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Zonal polynomials and domino tableaux

Nantel Bergeron and Adriano M. Garsia*

Department of Mathematics, C-012, University of California, San Diego, La Jolla, CA 92093, USA

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Abstract

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Let H be a subgroup of a finite group G. Define the element Θ of the group algebra $\mathcal{A}(G)$ by $\Theta=\sum_{h\in H} h/|H|$. This element is an idempotent which may be used to project from $\mathcal{A}(G)$ to the linear span of the left cosets of H in G. If (H, G) is a Gelfand pair then the decomposition of Θ into minimal idempotents yields a useful basis for the Hecke algebra $\mathcal{H}(H, G)$. When this decomposition is applied to the pair (B_n, S_{2n}) the resulting minimal idempotents are intimately related to the zonal polynomials. In fact, the latter are the images of the minimal idempotents under an analogue of the Frobenius map. We show here that the Fourier transform of the minimal idempotents is supported by standard *domino tableaux*. We also give a multiplication algorithm for the zonal polynomials and relate the expansion coefficients to the Littlewood–Richardson's coefficients.

0. Introduction

Let G be a finite group and $H<G$ a subgroup of G. Let $\mathcal{A}(G)$ denote the group algebra of G. Let $(\tau_i)_{i=1\cdots k}$ be a system of representatives for the left cosets of H in G. We define the G-module V^H by setting

$$V^H=\mathbb{C}[\tau_i\Theta \mid i=1\cdots k]$$

where

$$\Theta=\sum_{h\in H}\frac{h}{|H|}. \tag{0.1}$$

The action of G on V^H is defined by left multiplication. The matrices $A(g)$, $g\in G$, for this action are simply

$$A(g)=\|\chi(\tau_i^{-1}g\tau_j\in H)\|$$

where $\chi(P)$ is 1 if P is true and 0 otherwise.

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The *Hecke algebra* $\mathscr{H}(H, G)$ is the algebra of all G-endomorphisms of V^H. We can describe this algebra as

$$\mathrm{com}(A) = \{T \in M_{k\times k} \mid TA(g) = A(g)T \text{ for all } g \in G\}.$$

Let $\{A_i\}$ be the irreducible representations occurring in A. Let d_i and n_i be respectively the degree and the multiplicity of A_i in A. From Schur's Lemma [1] we obtain

$$\mathscr{H}(H, G) = \left\{\bigoplus (B_{n_i} \otimes I_{d_i}) \mid B_{n_i} \in M_{n_i\times n_i}\right\}. \tag{0.2}$$

For H a subgroup of G, (H, G) *is a Gelfand pair* if and only if the algebra $\mathscr{H}(H, G)$ is commutative. It follows immediately from equation (0.2) that $\mathscr{H}(H, G)$ is commutative if and only if all n_i are 1 or 0. In this case, the representation A is in the form

$$A \cong \bigoplus_{i=1}^{d} A_i \tag{0.3}$$

with distinct A_i. In particular, for this Gelfand pair we have

$$\dim \mathscr{H}(H, G) = d \tag{0.4}$$

The paper starts with a review of the basic material concerning Gelfand pairs. This is done in Sections 1 and 2. More precisely, in Section 1, we introduce our notation and relate the idempotent Θ to the Hecke algebra $\mathscr{H}(H, G)$. In Section 2 we express the minimal idempotents decomposing Θ in terms of suitable unitary matrix units. The latter give us a basis of orthonormal idempotents for $\mathscr{H}(H, G)$ usually referred to as the *spherical functions* assoociated to the pair (H, G). In Section 3 we specialize to the pair (B_n, S_{2n}) where S_{2n} is the permutation group of $2n$ elements and B_n is the hyperoctahedral group imbedded in S_{2n}. This embedding is the realization of the Wreath product $S_n[S_2]$ $(=B_n)$ in S_{2n}. In this section we show how domino tableaux appear in the support of the Fourier coefficients of Θ.

The zonal polynomials Z_λ are images of the spherical associated to the pair (B_n, S_{2n}) under a map which sends $\mathscr{H}(B_n, S_{2n})$ onto the symmetric polynomials. In Section 4 we show that this map has properties strongly analogous to the classical Frobenius map which sends the irreducible characters of S_n onto the Schur functions and use this fact to give a multiplication rule for the zonal polynomials.

1. Hecke algebra $\mathscr{H}(H, G)$

Note first that since H is a subgroup, we shall necessarily have that

$$\Theta^2 = \Theta. \tag{1.1}$$

In fact, Θ projects $\mathscr{A}(G)$ into V^H. More precisely we have the following.

