



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: March 27, 2015

REVISED: May 29, 2015

ACCEPTED: July 31, 2015

PUBLISHED: September 10, 2015

# Realistic three-generation models from $SO(32)$ heterotic string theory

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**ABSTRACT:** We search for realistic supersymmetric standard-like models from  $SO(32)$  heterotic string theory on factorizable tori with multiple magnetic fluxes. Three chiral generations of quarks and leptons are derived from the adjoint and vector representations of  $SO(12)$  gauge groups embedded in  $SO(32)$  adjoint representation. Massless spectra of our models also include Higgs fields, which have desired Yukawa couplings to quarks and leptons at the tree-level.

**KEYWORDS:** Supersymmetry Phenomenology, Strings and branes phenomenology

**ARXIV EPRINT:** [1503.06770](https://arxiv.org/abs/1503.06770)

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## 1 Introduction

Superstring theory is a good candidate for the unified theory of the gauge and gravitational interactions, and quark, lepton and Higgs fields. Indeed, there have been several approaches to derive the realistic string vacua by comparing the theoretical predictions with the data of the cosmological observations as well as the collider experiments which are known as the subjects of string cosmology and phenomenology.

Beginning with the work of ref. [1], there are much progresses to find the standard-like models from the  $E_8 \times E_8$  heterotic string theory instead of the SO(32) heterotic string theory. (See for a review, e.g. [2].) This is because  $E_8$  gauge group involves several candidates of the grand unified groups such as  $E_6$ , SO(10) and SU(5) as the subgroups of  $E_8$  and the  $E_8$  adjoint representation includes matter representations such as **27** of  $E_6$ , **16** of SO(10) and **10** and **5** of SU(5). However, in SO(32) heterotic string theory, for example, the **16** spinor representation of SO(10) is not involved in the adjoint representation of SO(32). (In the framework of toroidal  $Z_N$  orbifold, there are some possibilities to obtain the spinor representation of SO groups as discussed in ref. [3, 4].) Therefore, as one of the procedures to find the realistic string vacua, we try to derive the (non-)supersymmetric standard-like models from the SO(32) heterotic string theory without going through the grand unified groups. This approach might be useful to search for the realistic standard model, because the standard-like model given through the decomposition of GUT groups

have the extra matters which should be decoupled from the low-energy dynamics in terms of some non-trivial mechanisms.

The standard model is a chiral theory. Thus, the key point to realize the standard model is how to realize a chiral theory. Toroidal compactification is simple, but it can not realize a chiral theory unless introducing additional backgrounds. Orbifold and Calabi-Yau compactifications can lead to a chiral theory. Toroidal compactification with magnetic fluxes can also lead to a chiral theory. Here, we study such a background. That is, our key ingredients are the multiple  $U(1)$  magnetic fluxes inserted into  $SO(32)$  gauge group. These magnetic fluxes are first discussed in ref. [1], where the  $SU(5)$  grand unified groups can be realized from the  $SO(32)$  heterotic string theory. Furthermore, there are much progresses on the resolved toroidal orbifold in [5] and on more general Calabi-Yau manifolds for  $E_8 \times E_8$  and/or  $SO(32)$  heterotic string theory via the spectral cover construction [6, 7] and the extension of it [8] (see e.g., refs. [9–12]).<sup>1</sup>

In this paper, we study  $SO(32)$  heterotic string theory on six-dimensional (6D) torus with magnetic fluxes, which is one of the simplest compactifications leading to a chiral theory. Then, we search the models, where the unbroken gauge group includes  $SU(3) \times SU(2) \times U(1)_Y$  and massless spectra correspond to three chiral generations of quarks and leptons.

The paper is organized as follows. In section 2, we show our set-up and typical theoretical constraints which are required from the consistency of heterotic string theory. For example, in the standard embedding scenario of the Calabi-Yau compactification, the internal gauge backgrounds are set to be equal to spin connections of the Calabi-Yau manifolds. On the other hand, in the non-standard embedding scenario, the gauge fields are not always identified as the spin connections of the internal manifold due to the existence of the fluxes. We discuss the consistency conditions for such fluxes on 6D torus in section 2.1. In addition, the  $U(1)_Y$  gauge boson should be massless, even if the consistent fluxes are inserted into  $SO(32)$  gauge groups in order to derive the standard-like model gauge groups. Generically,  $U(1)$  gauge bosons appeared in the low-energy effective theory couple to the universal and Kähler axions through the ten-dimensional (10D) Green-Schwarz term [15] which implies that the linear combination of  $U(1)$  gauge bosons may absorb these axions by their Stueckelberg couplings and become massive. Thus the axionic couplings of  $U(1)_Y$  gauge boson should be absent, otherwise  $U(1)_Y$  gauge boson would become massive as discussed in section 2.2. In addition to the merits of gauge symmetry breaking, the fluxes are important tools to realize the degenerate zero-modes, i.e., three generations of the elementary particles. In fact, in section 2.3, the chiral theory with degenerate zero-modes can be obtained from the considerations of zero-mode wavefunctions on tori. At the same time, the existence of four-dimensional (4D)  $\mathcal{N} = 1$  supersymmetry (SUSY) depends on the ansatz of  $U(1)$  fluxes due to the flux-induced Fayet-Iliopoulos terms.

In section 3.1, we discuss the concrete embeddings of the standard model gauge groups into  $SO(32)$  gauge group in terms of the multiple  $U(1)$  fluxes. The correct matter contents

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<sup>1</sup>Also the low-energy massless spectra were studied within the ten-dimensional  $E_8 \times E_8$  theory on torus with magnetic fluxes from the field-theoretical viewpoint [13, 14].

of the standard model are then derived from the adjoint and vector representations of  $SO(12)$  given by the subgroup of  $SO(32)$ . Since the number of generations corresponds to the number of  $U(1)$  fluxes, we search for the desired matter contents of the standard model satisfying the  $U(1)_Y$  massless conditions as well as the SUSY conditions as can be seen in section 3.2. In section 3.3, we further constrain the models by imposing the so-called K theory constraints. Finally, section 4 is devoted to the conclusion. The normalization of  $SO(32)$  generators and useful trace identities of  $SO(32)$  gauge group are summarized in appendices A and B, respectively.

## 2 $SO(32)$ heterotic string theory on tori with $U(1)$ magnetic fluxes

### 2.1 Low-energy description of $SO(32)$ heterotic string theory

We briefly review the  $SO(32)$  heterotic string theory on a general complex manifold with multiple  $U(1)$  magnetic fluxes. The notation is based on refs. [16–19]. The low-energy effective action of  $SO(32)$  heterotic string theory is given by

$$S_{\text{bos}} = \frac{1}{2\kappa_{10}^2} \int_{M^{(10)}} e^{-2\phi_{10}} \left[ R + 4d\phi_{10} \wedge *d\phi_{10} - \frac{1}{2}H \wedge *H \right] - \frac{1}{2g_{10}^2} \int_{M^{(10)}} e^{-2\phi_{10}} \text{tr}(F \wedge *F), \quad (2.1)$$

which is the bosonic part of the action at the string frame in the notation of [16]. The gravitational and Yang-Mills couplings are set by  $2\kappa_{10}^2 = (2\pi)^7(\alpha')^4$  and  $g_{10}^2 = 2(2\pi)^7(\alpha')^3$  and  $\phi_{10}$  denotes the ten-dimensional dilaton. Here the field-strength of  $SO(32)$  gauge groups  $F$  has the index of vector-representation. In what follows, “tr” and “Tr” represent for the trace in the vector and adjoint representation of the  $SO(32)$  gauge group, respectively. In addition,  $H$  denotes the heterotic three-form field strength defined by

$$H = dB^{(2)} - \frac{\alpha'}{4}(w_{\text{YM}} - w_L), \quad (2.2)$$

where  $w_{\text{YM}}$  and  $w_L$  are the gauge and gravitational Chern-Simons three-forms, respectively.

From the action given by eq. (2.1), the kinetic term of the B-field is extracted as

$$S_{\text{kin}} + S_{\text{WZ}} = -\frac{1}{4\kappa_{10}^2} \int_{M^{(10)}} dB^{(2)} \wedge *dB^{(2)} - \sum_a N_a T_5 \int_{\Gamma_a} B^{(6)} \\ = -\frac{1}{4\kappa_{10}^2} \int_{M^{(10)}} dB^{(2)} \wedge *dB^{(2)} - \sum_a N_a T_5 \int_{M^{(10)}} B^{(6)} \wedge \delta(\Gamma_a), \quad (2.3)$$

where we add the Wess-Zumino term which describes the magnetic sources for the Kalb-Ramond field  $B^{(6)}$ . Such sources correspond to the non-perturbative objects, i.e., the stacks of  $N_a$  five-branes which wrap the holomorphic two-cycles  $\Gamma_a$  and their tensions are given by  $T_5 = ((2\pi)^5(\alpha')^3)^{-1}$ . Here,  $\delta(\Gamma_a)$  denote the Poincaré dual four-form of the two-cycles  $\Gamma_a$ .

By employing the ten-dimensional Hodge duality, the Kalb-Ramond two-form  $B^{(2)}$  and six-form  $B^{(6)}$  are related as

$$*dB^{(2)} = e^{2\phi_{10}} dB^{(6)}, \quad (2.4)$$

and then the kinetic term of Kalb-Ramond field and Wess-Zumino term (2.3) are rewritten as

$$S_{\text{kin}} + S_{\text{WZ}} = -\frac{1}{4\kappa_{10}^2} \int_{M^{(10)}} e^{2\phi_{10}} dB^{(6)} \wedge *dB^{(6)} + \frac{\alpha'}{8\kappa_{10}^2} \int_{M^{(10)}} B^{(6)} \wedge \left( \text{tr} F^2 - \text{tr} R^2 - 4(2\pi)^2 \sum_a N_a \delta(\Gamma_a) \right), \quad (2.5)$$

where  $N_a = \pm 1$  represent for the contributions of heterotic and anti-heterotic five-brane, respectively. The equation of motion of  $B^{(6)}$  leads to the following tadpole condition of the NS-NS fluxes in the presence of five-branes,

$$d(e^{2\phi_{10}} * dB^{(6)}) = -\frac{\alpha'}{4} \left( \text{tr} \bar{F}^2 - \text{tr} \bar{R}^2 - 4(2\pi)^2 \sum_a N_a \delta(\Gamma_a) \right) = 0, \quad (2.6)$$

in cohomology and where  $\bar{F}$  stand for the gauge field strengths of the internal gauge fields whose gauge groups are embedded in  $\text{SO}(32)$ . When the extra-dimension is compactified on the flat space such as three 2-tori,  $(T^2)_1 \times (T^2)_2 \times (T^2)_3$ , the tadpole cancellation requires the following consistency conditions,

$$\int_{(T^2)_i \times (T^2)_j} \left( \text{tr} \bar{F}^2 - 4(2\pi)^2 \sum_a N_a \delta(\Gamma_a) \right) = 0, \quad (2.7)$$

which should be satisfied on  $(T^2)_i \times (T^2)_j$  with  $i \neq j$ ,  $i, j = 1, 2, 3$ . Thus if the nonvanishing fluxes are not canceled by themselves, the non-perturbative objects would contribute to the cancellation of anomalies. It suggests that the modular invariance of heterotic string theory is recovered by the existence of these non-perturbative objects [20, 21] which can be also realized in the framework of heterotic orbifold [22].<sup>2</sup>

## 2.2 Generalized Green-Schwarz mechanism

In addition to the consistency condition as discussed in section 2.1, it must be ensured that our models do not have gauge and gravitational anomalies. In heterotic string theory, it is known that some gauge and gravitational anomalies are canceled by considering the following one-loop Green-Schwarz term at the string frame [15],

$$S_{\text{GS}} = \frac{1}{24(2\pi)^5 \alpha'} \int B^{(2)} \wedge X_8, \quad (2.8)$$

whose normalization factor is determined by the S-dual type I theory as shown in appendix of [25] and the eight-form  $X_8$  reads,

$$X_8 = \frac{1}{24} \text{Tr} F^4 - \frac{1}{7200} (\text{Tr} F^2)^2 - \frac{1}{240} (\text{Tr} F^2)(\text{tr} R^2) + \frac{1}{8} \text{tr} R^4 + \frac{1}{32} (\text{tr} R^2)^2. \quad (2.9)$$

Although the gauge and gravitational anomalies for the non-Abelian gauge groups are canceled by the above Green-Schwarz term (2.8) and the tadpole condition (2.6) as shown

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<sup>2</sup>Even if the consistency condition is satisfied at the non-perturbative level, we have to care about the anomaly on heterotic five-branes and the global Witten anomaly is absent if the number of chiral fermions on the heterotic five branes is even [23, 24].

in ref. [1], the anomalies relevant to the multiple Abelian gauge groups, which appear in low-energy effective theory, can be also canceled by same Green-Schwarz mechanism, for more details see refs. [17, 18]. In fact, since we derive just the three-generation standard-like models, our phenomenological models do not receive these anomalies. However, as pointed out in refs. [17, 18], even if the Abelian gauge symmetries are anomaly-free, the Abelian gauge bosons may become massive due to the Green-Schwarz coupling given by eq. (2.8). In order to ensure that the hypercharge gauge boson is massless, they should not couple to the axions which is hodge dual to the Kalb-Ramond field.

