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Unsteady Couette flow of viscous fluid under a non-uniform magnetic field

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ABSTRACT

The aim of this letter is to construct the analytic solution for unsteady Couette flow in the presence of an arbitrary non-uniform applied magnetic field. The flow is induced by a generalized velocity given to the lower plate. The perturbed eigenfunction expansion method is employed to develop a series solution for small magnetic field.

1. Introduction

There has been increasing interest from researchers in the flows of an electrically conducting fluid during the past few decades. Interest in such flows arises because of their occurrence in many branches of science and engineering. These fluids are specifically encountered in magnetohydrodynamic (MHD) generators, plasma flows, nuclear reactor dynamics, geothermal energy extraction, electromagnetic propulsion, boundary layer control in aerodynamics, metallurgical processes etc. The MHD flow in a channel is thus studied extensively and we would like to refer to a few recent studies [1–6]. A good survey of relevant studies in magnetohydrodynamics can be found in Moreau [7].

In the literature, much attention has been devoted to the flow problems in the presence of a uniform applied magnetic field. But this assumption is not appropriate in several engineering applications. Therefore, it is desirable to discuss magnetohydrodynamic (MHD) flow in the presence of variable magnetic field. Abdel-Malek [8] examined the Rayleigh problem for a power law fluid in the presence of an arbitrary time dependent magnetic field and presented an analytic solution for when the magnetic field strength is proportional to t and $t^{-1/2}$. Arbitrary time dependent magnetic fields were addressed by Wafo-Soh [9]. Hayat and Kara [10] examined the flow of a third-grade fluid with a variable time dependent magnetic field. It is apparent that MHD flow in the presence of a non-uniform space dependent magnetic field has received less attention. To our knowledge, Chiam [11] has addressed the steady stagnation point flow near a stretching sheet in the presence of a non-uniform applied magnetic field. He successfully obtained the similarity solution by adopting a special form of the space dependent magnetic field. Nadeem and Akbar [12] discussed peristaltic flow with heat and mass transfer in an annulus influenced by a radially varying magnetic field.

The main objective of the present letter is to move further on MHD flows under a non-uniform applied magnetic field. For that, we choose to present the analytical solution for unsteady Couette flow in the presence of an arbitrary non-uniform space dependent applied magnetic field. Further generalization is achieved by giving arbitrary velocity to one of the plates. An analytical solution is obtained using a novel application of the perturbed eigenfunction expansion method for small magnetic parameter.

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2. The problem definition

We consider an electrically conducting viscous fluid between two plates a distance L apart. The fluid is electrically conducting and a non-uniform magnetic field is applied in the direction transverse to the flow. The upper plate is stationary while the lower plate suddenly starts moving with a generalized velocity $kf(t)$, where k is the dimensional constant. The problem statement can be written as

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{\upsilon}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\sigma}{\rho}B^2(\bar{y})\bar{u} \tag{1}$$

$$\begin{aligned} &\bar{u}\left(L, \bar{t}\right) = 0, \quad \bar{t} \geq 0, \\ &\bar{u}\left(0, \bar{t}\right) = kf(\bar{t}), \quad t \geq 0, \\ &\bar{u}(\bar{y}, 0) = 0, \quad 0 \leq \bar{y} \leq L. \end{aligned} \tag{2}$$

We define the non-dimensional variables

$$u = \frac{\bar{u}}{u_0}, \qquad y = \frac{\bar{y}}{L} \quad \text{and} \quad t = c\bar{t}. \tag{3}$$

Eqs. (1) and (2) in the new variables are

$$\frac{\partial u}{\partial t} = \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{c\rho}B^2(y)u \tag{4}$$

$$\begin{aligned} &u(1, t) = 0 \quad t \geq 0, \\ &u(0, t) = Kf(t) \quad t \geq 0, \\ &u(y, 0) = 0 \quad 0 \leq y \leq 1, \end{aligned} \tag{5}$$

where $\upsilon = \frac{\bar{\upsilon}}{cL^2}$ and $\frac{\sigma}{\rho c}B^2(y)$ are non-dimensional viscosity and magnetic field parameters. Writing $B^2(y) = B_0^2 r(y)$, where $r(y)$ is a non-dimensional function, Eq. (4) can be rewritten as

$$\frac{\partial u}{\partial t} = \upsilon\frac{\partial^2 u}{\partial y^2} - \varepsilon r(y)u \tag{6}$$

where $\varepsilon = \frac{\sigma B_0^2}{\rho c}$ is a small parameter (for small magnetic parameter). We decompose the function $u(y, t)$ as

$$u(y, t) = V(y, t) + W(y, t), \tag{7}$$

where $W(y, t)$ satisfies

$$\frac{\partial^2 W}{\partial y^2} = 0, \tag{8}$$

$$\begin{aligned} &W(1, t) = 0 \quad t \geq 0, \\ &W(0, t) = Kf(t) \quad t \geq 0. \end{aligned} \tag{9}$$