Lemma 1.1. *For* $f \in \mathscr{A}(G)$, $f \in V^H \Leftrightarrow f\Theta = f$.

Proof. Let $(\tau_i)_{i=1\cdots k}$ be a system of representatives for the left cosets of H in G. The product $f\Theta$ is then

$$f\Theta = \sum_{i=1}^{k} \Big(\sum_{h \in H} f(\tau_i h) \Big) \tau_i \Theta$$

and we immediately see both implications. □

Let us describe more precisely the algebra $\mathscr{H}(H, G)$. An element T of $\mathscr{H}(H, G)$ is an endomorphism $T: V^H \to V^H$ such that for any $g \in G$, $T(gf) = gTf$. Now for any $f \in V^H$, by Lemma 1.1, we have

$$Tf = T(f\Theta) = fT\Theta.$$

Hence T is entirely determined by its evaluation at Θ. Set $T\Theta = \omega$, and note that since $\omega \in V^H$ then for any $f \in V^H$ we have

$$Tf = f\omega = f\Theta\omega\Theta.$$

For $\omega \in \mathscr{A}(G)$ define $T_\omega: V^H \to V^H$ by $T_\omega f = f\Theta\omega\Theta$. The next proposition gives a characterization of G-endomorphisms by elements ω in $\mathscr{A}(G)$.

Proposition 1.2. $\mathscr{H}(H, G) = \{T_\omega \mid \omega \in \mathscr{A}(G)\}$.

Proof. We have seen that for any $T \in \mathscr{H}(H, G)$ there exits an element ω in $\mathscr{A}(G)$ sucha that $T = T_\omega$.

Conversely, we need to show that for any element ω of $\mathscr{A}(G)$, T_ω is an element of $\mathscr{H}(H, G)$. But this is immediate since for any $f \in V^H$ and any $g \in G$ we have

$$T_\omega(gf) = (gf)\Theta\omega\Theta = g(f\Theta\omega\Theta) = gT_\omega f \in V^H. \quad \square$$

Let $\Omega = \{HgH \mid g \in G\}$. Note that if $(v_i)_{i=1\cdots d}$ is a system of representatives for the double cosets of H in G then

$$\Omega = \{Hv_iH \mid 1 \leq i \leq d\}.$$

Moreover, we have the following.

Proposition 1.3. *The set* $\{T_{v_i} \mid 1 \leq i \leq n\}$ *is a basis for* $\mathscr{H}(H, G)$, *in particular*

$$\dim(\mathscr{H}(H, G)) = |\Omega|.$$

Proof. It is clear from Proposition 1.2 that the set $\{T_{v_i} \mid 1 \leq i \leq n\}$ spans the space $\mathscr{H}(H, G)$. The independence follows from the fact that the double cosets are disjoint. □

The following criterion is helpful in checking if a pair (H, G) is Gelfand.

Proposition 1.4. *If $HgH = Hg^{-1}H$ for any $g \in G$ then $\mathcal{H}(H, G)$ is commutative.*

Proof. From Proposition 1.2, we only have to check the commutativity of the set of generators $\{T_g \mid g \in G\}$. For convenience, let us set

$$\downarrow f = \sum_{g \in G} f(g) g^{-1}.$$

Using twice the hypothesis we have

$$\begin{aligned} T_g T_h &= \Theta g \Theta \Theta h \Theta = \Theta \downarrow (g \Theta \Theta h) \Theta \\ &= \Theta h^{-1} \Theta \Theta g^{-1} \Theta = T_h T_g. \qquad \square \end{aligned}$$

The converse of Proposition 1.4 has been shown in [2] for the case where all the representations of G are real.

2. The Fourier transform of the idempotent Θ in a Gelfand pair (H, G)

Let $(U^\lambda)_{\lambda \in \Lambda}$ be a complete family of irreducible unitary representations of G. Let d^λ be the dimension of the representations U^λ. From representation theory [9–10], one can associate to these representations the basis

$$e_{ij}^\lambda \in \mathcal{A}(G)$$

where

$$e_{ij}^\lambda(g) = \frac{1}{h_\lambda} \bar{u}_{ij}^\lambda(g), \qquad h_\lambda = \frac{|G|}{d^\lambda}, \qquad U^\lambda(g) = \|u_{ij}^\lambda(g)\|,$$

and for $f \in \mathcal{A}(G)$ let

$$U^\lambda(f) = \sum_{g \in G} f(g) U^\lambda(g).$$

This basis has the following property,

$$f = \sum_{i,j,\lambda} u_{ij}^\lambda e_{ij}^\lambda \Leftrightarrow U^\lambda(f) = \|u_{ij}^\lambda\| \quad \forall \lambda \in \Lambda. \tag{2.1}$$

From (2.1), we define the so called *Fourier transform*

$$\mathcal{F} : \mathcal{A}(G) \to \bigoplus M_{d^\lambda \times d^\lambda}(\mathbb{C}),$$

$$f \to \bigoplus_{\lambda \in \Lambda} U^\lambda(f).$$

We know that $\mathcal{F}$ is an isomorphism of algebras.