For completeness, we define the hypercharge gauge group as the subgroup of SO(32) as follows. The decomposition of the SO(32) gauge group can be realized by inserting the multiple U(1) constant magnetic fluxes as those satisfying

$$\text{SO}(32) \rightarrow \text{SU}(3)_C \otimes \text{SU}(2)_L \otimes_{a=1}^{13} \text{U}(1)_a. \tag{2.10}$$

Totally, SO(32) has 16 Cartan elements,  $H_i$  ( $i = 1, \dots, 16$ ). We take the Cartan elements of SU(3) along  $H_1 - H_2$ ,  $H_1 + H_2 - 2H_3$  and Cartan element of SU(2) as  $H_5 - H_6$ . The other Cartan directions of SO(32) are chosen as,

$$\begin{aligned} \text{U}(1)_1 &: (0, 0, 0, 0, 1, 1; 0, 0, \dots, 0), \\ \text{U}(1)_2 &: (1, 1, 1, 1, 0, 0; 0, 0, \dots, 0), \\ \text{U}(1)_3 &: (1, 1, 1, -3, 0, 0; 0, 0, \dots, 0), \\ \text{U}(1)_4 &: (0, 0, 0, 0, 0, 0; 1, 0, \dots, 0), \\ \text{U}(1)_5 &: (0, 0, 0, 0, 0, 0; 0, 1, \dots, 0), \\ &\vdots \\ \text{U}(1)_{13} &: (0, 0, 0, 0, 0, 0; 0, 0, \dots, 1), \end{aligned} \tag{2.11}$$

in the basis  $H_i$ . Then, we use the basis that non-zero roots have charge

$$(\underline{\pm 1}, \underline{\pm 1}, 0, \dots, 0), \tag{2.12}$$

under  $H_i$  ( $i = 1, \dots, 16$ ), where the underline means any possible permutations. The normalization of the Abelian gauge groups are discussed in the appendix A and the concrete identification of standard model gauge groups and its representations are shown in section 3. Note that some gauge groups would be enhanced to the larger one if any of U(1) fluxes are absent or degenerate.

When the U(1) fluxes are inserted along the Cartan direction of SO(32), the field strengths of U(1)s,  $f$  are decomposed into the four-dimensional parts  $f$  and extra-dimensional parts  $\bar{f}$ ,

$$f \rightarrow f + \bar{f}, \tag{2.13}$$

and then we can dimensionally reduce the one-loop Green-Schwarz term (2.8) to

$$S_{\text{GS}} = \frac{1}{(2\pi)^3 l_s^2} \int_{M^{(10)}} B^{(2)} \wedge \frac{1}{144} (\text{Tr } F \bar{f}^3) \quad (2.14)$$

$$- \frac{1}{(2\pi)^3 l_s^2} \int_{M^{(10)}} B^{(2)} \wedge \frac{1}{2880} (\text{Tr } F \bar{f}) \wedge \left( \frac{1}{15} \text{Tr } \bar{f}^2 + \text{tr } \bar{R}^2 \right) \quad (2.15)$$

$$+ \frac{1}{(2\pi)^3 l_s^2} \int_{M^{(10)}} B^{(2)} \wedge \left[ \frac{1}{96} (\text{Tr } F^2 \bar{f}^2) - \frac{1}{43200} (\text{Tr } F \bar{f})^2 \right] \quad (2.16)$$

$$- \frac{1}{(2\pi)^3 l_s^2} \int_{M^{(10)}} B^{(2)} \wedge \frac{1}{5760} (\text{Tr } F^2) \wedge \left( \frac{1}{15} \text{Tr } \bar{f}^2 + \text{tr } \bar{R}^2 \right) \quad (2.17)$$

$$+ \frac{1}{(2\pi)^3 l_s^2} \int_{M^{(10)}} B^{(2)} \wedge \frac{1}{384} (\text{tr } R^2) \wedge \left( \text{tr } \bar{R}^2 - \frac{1}{15} \text{Tr } \bar{f}^2 \right) \quad (2.18)$$

where  $l_s = 2\pi\sqrt{\alpha'}$ ,  $F$  denote the field strengths of  $\text{SU}(3)_C$ ,  $\text{SU}(2)_L$ ,  $\text{U}(1)_Y$ . The explicit forms of traces appeared in eqs. (2.14)–(2.18) are shown in appendix B.

Before evaluating the mass term of  $\text{U}(1)$  gauge bosons, for completeness, we show the definition of three 2-tori  $(T^2)_i \simeq \mathbf{C}/\Lambda_i$  with  $i = 1, 2, 3$ , where the lattices  $\Lambda_i$  are generated by two vectors  $e_i = 2\pi R_i$  and  $e_i = 2\pi R_i \tau_i$ . Here,  $R_i$  and  $\tau_i$  are the radii and complex structure moduli of  $(T^2)_i$ , respectively. The metrics of three 2-tori are then given by

$$ds_6^2 = g_{mn} dx^m dx^n = 2h_{i\bar{j}} dz^i dz^{\bar{j}}, \quad (2.19)$$

$$g_{mn} = \begin{pmatrix} g^{(1)} & 0 & 0 \\ 0 & g^{(2)} & 0 \\ 0 & 0 & g^{(3)} \end{pmatrix}, \quad h_{i\bar{j}} = \begin{pmatrix} h^{(1)} & 0 & 0 \\ 0 & h^{(2)} & 0 \\ 0 & 0 & h^{(3)} \end{pmatrix}, \quad (2.20)$$

where  $x^m$  are the coordinates of  $T^2$  with  $m, n = 4, 5, 6, 7, 8, 9$ ,  $z^i = x^{2+2i} + \tau^i x^{3+2i}$  and the rank 2 diagonal matrices  $g^{(i)}$  and  $h^{(i)}$  are given by

$$g^{(i)} = (2\pi R_i)^2 \begin{pmatrix} 1 & \text{Re } \tau_i \\ \text{Re } \tau_i & |\tau_i|^2 \end{pmatrix}, \quad h^{(i)} = (2\pi R_i)^2 \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \quad (2.21)$$

From this expression, we expand the Kalb-Ramond field  $B^{(2)}$  and internal  $\text{U}(1)_a$  field strengths  $\bar{f}_a$ , ( $a = 1, \dots, 13$ ) in the basis of Kähler forms,  $w_i = idz^i \wedge d\bar{z}^i / (2 \text{Im } \tau^{(i)})$  on tori  $(T^2)_i$  derived from the metrics (2.21),

$$B^{(2)} = b_S^{(2)} + l_s^2 \sum_{i=1}^3 b_i^{(0)} w_i, \quad (2.22)$$

$$\bar{f}_a = 2\pi \sum_{i=1}^3 m_a^{(i)} w_i,$$

where  $m_a^{(i)}$  are the integers or half-integers determined by Dirac quantization condition. Since Dirac quantization is satisfied in the adjoint representation of  $\text{SO}(32)$ , the fractional numbers of  $m_a^{(i)}$  can be allowed as pointed out in ref. [1]. From the eqs. (2.14) and (2.15),

we can extract the Stueckelberg couplings,

$$\frac{1}{3(2\pi)^3 l_s^2} \int b_S^{(2)} \wedge \left[ \text{tr } T_1^4 \bar{f}_1^3 f_1 + (\text{tr } T_2^4 \bar{f}_2^3 + 3(\text{tr } T_2^2 T_3^2) \bar{f}_2 \bar{f}_3^2 + (\text{tr } T_2 T_3^3) \bar{f}_3^3) f_2 \right. \\ \left. + (\text{tr } T_3^4 \bar{f}_3^3 + 3(\text{tr } T_2 T_3^3) \bar{f}_2 \bar{f}_3^2 + 3(\text{tr } T_2^2 T_3^2) \bar{f}_2^2 \bar{f}_3) f_3 + \sum_{c=4}^{13} \text{tr } T_c^4 \bar{f}_c^3 f_c \right], \quad (2.23)$$

where the trace identities are employed as shown in appendix B. If the U(1) gauge fields couple to the universal axion  $b_S^{(0)}$  which is the hodge dual of the tensor field  $b_S^{(2)}$ , one of the multiple U(1) gauge fields absorbs the universal axion and become massive. In our model, since the hypercharge U(1)<sub>Y</sub> is identified as the linear combinations of multiple U(1)s, i.e., U(1)<sub>Y</sub> =  $\frac{1}{6}$ (U(1)<sub>3</sub> + 3∑<sub>c</sub> U(1)<sub>c</sub>) as shall be discussed in section 3,<sup>3</sup> the U(1)<sub>Y</sub> gauge field becomes massless under the condition

$$6 \text{tr}(T_3^4) m_3^{(1)} m_3^{(2)} m_3^{(3)} + 3 \text{tr}(T_2 T_3^3) d_{ijk} m_2^{(i)} m_3^{(j)} m_3^{(k)} + 3 \text{tr}(T_2^2 T_3^2) d_{ijk} m_2^{(i)} m_2^{(j)} m_3^{(k)} \\ + 18 \sum_c \text{tr}(T_c^4) m_c^{(1)} m_c^{(2)} m_c^{(3)} = 0, \quad (2.24)$$

which means no interaction between U(1)<sub>Y</sub> and the universal axion  $b_S^{(0)}$ . Here the following formulas are satisfied  $\int_{T^2 \times T^2 \times T^2} \bar{f}_a^3 = (2\pi)^3 d_{ijk} m_a^{(i)} m_a^{(j)} m_a^{(k)} = 6(2\pi)^3 m_a^{(1)} m_a^{(2)} m_a^{(3)}$  with the non-vanishing intersection numbers of 2-tori,  $d_{ijk} = 1$  ( $i \neq j \neq k$ ).

Except for the universal axion, there are other axions associated with the internal cycles, that is, Kähler axions which couple to the U(1) gauge bosons originated from the action given by eq. (2.5). Along with the Kalb-Ramond field  $B^{(2)}$ , we expand the dual field  $B^{(6)}$  as

$$B^{(6)} = l_s^6 b_0^{(0)} \text{vol}_6 + l_s^4 \sum_{k=1}^3 b_k^{(2)} \hat{w}_k, \quad (2.25)$$

where  $\hat{w}_k$  are the Hodge dual four-forms of the Kähler forms,

$$\hat{w}_k = \frac{d_{kij}}{2} i \frac{dz^i \wedge d\bar{z}^i}{2 \text{Im } \tau^{(i)}} \wedge i \frac{dz^j \wedge d\bar{z}^j}{2 \text{Im } \tau^{(j)}}, \quad (2.26)$$

which are defined as those satisfying  $\int_{T^2 \times T^2 \times T^2} w_i \wedge \hat{w}_j = \delta_{ij}$ . After inserting these expressions into the action given by eq. (2.5), we can extract the mass terms of the U(1) gauge bosons,

$$\frac{1}{l_s^2} \int b_i^{(2)} \wedge \sum_{a=1}^{13} \text{tr}(T_a^2) f_a m_a^{(i)}. \quad (2.27)$$

In the same way as the case of universal axion, the U(1)<sub>Y</sub> gauge field should not couple to the Kähler axions, otherwise it becomes massive. Thus the U(1)<sub>Y</sub> gauge boson is massless under the following condition,

$$\text{tr}(T_3^2) m_3^{(i)} + 3 \sum_{c=4}^{13} \text{tr}(T_c^2) m_c^{(i)} = 0, \quad (2.28)$$

with  $i = 1, 2, 3$ .

