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AN ASSEMBLY CELL WITH AN AUTOMATED QUALITY CONTROL STATION: A FINITE CAPACITY AND GENERALLY DISTRIBUTED PROCESSING TIMES

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Abstract—The performance of an assembly cell consisting of a set of machines, a finite capacity local storage, an automated quality control station, a loading station and an unloading station is modeled by an $M/G/1K$ queuing system with a Poisson input, a single server, generally distributed processing times, a buffer of size K, the first come first served queuing discipline, and with a fixed delay Bernoulli feedback mechanism, in steady state. In a queuing system with a fixed delay Bernoulli feedback mechanism, a fraction of the departing units will merge with the incoming arrival process to be reprocessed, after being delayed for a fixed length of time. The performance this system is approximated by a recursive algorithm. Furthermore, approximation outcomes are compared against those from a simulation study.

1. INTRODUCTION

The primary objectives of this paper are to develop a model and test an approximation algorithm for analyzing the performance of a finite capacity assembly cell (see Fig. 1). This model could then be used to quantify the effect of the quality control mechanism on the performance of the assembly cell. In this system, the assembly cell consists of a set of machines, capable of processing workpieces belonging to the same family of parts, contains a local storage, and is placed adjacent to an inspection station. In this paper, it is assumed that no matter how many stages of operation each workpiece requires at different machines of the assembly cell, the aggregated duration of the processing time at the work station can be approximated by a generally distributed random variable. This aggregation assumption is based on the group technology concept. That is, the assembly cell is intelligently designed such that a compatible group of machines are placed in it for processing various states of operation of a compatible group of workpieces. For a review of the literature of the group technology, see Waghodekar and Sahu [1]. Additionally, the inspection station is used to identify the defective parts. In general, the workpieces which are transported to the assembly cell, after being processed, leave the assembly cell and arrive at the inspection station; there after being inspected, the non-defective workpieces leave the system and the defective workpieces after a fixed inspection time are rerouted to the assembly cell to be reprocessed.

To develop the model, the performances of the assembly cell and the inspection station are jointly modeled by a finite capacity $M/G/1$ queuing system with a Poisson input, a single server, generally distributed processing times, the first come first served queuing discipline (e.g. dispatching rule), and a Bernoulli feedback mechanism. That is, the performance of the inspection station is modeled by a Bernoulli filter which operates as a binary mechanism for identifying defective or non-defective workpieces. In a queuing system with a fixed delay Bernoulli feedback mechanism, a fraction of the departing units will merge with the incoming arrival units to be reprocessed, after being delayed for a fixed length of time. It is noted that this problem has never been considered in the literature. However, Pourbabai [2] has considered a special case of this problem with an infinite capacity local storage.

The motivation behind developing the proposed algorithm is that the performance of the above system is too complex and its performance cannot be analytically quantified. Hence, to be able to understand how this system behaves, a heuristic algorithm is proposed.

Queuing systems with a Bernoulli feedback mechanism have previously been considered in the literature. For the characterization of an $M/M/1$ queuing system with an instantaneous Bernoulli feedback mechanism, see Burke [3]. For an $M/G/1$ queuing system with an instantaneous Bernoulli feedback mechanism, see Disney *et al.* [4]. It is noted that in the literature, there exist no exact or heuristic algorithm for analyzing finite capacity queuing systems with Bernoulli feedback. For a review of the performance modeling literature of the manufacturing systems, see Buzacott and Yao [5].

The organization of this paper is as follows. In Section 2, the components of the model are delineated. In Section 3, numerical results are presented. Finally, in Section 4, the concluding remarks are discussed.

2. THE MODEL

2.1. Assumptions

In this paper, the following assumptions are made. First, no matter how many stages of operation each workpiece requires at different machines of the assembly cell, the aggregated duration of the processing time at the assembly cell can be approximated by a generally distributed random variable. This aggregation assumption is based on the group technology concept. Second, The loading process is a Poisson process. Third, the non-renewal superposition arrival process can be approximated with a Poisson process. Fourth, the flow of the defective parts can be approximated with a thinned Poisson process. Fifth, the fixed inspection time does not influence the performance of the system in a steady state. Sixth, the workpieces which find the assembly system full are permanently discarded. That is, the blocked (e.g. the overflow) units leave the system. Seventh, the thinning (e.g. defective) probability is sufficiently small (e.g. less than 0.3).