The solution of Eqs. (8) and (9) is

$$W(y, t) = Kf(t)(1 - y). \tag{10}$$

Now, using Eq. (10) in Eqs. (6) and (7) and writing the resulting equation for $V(y, t)$ we obtain

$$\frac{\partial V}{\partial t} = \upsilon\frac{\partial^2 V}{\partial y^2} - \varepsilon r(y)V - H(y, t), \tag{11}$$

$$\begin{aligned} &V(1, t) = 0 \quad t \geq 0, \\ &V(0, t) = 0 \quad t \geq 0, \\ &V(y, 0) = -W(y, 0) \quad -1 < y < 1 \end{aligned} \tag{12}$$

where

$$H(y, t) = \frac{\partial W}{\partial t} + \varepsilon r(y)W. \tag{13}$$

3. Perturbative eigenvalues and eigenfunctions

To proceed further, we define the related Sturm–Liouville (S–L) boundary value problem as

$$\frac{\partial^2\phi}{\partial y^2} + (\lambda - \varepsilon r(y))\,\phi = 0, \tag{14}$$

$$\varphi(1) = 0, \qquad \varphi(0) = 0. \tag{15}$$

It is important to note that the above boundary value problem constitutes a perturbed eigenvalue problem. We expand ϕ and λ in power series in ε:

$$\begin{aligned}\phi &= \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots\\ \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots.\end{aligned} \tag{16}$$

Substituting Eq. (16) in Eqs. (14) and (15), the leading order eigenvalue problem is

$$\begin{aligned}&\frac{\partial^2\phi_0}{\partial y^2} + \lambda_0\phi_0 = 0,\\ &\phi_0(0) = 0, \qquad \phi_0(1) = 0.\end{aligned} \tag{17}$$

The corresponding eigenvalues and eigenfunction are

$$\lambda_{0n} = (n\pi)^2 \quad \text{with } \varphi_{0n}(y) = \sin(n\pi y) \ \text{ for } n = 1, 2, 3, \ldots. \tag{18}$$

The first-order system is given by

$$\begin{aligned}&\frac{\partial^2\phi_1}{\partial y^2} - r(y)\phi_0 + \lambda_0\phi_1 + \lambda_1\phi_0 = 0\\ &\phi_1(0) = 0, \qquad \phi_1(1) = 0.\end{aligned} \tag{19}$$

Now let us expand ϕ_1 in terms of the eigenfunctions $\{\phi_{0n}\}$ of the S–L problem (17)

$$\phi_{1n} = \sum_{m=1}^{\infty} a_{mn}\phi_{0n}. \tag{20}$$

Using Eq. (20) in Eq. (19) yields

$$-\sum_{m=1}^{\infty} a_{mn}\phi_{0m}\lambda_{0m} + -\lambda_{0n}\sum_{m=1}^{\infty} a_{mn}\phi_{0n} + \lambda_{1n}\phi_{0n} - r(y)\phi_{0n} = 0. \tag{21}$$

Multiplying both sides of Eq. (21) by ϕ_{0k} and integrating from 0 to 1, we get

$$-a_{kn}\lambda_{0k} + \lambda_{0n}a_{kn} + \lambda_{1n}\delta_{nk} - F_{nk} = 0, \tag{22}$$

where

$$F_{nk} = \int_0^1 r(y)\phi_{0n}\phi_{0k}dy \tag{23}$$

$$\delta_{nk} = \int_0^1 \phi_{0n}\phi_{0k}dy. \tag{24}$$

Eq. (22) gives

$$\lambda_{1n} = F_{nn} \quad \text{for } n = k, \tag{25}$$

$$a_{kn} = \frac{F_{kn}}{\lambda_{0n} - \lambda_{0k}} \quad \text{for } n \neq k. \tag{26}$$

Eq. (20) together with Eqs. (25) and (26) yields

$$\phi_{1n} = \sum_{k\neq n}\left(\frac{F_{kn}}{\lambda_{0n} - \lambda_{0k}}\right)\phi_{0k} + a_{nn}\phi_{0n}. \tag{27}$$