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<sup>3</sup>In the definition of U(1)<sub>Y</sub>, the summation over  $c$  depends on the models as shown in section 3.



As a step to realize the realistic models, the massless conditions for  $U(1)_Y$  gauge boson given by eqs. (2.24) and (2.28) should be satisfied. It is remarkable that these  $U(1)$  fluxes are sensitive to the consistency condition given by eq. (2.7) as shown in the section 2.1. When the heterotic five-branes are absent in our system, the following conditions,

$$\sum_{a=1}^{13} \text{tr}(T_a^2) m_a^{(i)} m_a^{(j)} = 0, \quad i \neq j, \quad (i, j = 1, 2, 3), \quad (2.29)$$

are required from the consistencies of heterotic string theory, otherwise the NS-NS tadpole could be canceled by the existence of heterotic five-branes.

### 2.3 The chiral fermions and degenerate zero-modes

The heterotic string theory on three 2-tori has  $\mathcal{N} = 4$  supersymmetry in the language of 4D supercharges which have to be broken to at least  $\mathcal{N} = 1$  supersymmetry in the four-dimension, otherwise the chiral matters do not appear in the low-energy effective theory. Although it is known that there are much progresses in the framework of toroidal orbifold, in this paper, we focus on the realization of chiral fermions by employing the multiple  $U(1)$  fluxes as discussed in this section.<sup>4</sup>

First we define the 10D Majorana-Weyl spinor  $\lambda$  which satisfies the Majorana-Weyl condition,

$$\Gamma \lambda = \lambda, \quad (2.30)$$

where  $\Gamma$  is the 10D chirality matrix. The following analysis is based on ref. [26]. In order to discuss the 4D chirality, we decompose the 10D Majorana-Weyl spinor  $\lambda$  into four 4D Weyl spinors  $\lambda_0$  and  $\lambda_i$  with  $i = 1, 2, 3$  as the representation of  $SU(4) \simeq SO(6)$ . The 10D chirality matrix  $\Gamma$  is also decomposed into the product of three 2D chirality operators,  $\Gamma_i = -i\Gamma_i^1 \Gamma_i^2$  on  $(T^2)_i$ , where

$$\Gamma_i^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_i^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (2.31)$$

satisfying the Clifford algebra. Then the 4D chirality is fixed as

$$\Gamma^i \lambda_0 = \lambda_0, \quad \Gamma^i \lambda_j = \begin{cases} +\lambda_j & (i = j), \\ -\lambda_j & (i \neq j), \end{cases} \quad (2.32)$$

which lead to the following 4D Weyl spinors,

$$\lambda_0 = \lambda_{+++}, \quad \lambda_1 = \lambda_{+--}, \quad \lambda_2 = \lambda_{-+-}, \quad \lambda_3 = \lambda_{--+}, \quad (2.33)$$

where the subscript indexes denote the eigenvalues of  $\Gamma^i$  with  $i = 1, 2, 3$ . When we insert the magnetic fluxes on three 2-tori, one of the four 4D Weyl spinors would be chosen.

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<sup>4</sup>Although the gauge sector still remains 4D  $\mathcal{N} = 4$  SUSY, it could be broken to  $\mathcal{N} = 1$  SUSY by extending our system to the toroidal orbifold with trivial gauge embedding. The (anti-) heterotic five-branes would break (all) partial SUSY. It is then expected that the heterotic five branes compensate the moduli invariance even if the moduli invariance is violated at the string tree-level.

In order to prove the above statements, we show the zero-mode wavefunction of fermions originating from the 10D gaugino field by solving their Dirac equations.

The zero-modes of 10D gaugino field  $\lambda$  and gauge field  $A_M$  are defined through the following decompositions,

$$\begin{aligned}\lambda(x^\mu, z^i) &= \sum_n \chi_n(x^\mu) \otimes \psi_n^{(1)}(z^1) \otimes \psi_n^{(2)}(z^2) \otimes \psi_n^{(3)}(z^3), \\ A_M(x^\mu, z^i) &= \sum_n \varphi_{n,M}(x^\mu) \otimes \phi_{n,M}^{(1)}(z^1) \otimes \phi_{n,M}^{(2)}(z^2) \otimes \phi_{n,M}^{(3)}(z^3),\end{aligned}\tag{2.34}$$

where  $M = 0, 1, \dots, 9$  and  $x^\mu, \mu = 0, 1, 2, 3$  are the coordinates of the 4D spacetime. The zero-modes of gaugino fields,  $\psi_0^{(i)}(z^i)$  are expressed as

$$\psi_0^{(i)}(z^i) = \begin{pmatrix} \psi_+^{(i)}(z^i) \\ \psi_-^{(i)}(z^i) \end{pmatrix},\tag{2.35}$$

where hereafter we omit the subscript 0 of the zero-modes, that is,  $\psi^{(i)}(z^i) = \psi_0^{(i)}(z^i)$ . On the other hand, the extra dimensional components of  $U(1)_a$  gauge backgrounds  $A_a^{(i)}(z^i)$  ( $a = 1, 2, \dots, 13$ ) are given by

$$A_a^{(i)}(z^i) = \frac{\pi m_a^{(i)}}{\text{Im } \tau_i} \text{Im}(\bar{z}_i dz_i),\tag{2.36}$$

which lead to the magnetic fluxes given by eq. (2.22) along the Cartan direction of  $SO(32)$ . Here and hereafter, we multiply the  $U(1)_a$  magnetic fluxes  $m_a^{(i)}$  by their corresponding normalization factors.

Then the zero-mode equations of fermions  $\psi^{(i)}(z^i)$  with the  $U(1)_a$  charge  $q_a$  are given by

$$\mathcal{D}_i \psi^{(i)}(z^i) = (\Gamma^{z^i} \nabla_{z^i} + \Gamma^{\bar{z}^i} \nabla_{\bar{z}^i}) \psi^{(i)}(z^i) = 0\tag{2.37}$$

where the Gamma matrices and covariant derivatives in terms of the complex coordinates,  $(z^i, \bar{z}^i)$  are defined as

$$\Gamma^{z^i} = \frac{1}{2\pi R_i} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \Gamma^{\bar{z}^i} = \frac{1}{2\pi R_i} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},\tag{2.38}$$

which can be derived from the Gamma matrices in flat space (2.31) and the metric of torus (2.21) and

$$\begin{aligned}\nabla_{z^i} &= \partial_{z^i} - iq_a (A_a^{(i)})_{z^i}, \\ \nabla_{\bar{z}^i} &= \partial_{\bar{z}^i} - iq_a (A_a^{(i)})_{\bar{z}^i}.\end{aligned}\tag{2.39}$$

The spin connections are vanishing due to the topology of tori. Thus the Dirac equations on  $(T^2)_i$  are rewritten as

$$\begin{aligned}\left(\bar{\partial}_{\bar{z}^i} + \frac{\pi q_a m_a^i}{2 \text{Im } \tau_i} z^i\right) \psi_+^{(i)}(z^i, \bar{z}^i) &= 0, \\ \left(\partial_{z^i} - \frac{\pi q_a m_a^i}{2 \text{Im } \tau_i} \bar{z}^i\right) \psi_-^{(i)}(z^i, \bar{z}^i) &= 0.\end{aligned}\tag{2.40}$$

Then  $\psi_+^{(i)}(z^i, \bar{z}^i)$  has zero-modes only if  $M^i = q_a m_a^i > 0$ , whereas  $\psi_-^{(i)}(z^i, \bar{z}^i)$  has zero-modes only if  $M^i < 0$ . In both cases, the wavefunctions have  $|M^i|$  independent solutions as the solution of Dirac equations (2.40). Hence the number of generations of zero-modes,  $M$  is given by the product of  $|M^i|$ , that is,  $M = |M^1||M^2||M^3|$ . (This result is consistent with that of the index theorem.) Since the nonvanishing fluxes  $|M^i|$  select one of the two chiralities on  $(T^2)_i$ , i.e.,  $\psi_+^{(i)}$  or  $\psi_-^{(i)}$ , non vanishing fluxes on three 2-tori lead to the chiral spectrum as can be seen in eq. (2.33).

However, such magnetic fluxes may break all  $\mathcal{N} = 4$  SUSY through the D-terms or Fayet-Iliopoulos terms in the language of 4D  $\mathcal{N} = 1$  SUSY. When  $\mathcal{N} = 1$  SUSY is preserved in the system, the vanishing D-terms imply that the hermitian Yang-Mills equations for the  $U(1)_a$  field strengths should be satisfied at the vacuum,

$$g^{i\bar{j}}(\bar{f}_a)_{i\bar{j}} = 0. \tag{2.41}$$

In our set-up, these conditions are equal to

$$\sum_{i=1}^3 \frac{m_a^i}{\mathcal{A}_i} = 0, \tag{2.42}$$

where  $\mathcal{A}_i = (2\pi R_i)^2 \text{Im } \tau_i$  are the areas of tori,  $(T^2)_i$ . Indeed, when these conditions are satisfied, massless scalar fields appear for  $A_M$  ( $M = 4, \dots, 9$ ), and they correspond to superpartners of the above massless fermions. At the perturbative level, the D-term conditions receive at most one-loop corrections [27] which have the dilaton dependence.

Finally, we comment on the Wilson lines which play a role of breaking the gauge group into its subgroups without changing the rank of gauge groups. In fact, when we introduce the Wilson lines  $\zeta_a^{(i)}$ , along the  $U(1)_a$  directions, the internal components of  $U(1)_a$  gauge fields take the following shifts compared to eq. (2.36),

$$A_a(z^i) = \frac{\pi m_a^{(i)}}{\text{Im } \tau_i} \text{Im}((\bar{z}_i + \bar{\zeta}_a^{(i)}) dz_i), \tag{2.43}$$

which modify the zero-mode wavefunctions determined by the Dirac equations (2.40), whereas the number of zero-modes and  $U(1)$  fluxes are not modified. When we evaluate the values of Yukawa couplings, such Wilson lines would give significant effects.

### 3 Three-generation models in the $SO(32)$ heterotic string theory

#### 3.1 Matter content

In this section, we show the concrete decomposition of  $SO(32)$  gauge group into the standard model gauge groups and then the parts of adjoint representation of  $SO(32)$  are identified as the matter contents of the standard model. As the first step to obtain the standard model gauge groups, we consider the decomposition of  $SO(32)$  illustrated as

$$\begin{aligned} SO(32) &\rightarrow SO(12) \otimes SO(20), \\ 496 &\rightarrow (1, 190) \oplus (12_v, 20_v) \oplus (66, 1), \end{aligned} \tag{3.1}$$

where the multiple  $U(1)$  fluxes are assumed along the Cartan directions of  $SO(32)$ .