2.2. Notation

Throughout the paper, the following notation will be used. Let j be the iteration (e.g. superposition) index. It is noted that in the proposed algorithm, the iteration index counts the number of times a fraction of the departing units will merge with the arrival units; μ be the service rate; λ^a be the arrival rate; c by the coefficient of variation (c.v.) of the distribution of the service time, which is equal to the product of the service rate and the standard deviation of the service times; λ_j^o be the overflow rate at iteration j; λ_j^s be the superposition arrival rate at iteration j; λ_j^d be the departure (e.g. throughput) rate at iteration j; $\hat{\lambda}_j^d$ be the departure rate of the thinned departure process which will superimpose with the incoming arrival process at iteration j; c_j^d be the c.v. of the distribution of the interdeparture time of the net departure process at iteration j; $\tilde{\lambda}_j^d$ and $\tilde{c}_j^d$ be the actual depare rate from the system and the corresponding c.v. of the distribution

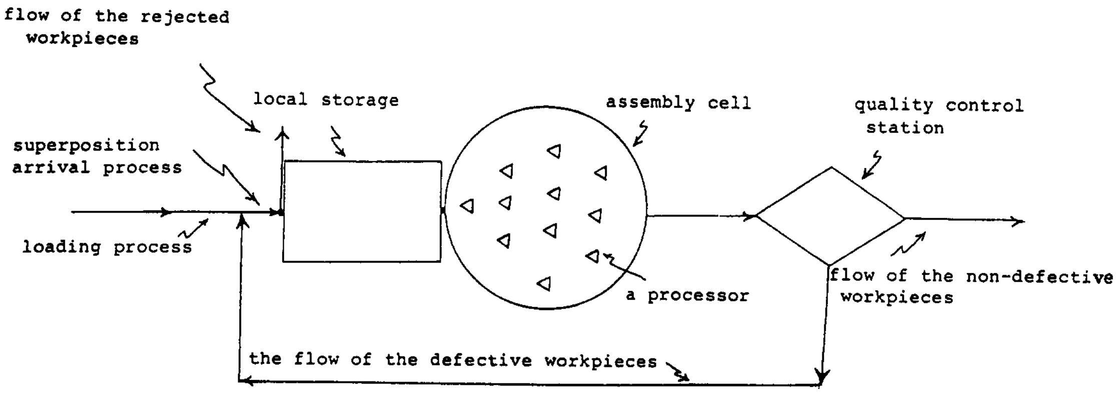

Fig. 1. An assembly system consisting of an assembly cell and a quality control station.

of the interdepature time at iteration j; K be the number of waiting spaces; p be the fraction of the departing units which will merge with the incoming arrival units; ρ_j be the utilization factor at iteration j; q_j be the blocking probability (i.e. the probability that an arbitrary unit at the instant of its arrival at the system finds it full) at iteration j; and z be the length of the inspection time. Also, let λ_j^d, C_j^d, and $D_j(t)$ be the departure rate, c.v. of the distribution of the interdeparture time, and the distribution of the interdeparture time at iteration j, respectively.

2.3. The algorithm

Before describing the algorithms, at every iteration, a fraction of departing units will merge with the incoming arrival units to form a new arrival process. We call the latter process the superposition arrival process.

To approximate the performance of an $M/G/1/K$ queuing system with a fixed delay Bernoulli feedback mechanism in steady state, the following steps will be implemented.

Step 1. At iteration j, the departure process will be approximated based on at least the first two moments of the distribution of the interdeparture time, and the departure process will be treated as a compatible renewal process. That is, the dependencies among the interdeparture times will be ignored, and the distribution of the interdeparture time will be approximated with a compatible (e.g. a phase type) distribution function. For further details, see Appendices A and B.

Step 2. At iteration j, the non-renewal departure process corresponding to the departing units which will merge with the incoming arrival units will be approximated by a Poisson process, based on Gnedenko and Kovalenko's [6] results on thinning of a renewal process. The parameter of $\hat{\lambda}_j^d$ of the thinned departure process which will superimpose with the incoming arrival process can be obtained, as follows. That is, the dependencies among the interdeparture times of the thinned departure process will be ignored and an exponential distribution will be used to approximate the distribution of the interdepature time of the thinned departure process:

