Representation of Systems Disturbed by Wide Band Noise

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Abstract—In this paper, a method of handling and working with wide band noise is developed. We represent wide band noise as a distributed delay of white noise and use it to reduce a nonlinear system disturbed by wide band noise to a nonlinear system disturbed by white noise. An application of this reduction to a nonlinear filtering problem under a wide band noise disturbance is discussed.

Keywords—Wide band noise, White noise, Linear stochastic systems.

1. INTRODUCTION

The actual noise processes, corrupting the systems in reality, are marked with a property that their values are correlated within a small time interval. At the same time the white noise model of actual noise ignores this property. As a result in engineering, it often happens that the optimal controls and the optimal estimators, calculated in accordance with the theoretical results for the white noise model, are not optimal and, indeed, might be quite far from being optimal. It becomes important to develop the control and estimation theories for the systems disturbed by noise models which describe actual noise more adequately. Such noise model is the so-called wide band noise model (see [1, p. 126]). There are different techniques of handling and working with the wide band noise processes. For example, in [2] the approximation approach is used. Another
approach based on a certain integral representation is suggested in [3] and its applications to space engineering and gravimetry are discussed in [4,5].

In [3,6] the Kalman estimation results are modified to a linear stochastic system disturbed by wide band noise represented in the integral form. The proofs of these results are based on the duality principle and, technically, they are routine being useless if one tries to modify them to nonlinear systems. It becomes important to develop a more handle technique of working with wide band noise processes represented in the integral form. Motivated from the representation theory of infinite-dimensional systems (see [7]), in this paper, we establish a technique of reduction of a system disturbed by wide band noise in the integral form to a system disturbed by white noise. Such reduction allows to get results regarding wide band noise from the respective results regarding white noise. As an application of this reduction, we discuss a nonlinear filtering problem under a wide band noise disturbance.

Additionally, in this paper, we prove that at least in one-dimensional case for any autocovariance function satisfying some suitable conditions there exists an infinite number of stationary (starting from a small time moment) wide band noise process represented in the integral form. This result shows that the class of the wide band noise processes represented in the integral form is rather wide.

2. WIDE BAND NOISE

We denote by $L_2(G, X)$ the space of $X$-valued square integrable functions defined on the set $G$. If $X$ is the real line, then for the above space the brief symbol $L_2(G)$ is used.

**Definition 1.** A random process $\psi(t)$, $t \geq 0$, is called a wide band noise process if

$$\text{cov}(\psi(t + s), \psi(t)) = \lambda(t, s) \neq 0, \quad \text{for } 0 \leq s < \varepsilon \quad \text{and} \quad \text{cov}(\psi(t + s), \psi(t)) = 0, \quad \text{for } s \geq \varepsilon,$$

where $\varepsilon > 0$ is a small value. If, additionally, $\lambda$ depends only on $s$, then $\psi$ is said to be stationary (in wide sense). The function $\lambda$ is called the autocovariance function of $\psi$.

Consider the random process

$$\varphi(t) = \int_{\max(0, t-\varepsilon)}^{t} \phi(s - t) \, dw(s), \quad t \geq 0,$$

(1)

where $w$ is a stationary process with orthogonal increments, $\text{cov}w(t) = \varepsilon > 0$ and $\phi \in L_2(-\varepsilon, 0)$. One can easily verify that the random process $\varphi$, as defined by (1), is a wide band noise process which becomes stationary starting the time moment $t = \varepsilon$ with the autocovariance function

$$\lambda(s) = \int_{-\varepsilon}^{-s} \phi(r)\phi(r + s) \, dr, \quad 0 \leq s \leq \varepsilon.$$

(2)

Formally, the wide band noise process (1) can be interpreted as vibration. At moment $t$ a vibration that is formed by the action of white noise during the time between $t - \varepsilon$ and $t$ affects the system. Values of white noise until $t - \varepsilon$ do not take part in the formation of this vibration at moment $t$, because their weight is sufficiently small and we can neglect them in model (1). Consequently, the function $\phi$ stands for the coefficient of relaxing the initial effect of white noise at different time moments. By this interpretation, model (1) corresponds to real cases when the vibration generated by white noise stands to affect a system starting with the initial time $t = 0$ (a reason for that may be the switching the system on from resting state to dynamic state which changes its sensitivity to random disturbances). Hence, when $0 < t < \varepsilon$ the wide band noise process (1) is formed by values of white noise on $[0, t]$.