To find a_{nn}, we normalize ϕ as

$$\int_0^1 (\phi_{0n} + \varepsilon\phi_{1n} + \cdots)^2\, dy = 1 \tag{28}$$

which at the $O(\varepsilon)$ gives

$$\int_0^1 \phi_{0n}\phi_{1n}dy = 0. \tag{29}$$

Using (27) and the property of orthogonality of the eigenfunctions, we find

$$a_{nn} = 0. \tag{30}$$

Using Eq. (30) in Eq. (27), we get

$$\phi_{1n} = \sum_{k\neq n}\left(\frac{F_{kn}}{\lambda_{0n} - \lambda_{0k}}\right)\phi_{0k}. \tag{31}$$

From Eqs. (18) and (31), the two-term perturbation expansion for the eigenfunctions $\{\phi_n\}$ is expressed as

$$\begin{aligned}\phi_n &= \phi_{0n} + \varepsilon\phi_{1n} \\ &= \sin(n\pi y) + \varepsilon\sum_{\substack{k\neq n\\ k=1}}^{\infty}\left(\frac{F_{kn}}{\lambda_{0n} - \lambda_{0k}}\right)\phi_{0k} + O(\varepsilon^2).\end{aligned} \tag{32}$$

4. The solution of the problem

Having found the eigenfunctions (Eq. (31)) of the related eigenvalue problem, we revert back to Eq. (11). The function $H(y, t)$ and $V(y, t)$ can be expanded as

$$H(y,t) = \sum_{n=1}^{\infty} b_n(t)\varphi_n(y), \tag{33}$$

$$V(y,t) = \sum_{n=1}^{\infty} A_n(t)\varphi_n(y). \tag{34}$$

The coefficients in Eqs. (33) and (34) are conveniently written as

$$b_n(t) = b_{0n}(t) + \varepsilon b_{1n}(t) + O(\varepsilon^2) \tag{35}$$

$$A_n(t) = A_{0n}(t) + \varepsilon A_{1n}(t) + O(\varepsilon^2). \tag{36}$$

Taking $H\ (y, t) = H_0(y, t) + \varepsilon H_1(y, t)$ and using Eqs. (32) and (35) in Eq. (33), we get

$$\begin{aligned}H_0(y,t) + \varepsilon H_1(y,t) &= \sum_{n=1}^{\infty}(b_{0n}(t) + \varepsilon b_{1n}(t))(\varphi_{0n}(y) + \varepsilon\varphi_{1n}(y)) \\ &= \sum_{n=1}^{\infty} b_{0n}(t)\varphi_{0n}(y) + \varepsilon\sum_{n=1}^{\infty}(b_{0n}(t)\varphi_{1n}(y) + b_{1n}(t)\varphi_{0n}(y)) + O(\varepsilon^2).\end{aligned} \tag{37}$$

Comparing the coefficients of ε in Eq. (37), we have

$$H_0(y,t) = \sum_{n=1}^{\infty} b_{0n}(t)\varphi_{0n}(y) \tag{38}$$

and

$$H_1(y,t) = \sum_{n=1}^{\infty}\left(b_{0n}(t)\varphi_{1n}(y) + b_{1n}(t)\varphi_{0n}(y)\right). \tag{39}$$

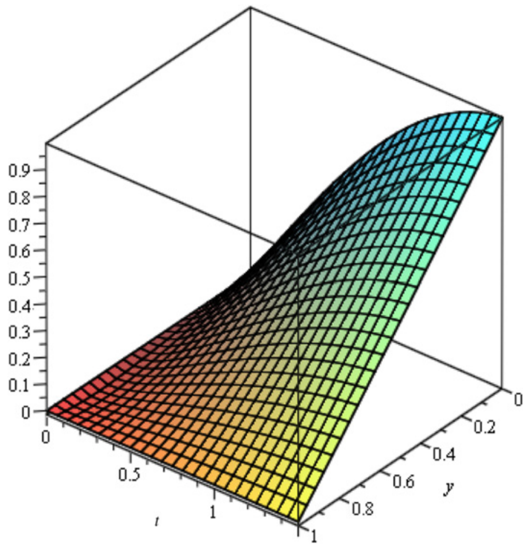
The coefficients in Eq. (35) can be determined using the orthogonality of the sets $\{\varphi_{0n}\}$ and $\{\varphi_{1n}\}$:

$$b_{0n}(t) = \frac{\int_0^1 H_0\ (y,t)\ \varphi_{0n}\ (y)\ dy}{\int_0^1 \varphi_{0n}^2\ (y)\ dy}, \tag{40}$$

