \to \Sigma t_H \to \Sigma t_k,
\end{align*}
\]

and from the above we know that \( t_H \) and \( t_k \) both lie in \( \mathcal{S}_k \). The triangle tells us that \( t \) must also lie in \( \mathcal{S}_k \). In other words, \( k \otimes t = 0 \). But since the ring \( R \) is regular and \( m \) is the maximal ideal, we may choose a regular sequence of generators for \( m \). That is, we may choose \( a, b \in m \) so that

\[
\begin{align*}
R/m = \left\{ \frac{R}{Ra} \right\} \otimes \left\{ \frac{R}{Rb} \right\}.
\end{align*}
\]

This expression for \( R/m = k \) allows us to identify, for every \( y \in \mathcal{I} \),

\[
\begin{align*}
y \otimes k = \Sigma^{-1} \text{RHom}(k, y).
\end{align*}
\]

Letting \( y \) be the \( t \) in the triangle

\[
\begin{align*}
t \to x_H \xrightarrow{g} x_k \to \Sigma t
\end{align*}
\]

we discover that \( \text{RHom}(k, t) = 0 \). In other words, for the unique prime ideal of height 2, \( \text{RHom}(k(p), t) = 0 \).

If \( p \) is a prime ideal of height 1, then \( k(p) \) lies in \( \mathcal{S}_H \subseteq \mathcal{S}_k \). Hence,

\[
\begin{align*}
\text{RHom}(k(p), x_H) = 0 \quad \text{and} \quad \text{RHom}(k(p), x_k) = 0
\end{align*}
\]

since \( x_H \) is \( \mathcal{S}_H \)-local, and \( x_k \) is \( \mathcal{S}_k \)-local. From the triangle we deduce that \( \text{RHom}(k(p), t) = 0 \).

We summarise this:
Lemma 3.1. In the triangle
\[ t \rightarrow x_H \rightarrow x_k \rightarrow \Sigma t \]
the object \( t \) is a direct sum of suspensions of \( K \).

Proof. We already know that for any prime ideal \( p \subset R \) of height \( > 0 \), \( \text{RHom}(k(p), t) = 0 \). This means that for any prime ideal other than 0, \( \text{RHom}(k(p), t) = 0 \). Thus, \( t \) is local for \( \mathcal{S}_k \), the category defined by \( K \otimes H = 0 \). But in this case, the local objects are of the form \( x \otimes K \), and hence are coproducts of suspensions of \( K \).

Lemma 3.2. In the triangle
\[ t \rightarrow x_H \rightarrow x_k \rightarrow \Sigma t \]
the map \( \phi \) may be identified with
\[ x_k \rightarrow x_k \otimes K \rightarrow \Sigma t_k \otimes K, \]
where \( x_k \otimes K \rightarrow \Sigma t_k \otimes K \) is obtained from the map \( x_k \rightarrow \Sigma t_k \) of the triangle
\[ t_k \rightarrow x \rightarrow x_k \rightarrow \Sigma t_k, \]
by tensoring with \( K \).

Proof. Recall the triangle
\[ t_k \rightarrow t \rightarrow \Sigma t_H \rightarrow \Sigma t_k \]
given by the octahedral axiom. Recalling that \( t_H \) lies in \( \mathcal{S}_H \) and hence has a vanishing tensor product with \( K \), we have that the natural map
\[ t_k \otimes K \rightarrow t \otimes K \]
is an isomorphism. On the other hand, by Lemma 3.1, \( t = t \otimes K \). The natural map \( x_k \rightarrow t \) in the octahedron is identified, therefore, with
\[ x_k \rightarrow x_k \otimes K \rightarrow \Sigma t_k \otimes K. \]

Lemma 3.3. Let \( y_i, i \in \mathbb{N} \) be a countable set of \( \mathcal{S}_k \) local objects. Let
\[ x = \prod_{i \in \mathbb{N}} y_i. \]
Suppose \( c \) is an object of \( \mathcal{T} \) which is compact, viewed as an object of \( \mathcal{T}/\mathcal{S}_H \). Then the map
\[ \mathcal{T}(c, x_k) \rightarrow \mathcal{T}(c, \Sigma t_k \otimes K) \]
is surjective.

Proof. The map \( x_k \rightarrow \Sigma t_k \otimes K \) was identified in Lemma 3.2 with the differential \( \phi \) in the triangle
\[ t \rightarrow x_H \rightarrow x_k \rightarrow \Sigma t. \]
We have an exact sequence
\[ \mathcal{T}(c, x_k) \rightarrow \mathcal{T}(c, \Sigma t) \rightarrow \mathcal{T}(c, \Sigma x_H) \rightarrow \mathcal{T}(c, \Sigma x_k). \]
and to prove that $\phi$ is surjective is equivalent to proving $\psi$ injective. Desuspending, pick a map

$$c \to x_H$$

so that the composite

$$c \to x_H \to x_k$$

vanishes. Now, $x_H = j_* j^* x$, and a map $c \to j_* j^* x$ in $\mathcal{F}$ is the same as a map $j^* c \to j^* x$ in $\mathcal{F}/\mathcal{F}_H$. But in $\mathcal{F}/\mathcal{F}_H$ we are assuming the object $c$ compact. The map therefore factors through a finite direct summand, which up to reordering is just

$$\bigoplus_{i=1}^n y_i \subset x = \prod_{i \in \mathbb{N}} y_i.$$ 

By hypothesis, the $y_i$'s are all $\mathcal{S}_k$ local. Hence, so is any finite coproduct of them. Therefore, the natural projection $x \to \bigoplus_{i=1}^n y_i$ localises to give a map

$$x_k \to \left\{ \bigoplus_{i=1}^n y_i \right\}_k = \bigoplus_{i=1}^n y_i.$$ 

The composite

$$c \to x_H \to x_k \to \bigoplus_{i=1}^n y_i$$

vanishes because $c \to x_H \to x_k$ does. On the other hand, $c \to x_H$ can be written as the composite

$$c \to x_H \to x_k \to \bigoplus_{i=1}^n y_i \to x_H,$$

and therefore must also vanish.