In practice, wide band noise is identified by its autocovariance function. Therefore, to get representation (1) one has to solve equation (2) in $\phi$ under given $\lambda$. For this, we present the following.
THEOREM 1. Let $\varepsilon > 0$ and let $\lambda \in L_2(0, \varepsilon)$. Define the function $\lambda^*$ as the even extension of $\lambda$ to the real line vanishing outside of $[-\varepsilon, \varepsilon]$ and assume that $\lambda^*$ is positive definite and $\mathcal{F}(\lambda^*)^{1/2} \in L_2(-\infty, \infty)$, where $\mathcal{F}(\lambda^*)$ is the Fourier transformation of $\lambda^*$. Then there exists a solution of equation (2) in $L_2(-\varepsilon, 0)$. If $\lambda$ is a nonzero function, then the number of distinct solutions of equation (2) is infinite.

PROOF. With each $\phi \in L_2(-\varepsilon, 0)$, associate two functions $\phi^*$ and $\phi^{**}$. Let $\phi^*$ be the extension of $\phi$ to the real line vanishing outside of $[-\varepsilon, 0]$ and define $\phi^{**}$ by $\phi^{**}(s) = \phi^*(-s)$ for $-\infty < s < \infty$. In terms of $\lambda^*, \phi^*$, and $\phi^{**}$ equation (2) has the form $\lambda^* = \phi^* \phi^{**}$ or $\mathcal{F}(\lambda^*) = \mathcal{F}(\phi^*) \mathcal{F}(\phi^{**})$ or $\mathcal{F}(\lambda^*) = \mathcal{F}(\phi^*) \bar{\mathcal{F}}(\phi^{**})$, where $f \ast g$ is the convolution of $f$ and $g$ and $\bar{a}$ is the conjugate of the complex number $a$. From properties of the Fourier integral, $\mathcal{F}(\lambda^*)$ is a nonnegative-valued even function of real variable since $\lambda^*$ is even and positive definite. If

$$\mathcal{F}(\phi^*) (\omega) = x(\omega) + iy(\omega),$$

where $x$ and $y$ are unknown real-valued functions and $i$ is the imaginary unit, then

$$\mathcal{F}(\lambda^*) (\omega) = x(\omega)^2 + y(\omega)^2.$$  \hfill (3)

Now in order to obtain $\phi^*$ to be real-valued and square integrable, we have to find an even function $x$ and an odd function $y$, both from $L_2(-\infty, \infty)$, satisfying (3). This can be done in the following way. Let $0 \leq \alpha \leq 1$. Write $\mathcal{F}(\lambda^*) (\omega) = u(\omega) + v(\omega)$, where $u(\omega) = \alpha \mathcal{F}(\lambda^*) (\omega)$ and $v(\omega) = (1 - \alpha) \mathcal{F}(\lambda^*) (\omega)$. Then construct a measurable even function $x$ and a measurable odd function $y$ satisfying $x(\omega)^2 = u(\omega)$ and $y(\omega)^2 = v(\omega)$. This can be done easily by considering different branches of square root. By the condition $\mathcal{F}(\lambda^*)^{1/2} \in L_2(-\infty, \infty)$, for each such pair $(x, y)$, the inverse Fourier transformation

$$\phi^* = \mathcal{F}^{-1}(\mathcal{F}(\phi^*)) = \mathcal{F}^{-1}(x + iy),$$

exists, it is real-valued and belongs to $L_2(-\infty, \infty)$ vanishing outside of $[-\varepsilon, 0]$ (otherwise, $\lambda^*$ will take nonzero values for $|s| > \varepsilon$). The restriction of each $\phi^*$, constructed in the above-mentioned way, to $[-\varepsilon, 0]$ is a solution of equation (2). From the above construction of $\phi^*$, it is clear that the number of solutions of equation (2) is infinite if $\lambda$ is a nonzero function.