$$\hat{\lambda}_j^d = p\lambda_j^d; \quad j > 1. \tag{1}$$

Step 3. At iteration j, the length of the inspection time will be set equal to zero (e.g. $z = 0$), and the thinned departure process will be superimposed with the incoming Poisson arrival process to form a new superposition Poisson arrival process at iteration $j+1$. That is, the dependencies among the interarrival times will be ignored, and the c.v. of the distribution of the interarrival time will be set equal to one. Based on a simulation analysis, as will be shown in the next section, it has been demonstrated that the value of z does not influence the approximation results in steady state. Notice that because the thinned departure process is a non-renewal process and because of the fixed delay feedback time, the superposition arrival process becomes a non-renewal process. However, it will be approximated with a Poisson process. That is, the distribution of the interarrival time will be approximated with a compatible exponential distribution function and the dependencies among the interarrival times will be ignored. The resulted superposition Poisson arrival process can be approximated based on the following parameter (notice that at iteration $j = 1$, $\lambda_1^s = \lambda^a$):

$$\lambda_{j+1}^s = \hat{\lambda}_j^d + \lambda^a; \quad j > 1. \tag{2}$$

Step 4. The previous steps will be repeated until a steady state is reached at iteration e. The steady state is identified as follows. Let δ be a sufficiently small value (e.g. 1×10^{-3}) and

$$e = \min(j; \lambda_{j+1}^s - \lambda_j^s \leqslant \delta). \tag{3}$$

Step 5. After reaching the steady state, the performance of the system can be evaluated based on the parameters of the overflow rate, the blocking probability, the service utilization factor, the departure rate, the thinned departure rate, and the c.v. of the distribution of the interdeparture time, as follows:

$$\lambda_e^{\mathrm{o}} = \lambda_e^{\mathrm{s}} - \lambda_e^{\mathrm{d}}, \tag{4}$$

$$q_e = \lambda_e^{\mathrm{o}}/\lambda_e^{\mathrm{s}}, \tag{5}$$

$$\rho_e = \lambda_e^{\mathrm{d}}/\mu, \tag{6}$$

$$\tilde{\lambda}_e^{\mathrm{d}} = (1-p)\,\lambda_e^{\mathrm{d}}, \tag{7}$$

$$\tilde{c}_j^{\mathrm{d}} = [p + (1-p)\,(c_e^{\mathrm{d}})^2]^{1/2}. \tag{8}$$

The c.v. of the distribution of the interdeparture time can be obtained from Appendices A and B. Also, for a summary of the approximation steps, see Fig. 2.

3. NUMERICAL RESULTS

In this section, several examples are presented for $K = 2$, $\mu = 1.0$, $p = 0.1$. It is noted that $K = 2$ was selected because the cost of simulation was expensive and consequently it was decided that it is more interesting to observe the performance of the system with a small buffer rather than a large buffer. Furthermore, the approximation outcomes are compared against those from a simulation study. The numerical results are presented in Tables 1 and 2. In addition, to investigate the accuracy of the approximation outcomes, a simulation model is also developed, using the SLAM simulation package of Pritsker and Pegden [7]. Each simulation outcome is obtained based on ten independent runs. To generate the simulation outcomes, the service times were assumed to have been generated based on one of the following three distributions: a hyperexponential distribution with two parameters and balanced means [i.e. expressions (A.1)–(A.4)], or an exponential distribution, or a shifted exponential distribution [(i.e. expressions (A.5)–(A.7)]. The approx. 95% confidence intervals for the reported measures are obtained based on a replication method and are also given in Tables 1 and 2. In Tables 1 and 2, for λ^{a} and $K = 2$, three sets of

Table 1. Approximation and simulation outcomes for $K = 2$, $\mu = 1.0$, and $p = 0.1$