Since Θ satisfies $\Theta^2 = \Theta$ and $\Theta^* = \Theta$, the matrices $U^\lambda(\Theta)$ are idempotent and unitary. Upon replacing the U^λ by similar representations if necessary, we can assume in this section that $U^\lambda(\Theta)$ is a 0, 1 diagonal matrix. For the particular case of a Gelfand pair, the elements of the form $\Theta f \Theta$, $f \in \mathcal{A}(G)$, commute with each other. Hence the corresponding matrices $U^\lambda(\Theta f \Theta)$ must also commute with each other. This implies that $U^\lambda(\Theta)$ has rank at most 1 and we can thus assume that

$$u_{ij}^\lambda(\Theta)=\begin{cases}1 & \text{if } i=j=1,\\ 0 & \text{otherwise.}\end{cases}$$

Consequently, from equation (0.3), we may state the following.

Proposition 2.1. *If* (H, G) *is a Gelfand pair then there exists a subset* K *of* Λ *such that*

$$U^\lambda(\Theta)\neq 0 \Leftrightarrow \lambda\in K.$$

In particular, from (0.4), *the cardinality of* K *is exactly the dimension of* $\mathcal{H}(H, G)$.

For $\lambda\in K$, let us define

$$\Theta^\lambda=\mathcal{F}^{-1}\Big(\bigoplus_{\mu\in\Lambda}\delta_{\nu\lambda}U^\mu(\Theta)\Big)=e_{11}^\lambda,$$

where δ is the Kronecker symbol. Equivalently,

$$\Theta^\lambda=\frac{\Theta\chi^\lambda}{h_\lambda} \tag{2.2}$$

where χ^λ is the irreducible character corresponding to λ. Since the $U^\lambda(\Theta)$ are idempotent and since $\mathcal{F}$ is an isomorphism, we have

$$\Theta^\lambda\Theta^\mu=\delta_{\lambda\mu}\Theta^\lambda. \tag{2.3}$$

The Θ^λ are thus independent and idempotent. Moreover,

$$\Theta=\sum_{\lambda\in K}\Theta^\lambda. \tag{2.4}$$

From (2.3) and (2.4), we get the identity $\Theta\Theta^\lambda\Theta=\Theta^\lambda$. Hence the Θ^λ are elements of $\mathcal{H}(H, G)$. Using the usual scalar product in $\mathcal{A}(G)$ we get

$$\langle\Theta^\lambda,\Theta^\mu\rangle_G=\frac{1}{|G|}\sum_{g\in G}\Theta^\lambda(g)\Theta^\mu(g^{-1})=\frac{1}{|G|}(\Theta^\lambda\Theta^\mu)(\varepsilon)=\delta^{\lambda\mu}\Theta^\lambda(\varepsilon)\frac{1}{|G|}. \tag{2.5}$$

The elements Θ^λ are thus orthogonal. Moreover, from representation theory [9–10],

$$\Theta^\lambda(\varepsilon)=e_{11}^\lambda(\varepsilon)=\frac{1}{h_\lambda}.$$

The elements $h_\lambda\Theta^\lambda$ are known as the *spherical functions* of $\mathcal{H}(H, G)$. In summary we have the following.

Theorem 2.2. *For* (H, G) *a Gelfand pair, the elements* $T^\lambda=T_{\Theta^\lambda}$ *form an ortho-idempotent basis of* $\mathcal{H}(H, G)$.

3. The Gelfand pair (B_n, S_{2n})

Let us now consider the algebra $\mathscr{H}(B_n, S_{2n})$. We shall start by describing the double cosets of B_n in S_{2n}. We can associate an invariant to each double coset. More precisely we construct a graph for each permutation σ of S_{2n} which characterizes the double coset to which σ belongs.