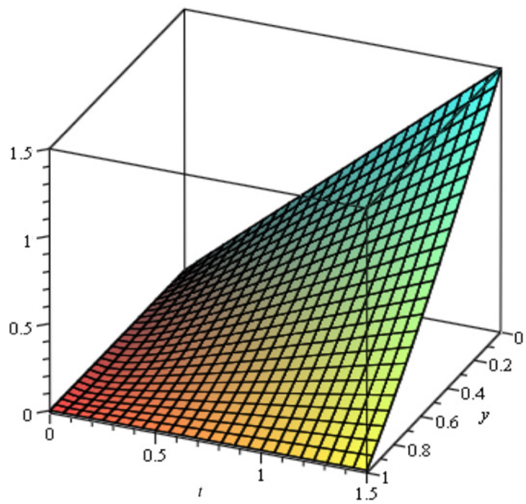
$$b_{1n}(t) = \frac{\int_0^1 H_2\ (y,t)\ \varphi_{0n}\ (y)\ dx}{\int_0^1 \varphi_{0n}^2\ (y)\ dx}. \tag{41}$$

In Eq. (41), $H_2(y, t)$ is

$$H_2(y,t) = H_1(y,t) + \sum_{n=1}^{\infty} b_{0n}(t)\varphi_{1n}(y). \tag{42}$$



(a) Velocity profile for $f(t) = \sin(t)$, $r(y) = e^{-y}$, $K = 1$, $\upsilon = 1$ and $n = 3$.



(b) Velocity profile for $f(t) = t$, $r(y) = y^2$, $K = 1$, $\upsilon = 1$ and $n = 3$.

Fig. 1. Velocity profiles for different plate velocities and magnetic fields.

Expressing $V(y, t)$ in terms of the eigenfunctions (Eq. (32)) gives

$$V = \sum_{n=1}^{\infty} (A_{0n}(t) + \varepsilon A_{1n}(t))(\varphi_{0n}(y) + \varepsilon\varphi_{1n}(y)). \tag{43}$$

In what follows, we will calculate the constants appearing in Eq. (43) to reach the final destination $V(y, t)$. Putting Eq. (43) in Eq. (11), using Eqs. (17) and (19) in the resulting equation and taking the leading order term we arrive at

$$\frac{dA_{0n}}{dt} + \lambda_{0n}\upsilon A_{0n} = b_{0n}. \tag{44}$$

The solution of Eq. (44) gives the coefficients $A_{0n}(t)$:

$$A_{0n}(t) = e^{-\lambda_{0n}\upsilon t}\left(A_{0n}(0) + \int_0^t b_{0n}(\tau)e^{\lambda_{0n}\upsilon\tau}d\tau\right) \tag{45}$$

where

$$A_{0n}(0) = -2\int_0^1 W(y, 0)\phi_{0n}(y)dy.$$

Following the same steps as for Eq. (45), the first-order term becomes

$$\left(\frac{dA_{1n}}{dt} + \lambda_{0n}\upsilon A_{1n}\right)\phi_{0n} = A_{0n}\upsilon(\lambda_{0n}\phi_{1n} + \lambda_{1n}\phi_{0n}) + b_{1n}\phi_{0n} + b_{0n}\phi_{1n} - \phi_{1n}\frac{dA_{0n}}{dt}. \tag{46}$$

Multiplying both sides of Eq. (46) by ϕ_{0n} and integrating from 0 to 1, we get

$$\frac{dA_{1n}}{dt} + \lambda_{0n}\upsilon A_{1n} = F(t) \tag{47}$$

where

$$F(t) = A_{0n}\upsilon\lambda_{1n} + b_{1n}. \tag{48}$$

The solution of Eq. (38) gives

$$A_{1n}(t) = -e^{-\lambda_{0n}\upsilon t}\int_0^t F(t)e^{\lambda_{0n}\upsilon t}dt. \tag{49}$$

The solution $V(y, t)$ can now be constructed from Eq. (43) with the help of Eqs. (45) and (49) which in turn gives the solution $u(y, t)$ through Eqs. (7) and (10). It is important to note that in the final solution all the functions are known once $f(t)$ and $r(y)$ are specified. As an example, we draw the graph of $u(y, t)$ by choosing two specific values of $f(t)$ and $r(y)$ in Fig. 1. The plate velocity and the magnetic field being arbitrary, the other choices can be treated in a similar fashion.

5. Closing remark

The perturbed eigenfunction method for small magnetic parameter is employed to develop an analytical solution for the unsteady Couette flow. The fluid is subjected to an arbitrary space dependent magnetic field and the flow is generated by a generalized velocity given to the lower plate. To our knowledge, the consideration of unsteady Couette flow for an arbitrary space dependent magnetic field and the concept of perturbed eigenfunctions have not been addressed in the literature. We hope that the idea of perturbed eigenfunctions will be useful in some future studies.

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