In order to derive the matter contents of the standard model, we examine whether the adjoint representation of  $SO(12)$  involves the candidates of elementary particles or not. When we put three  $U(1)_{1,2,3}$  fluxes along the Cartan directions of  $SO(12)$  gauge group, it is found that  $SO(12)$  involves the candidates of  $SU(3)_C$  and  $SU(2)_L$ ,

$$\begin{aligned} SO(12) &\rightarrow SO(8) \otimes SU(2)_L \otimes U(1)_1 \rightarrow SU(4) \otimes U(1)_2 \otimes SU(2)_L \otimes U(1)_1 \\ &\rightarrow SU(3)_C \otimes U(1)_3 \otimes U(1)_2 \otimes SU(2)_L \otimes U(1)_1, \end{aligned} \quad (3.2)$$

where the Cartan directions of  $U(1)_{1,2,3}$  are given by eq. (2.11). Then the adjoint representation of  $SO(12)$  is decomposed as

$$66 \left\{ \begin{array}{l} (28, 1)_0 \left\{ \begin{array}{l} (15, 1)_{0,0} \left\{ \begin{array}{l} (8, 1)_{0,0,0} \\ (3, 1)_{0,0,4} \\ (\bar{3}, 1)_{0,0,-4} \\ (1, 1)_{0,0,0} \end{array} \right. \\ (6, 1)_{0,2} \left\{ \begin{array}{l} (3, 1)_{0,2,-2} \\ (\bar{3}, 1)_{0,2,2} \end{array} \right. \\ (\bar{6}, 1)_{0,-2} \left\{ \begin{array}{l} (3, 1)_{0,-2,-2} \\ (\bar{3}, 1)_{0,-2,2} \end{array} \right. \\ (1, 1)_{0,0,0} \end{array} \right. \\ (8_v, 2)_1 \left\{ \begin{array}{l} (4, 2)_{1,1} \left\{ \begin{array}{l} (3, 2)_{1,1,1} \\ (1, 2)_{1,1,-3} \end{array} \right. \\ (\bar{4}, 2)_{1,-1} \left\{ \begin{array}{l} (\bar{3}, 2)_{1,-1,-1} \\ (1, 2)_{1,-1,3} \end{array} \right. \end{array} \right. \\ (8_v, 2)_{-1} \left\{ \begin{array}{l} (4, 2)_{-1,1} \left\{ \begin{array}{l} (3, 2)_{-1,1,1} \\ (1, 2)_{-1,1,-3} \end{array} \right. \\ (\bar{4}, 2)_{-1,-1} \left\{ \begin{array}{l} (\bar{3}, 2)_{-1,-1,-1} \\ (1, 2)_{-1,-1,3} \end{array} \right. \end{array} \right. \\ (1, 3)_{0,0,0} \\ (1, 1)_{2,0,0} \\ (1, 1)_{-2,0,0} \\ (1, 1)_{0,0,0} \end{array} \right. , \quad (3.3)$$

which are singlets of  $SO(20)$ , where the subscript indices denote the  $U(1)_{1,2,3}$  charge  $q_{1,2,3}$ . The normalization of  $U(1)$  generators are given by appendix A. Thus when we identify the hypercharge as  $U(1)_Y = U(1)_3/6$ , we can extract the candidates of the quarks, charged

leptons and/or Higgs,

$$\begin{aligned}
 Q : \begin{cases} Q_1 = (3, 2)_{1,1,1} \\ Q_2 = (3, 2)_{-1,1,1} \end{cases}, \quad L : \begin{cases} L_1 = (1, 2)_{1,1,-3} \\ L_2 = (1, 2)_{-1,1,-3} \end{cases}, \quad u_R^c : \begin{cases} u_{R_1}^c = (\bar{3}, 1)_{0,0,-4} \end{cases}, \\
 d_R^c : \begin{cases} d_{R_1}^c = (\bar{3}, 1)_{0,2,2} \\ d_{R_2}^c = (\bar{3}, 1)_{0,-2,2} \end{cases}, \quad n_1 = (1, 1)_{2,0,0}.
 \end{aligned} \tag{3.4}$$

As shown in the above analysis, the adjoint representation of SO(12), 66 does not involve the candidate of right-handed leptons. Therefore, we further decompose the SO(20) gauge group into  $U(1)_{4,5,\dots,13}$  gauge groups,

$$SO(20) \rightarrow U(1)_4 \otimes \cdots \otimes U(1)_{13}, \tag{3.5}$$

where the nonvanishing U(1) fluxes along all  $U(1)_{4,\dots,13}$  directions are inserted shown in eq. (2.11). Now SO(2) is identified as U(1). The vector representation and the singlet of SO(12),  $12_v$  and 1 give the suitable matter contents, i.e., right-handed quarks and leptons, charged-leptons and/or Higgs,

$$\begin{aligned}
 (12_v, 20_v) \rightarrow \begin{cases} L_3^a = (1, 2)_{1,0,0;\underline{-1,0,\dots,0}} \\ L_4^a = (1, 2)_{-1,0,0;\underline{-1,0,\dots,0}} \\ u_{R_2}^c = (\bar{3}, 1)_{0,-1,-1;\underline{-1,0,\dots,0}} \\ d_{R_3}^c = (\bar{3}, 1)_{0,-1,-1;\underline{1,0,\dots,0}} \\ e_{R_1}^c = (1, 1)_{0,-1,3;\underline{1,0,\dots,0}} \\ n_2^c = (1, 1)_{0,-1,3;\underline{-1,0,\dots,0}} \end{cases}, \quad (a = 4, 5, \dots, 13), \\
 (1, 190) \rightarrow \begin{cases} e_{R_2}^{c\ ab} = (1, 1)_{0,0,0;\underline{1,1,0,\dots,0}} \\ n_3^{c\ ab} = (1, 1)_{0,0,0;\underline{1,-1,0,\dots,0}} \end{cases}, \quad (a, b = 4, 5, \dots, 13, a \neq b),
 \end{aligned} \tag{3.6}$$

where the underlines for  $U(1)_{4,5,\dots,13}$  charge  $q_{4,5,\dots,13}$  denote all the possible permutations. It is remarkable that the correct  $U(1)_Y$  charge can be also realized as

$$U(1)_Y = \frac{1}{6} \left( U(1)_3 + 3 \sum_{c=4}^{13} U(1)_c \right). \tag{3.7}$$

### 3.2 Three-generation models

Since the matter contents of the standard model are correctly identified in the previous section, we show the number of generations for each representation in this section.

As discussed in section 2.3, the U(1) fluxes generate the degenerate zero-modes if these zero-modes have U(1) charges. It implies that the number of generations for the representations embedded in the adjoint and vector representations of SO(12), 66 and  $12_v$

are determined by the following formulas,

$$\begin{aligned}
 m_{Q_1} &= \prod_{i=1}^3 m_{Q_1}^i = \prod_{i=1}^3 (m_1^i + m_2^i + m_3^i), & m_{Q_2} &= \prod_{i=1}^3 m_{Q_2}^i = \prod_{i=1}^3 (-m_1^i + m_2^i + m_3^i), \\
 m_{L_1} &= \prod_{i=1}^3 m_{L_1}^i = \prod_{i=1}^3 (m_1^i + m_2^i - 3m_3^i), & m_{L_2} &= \prod_{i=1}^3 m_{L_2}^i = \prod_{i=1}^3 (-m_1^i + m_2^i - 3m_3^i), \\
 m_{u_{R_1}^c} &= \prod_{i=1}^3 m_{u_{R_1}^c}^i = \prod_{i=1}^3 (-4m_3^i), & m_{n_1} &= \prod_{i=1}^3 m_{n_1}^i = \prod_{i=1}^3 (2m_1^i), \\
 m_{d_{R_1}^c} &= \prod_{i=1}^3 m_{d_{R_1}^c}^i = \prod_{i=1}^3 (2m_2^i + 2m_3^i), & m_{d_{R_2}^c} &= \prod_{i=1}^3 m_{d_{R_2}^c}^i = \prod_{i=1}^3 (-2m_2^i + 2m_3^i),
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
 m_{L_3^a} &= \prod_{i=1}^3 m_{L_3^a}^i = \prod_{i=1}^3 (m_1^i - m_a^i), & m_{L_4^a} &= \prod_{i=1}^3 m_{L_4^a}^i = \prod_{i=1}^3 (-m_1^i - m_a^i), \\
 m_{u_{R_2}^{c_a}} &= \prod_{i=1}^3 m_{u_{R_2}^{c_a}}^i = \prod_{i=1}^3 (-m_2^i - m_3^i - m_a^i), & m_{d_{R_3}^{c_a}} &= \prod_{i=1}^3 m_{d_{R_3}^{c_a}}^i = \prod_{i=1}^3 (-m_2^i - m_3^i + m_a^i), \\
 m_{e_{R_1}^{c_a}} &= \prod_{i=1}^3 m_{e_{R_1}^{c_a}}^i = \prod_{i=1}^3 (-m_2^i + 3m_3^i + m_a^i), & m_{n_2^a} &= \prod_{i=1}^3 m_{n_2^a}^i = \prod_{i=1}^3 (-m_2^i + 3m_3^i - m_a^i),
 \end{aligned} \tag{3.9}$$

respectively.

Now we are ready to search for the realistic three-generation models in the framework of SO(32) heterotic string theory. In the light of U(1)<sub>Y</sub> massless conditions given by eqs. (2.24) and (2.28), the nonvanishing U(1)<sub>3</sub> fluxes seem to violate these massless conditions. Therefore, in this paper, we restrict ourselves to the case that U(1)<sub>3</sub> fluxes are absent in our system, which lead to no chiral generations of right-handed quarks,  $u_R^c$  and  $d_R^c$  from the adjoint representation of SO(12) as can be seen in eq. (3.8). Only left-handed quarks  $Q$  and charged-leptons  $L$  are then generated from the adjoint representation of SO(12). As for the left-handed quarks,  $Q$ , there are two possibilities to reproduce the three generations of  $Q$ ,

$$\text{Type A: } (m_{Q_1}, m_{Q_2}) = (2, 1), \quad \text{Type B: } (m_{Q_1}, m_{Q_2}) = (3, 0), \tag{3.10}$$

without loss of generality, because we can exchange  $m_{Q_1}$  and  $m_{Q_2}$  under flipping the sign of  $m_1^i$  with  $i = 1, 2, 3$ . In both cases, the possible U(1) fluxes are summarized in tables 1 and 2 and in the case of Type B, it is restricted within the range of  $-2 \leq m_{Q_2}^i \leq 2$ ,  $i = 1, 2, 3$ , for simplicity. In both tables, possible permutations among the first, second and third 2-tori are omitted. Also, when we flip signs of magnetic fluxes in two of three 2-tori, we obtain the same generation number. For example the magnetic fluxes,  $(m_1^1, m_1^2, m_1^3) = (-3/2, 0, 1)$   $(m_2^1, m_2^2, m_2^3) = (-1/2, -1, 0)$ , are obtained by flipping the signs of magnetic fluxes in the first and second 2-tori from ones in table 1 and they lead to the same generation numbers. We omit such possibilities in both tables.

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$
$(\frac{1}{2}, 0, 0)$	$(\frac{3}{2}, 1, 1)$
$(1, 1, \frac{1}{2})$	$(0, 0, \frac{3}{2})$
$(\frac{3}{2}, 1, 0)$	$(\frac{1}{2}, 0, 1)$

**Table 1.** The possible magnetic fluxes in Type A. Possible permutations among the three 2-tori are omitted. Certain types of sign flipping are also omitted.