Approximation					Simulation		
λ^{a}	c	λ_e^{d}	c_e^{d}	λ^{0}	λ_e^{d}	c_e^{d}	λ_e^{0}
0.3	0.5	0.329	0.950	0.004	0.328 ± 0.003*	1.024 ± 0.009	0.004 ± 0.001
					0.328 ± 0.001**	0.953 ± 0.002	0.004 ± 0.001
					0.328 ± 0.001***	0.949 ± 0.003	0.004 ± 0.001
0.5	0.5	0.525	0.864	0.027	0.525 ± 0.001	0.919 ± 0.001	0.027 ± 0.001
					0.524 ± 0.001	0.864 ± 0.002	0.027 ± 0.001
					0.524 ± 0.003	0.865 ± 0.002	0.025 ± 0.002
0.7	0.5	0.681	0.761	0.087	0.682 ± 0.004	0.799 ± 0.007	0.086 ± 0.003
					0.681 ± 0.001	0.760 ± 0.002	0.086 ± 0.002
					0.681 ± 0.003	0.761 ± 0.001	0.086 ± 0.001
0.3	1.0	0.324	0.983	0.008	0.324 ± 0.001	1.055 ± 0.007	0.007 ± 0.001
					0.324 ± 0.001	1.037 ± 0.001	0.008 ± 0.001
					0.324 ± 0.001	0.983 ± 0.002	0.008 ± 0.001
0.5	1.0	0.505	0.958	0.045	0.508 ± 0.001	1.008 ± 0.002	0.042 ± 0.001
					0.505 ± 0.001	0.958 ± 0.002	0.045 ± 0.001
					0.504 ± 0.001	0.957 ± 0.002	0.045 ± 0.001
0.7	1.0	0.642	0.941	0.112	0.649 ± 0.001	0.975 ± 0.003	0.116 ± 0.004
					0.642 ± 0.002	0.941 ± 0.002	0.121 ± 0.002
					0.642 ± 0.002	0.941 ± 0.002	0.122 ± 0.002
0.3	1.5	0.317	1.032	0.014	0.318 ± 0.001	1.102 ± 0.001	0.013 ± 0.001
					0.317 ± 0.001	1.032 ± 0.002	0.014 ± 0.001
					0.317 ± 0.001	1.032 ± 0.002	0.014 ± 0.001
0.5	1.5	0.485	1.084	0.063	0.491 ± 0.003	1.133 ± 0.002	0.057 ± 0.001
					0.485 ± 0.001	1.083 ± 0.003	0.063 ± 0.002
					0.484 ± 0.001	1.085 ± 0.003	0.063 ± 0.001
0.7	1.5	0.612	1.147	0.149	0.621 ± 0.002	1.183 ± 0.003	0.140 ± 0.001
					0.611 ± 0.001	1.147 ± 0.004	0.150 ± 0.002
					0.612 ± 0.002	1.146 ± 0.002	0.150 ± 0.001

*$z = 0$; **$z = 10$; ***$z = 100$.

Table 2. Approximation and simulation outcomes for $K = 2$, $\mu = 1.0$, and $p = 0.1$

Approximation					Simulation		
λ^a	c	λ_e^d	c_e^d	λ_e^o	λ_e^d	c_e^d	λ_e^o
0.3	2.0	0.310	1.096	0.020	0.312 ± 0.001*	1.162 ± 0.002	0.018 ± 0.001
					0.310 ± 0.001**	1.098 ± 0.002	0.020 ± 0.001
					0.309 ± 0.001***	1.098 ± 0.002	0.021 ± 0.001
0.5	2.0	0.469	1.228	0.078	0.468 ± 0.001	1.229 ± 0.005	0.078 ± 0.001
					0.469 ± 0.001	1.229 ± 0.005	0.077 ± 0.001
					0.469 ± 0.001	1.229 ± 0.005	0.077 ± 0.001
0.7	2.0	0.590	1.364	0.169	0.600 ± 0.002	1.404 ± 0.008	0.158 ± 0.001
					0.590 ± 0.002	1.364 ± 0.008	0.168 ± 0.001
					0.589 ± 0.001	1.363 ± 0.009	0.168 ± 0.001
0.3	2.5	0.304	1.173	0.026	0.305 ± 0.001	1.174 ± 0.003	0.025 ± 0.001
					0.305 ± 0.001	1.174 ± 0.003	0.025 ± 0.001
					0.304 ± 0.001	1.172 ± 0.004	0.026 ± 0.001
0.5	2.5	0.457	1.386	0.088	0.463 ± 0.001	1.435 ± 0.010	0.082 ± 0.001
					0.459 ± 0.001	1.391 ± 0.012	0.086 ± 0.001
					0.457 ± 0.001	1.388 ± 0.010	0.088 ± 0.001
0.7	2.5	0.575	1.591	0.182	0.586 ± 0.002	1.633 ± 0.016	0.171 ± 0.002
					0.576 ± 0.002	1.592 ± 0.016	0.180 ± 0.001
					0.574 ± 0.002	1.589 ± 0.014	0.182 ± 0.002
0.3	3.0	0.300	1.261	0.030	0.302 ± 0.001	1.321 ± 0.005	0.027 ± 0.001
					0.300 ± 0.001	1.264 ± 0.005	0.029 ± 0.001
					0.299 ± 0.001	1.262 ± 0.005	0.030 ± 0.001
0.5	3.0	0.449	1.555	0.096	0.456 ± 0.001	1.606 ± 0.010	0.089 ± 0.001
					0.450 ± 0.001	1.564 ± 0.013	0.094 ± 0.001
					0.447 ± 0.001	1.556 ± 0.012	0.096 ± 0.001
0.7	3.0	0.564	1.825	0.192	0.563 ± 0.001	1.823 ± 0.012	0.192 ± 0.002
					0.576 ± 0.001	1.830 ± 0.014	0.190 ± 0.002
					0.576 ± 0.001	1.829 ± 0.013	0.109 ± 0.001