\[\square\]

Remark 3.4. Lemma 3.3 tells us that if $c$ is compact, then

$$\mathcal{F}(c, x_k) \to \mathcal{F}(c, \Sigma t_k \otimes K)$$

is surjective. But note that to give a map

$$c \to \Sigma t_k \otimes K$$

is the same as giving a map

$$c \otimes K \to \Sigma t_k \otimes K.$$ 

This is a map between sums of suspensions of $K$; such a map is entirely determined by what it does in homology. In other words, to give a map

$$c \to \Sigma t_k \otimes K$$

is nothing other than to give a map of graded groups

$$H(c) \to H(\Sigma t_k) \otimes K.$$ 

To say that any such map lifts, in the derived category, to a map

$$c \to x_k$$
asserts, among other things, that it lifts as
\[ H(c) \rightarrow H(x_k) \rightarrow H(\Sigma t_k) \otimes K. \]

**Lemma 3.5.** Let \( c \) be an object of \( \mathcal{F} \), compact when viewed as an object of \( \mathcal{F}/\mathcal{H} \). Then \( c \otimes K = 0 \).

**Proof.** In the notation of Lemma 3.3, let each \( y_i, i \in \mathbb{N} \) be \( y_i = \hat{R} \), the completion of \( R \) at the ideal \( m \). Then \( x \) is the coproduct of the \( y_i \)'s, and \( x_k \) its completion. It is the \( \text{Holim} \) of the sequence \( x \otimes R/m^n \). Since the sequence is Mittag–Loeffer, the \( \text{lim}^1 \) vanishes. Thus, \( x_k \) is an ordinary \( R \)-module, concentrated in degree 0. Furthermore, the map \( x \rightarrow x_k \) of \( x \) to the inverse limit is clearly injective. We get a short exact sequence of ordinary modules
\[ 0 \rightarrow x \rightarrow x_k \rightarrow \Sigma t_k \rightarrow 0. \]

It is possible to describe \( x_k \) very concretely. It embeds in the product
\[ x_k \subseteq \prod_{i \in \mathbb{N}} y_i \]
and an element of the product lies in \( x_k \) if and only if, for every \( n > 0 \), modulo \( m^n \) only finitely many terms are non-zero. Let \( a \) and \( b \) be generators for the maximal ideal \( m \subset R \). Then the element
\[ \prod_{i \in \mathbb{N}} a^i \in \prod_{i \in \mathbb{N}} y_i \]
is an example of an element in \( x_k \) which is not in the image of \( x \). Furthermore, no multiple of it is. It defines a non-zero element \( \pi \in \Sigma t_k \otimes K \).

Suppose now that \( c \otimes K \neq 0 \). Suspending if necessary, we may assume \( H^0(c) \otimes K \neq 0 \). Choose any non-zero map \( H^0(c) \otimes K \rightarrow K \), and replacing it by a multiple if necessary, assume that 1 lies in the image of the composite
\[ H^0(c) \rightarrow H^0(c) \otimes K \rightarrow K. \]

Now, define the map \( K \rightarrow \Sigma t_k \otimes K \) by sending 1 to \( \pi/b \). The composite map is
\[ H^0(c) \rightarrow \Sigma t_k \otimes K, \]
and \( \pi/b \) is in the image. By Remark 3.4 we must have a factorisation
\[ H^0(c) \rightarrow H^0(x_k) = x_k \rightarrow \Sigma t_k \otimes K, \]
and therefore \( \pi/b \) would lie in the image of \( x_k \rightarrow \Sigma t_k \otimes K \). To say that \( \pi/b \) lies in the image is to say that, except for finitely many terms, the elements \( a^i/b \) lie in \( \hat{R} \). This is obviously not the case, since \( \hat{R} \) is a unique factorisation domain and \( a \) and \( b \) are distinct primes. \( \Box \)

**Proposition 3.6.** The category \( \mathcal{F}/\mathcal{H} \) is not compactly generated.

**Proof.** Let \( c \) be an object of \( \mathcal{F} \), compact in \( \mathcal{F}/\mathcal{H} \). The object \( K \) is \( \mathcal{H} \) local. That is, \( K = j_* j^* K \). To give a map in \( \mathcal{F}/\mathcal{H} \) of the form \( j^* c \rightarrow j^* K \) is to give a map in \( \mathcal{F} \) of the form \( c \rightarrow j_* j^* K = K \). But this is equivalent to giving a map \( c \otimes K \rightarrow K \), and since by Lemma 3.5 \( c \otimes K = 0 \), all such maps vanish. There is no non-zero map \( c \rightarrow K \), for any compact object \( c \in \mathcal{F}/\mathcal{H} \). The category is not compactly generated. \( \Box \)
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