Under the constrained magnetic fluxes in tables 1 and 2, we further search for the realistic three generations of  $u_R^c$ ,  $d_R^c$  and  $e_R^c$  satisfying the  $U(1)_Y$  massless conditions (2.24), (2.28) as well as the SUSY conditions (2.42).<sup>5</sup> As a result, within the range of  $-10 \leq m_{u_{R_2}^{c a}}^i \leq 10$ , there are three choices for the  $U(1)$  fluxes as follows,

$$\begin{aligned}
 \text{Case I} \quad & m_4^i = m_5^i = m_6^i = -m_7^i = -m_8^i = -m_9^i, \\
 & m_{10}^i = m_{11}^i = -m_{12}^i = -m_{13}^i, \\
 & (m_{u_{R_2}^{c a}}, m_{d_{R_3}^{c a}}, m_{e_R^{c a}}, m_{n_2^a}) = (1, 0, 0, 1), \quad (a = 4, 5, 6), \\
 & (m_{u_{R_2}^{c b}}, m_{d_{R_3}^{c b}}, m_{e_R^{c b}}, m_{n_2^b}) = (0, 1, 1, 0), \quad (b = 7, 8, 9), \\
 & (m_{u_{R_2}^{c d}}, m_{d_{R_3}^{c d}}, m_{e_R^{c d}}, m_{n_2^d}) = (0, 0, 0, 0), \quad (d = 10, 11, 12, 13),
 \end{aligned} \tag{3.11}$$

$$\begin{aligned}
 \text{Case II} \quad & m_4^i = -m_5^i, \\
 & m_6^i = m_7^i = m_8^i = m_9^i = -m_{10}^i = -m_{11}^i = -m_{12}^i = -m_{13}^i, \\
 & (m_{u_{R_2}^{c 4}}, m_{d_{R_3}^{c 4}}, m_{e_R^{c 4}}, m_{n_2^4}) = (3, 0, 0, 3), \\
 & (m_{u_{R_2}^{c 5}}, m_{d_{R_3}^{c 5}}, m_{e_R^{c 5}}, m_{n_2^5}) = (0, 3, 3, 0), \\
 & (m_{u_{R_2}^{c a}}, m_{d_{R_3}^{c a}}, m_{e_R^{c a}}, m_{n_2^a}) = (0, 0, 0, 0), \quad (a = 6, 7, 8, 9, 10, 11, 12, 13).
 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
 \text{Case III} \quad & m_4^i = -m_5^i, \quad m_6^i = -m_7^i, \\
 & m_8^i = m_9^i = m_{10}^i = -m_{11}^i = -m_{12}^i = -m_{13}^i, \\
 & (m_{u_{R_2}^{c 4}}, m_{d_{R_3}^{c 4}}, m_{e_R^{c 4}}, m_{n_2^4}) = (2, 0, 0, 2), \\
 & (m_{u_{R_2}^{c 5}}, m_{d_{R_3}^{c 5}}, m_{e_R^{c 5}}, m_{n_2^5}) = (0, 2, 2, 0), \\
 & (m_{u_{R_2}^{c 6}}, m_{d_{R_3}^{c 6}}, m_{e_R^{c 6}}, m_{n_2^6}) = (1, 0, 0, 1), \\
 & (m_{u_{R_2}^{c 7}}, m_{d_{R_3}^{c 7}}, m_{e_R^{c 7}}, m_{n_2^7}) = (0, 1, 1, 0), \\
 & (m_{u_{R_2}^{c a}}, m_{d_{R_3}^{c a}}, m_{e_R^{c a}}, m_{n_2^a}) = (0, 0, 0, 0), \quad (a = 8, 9, 10, 11, 12, 13).
 \end{aligned} \tag{3.13}$$

In the case of Type A, only ‘‘Case I’’ is allowed as the realistic three-generation models. The typical  $U(1)$  fluxes and the number of generations of matters are given by tables 3 and 4. Under the  $U(1)$  gauge symmetries, the following Yukawa couplings of quarks and

<sup>5</sup>Here we do not constrain the number of charged-leptons,  $L$ , because some of them may be identified as higgsino fields.

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$
$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(\frac{5}{2}, \frac{1}{2}, \frac{3}{2})$
$(\frac{1}{2}, \frac{1}{2}, 0)$	$(\frac{5}{2}, \frac{1}{2}, 1)$
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$
$(1, \frac{1}{2}, -\frac{1}{2})$	$(2, \frac{1}{2}, \frac{3}{2})$
$(1, \frac{1}{2}, 0)$	$(2, \frac{1}{2}, 1)$
$(1, \frac{1}{2}, \frac{1}{2})$	$(0, \frac{5}{2}, \frac{1}{2})$
$(1, \frac{1}{2}, \frac{1}{2})$	$(2, \frac{1}{2}, \frac{1}{2})$
$(1, 1, \frac{1}{2})$	$(2, 0, \frac{1}{2})$
$(\frac{3}{2}, 0, -\frac{1}{2})$	$(\frac{3}{2}, 1, \frac{3}{2})$
$(\frac{3}{2}, 0, 0)$	$(\frac{3}{2}, 1, 1)$
$(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{3}{2})$
$(\frac{3}{2}, \frac{1}{2}, 0)$	$(\frac{3}{2}, \frac{1}{2}, 1)$
$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, \frac{5}{2}, \frac{1}{2})$
$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2})$
$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
$(\frac{3}{2}, 1, -\frac{1}{2})$	$(\frac{3}{2}, 0, \frac{3}{2})$
$(\frac{3}{2}, 1, 0)$	$(\frac{3}{2}, 0, 1)$
$(\frac{3}{2}, 1, \frac{1}{2})$	$(-\frac{1}{2}, 2, \frac{1}{2})$
$(\frac{3}{2}, 1, \frac{1}{2})$	$(\frac{3}{2}, 0, \frac{1}{2})$
$(\frac{3}{2}, 1, 1)$	$(\frac{3}{2}, 0, 0)$
$(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2})$
$(\frac{3}{2}, \frac{3}{2}, 0)$	$(\frac{3}{2}, -\frac{1}{2}, 1)$
$(\frac{3}{2}, \frac{3}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$
$(\frac{3}{2}, \frac{3}{2}, 1)$	$(\frac{3}{2}, -\frac{1}{2}, 0)$
$(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$
$(2, \frac{1}{2}, -\frac{1}{2})$	$(1, \frac{1}{2}, \frac{3}{2})$
$(2, \frac{1}{2}, 0)$	$(1, \frac{1}{2}, 1)$
$(2, \frac{1}{2}, \frac{1}{2})$	$(1, \frac{1}{2}, \frac{1}{2})$
$(2, 1, \frac{1}{2})$	$(1, 0, \frac{1}{2})$
$(2, \frac{3}{2}, \frac{1}{2})$	$(1, -\frac{1}{2}, \frac{1}{2})$
$(\frac{5}{2}, \frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$
$(\frac{5}{2}, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 1)$
$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$(\frac{5}{2}, 1, \frac{1}{2})$	$(\frac{1}{2}, 0, \frac{1}{2})$
$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$

**Table 2.** The possible magnetic fluxes in Type B within the range of  $-2 \leq m_{Q_2}^i \leq 2$ ,  $i = 1, 2, 3$ . Possible permutations among the three 2-tori are omitted. Certain types of sign flipping are also omitted.



$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_{10}^1, m_{10}^2, m_{10}^3)$
$(\frac{3}{2}, 0, 1)$	$(\frac{1}{2}, 1, 0)$	$(0, 0, 0)$	$(\frac{1}{2}, -2, 1)$	$(\frac{1}{2}, 1, 0)$

**Table 3.** The typical values of U(1) fluxes in the model of type A and “Case I” given by eqs. (3.10) and (3.11).

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(2, 1, 2, 1, 0, 0, 0, 0)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(0, 8, 1, 0, 0, 1)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(0, 8, 1, 0, 0, 1)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(0, 8, 1, 0, 0, 1)$
$(L_3^7, L_4^7, u_{R_2}^{c7}, d_{R_3}^{c7}, e_{R_1}^{c7}, n_2^7)$	$(-8, 0, 0, 1, 1, 0)$
$(L_3^8, L_4^8, u_{R_2}^{c8}, d_{R_3}^{c8}, e_{R_1}^{c8}, n_2^8)$	$(-8, 0, 0, 1, 1, 0)$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(-8, 0, 0, 1, 1, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(-1, -2, 0, 0, 0, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{c11}, d_{R_3}^{c11}, e_{R_1}^{c11}, n_2^{11})$	$(-1, -2, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{c12}, d_{R_3}^{c12}, e_{R_1}^{c12}, n_2^{12})$	$(2, 1, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(2, 1, 0, 0, 0, 0)$

**Table 4.** The number of generations for the representations defined in the model of type A and “Case I” given by eqs. (3.10) and (3.11).

leptons are allowed in terms of the renormalizable operators,

$$\begin{aligned}
 & (Q_1, \bar{L}_3^4, u_{R_2}^{c4}), (Q_2, \bar{L}_4^4, u_{R_2}^{c4}), (L_1, \bar{L}_3^4, n_2^4), (L_2, \bar{L}_4^4, n_2^4), \\
 & (Q_1, \bar{L}_3^5, u_{R_2}^{c5}), (Q_2, \bar{L}_4^5, u_{R_2}^{c5}), (L_1, \bar{L}_3^5, n_2^5), (L_2, \bar{L}_4^5, n_2^5), \\
 & (Q_1, \bar{L}_3^6, u_{R_2}^{c6}), (Q_2, \bar{L}_4^6, u_{R_2}^{c6}), (L_1, \bar{L}_3^6, n_2^6), (L_2, \bar{L}_4^6, n_2^6), \\
 & (Q_1, L_4^7, d_{R_3}^{c7}), (Q_2, L_3^7, d_{R_3}^{c7}), (L_1, L_4^7, e_{R_1}^{c7}), (L_2, L_3^7, e_{R_1}^{c7}), \\
 & (Q_1, L_4^8, d_{R_3}^{c8}), (Q_2, L_3^8, d_{R_3}^{c8}), (L_1, L_4^8, e_{R_1}^{c8}), (L_2, L_3^8, e_{R_1}^{c8}), \\
 & (Q_1, L_4^9, d_{R_3}^{c9}), (Q_2, L_3^9, d_{R_3}^{c9}), (L_1, L_4^9, e_{R_1}^{c9}), (L_2, L_3^9, e_{R_1}^{c9}).
 \end{aligned} \tag{3.14}$$

These include useful Yukawa couplings to give all of the quarks and leptons masses when  $\bar{L}_3^a, \bar{L}_4^a, L_3^b, L_4^b$  with  $a = 4, 5, 6$  and  $b = 7, 8, 9$  are identified as Higgs doublets and  $\bar{L}_{3,4}^a$  denote conjugate representations of  $L_{3,4}^a$ .

Next, we consider the case of Type B. As the supersymmetric three-generation models, both “Case I” and “Case II” are allowed and they are then categorized as the four types of models,

$$\begin{aligned}
 & \text{BI : “Case I” in type B,} \\
 & \text{BII : “Case II” in type B with } m_{n_1} = 0, \\
 & \text{BIII : “Case II” in type B with } m_{n_1} \neq 0, \\
 & \text{BIV : “Case III” in type B.}
 \end{aligned} \tag{3.15}$$

For each model, the typical U(1) fluxes and the number of generations of matters are summarized in tables 5, 6, 7, 8, 9 and 10. In the type BI model summarized in tables 5 and 6, the following Yukawa couplings of quarks and leptons are allowed in terms of the

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_{10}^1, m_{10}^2, m_{10}^3)$
$(1, 0, \frac{1}{2})$	$(2, 1, \frac{1}{2})$	$(0, 0, 0)$	$(-1, -2, \frac{1}{2})$	$(0, 1, -\frac{1}{2})$

**Table 5.** The typical values of U(1) fluxes in the type BI model given by eq. (3.15).

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(3, 0, 3, 0, 0, 8, -8, 0)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(0, 0, 1, 0, 0, 1)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(0, 0, 1, 0, 0, 1)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(0, 0, 1, 0, 0, 1)$
$(L_3^7, L_4^7, u_{R_2}^{c7}, d_{R_3}^{c7}, e_{R_1}^{c7}, n_2^7)$	$(0, 0, 0, 1, 1, 0)$
$(L_3^8, L_4^8, u_{R_2}^{c8}, d_{R_3}^{c8}, e_{R_1}^{c8}, n_2^8)$	$(0, 0, 0, 1, 1, 0)$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(0, 0, 0, 1, 1, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(-1, 0, 0, 0, 0, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{c11}, d_{R_3}^{c11}, e_{R_1}^{c11}, n_2^{11})$	$(-1, 0, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{c12}, d_{R_3}^{c12}, e_{R_1}^{c12}, n_2^{12})$	$(0, 1, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(0, 1, 0, 0, 0, 0)$

**Table 6.** The number of generations for the representations in the type BI model given by eq. (3.15).

renormalizable operators,

$$\begin{aligned}
 & (Q_1, \bar{L}_3^4, u_{R_2}^{c4}), (Q_1, \bar{L}_3^5, u_{R_2}^{c5}), (Q_1, \bar{L}_3^6, u_{R_2}^{c6}), (L_1, \bar{L}_3^4, n_2^4), (L_1, \bar{L}_3^5, n_2^5), (L_1, \bar{L}_3^6, n_2^6), \\
 & (Q_1, \bar{L}_4^7, d_{R_3}^{c7}), (Q_1, \bar{L}_4^8, d_{R_3}^{c8}), (Q_1, \bar{L}_4^9, d_{R_3}^{c9}), (L_1, \bar{L}_4^7, e_{R_1}^{c7}), (L_1, \bar{L}_4^8, e_{R_1}^{c8}), (L_1, \bar{L}_4^9, e_{R_1}^{c9}).
 \end{aligned}
 \tag{3.16}$$

These also include useful Yukawa couplings when  $\bar{L}_3^a, L_4^b$  with  $a = 4, 5, 6$  and  $b = 7, 8, 9$  are identified as Higgs doublets. In both type BII and type BIII models summarized in tables 7, 8, 9 and 10, the useful Yukawa couplings of quarks and leptons are allowed in terms of the renormalizable operators,

$$(Q_1, \bar{L}_3^4, u_{R_2}^{c4}), (Q_1, \bar{L}_4^5, d_{R_3}^{c5}), (L_1, \bar{L}_3^4, n_2^4), (L_1, \bar{L}_4^5, e_{R_1}^{c5}),
 \tag{3.17}$$

where  $\bar{L}_3^4, \bar{L}_4^5$  are identified as Higgs doublets. Finally, in type BIV model summarized in tables 11 and 12, the useful Yukawa couplings of quarks and leptons are allowed in terms of the renormalizable operators,

$$\begin{aligned}
 & (Q_1, \bar{L}_3^4, u_{R_2}^{c4}), (Q_1, \bar{L}_3^6, u_{R_2}^{c6}), (L_1, \bar{L}_3^4, n_2^4), (L_1, \bar{L}_3^6, n_2^6), \\
 & (Q_1, \bar{L}_4^5, d_{R_3}^{c5}), (Q_1, \bar{L}_4^7, d_{R_3}^{c7}), (L_1, \bar{L}_4^5, e_{R_1}^{c5}), (L_1, \bar{L}_4^7, e_{R_1}^{c7}),
 \end{aligned}
 \tag{3.18}$$

where  $\bar{L}_3^{4,6}, \bar{L}_4^{5,7}$  are identified as Higgs doublets.