We shall use the two-line notation to represent a permutation. For each permutation of S_{2n}, let us draw an edge between the numbers $2i-1$ and $2i$ for all $1 \leq i \leq n$ in both the top and bottom lines representing the permutation. This operation constructs a graph consisting of cycles of even length. As example, see Fig. 1.

The resulting graph completely characterizes the double coset. This is because left or right multiplication of σ by an element of B_n permutes the edges but does not affect the structure of the graph itself. The number of double cosets is hence equal to the number of partitions of $2n$ with even parts. Let us denote by $\Lambda[n]$ the set of partitions of the integer n. The partitions of $2n$ with even parts are obtained from $\Lambda[n]$ by multiplying the parts of the partitions in $\Lambda[n]$ by 2. We shortly write 2λ for this last operation. A double coset whose associated graph has cycle length given by 2λ will be said of type λ. We can thus state the next proposition.

Proposition 3.1. $\dim(\mathscr{H}(B_n, S_{2n})) = |\Lambda[n]|$.

Moreover, we have the following.

Proposition 3.2. *(B_n, S_{2n}) is a Gelfand pair.*

Proof. Using Proposition 1.4, we only have to show that

$$B_n \sigma B_n = B_n \sigma^{-1} B_n.$$

But σ and σ^{-1} have the same characteristic graph. Just a reflection across the horizontal line sends one onto the other. The two permutations are thus in the same double coset. □

Let Φ be the character associated to the S_{2n}-module V^{B_n}. Note that Φ is the character of the representation obtained by inducing the trivial character from B_n to S_{2n}. The Frobenius image of Φ turns out to be the Pólya enumerator $\mathscr{P}_{B_n}$ for

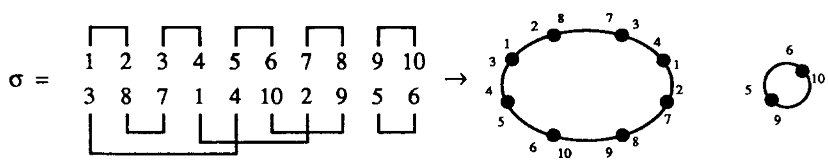

Fig. 1.

the subgroup B_n. This is an immediate consequence of Frobenius reciprocity, indeed we have

$$\langle \mathrm{ind}(\mathbb{1})_{B_n}^{S_{2n}}, \Psi \rangle_{S_{2n}} = \langle \mathbb{1}, \Psi|_{B_n} \rangle_{B_n} = \mathcal{P}_{B_n}(X)$$

where the Ψ's are the power sum symmetric functions given by the cycle type of the permutations.

On the other hand, because B_n is a Wreath product of two symmetric groups, it follows that

$$\sum_{n \geq 0} \mathcal{P}_{B_n}(X) t^n = \prod_{i \leq j} \frac{1}{1 - x_i x_j t}.$$

In [8], Macdonald gives the identity

$$\sum_{n \geq 0} \mathcal{P}_{B_n}(X) t^n = \prod_{i \leq j} \frac{1}{1 - x_i x_j t} = \sum_{n \geq 0} \Big(\sum_{\lambda \in \Lambda[n]} S_{2\lambda} \Big) t^n.$$

We hence have

$$\Phi = \bigoplus_{\lambda \in \Lambda[n]} \chi^{2\lambda}. \tag{3.1}$$

For the pair (B_n, S_{2n}), (3.1) and Proposition 2.1 imply the following.

Corollary 3.3. *$U^\mu(\Theta) \neq 0 \Leftrightarrow \mu = 2\lambda$ for $\lambda \in \Lambda[n]$.*

The spherical functions for (B_n, S_{2n}) are defined using (2.2). That is for $\lambda \in \Lambda[n]$ we set

$$\Theta^\lambda = \frac{\Theta \chi^{2\lambda}}{h_{2\lambda}} \tag{3.2}$$

It develops that when we use Young's orthogonal representations [9, 10] for the U in Corollary 3.3, a surprising fact involving the *domino tableaux* appears. To state this we need to define the standard domino tableaux of shape λ (D^λ-tableaux). Denote by $\mathbb{T}^\lambda[n]$ the set of all standard tableaux of shape $\lambda \in \Lambda[n]$. That is all the injective fillings of the Ferrer diagram of shape λ with the numbers $1 \cdots n$, strictly increasing in the rows and the columns. Define also the map

$$D^\lambda : \mathbb{T}^\lambda[n] \to \mathbb{T}^{2\lambda}[n]$$

which sends each square i into the horizontal domino $2i-1$, $2i$, as, for example, in Fig. 2. Then the set of D^λ-tableaux is the image of the map D^λ.