Note that the condition on positive definiteness of $\lambda^*$ is ordinary since $\lambda$ is an autocovariance function. Thus, given a suitable function $\lambda$, there is a variety of wide band noise processes in form (2) which have the same autocovariance function $\lambda$ starting the time moment $\varepsilon$. This variety is reached by different solutions of equation (2) and also by different processes $w$.

3. REDUCTION

Let $\{\mathcal{F}_t\}$ be a complete and right-continuous filtration on a fixed complete probability space, $X$ and $H$ be separable Hilbert spaces and let $\mathcal{L}(H, X)$ be the space of linear bounded operators from $H$ to $X$. From the discussion given in Section 2, it is reasonable to define a Hilbert space version of representation (1), and hence, we let $\varphi$ be $X$-valued wide band noise process in the form

$$\varphi(t) = \int_{\max(0,t-\varepsilon)}^{t} \Phi(s - t) dw(s), \quad t \geq 0,$$  \hfill (4)

where $\varepsilon > 0$, $\Phi \in B_2(-\varepsilon, 0; \mathcal{L}(H, X))$ (the space of strongly measurable and square integrable operator-valued functions from $[-\varepsilon, 0]$ to $\mathcal{L}(H, X)$), $w$ is an $H$-valued Wiener process with respect to the filtration $\{\mathcal{F}_t\}$. Consider the equation

$$\frac{d}{dt} x(t) = Ax(t) + f(x(t)) + \varphi(t), \quad x(0) = x_0, \quad t \geq 0,$$  \hfill (5)
where $A$ is the infinitesimal generator of a strongly continuous semigroup $\mathcal{U}$ of bounded linear operators on $X$ (for the latter we use the symbol $\mathcal{U} \in \mathcal{S}(X)$), $f : X \to X$ satisfies the Lipschitz and linear growth conditions, $\varphi$ is defined by (4) and $x_0$ is an $\mathcal{F}_0$-measurable and square integrable random variable with the values in $X$.

Different concepts of solution can be defined for stochastic differential equations (see [8]). In particular, the random process $x$ is said to be a mild solution of equation (5) if it satisfies

$$x(t) = \mathcal{U}(t)x_0 + \int_0^t \mathcal{U}(t-s)(f(x(s)) + \varphi(s)) \, ds, \quad t \geq 0. \tag{6}$$

It is an immediate consequence from the contraction mapping principle that equation (5) has a unique (up to modification) $\mathcal{F}_t$-adapted mild solution.

Equation (5) is not adapted to the existing theory of nonlinear filtering since the noise process $\varphi$ is wide band. In this section, we will reduce equation (5) to an equation disturbed by white noise. For this, let $\tilde{X} = L_2(-\varepsilon, 0; X)$. Consider the semigroup of right translation $T \in \mathcal{S}(\tilde{X})$ defined by

$$[T(t)h](\theta) = \begin{cases} h(\theta - t), & \theta - t \geq -\varepsilon, \\ 0, & \theta - t < -\varepsilon \end{cases}, \quad -\varepsilon \leq \theta \leq 0, \quad t \geq 0, \quad h \in \tilde{X}. \tag{7}$$

Let

$$\tilde{\mathcal{U}}(t) = \begin{bmatrix} \mathcal{U}(t) & \mathcal{E}(t) \\ 0 & T(t) \end{bmatrix}, \quad t \geq 0, \tag{8}$$

where $\mathcal{E} : [0, \infty) \to \mathcal{L}(\tilde{X}, X)$ is the function defined by

$$\mathcal{E}(t)h = \int_{\max(-\varepsilon, -t)}^0 \mathcal{U}(t + r)h(r) \, dr, \quad t \geq 0, \quad h \in \tilde{X}. \tag{9}$$

One can easily verify that $\tilde{\mathcal{U}} \in \mathcal{S}(X \times \tilde{X})$ (see, for example, [7]).