*$z = 0$; **$z = 10$; ***$z = 100$.

simulation outcomes are reported, based on 100,000 observations. Notice that in Tables 1 and 2 the service rate is set equal to one. Hence, the values of the departure rate from the system and the service utilization factor are identical [see expression (9)]. Also, from the reported performance measures the value of the blocking probability can also be evaluated [see expression (8)]. Furthermore, the expressions which were used to obtain the second moment of the distribution of the interdeparture time for every one of the examples in this section are presented in Appendix B, which in conjunction with expression (B.2) result in the expression for the c.v. of the distribution of the interdeparture time.

From Tables 1 and 2, it is evident that the approximation outcomes closely match the simulation results. Furthermore, the following results are observed from Tables 1 and 2, for $K = 2$: firstly, the length of time required for a departing unit to travel prior to merging with the incoming arrival units does not significantly influence the values of none of the performance measures; secondly, for each value of the c.v. of the distribution of the service time, as the arrival rate increases, the values of both λ_e^o and λ_e^d increase, but the value of c_e^d decreases; thirdly, for each value of the arrival rate, as the c.v. of the distribution of the service time increases, the value of λ_e^d decreases, but the values of both λ_e^o and c_e^d increase.

4. CONCLUDING REMARKS

In this paper a performance model is presented for analyzing the effect of the quality control on the performance of an automated assembly cell. For this purpose, an efficient algorithm for approximating the performance of an $M/G/1/K$ queuing system with a fixed delay Bernoulli feedback mechanism is suggested. Furthermore, the parameters of the departure process of the queuing system are also accurately approximated, which can be used to study the tandem behavior of the system.

It is reminded that in an $M/G/1/K$ queuing system with an instantaneous Bernoulli feedback mechanism, the superposition arrival process is a Poisson process. Hence, all the reported results in this paper could have been obtained by exact analysis. But, in an $M/G/1/K$ queuing system with a fixed delay Bernoulli feedback mechanism, the superposition arrival process is not even a renewal process. Thus, developing an approximation algorithm seems to be the only available

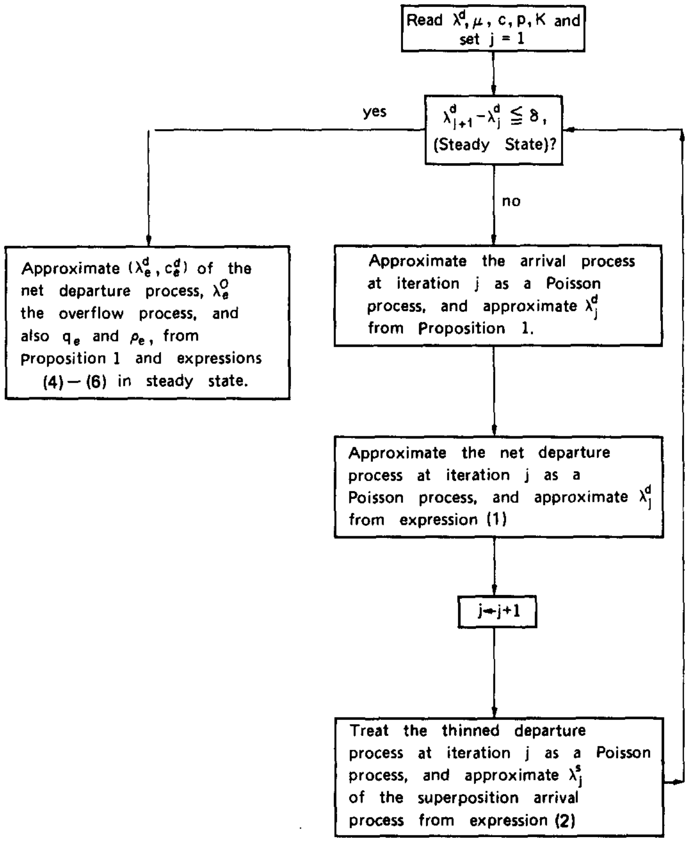

Fig. 2. Flowchart of the proposed algorithm.