Note that in our models, the consistency conditions given by eq. (2.7) are not satisfied without introducing the heterotic five-branes. In this case, we have to take care of the Witten anomaly [23, 24] on the heterotic five-branes with  $\text{Sp}(2N)$  gauge groups which is the case that the number of heterotic five-branes is  $N$ . In order to avoid the Witten anomaly, the number of chiral fermions under the fundamental representations of  $\text{Sp}(2N)$  are even [23, 24]. These fundamental representations of  $(32, 2N)$  under  $\text{SO}(32) \otimes \text{Sp}(2N)$

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_6^1, m_6^2, m_6^3)$
$(\frac{5}{2}, 0, \frac{1}{2})$	$(\frac{1}{2}, 1, \frac{1}{2})$	$(0, 0, 0)$	$(-\frac{3}{2}, 2, \frac{1}{2})$	$(\frac{9}{2}, -1, \frac{1}{2})$

**Table 7.** The typical values of U(1) fluxes in the type BII model given by eq. (3.15).

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(3, 0, 3, 0, 0, 2, -2, 0)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(0, -2, 3, 0, 0, 3)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(2, 0, 0, 3, 3, 0)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(0, 7, 0, 0, 0, 0)$
$(L_3^7, L_4^7, u_{R_2}^{c7}, d_{R_3}^{c7}, e_{R_1}^{c7}, n_2^7)$	$(0, 7, 0, 0, 0, 0)$
$(L_3^8, L_4^8, u_{R_2}^{c8}, d_{R_3}^{c8}, e_{R_1}^{c8}, n_2^8)$	$(0, 7, 0, 0, 0, 0)$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(0, 7, 0, 0, 0, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(-7, 0, 0, 0, 0, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{c11}, d_{R_3}^{c11}, e_{R_1}^{c11}, n_2^{11})$	$(-7, 0, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{c12}, d_{R_3}^{c12}, e_{R_1}^{c12}, n_2^{12})$	$(-7, 0, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(-7, 0, 0, 0, 0, 0)$

**Table 8.** The number of generations for the representations in the type BII model given by eq. (3.15).

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_6^1, m_6^2, m_6^3)$
$(1, -\frac{1}{2}, \frac{1}{2})$	$(2, \frac{3}{2}, \frac{1}{2})$	$(0, 0, 0)$	$(-1, -\frac{9}{2}, \frac{1}{2})$	$(2, \frac{13}{2}, -\frac{1}{2})$

**Table 9.** The typical values of U(1) fluxes in the type BIII model given by eq. (3.15).

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(3, 0, 3, 0, 0, 12, -12, 2)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(0, 0, 3, 0, 0, 3)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(0, 0, 0, 3, 3, 0)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(7, 0, 0, 0, 0, 0)$
$\vdots$	$\vdots$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(7, 0, 0, 0, 0, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(0, -7, 0, 0, 0, 0)$
$\vdots$	$\vdots$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(0, -7, 0, 0, 0, 0)$

**Table 10.** The number of generations for the representations in the type BIII model given by eq. (3.15).

can be read in the type I string with D5-and D9-brane system which is expected as the S-dual of the SO(32) heterotic string. The generations of the chiral fermions included in  $(12, 2N)$  under  $SO(12) \otimes Sp(2N)$  and  $(20, 2N)$  under  $SO(20) \otimes Sp(2N)$  are determined by  $\pm \prod_{i=1}^3 m_a^{(i)}$  for  $a = 1, 2, 4, \dots, 13$ , in the case  $m_3^{(i)} = 0$  with  $i = 1, 2, 3$ . In our most supersymmetric models, the chiral fermions arise from  $(32, 2N)$  under  $SO(32) \otimes Sp(2N)$ . Thus we require the non-trivial mechanism to obtain the even number of chiral fermions such as U(1) fluxes on the heterotic five-branes in order to avoid the Witten anomaly.

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_6^1, m_6^2, m_6^3)$	$(m_8^1, m_8^2, m_8^3)$
$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 0)$	$(-\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

**Table 11.** The typical values of U(1) fluxes in the type BIV model given by eq. (3.15).

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(3, 0, 3, 0, 0, 1, -1, 5)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(0, 0, 2, 0, 0, 2)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(0, 0, 0, 2, 2, 0)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(0, -1, 1, 0, 0, 1)$
$(L_3^7, L_4^7, u_{R_2}^{c7}, d_{R_3}^{c7}, e_{R_1}^{c7}, n_2^7)$	$(0, -1, 1, 0, 0, 1)$
$(L_3^8, L_4^8, u_{R_2}^{c8}, d_{R_3}^{c8}, e_{R_1}^{c8}, n_2^8)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(0, -2, 0, 0, 0, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{c11}, d_{R_3}^{c11}, e_{R_1}^{c11}, n_2^{11})$	$(2, 0, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{c12}, d_{R_3}^{c12}, e_{R_1}^{c12}, n_2^{12})$	$(2, 0, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(2, 0, 0, 0, 0, 0)$

**Table 12.** The number of generations for the representations in the type BIV model given by eq. (3.15).

Finally we comment on the gauge enhancements induced by vanishing fluxes. In this paper, we focus on the case  $m_3^i = 0$ ,  $i = 1, 2, 3$  in the light of U(1)<sub>Y</sub> massless conditions given by eqs. (2.24) and (2.28). These vanishing fluxes cause the gauge enhancement,  $SU(3)_C \times U(1)_3 \rightarrow SU(4)$ . Moreover it requires the Wilson-lines into the internal component of U(1)<sub>3</sub> to break down SU(4) into SU(3). Our models have other gauge enhancements. The realistic three-generation models are summarized in three cases, “Case I”, “Case II” and “Case III” in eqs. (3.11), (3.12) and (3.13), respectively. In both cases, most magnetic fluxes are related to each other due to the U(1)<sub>Y</sub> massless conditions given by eqs. (2.24) and (2.28). For example, the invariant simple roots under the existences of fluxes read

$$\begin{aligned}
 \text{Case I} \quad & \alpha_1 = (0, 0, 0, 0, 0, 0; 1, -1, 0, \dots, 0), \\
 & \alpha_2 = (0, 0, 0, 0, 0, 0; 0, 1, -1, 0, \dots, 0), \\
 & \alpha_3 = (0, 0, 0, 0, 0, 0; 0, 0, 0, 1, -1, 0, \dots, 0), \\
 & \alpha_4 = (0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 1, -1, 0, \dots, 0), \\
 & \alpha_5 = (0, 0, 0, 0, 0, 0; 0, 0, 1, 0, 0, 1, 0, \dots, 0), \\
 \text{Case II} \quad & \alpha_1 = (0, 0, 0, 0, 0, 0; 1, 1, 0, \dots, 0), \\
 \text{Case III} \quad & \alpha_1 = (0, 0, 0, 0, 0, 0; 1, 1, 0, \dots, 0), \\
 & \alpha_2 = (0, 0, 0, 0, 0, 0; 0, 0, 1, 1, 0, \dots, 0),
 \end{aligned} \tag{3.19}$$

which implies the SU(6), SU(2) and SU(2) × SU(2) gauge symmetries, respectively. All of them include SU(2)<sub>R</sub>. Furthermore, SU(3) of SU(6) is a flavor symmetry of right-handed matter fields, and the three right-handed matter generations is a triplet under SU(3) flavor

symmetry, while the left-handed matter fields are singlets. We introduce Wilson lines to break these symmetries.

### 3.3 Three-generation models with K-theory constraints

So far, we have not considered the so-called K-theory constraints which are formulated in the S-dual to the SO(32) heterotic string theory, i.e., Type I string theory. In the SO(32) heterotic string theory, the total number of magnetic fluxes is further constrained as

$$\sum_{a=1}^{13} m_a^i = 0 \pmod{2}, \tag{3.20}$$

for  $i = 1, 2, 3$ , as stated in ref. [17, 18]. Such a condition allows for the well-defined spinor representation of the gauge bundle, otherwise its wavefunction is not single-valued.

When we assume that the SO(32) heterotic string theory on our gauge background is described as its S-dual theory, i.e., Type I string theory, the above condition (3.20) may correspond to the K-theory constraints [31] which cannot be classified in terms of a homology. These constraints can be understood by introducing all the possible probe D-branes [32], and then they show the existence of several stable non-BPS branes with the discrete K-theory charge, i.e.,  $Z_2$ -charge. In the case of  $N$  stacks of heterotic five-brane with  $Sp(2N)$  gauge group, they require the condition (3.20) in order to avoid the Witten anomaly [23, 24].<sup>6</sup> Furthermore, in type I string, the fractional fluxes are allowed due to multiple wrapping numbers of D-branes. Although such a degree of freedom is expected to appear in the heterotic string side, we do not consider these possibilities, which we leave for future works. Since all the models discussed in section 3.2 do not satisfy the K-theory condition, in this section, we further search for the possibilities of three-generation models under these assumptions.

First of all, in the light of  $U(1)_Y$  massless condition, we impose the constraints for U(1) fluxes as,

$$m_3^i = 0, \quad m_{a+3}^i = -m_{a+8}^i \quad (a = 1, 2, 3, 4, 5), \tag{3.21}$$

with  $i = 1, 2, 3$ , which simplify the K-theory condition as

$$\sum_{a=1}^2 m_a^i = 0 \pmod{2}. \tag{3.22}$$

From the fact that all the possible candidates for left-handed quarks  $Q$  and charged leptons  $L$  are involved in the adjoint representation of SO(12), three generations of  $Q$  and  $L$  have to be realized from such a representation. Then, their fluxes are constrained as

$$\begin{aligned} \text{Type A: } (m_{Q_1}, m_{Q_2}) &= (2, 1), & \text{Type A': } (m_{Q_1}, m_{Q_2}) &= (1, 2), \\ \text{Type B: } (m_{Q_1}, m_{Q_2}) &= (3, 0), & \text{Type B': } (m_{Q_1}, m_{Q_2}) &= (0, 3), \end{aligned} \tag{3.23}$$

---

<sup>6</sup>In the heterotic string, the K-theory may be understood in terms of closed string tachyon [33] based on supercritical string [34].