**Lemma 1.** Let $\tilde{\varphi}$ be the $\tilde{X}$-valued random process defined by

$$[\tilde{\varphi}(t)](\theta) = \int_{\max(0, t+\varepsilon-\theta)}^t \Phi(s - t + \theta) \, dw(s), \quad -\varepsilon \leq \theta \leq 0, \quad t > 0. \tag{10}$$

Then

$$\tilde{\varphi}(t) = \int_0^t T(t-s)\Phi \, dw(s), \quad t \geq 0, \tag{11}$$

where $T$ is defined by (7) and the operator $\Phi \in \mathcal{L}(H, \tilde{X})$ is defined by

$$[\Phi h] (\theta) = \Phi(\theta)h, \quad -\varepsilon \leq \theta \leq 0, \quad h \in H. \tag{12}$$

**Proof.** By (7) and (12), and for $h \in H$, we have

$$[T(t-s)\Phi h](\theta) = \begin{cases} \Phi(\theta - t + s)h, & \theta - t + s \geq -\varepsilon, \\ 0, & \theta - t + s < -\varepsilon \end{cases}. \tag{13}$$

Therefore, by (10),

$$[\tilde{\varphi}(t)](\theta) = \int_{\max(0, t-\varepsilon-\theta)}^t \Phi(s - t + \theta) \, dw(s) = \int_0^t T(t-s)\Phi \, dw(s)](\theta),$$

and consequently, representation (11) for $\tilde{\varphi}$ holds.
LEMA 2. Let the random process $x$ be defined by (6) and (4). Then

$$x(t) = U(t)x_0 + \int_0^t U(t-s)f(x(s)) \, ds + \int_0^t \mathcal{E}(t-s)\Phi \, dw(s), \quad t \geq 0,$$

(13)

where $\mathcal{E}$ is defined by (9) and $\Phi \in \mathcal{L}(H, \tilde{X})$ is defined by (12).

PROOF. Substituting $\varphi$ from (4) to the last term in the right-hand side of (6), we obtain

$$\int_0^t U(t-s)\varphi(s) \, ds - \int_0^t \int_{\max(0, s-\epsilon)}^s U(t-s)\Phi(r-s) \, dw(r) \, ds$$

$$= \int_0^t \int_{\min(1, r+\epsilon)}^r U(t-s)\Phi(r-s) \, dw(r) \, ds$$

$$= \int_0^t \int_{\max(-\epsilon, r-t)}^0 U(t-s)\Phi(s) \, ds \, dw(r).$$

For $h \in H$, from (9) and (12),

$$\mathcal{E}(t-r)\Phi h = \int_{\max(-\epsilon, r-t)}^0 U(t-s)\Phi(s) \, ds \, dw(r) = \left( \int_{\max(-\epsilon, r-t)}^0 U(t-s) \Phi(s) \, ds \right) h.$$

Hence,

$$\int_0^t U(t-s)\varphi(s) \, ds = \int_0^t \int_{\max(-\epsilon, r-t)}^0 U(t-s)\Phi(s) \, ds \, dw(r) = \int_0^t \mathcal{E}(t-r)\Phi \, dw(r).$$

Thus, $x$ satisfies (13).

THEOREM 2. Define the $X \times \tilde{X}$-valued random process $\tilde{x}$ by

$$\tilde{x}(t) = \begin{bmatrix} x(t) \\ \tilde{\varphi}(t) \end{bmatrix}, \quad t \geq 0, \quad \text{with } \tilde{x}(0) = \tilde{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

where $x$ is defined by (6) and (4) and $\tilde{\varphi}$ defined by (10). Then

$$\tilde{x}(t) = \tilde{U}(t)\tilde{x}_0 + \int_0^t \tilde{U}(t-s)F(\tilde{x}(s)) \, ds + \int_0^t \tilde{U}(t-s)\tilde{I}\Phi \, dw(s), \quad t \geq 0,$$

(14)

where $\tilde{U} \in \mathcal{S}(X \times \tilde{X})$ is defined by (8), $\tilde{\Phi}$ is defined by (12),

$$\tilde{I} = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathcal{L}(\tilde{X}, X \times \tilde{X}),$$

$I$ is the identity operator on $\tilde{X}$ and $F : X \times \tilde{X} \to X \times \tilde{X}$ is a nonlinear function defined by

$$F(\tilde{x}) = F\left( \begin{bmatrix} x \\ \tilde{\varphi} \end{bmatrix} \right) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix}.$$

PROOF. This is an immediate consequence from Lemmas 1 and 2.