solution technique. It is important to note that as numerically demonstrated in Tables 1 and 2, the performances of the $M/G/1/K$ queuing systems with an instantaneous or a fixed delay Bernoulli feedback mechanism are approximately identical, in steady state.

It is interesting to note that approximating the superposition arrival process by a Poisson process has not significantly influenced the approximation accuracy of our algorithm. If in the future, the distribution or the interdeparture time of a $G/G/1/K$ queuing system with generally distributed interarrival time is characterized, then the proposed algorithm can be improved by using at least the first two moments of both the respective distributions of the thinned departure process and the original arrival process.

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APPENDIX A

As discussed in Kuehn [8] and Whitt [9], to approximate the distribution of the interarrival time of a generic stationary arrival process with the arrival rate λ and the c.v. of the distribution of the interarrival time $c \geqslant 1$, the following hyperexponential distribution function with two parameters and balanced means can be used:

$$H_2(\theta, \bar{\gamma}; t) = 1 - \theta\, e^{-\gamma_1 t} - (1-\theta)\, e^{-\gamma_2 t}, \quad t \geqslant 0, \tag{A.1}$$

where the shape parameter is

$$\theta = \left[\left(\frac{c^2-1}{c^2+1}\right)^{1/2} + 1\right] \Big/ 2 \tag{A.2}$$

and the intensity parameters are

$$\gamma_1 = 2\theta\lambda, \tag{A.3}$$

$$\gamma_2 = 2(1-\theta)\lambda. \tag{A.4}$$

On the other hand, when $0 \leqslant c \leqslant 1$, the following shifted exponential distribution function can be used:

$$M'(b, \beta; t) = 1 - e^{-\beta(t-b)}, \quad t \geqslant b, \tag{A.5}$$

where the intensity parameter is

$$\beta = \frac{\lambda}{c} \tag{A.6}$$

and the shift parameter is

$$b = \frac{1}{\lambda} - \frac{1}{\beta}. \tag{A.7}$$

APPENDIX B

These results are also discussed in Pourbabai [10]. Let the service times $\{S_n\}$ be independent and identically distributed, where, $\Pr\{S_n \leqslant t\} = B(t)$, $t \geqslant 0$ and for all n. Potential units arrive at successive epochs of a stationary counting arrival process with interarrival times $\{t_i\}$, where $\Pr\{T_i \leqslant t\} = A(t)$, $A(0+) = 0$. An arrival finding a queue with K units waiting, does not enter the queuing system, and is considered lost unit (e.g. such a unit is not considered as part of the departure process). Furthermore, let $\{Q_n\}$ be the queue sizes at successive service completion epochs, denoting its stationary distribution by $\pi_j = P(Q_n = j)$, $0 \leqslant j \leqslant K$, for all n. Also, let μ, c, $D(t)$, λ^d, and c^d be the service rate and the c.v. of the distribution of the service time, the distribution of the interdeparture time, the departure rate, and the cv. of the distribution of the interdeparture time, respectively. Furthermore, let $D(t)$, $D^*(s)$, and $(-1)^m D^{*m}(0)$ be the stationary distribution function of the interdeparture time, its Laplace–Stieltjes transform, and its m^{th} moment, respectively. Then,

$$\lambda^d = -[D^{*\prime}(0)]^{-1}, \tag{B.1}$$

$$c^d = \lambda^d\{D^{*\prime\prime}(0) - [D^{*\prime}(0)]^2\}^{1/2}, \tag{B.2}$$

where

$$D^*(s) = \int_0^\infty e^{-st}\, dD(t). \tag{B.3}$$

Proposition 1

Consider an $M/G/1/K$ queuing system. Let D_n be the interdeparture interval between the epochs defining Q_n and Q_{n+1}, and let X be a generic interarrival random variable, exponentially distributed with mean $1/\lambda'$ independent of Q_n and S_{n+1}. Then,

$$D_n \triangleq \begin{cases} S_{n+1}, & \text{if } Q_n > 0, \quad \text{(B.4)} \\ S_{n+1} + X, & \text{if } Q_n = 0. \quad \text{(B.5)} \end{cases}$$