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$
$(-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2})$	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$
$(-\frac{5}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
$(-\frac{5}{2}, -\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$
$(-\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2})$	$(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$
$(-\frac{3}{2}, -\frac{3}{2}, -\frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$
$(-\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2})$
$(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$
$(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{3}{2})$
$(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$
$(-\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$	$(\frac{5}{2}, -\frac{1}{2}, \frac{1}{2})$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$
$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$(\frac{5}{2}, \frac{1}{2}, \frac{3}{2})$

**Table 13.** The possible magnetic fluxes in Type B' within the range of  $-2 \leq m_{Q_1}^i \leq 2$  for  $m_{Q_2}^i = 0$  and  $-2 \leq m_{Q_2}^i \leq 2$  for  $m_{Q_1}^i = 0$ , where  $i = 1, 2, 3$ .

where  $m_{Q_{1,2}}^i$ , and hereafter we focus on the case that the right-handed quarks  $d_R^c$  are generated from the vector representation of  $SO(12)$ , for simplicity. In such cases, we find that only Type B' in eq. (3.23) satisfies the K-theory condition (3.22) and the SUSY condition (2.42) yielding three generations of  $Q$  and  $L$ . The possible  $U(1)_{1,2}$  fluxes are summarized in table 13.

Next, we consider the remaining matter contents in the standard model, that is,  $u_R^c$ ,  $d_R^c$  and  $e_R^c$ . Among the constrained magnetic fluxes listed in table 13, we further search for those yield three generations of  $u_R^c$ ,  $d_R^c$  and  $e_R^c$ , satisfying the  $U(1)_Y$  massless condition (3.21) as well as the SUSY condition (2.42). Note that the K-theory condition is already satisfied under the constraints (3.21) and (3.22). As a result, within the range of  $-5 \leq m_{u_{R_2}^{c_a}}^i \leq 5$ , there are three allowed choices for the  $U(1)$  fluxes as follows,

$$\begin{aligned}
 \text{“Case I”} \quad (m_{u_{R_2}^{c_4}}, m_{u_{R_2}^{c_5}}, \dots, m_{u_{R_2}^{c_{13}}}) &= (1, 1, 1, 0, 0, 0, 0, 0, 0, 0), \\
 (m_{d_{R_2}^{c_4}}, m_{d_{R_2}^{c_5}}, \dots, m_{d_{R_2}^{c_{13}}}) &= (0, 0, 0, 0, 0, 1, 1, 1, 0, 0),
 \end{aligned} \tag{3.24}$$

$$\begin{aligned}
 \text{“Case II”} \quad (m_{u_{R_2}^{c_4}}, m_{u_{R_2}^{c_5}}, \dots, m_{u_{R_2}^{c_{13}}}) &= (3, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\
 (m_{d_{R_2}^{c_4}}, m_{d_{R_2}^{c_5}}, \dots, m_{d_{R_2}^{c_{13}}}) &= (0, 0, 0, 0, 0, 3, 0, 0, 0, 0),
 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
 \text{“Case III”} \quad (m_{u_{R_2}^{c_4}}, m_{u_{R_2}^{c_5}}, \dots, m_{u_{R_2}^{c_{13}}}) &= (2, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\
 (m_{d_{R_2}^{c_4}}, m_{d_{R_2}^{c_5}}, \dots, m_{d_{R_2}^{c_{13}}}) &= (0, 0, 0, 0, 0, 2, 1, 0, 0, 0).
 \end{aligned} \tag{3.26}$$

For each model, the typical  $U(1)$  fluxes and the number of generations of matters are summarized in tables 14, 15, 16, 17, 18 and 19. In the “Case I” summarized in tables 14 and 15, non-vanishing Yukawa coupling terms involving the following combinations of quarks and

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_7^1, m_7^2, m_7^3)$
$(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 0)$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

**Table 14.** The typical values of U(1) fluxes in the “Case I” given by eq. (3.24). The other U(1) fluxes are constrained to be  $m_4^i = m_5^i = m_6^i = -m_9^i = -m_{10}^i = -m_{11}^i$  and  $m_7^i = m_8^i = -m_{12}^i = -m_{13}^i$  with  $i = 1, 2, 3$ .

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(0, 3, 0, 3, 0, 3, -3, 3)$
$(L_3^4, L_4^4, u_{R_2}^{c4}, d_{R_3}^{c4}, e_{R_1}^{c4}, n_2^4)$	$(1, 0, 1, 0, 0, 1)$
$(L_3^5, L_4^5, u_{R_2}^{c5}, d_{R_3}^{c5}, e_{R_1}^{c5}, n_2^5)$	$(1, 0, 1, 0, 0, 1)$
$(L_3^6, L_4^6, u_{R_2}^{c6}, d_{R_3}^{c6}, e_{R_1}^{c6}, n_2^6)$	$(1, 0, 1, 0, 0, 1)$
$(L_3^7, L_4^7, u_{R_2}^{c7}, d_{R_3}^{c7}, e_{R_1}^{c7}, n_2^7)$	$(0, 0, 0, 0, 0, 0)$
$(L_3^8, L_4^8, u_{R_2}^{c8}, d_{R_3}^{c8}, e_{R_1}^{c8}, n_2^8)$	$(0, 0, 0, 0, 0, 0)$
$(L_3^9, L_4^9, u_{R_2}^{c9}, d_{R_3}^{c9}, e_{R_1}^{c9}, n_2^9)$	$(0, -1, 0, 1, 1, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{c10}, d_{R_3}^{c10}, e_{R_1}^{c10}, n_2^{10})$	$(0, -1, 0, 1, 1, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{c11}, d_{R_3}^{c11}, e_{R_1}^{c11}, n_2^{11})$	$(0, -1, 0, 1, 1, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{c12}, d_{R_3}^{c12}, e_{R_1}^{c12}, n_2^{12})$	$(0, 0, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{c13}, d_{R_3}^{c13}, e_{R_1}^{c13}, n_2^{13})$	$(0, 0, 0, 0, 0, 0)$

**Table 15.** The number of generations for the representations in the “Case I” given by eq. (3.24).

leptons are allowed as the renormalizable operators,

$$(Q_2, \bar{L}_4^4, u_{R_2}^{c4}), (Q_2, \bar{L}_4^5, u_{R_2}^{c5}), (Q_2, \bar{L}_4^6, u_{R_2}^{c6}), (L_2, \bar{L}_4^4, n_2^4), (L_2, \bar{L}_4^5, n_2^5), (L_2, \bar{L}_4^6, n_2^6), \\ (Q_2, L_3^9, d_{R_3}^{c9}), (Q_2, L_3^{10}, d_{R_3}^{c10}), (Q_2, L_3^{11}, d_{R_3}^{c11}), (L_2, L_3^9, e_{R_1}^{c9}), (L_2, L_3^{10}, e_{R_1}^{c10}), (L_2, L_3^{11}, e_{R_1}^{c11}). \quad (3.27)$$

These include useful Yukawa couplings to give masses of all the quarks and leptons when  $\bar{L}_4^a, L_3^b$  with  $a = 4, 5, 6$  and  $b = 9, 10, 11$  are identified as Higgs doublets.

As for the “Case II” summarized in tables 16 and 17, the following combinations of quarks and leptons have renormalizable Yukawa coupling,

$$(Q_2, \bar{L}_4^4, u_{R_2}^{c4}), (Q_2, L_3^9, d_{R_3}^{c9}), (L_2, \bar{L}_4^4, n_2^4), (L_2, L_3^9, e_{R_1}^{c9}), \quad (3.28)$$

where  $\bar{L}_4^4, L_3^9$  are identified as Higgs doublets in order to be phenomenologically viable.

Next, in the “Case III” summarized in tables 18 and 19, the renormalizable Yukawa couplings are allowed for the following combinations of quarks and leptons,

$$(Q_2, \bar{L}_4^4, u_{R_2}^{c4}), (Q_2, \bar{L}_4^5, u_{R_2}^{c5}), (L_2, \bar{L}_4^4, n_2^4), (L_2, \bar{L}_4^5, n_2^5), \\ (Q_2, L_3^9, d_{R_3}^{c9}), (Q_2, L_3^{10}, d_{R_3}^{c10}), (L_2, L_3^9, e_{R_1}^{c9}), (L_2, L_3^{10}, e_{R_1}^{c10}), \quad (3.29)$$

where  $\bar{L}_4^{4,5}, L_3^{9,10}$  can be identified as Higgs doublets. Note that in the same way as in section 3.2, the consistency conditions given by eq. (2.7) are not satisfied without introducing the heterotic five-branes, in the supersymmetric case.

Finally we comment on the gauge enhancements induced by vanishing fluxes. As discussed in section 3.2, vanishing U(1)<sub>3</sub> fluxes require the existence of Wilson-lines for the internal component of U(1)<sub>3</sub> to break SU(4) down to SU(3). There are other gauge enhancements in three realistic models, “Case I”, “Case II” and “Case III”, where most

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_5^1, m_5^2, m_5^3)$
$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 0)$	$(-\frac{11}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{5}{2}, -\frac{1}{2}, -\frac{1}{2})$

**Table 16.** The typical values of U(1) fluxes in the ‘‘Case II’’ given by eq. (3.25). The other U(1) fluxes are constrained to be  $m_4^i = -m_9^i$  and  $m_5^i = m_6^i = m_7^i = m_8^i = -m_{10}^i = -m_{11}^i = -m_{12}^i = -m_{13}^i$  with  $i = 1, 2, 3$ .

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(0, 3, 0, 3, 0, 5, -5, 1)$
$(L_3^4, L_4^4, u_{R_2}^4, d_{R_3}^4, e_{R_1}^4, n_2^4)$	$(5, 0, 3, 0, 0, 3)$
$(L_3^5, L_4^5, u_{R_2}^5, d_{R_3}^5, e_{R_1}^5, n_2^5)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^6, L_4^6, u_{R_2}^6, d_{R_3}^6, e_{R_1}^6, n_2^6)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^7, L_4^7, u_{R_2}^7, d_{R_3}^7, e_{R_1}^7, n_2^7)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^8, L_4^8, u_{R_2}^8, d_{R_3}^8, e_{R_1}^8, n_2^8)$	$(0, -2, 0, 0, 0, 0)$
$(L_3^9, L_4^9, u_{R_2}^9, d_{R_3}^9, e_{R_1}^9, n_2^9)$	$(0, -5, 0, 3, 3, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{10}, d_{R_3}^{10}, e_{R_1}^{10}, n_2^{10})$	$(2, 0, 0, 0, 0, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{11}, d_{R_3}^{11}, e_{R_1}^{11}, n_2^{11})$	$(2, 0, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{12}, d_{R_3}^{12}, e_{R_1}^{12}, n_2^{12})$	$(2, 0, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{13}, d_{R_3}^{13}, e_{R_1}^{13}, n_2^{13})$	$(2, 0, 0, 0, 0, 0)$

**Table 17.** The number of generations for the representations in the ‘‘Case II’’ given by eq. (3.25).

$(m_1^1, m_1^2, m_1^3)$	$(m_2^1, m_2^2, m_2^3)$	$(m_3^1, m_3^2, m_3^3)$	$(m_4^1, m_4^2, m_4^3)$	$(m_5^1, m_5^2, m_5^3)$	$(m_6^1, m_6^2, m_6^3)$
$(-\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 0)$	$(-\frac{7}{2}, \frac{1}{2}, \frac{1}{2})$	$(-\frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2})$

**Table 18.** The typical values of U(1) fluxes in the ‘‘Case III’’ given by eq. (3.26). The other U(1) fluxes are constrained to be  $m_6^i = m_7^i = m_8^i = -m_{11}^i = -m_{12}^i = -m_{13}^i$  with  $i = 1, 2, 3$ .