Note that the infinitesimal generator of the semigroup $\tilde{U}$ defined by (8),(9) is

$$\tilde{A} = \begin{bmatrix} A & \Gamma \\ 0 & -\frac{d}{d\theta} \end{bmatrix},$$
which is a densely defined closed linear operator from

$$D(\tilde{A}) = \left\{ \begin{bmatrix} h \\ g \end{bmatrix} \in X \times \tilde{X} : h \in D(A), \ g' \in \tilde{X}, \ g(-\varepsilon) = 0 \right\}$$

to \(X \times \tilde{X}\), where \(D(A)\) is the domain of \(A\), \(\frac{\partial g}{\partial \theta}\) is the respective differential operator and \(\Gamma g = g(0)\).

Hence, the random process \(\tilde{x}\), as defined in Theorem 2, is a mild solution of the stochastic differential equation

$$d\tilde{x}(t) = \left(\tilde{A}\tilde{x}(t) + F(\tilde{x}(t))\right) dt + \tilde{I}\phi dw(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad t \geq 0. \quad (15)$$

Thus, system (5) disturbed by wide band noise is reduced to system (15) that is disturbed by white noise.

**REMARK 1.** Assume that equation (5) is equipped with a respective observation equation disturbed by white noise. Denote the respective observation process by \(\tilde{z}\) and consider the nonlinear filtering problem for the pair \((x, z)\). Using Theorem 2, one can reduce this problem to the nonlinear filtering problem for the pair \((\tilde{x}, \tilde{z})\). For the latter filtering problem, Zakai's equation of nonlinear filtering is derived in [9] and the conditions of existence of its solution are obtained there. Thus, we can get the same results for the pair \((x, z)\) as well.

**REMARK 2.** Assume that equation (5) is linear (that is \(f(x) \equiv 0\)). Then equation (15) is linear as well. Considering equations (5) and (15) as signal systems in linear filtering problems, we can find the respective state systems in the dual linear regulator problems. Denote them by (5') and (15'), respectively. The link between systems (5) and (15) must generate a certain link between systems (5') and (15'). It turns out that system (5)' is a linear differential equation with a distributed delay and system (15') is the well-known reduction of system (5') to a linear system without any delay by enlarging the state space.

**REMARK 3.** Motivated from the reduction described in Section 3, one can try to apply a similar reduction to an observation system disturbed by wide band noise. For this, consider the following linear observation system disturbed by the sum of white and wide band noise processes:

$$dz(t) = (C\tilde{x}(t) + \varphi(t)) dt + dv(t), \quad (16)$$

where \(C\) is a bounded linear operator, \(\varphi\) is the wide band noise process as defined by (4) and \(v\) is a Wiener process. From

$$\{\varphi(t)\} \{0\} = \int_{\max(0,t-\varepsilon)}^{t} \Phi(s-t) dw(s) = \varphi(t),$$

it follows that

$$dz(t) = \tilde{C}\tilde{x}(t) dt + dv(t), \quad (17)$$

where \(\tilde{C} = [CT]\) and \(\tilde{x}\) is as defined in Theorem 2. Thus, the observation system (16) is reduced to system (17) that is disturbed only by white noise. However, (17) is an unbounded observation system since \(\Gamma\) is an unbounded operator on \(\tilde{X}\). In order to understand the kind of this unboundedness, consider the linear filtering problem for the observation system (17) equipped with some linear signal system. One can verify that the respective dual linear regulator problem contains a distributed delay of the control. Thus, the difficulty of linear filtering problems for the observation process (17) are of the same kind with the difficulty arising in linear quadratic control problems with distributed delays in control (see [10]).
4. CONCLUSION

In this paper, a method of handling and working with wide band noise is presented. This method consists of two major steps. In the first step, a wide band noise process, that is given by its autocovariance function $\lambda$, is represented in the integral form (1) through a solution $\phi$ of equation (2). In the second step, an equation disturbed by wide band noise is reduced to an equation disturbed by white noise. Since the control and estimation theories are well developed for white noise disturbances the method determines a direct way of getting the respective results for wide band noise disturbances.

REFERENCES