The distribution of the interdeparture time is

$$\Pr(D_n \leqslant t) = B(t) - \pi_0 \int_0^t e^{-(t-u)}\, dB(u). \tag{B.6}$$

π_0 can be obtained by solving the following set of equations:

$$\pi_i = \begin{cases} \pi_0 k_i + \sum_{j=1}^{i+1} \pi_j k_{i-j+1}, & 0 \leqslant i \leqslant K-1, \quad \text{(B.7)} \\ 1 - \pi_0 \sum_{n=0}^{K-1} k_n - \sum_{j=1}^{K} \pi_j \sum_{n=0}^{K-j} k_n, & i = K, \quad \text{(B.8)} \end{cases}$$

where

$$k_i = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^i}{i!}\, dB(t), \quad 0 \leqslant i \leqslant K. \tag{B.9}$$

Proof. For expressions (B.6) and (B.7–B.8), see Daley and Shanbhag [11] and Gross and Harris [12, pp. 251, 229], respectively.

Proposition 2

Let

$$A(t) = 1 - e^{-\lambda t}, \quad t \geqslant 0, \tag{B.10}$$

$$B(t) = M'(\mu', b; t) = 1 - e^{-\mu'(t-b)}, \quad t \geqslant b, \tag{B.11}$$

$$\mu' = \frac{\mu}{c}, \tag{B.12}$$

$$b = \frac{1}{\mu} - \frac{1}{\mu'}. \tag{B.13}$$

Then, the first two moments of the distribution of the interdeparture times from an $M/M'/1/K$ queuing system are

$$-D^{*\prime}(0) = \frac{\pi_0}{\lambda} + \frac{1}{\mu'} + b, \tag{B.14}$$

$$D^{*\prime\prime}(0) = \frac{2\pi_0}{\lambda^2} + \frac{2\pi_0}{\lambda}\left(\frac{1}{\mu'} + b\right) + \frac{2}{\mu'^2} + \frac{2b}{\mu'} + b^2, \tag{B.15}$$

where π_0 is obtained from Proposition 1, and

$$k_i = \frac{\mu'\lambda^i}{(\lambda + \mu')^{i+1}} e^{-\mu' b} \sum_{i=0}^{i} \frac{[b(\lambda + \mu')]^i}{(i)!}, \quad 0 \leqslant i \leqslant K. \tag{B.16}$$

Proof. Directly obtained Proposition 1.

Proposition 3

Let

$$A(t) = 1 - e^{-\lambda t}, \quad t \geqslant 0, \tag{B.17}$$

$$B(t) = H_2(\tau, \bar{\beta}; t) = 1 - \tau e^{-\beta_1 t} - (1 - \tau) e^{-\beta_2 t}, \quad t \geqslant 0, \tag{B.18}$$

$$\tau = \left[\left(\frac{(c)^2 - 1}{(c)^2 + 1}\right)^{1/2} + 1\right] \Big/ 2, \tag{B.19}$$

$$\beta_1 = 2\tau\mu, \tag{B.20}$$

$$\beta_2 = 2(1 - \tau)\mu. \tag{B.21}$$

Then, the first two moments of the distribution of the interdeparture times from an $M/H_2/1/K$ queuing system are

$$-D^{*\prime}(0) = \frac{\pi_0}{\lambda} + \frac{\tau}{\beta_1} + \frac{(1-\tau)}{\beta_2}, \tag{B.22}$$

$$D^{*\prime\prime}(0) = \frac{2\pi_0}{\lambda^2} + \frac{2\pi_0}{\lambda}\left(\frac{\tau}{\beta_1} + \frac{1-\tau}{\beta_2}\right) + \frac{2\tau}{\beta_1^2} + \frac{2(1-\tau)}{\beta_2^2}, \tag{B.23}$$

where π_0 is obtained from Proposition 1, and

$$k_i = \frac{\tau\beta_1\lambda^i}{(\lambda + \beta_1)^{i+1}} + \frac{(1 - \tau)\beta_2\lambda^i}{(\lambda + \beta_2)^{i+1}}, \quad 0 \leqslant i \leqslant K. \tag{B.24}$$

Proof. Directly obtained from Proposition 1.