$(Q_1, Q_2, L_1, L_2, u_{R_1}^c, d_{R_1}^c, d_{R_2}^c, n_1)$	$(0, 3, 0, 3, 0, 3, -3, 3)$
$(L_3^4, L_4^4, u_{R_2}^4, d_{R_3}^4, e_{R_1}^4, n_2^4)$	$(2, 0, 2, 0, 0, 2)$
$(L_3^5, L_4^5, u_{R_2}^5, d_{R_3}^5, e_{R_1}^5, n_2^5)$	$(1, 0, 1, 0, 0, 1)$
$(L_3^6, L_4^6, u_{R_2}^6, d_{R_3}^6, e_{R_1}^6, n_2^6)$	$(0, 0, 0, 0, 0, 0)$
$(L_3^7, L_4^7, u_{R_2}^7, d_{R_3}^7, e_{R_1}^7, n_2^7)$	$(0, 0, 0, 0, 0, 0)$
$(L_3^8, L_4^8, u_{R_2}^8, d_{R_3}^8, e_{R_1}^8, n_2^8)$	$(0, 0, 0, 0, 0, 0)$
$(L_3^9, L_4^9, u_{R_2}^9, d_{R_3}^9, e_{R_1}^9, n_2^9)$	$(0, -2, 0, 2, 2, 0)$
$(L_3^{10}, L_4^{10}, u_{R_2}^{10}, d_{R_3}^{10}, e_{R_1}^{10}, n_2^{10})$	$(0, -1, 0, 1, 1, 0)$
$(L_3^{11}, L_4^{11}, u_{R_2}^{11}, d_{R_3}^{11}, e_{R_1}^{11}, n_2^{11})$	$(0, 0, 0, 0, 0, 0)$
$(L_3^{12}, L_4^{12}, u_{R_2}^{12}, d_{R_3}^{12}, e_{R_1}^{12}, n_2^{12})$	$(0, 0, 0, 0, 0, 0)$
$(L_3^{13}, L_4^{13}, u_{R_2}^{13}, d_{R_3}^{13}, e_{R_1}^{13}, n_2^{13})$	$(0, 0, 0, 0, 0, 0)$

**Table 19.** The number of generations for the representations in the ‘‘Case III’’ given by eq. (3.26).

magnetic fluxes are related to each other due to the  $U(1)_Y$  massless conditions (2.24), (2.28) and the K-theory condition (3.20). For example, there are invariant simple roots under the existences of fluxes such as SU(6), SU(2) and SU(2)  $\times$  SU(2) gauge symmetries for the



“Case I’”, “Case II’” and “Case III’”, respectively. We introduce Wilson-lines to break these gauge symmetries.

## 4 Conclusion

In this paper, we have derived the realistic standard model gauge groups from the framework of  $SO(32)$  heterotic string theory on three factorizable 2-tori with magnetic fluxes. Introducing magnetic fluxes as well as Wilson lines into Cartan directions of  $SO(32)$  break  $SO(32)$  to  $SU(3) \times SU(2) \times U(1)_Y$  and extra symmetries. These  $U(1)$  fluxes also lead to chiral fermions in the four dimensions if and only if the fluxes insert into all the three 2-tori. At the same time, the generations of chiral matters are determined by the numbers of fluxes. We have derived three chiral generations of quarks and leptons. Our models also include Higgs fields, which have Yukawa couplings to quarks and leptons at tree level.

Possible configurations of magnetic fluxes are severely constrained by the massless condition of  $U(1)_Y$  hypercharge gauge boson and the consistency condition of heterotic string theory. It is remarkable that in general, the ten-dimensional Green-Schwarz term induces the Stueckelberg couplings to multiple  $U(1)$  gauge bosons which might lead to the mass term of  $U(1)_Y$  hypercharge gauge boson. In this respect, the numbers of fluxes have been constrained by the massless condition of  $U(1)_Y$  gauge boson. Since the torus is flat, our models requires the existence of heterotic five-branes in order to satisfy the consistency conditions without introducing the extra Stueckelberg couplings to  $U(1)$  gauge boson, in contrast to the  $E_8 \times E_8$  heterotic string theory. At that time, the Witten anomaly cancellation constrains the number of  $U(1)$  fluxes due to the nature of symplectic groups on the heterotic five-branes. In fact, the chiral fermions under the fundamental representation of symplectic gauge groups do not arise in the parts of our models, whereas the other parts of our models requires the non-trivial mechanisms such as  $U(1)$  fluxes on heterotic five-branes to cause even number of these chiral fermions to avoid the Witten anomaly. We listed supersymmetric three-generation standard models with massless  $U(1)_Y$  gauge bosons and desirable Yukawa couplings of quarks, leptons and Higgs. The detailed phenomenological analysis of our models such as mass matrices would be studied in a separate work and the detail of this paper is applicable in the framework of type I string.

The unbroken gauge sector in our models has  $\mathcal{N} = 4$  supersymmetry, that is, three adjoint scalar fields and four types of gaugino fields. However, the existence of (anti-)heterotic five-branes would lead to the breaking of (all) partial breaking of supersymmetry in our model. Orbifolding would be useful to reduce  $\mathcal{N} = 4$  supersymmetry to  $\mathcal{N} = 1$ . Zero-mode wavefunctions have been also studied on orbifolds with magnetic fluxes [35–37]. Such extensions would be also interesting.

## Acknowledgments

H.O. would like to thank T. Higaki for useful discussions and comments. H.A. was supported in part by the Grant-in-Aid for Scientific Research No. 25800158 from the Ministry

of Education, Culture, Sports, Science and Technology (MEXT) in Japan. T.K. was supported in part by the Grant-in-Aid for Scientific Research No. 25400252 and No. 26247042 from the MEXT in Japan. H.O. was supported in part by a Grant-in-Aid for JSPS Fellows No. 26-7296.

## A Normalization of the SO(32) gauge group

In this appendix, we show the normalization of Abelian gauge groups embedded in SO(32) gauge group. (For more details, see refs. [16, 28, 29].) First, we comment on the normalization about the non-Abelian gauge groups in SO(32). The sum of each Coxeter labels associated with the simple roots of SO(32) are called as the Coxeter number  $h(g)$  which is related to the quadratic Casimir via the following relation,

$$\sum_{c,d} f^{acd} f^{bcd} = h(g)\psi^2 \delta^{ab}, \tag{A.1}$$

where  $h(g) = 30$ ,  $f^{abc}$  with  $a = 1, 2, \dots, 496$  are the structure constants of SO(32) and  $\psi^2$  denotes the length of the root which is normalized as two.

The normalization of the Abelian gauge groups are estimated by the current algebra or Kač-Moody algebra of SO(32) which is given by

$$[j_m^a, j_n^b] = i \sum_c f^{abc} j_{m+n}^c + \frac{2k}{\psi^2} m \delta^{ab} \delta_{m,-n}, \tag{A.2}$$

where  $k$  is the level of Kač-Moody algebra and  $j_m^a$  are the Laurent coefficients of the current  $j^a(z)$ ,

$$j^a(z) = \frac{1}{2} N(\psi^i T_{ij}^a \psi^j) = \sum_{m=-\infty}^{\infty} \frac{j_m^a}{z^{m+1}}, \tag{A.3}$$

with  $\psi^i$  and  $(T^a)_{ij}$  ( $i = 1, 2, \dots, 32$ ) being the 32 real fermions and generators in the vector representation of SO(32), respectively.  $N(\psi^i T_{ij}^a \psi^j)$  stands for the normal ordering of the operator,  $(\psi^i T_{ij}^a \psi^j)$ . When the level of Kač-Moody algebra is equal to one, we obtain the operator product expansion of the current

$$j^a(z) j^b(w) \sim \frac{2\delta^{ab}}{\psi^2(z-w)^2} + \frac{i f^{abc}}{z} j^c(w), \tag{A.4}$$

and then we can extract the normalization of  $(T^a)_{ij}$  as  $\text{tr}(T^a T^b) = 2\delta_{ab}$ .

In our model, the generators of  $U(1)_a$ ,  $T_a$  are normalized as

$$\begin{aligned}
T_1 &= \frac{1}{\sqrt{2}} \text{diag}(0, 0, 0, 0, 1, 1, 0, 0, \dots, 0), \\
T_2 &= \frac{1}{2} \text{diag}(1, 1, 1, 1, 0, 0, 0, 0, \dots, 0), \\
T_3 &= \frac{1}{\sqrt{12}} \text{diag}(1, 1, 1, -3, 0, 0, 0, 0, \dots, 0), \\
T_4 &= \text{diag}(0, 0, 0, 0, 0, 0, 1, 0, \dots, 0), \\
T_5 &= \text{diag}(0, 0, 0, 0, 0, 0, 0, 1, 0, \dots, 0), \\
&\vdots \\
T_{13} &= \text{diag}(0, 0, 0, 0, 0, 0, 0, 0, \dots, 1),
\end{aligned} \tag{A.5}$$

on the basis of  $U(16)$  which is the maximal subgroup of  $SO(32)$ . In general, the generators of  $U(N)$  can be identified as the part of  $SO(2N)$  generators. (See e.g. ref. [30].)

## B The trace identities

Here, we summarize the trace identities

$$\begin{aligned}
\text{Tr } F^2 &= 30 \text{tr } F^2 = 60 F_{\text{SU}(3)_C}^2 + 60 F_{\text{SU}(2)_L}^2 + 60 \sum_{a=1}^{13} f_a^2, \\
\text{Tr } \bar{F}^2 &= 30 \text{tr } \bar{F}^2 = 60 \sum_{a=1}^{13} \bar{f}_a^2, \\
\text{Tr } F \bar{F} &= 30 \text{tr } F \bar{F} = 60 \sum_{a=1}^{13} f_a \bar{f}_a, \\
\text{tr } F^2 \bar{F}^2 &= \left( \frac{1}{2} \text{tr}(T_2^2) \bar{f}_2^2 + \sqrt{\frac{\text{tr}(T_2^2) \text{tr}(T_3^2)}{3}} \bar{f}_2 \bar{f}_3 + \frac{1}{6} \text{tr}(T_3^2) \bar{f}_3^2 \right) \text{tr}(F_{\text{SU}(3)}^2) \\
&\quad + 2 \text{tr}(T_1^4) \bar{f}_1^2 f_1^2 + 2 \sum_{c=4}^{13} \text{tr}(T_c^4) f_c^2 \bar{f}_c^2 \\
&\quad + 2(\text{tr}(T_2^4) \bar{f}_2^2 + \text{tr}(T_2^2 T_3^2) \bar{f}_3^2) f_2^2 + 4(2 \text{tr}(T_2^2 T_3^2) \bar{f}_2 \bar{f}_3 + \text{tr}(T_2 T_3^3) \bar{f}_3^2) f_2 f_3 \\
&\quad + 2(\text{tr}(T_3^4) \bar{f}_3^2 + \text{tr}(T_2^2 T_3^2) \bar{f}_2^2 + 2 \text{tr}(T_2 T_3^3) \bar{f}_2 \bar{f}_3) f_3^2, \\
\text{tr } F \bar{F}^3 &= 2 \text{tr } T_1^4 \bar{f}_1^3 f_1 + 2(\text{tr } T_2^4 \bar{f}_2^3 + 3(\text{tr } T_2^2 T_3^2) \bar{f}_2 \bar{f}_3^2 + (\text{tr } T_2 T_3^3) \bar{f}_3^3) f_2 \\
&\quad + (\text{tr } T_3^4 \bar{f}_3^3 + 3(\text{tr } T_2 T_3^3) \bar{f}_2 \bar{f}_3^2 + 3(\text{tr } T_2^2 T_3^2) \bar{f}_2^2 \bar{f}_3) f_3 + 2 \sum_{c=4}^{13} \text{tr } T_c^4 \bar{f}_c^3 f_c, \tag{B.1}
\end{aligned}$$

where  $f_a$  and  $\bar{f}_a$  denote the four-dimensional and extra-dimensional field strengths of  $U(1)_a$  and we employ the trace identities such as

$$\begin{aligned}
 \text{Tr } F^2 &= 30 \text{tr } F^2, \\
 \text{Tr } F^4 &= 24 \text{tr } F^4 + 3(\text{tr } F^2)^2, \\
 \text{Tr } F\bar{F}^3 &= 24 \text{tr } F\bar{F}^3 + 3(\text{tr } F\bar{F})(\text{tr } \bar{F}^2), \\
 \text{Tr } F^2\bar{F}^2 &= 24 \text{tr } F^2\bar{F}^2 + 2(\text{tr } F\bar{F})^2 + (\text{tr } F^2)(\text{tr } \bar{F}^2), \\
 \text{tr } T_1^4 &= 1/2, \quad \text{tr } T_2^4 = 1/4, \quad \text{tr } T_3^4 = 7/12, \quad \text{tr } T_a^4 = 1 \quad (c=4, \dots, 13), \\
 \text{tr } T_2^2 T_3^2 &= 1/4, \quad \text{tr } T_2^3 T_3 = 0, \quad \text{tr } T_2 T_3 = -1/2\sqrt{3}.
 \end{aligned}
 \tag{B.2}$$

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