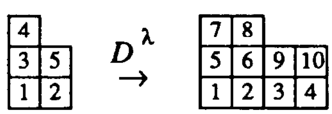

Fig. 2.

Theorem 3.4. *For Young's orthogonal representation* $U^{\mu}(\Theta)=\|u_{ij}^{\mu}\|$,

$$u_{ij}^{\mu}\neq 0 \Rightarrow \mu=2\lambda \text{ for } \lambda\in\Lambda[n]$$

and $T_i^{2\lambda}$, $T_j^{2\lambda}$ *are* D^{λ}*-tableaux.*

Proof. Young's basis [9–10] which yields the matrices U^{μ} is indexed by the standard tableaux $\mathbb{T}^{\mu}[2n]$. In that setting, a useful version of the branching rule for the representations of the symmetric groups can be stated as follows (see [9] for a proof).

Branching rule. *For each permutation* τ *in* $S_{2n-1}\hookrightarrow S_{2n}$

$$U^{\mu}(\tau)=\bigoplus_{v\in\mu^-}U^{v}(\tau)$$

where μ^- *is the set of all partitions obtained from* $\mu\in\Lambda[2n]$ *be decreasing one of the parts by* 1.

If we choose a system of representatives for the left and the right cosets of B_{n-1} in B_n we get the identity

$$U^{\mu}(\Theta_n)=\frac{1}{2n}U^{\mu}(\Theta_{n-1})R=\frac{1}{2n}LU^{\mu}(\Theta_{n-1}). \tag{3.3}$$

Here R is the matrix corresponding to the sum of the right coset representatives and L is the matrix corresponding to the left ones.

For $n=1$, we obtain the matrices

$$U^{\cdot\cdot}(\Theta_1)=[1] \quad\text{and}\quad U^{:}(\Theta_1)=[0].$$

We shall complete the proof by induction on n. Set for $\kappa\in\Lambda[2n-2]$

$$U^{\kappa}(\Theta_{n-1})=\|b_{ij}^{\kappa}\|.$$

By the induction hypothesis we have

$$b_{ij}^{\kappa}\neq 0 \Rightarrow \kappa=2v \quad\text{for } v\in\Lambda[n-1]$$

and T_i^{2v}, T_j^{2v} are D^{v}-tableaux. Set for $\mu\in\Lambda[2n]$,

$$U^{\mu}(\Theta_{n-1})=\|c_{ij}^{\mu}\|.$$

From Corollary 3.3, we can assume that $\mu=2\lambda$ ($\lambda\in\Lambda[n]$) in equation (3.3). Using twice the branching rule we get

$$U^{2\lambda}(\Theta_{n-1})=\bigoplus_{v\in(2\lambda)^-}\ \bigoplus_{\kappa\in v^-}U^{\kappa}(\Theta_{n-1}). \tag{3.4}$$

Note that by the induction hypothesis $U^{\kappa}(\Theta_{n-1})$ is different from the zero matrix only when $\kappa=2v$ for some $v\in\Lambda[n-1]$. Thus the only $U^{\kappa}(\Theta_{n-1})$ in (3.4) which contribute nonzero entries to $U^{2\lambda}(\Theta_{n-1})$ are those in which κ is obtained from 2λ

by removing twice a square from the same row. Thus we may conclude that

$$c_{ij}^{\mu} \neq 0 \Rightarrow \mu = 2\lambda \quad \text{for } \lambda \in \Lambda[n]$$

and $T_i^{2\lambda}$, $T_j^{2\lambda}$ are D^λ-tableaux. The equations (3.3) then yield that the same properties must be satisfied by the entries $\mu_{ij}^{\lambda}(\Theta_n)$ in $U^{\mu}(\Theta_n)$. This completes the induction. □

We conjecture that $u_{ij}^{2\lambda} \neq 0$ iff $T_i^{2\lambda}$ and $T_j^{2\lambda}$ are D^λ-tableaux. Moreover u_{ij}^{μ}, seems to be greater than or equal to zero in all cases.

4. Zonal polynomials

The zonal polynomials [5] are constructed in a way that reminds us of the construction of the Schur functions. That is we set

$$Z_\lambda = \sum_{\sigma \in S_{2n}} \Theta^\lambda(\sigma) \Psi_{\delta(\sigma)} = (2n)! \, \langle \Theta^\lambda, \Psi_{\delta(-)} \rangle_{S_{2n}}$$

where the Ψ's are the power sum symmetric functions and $\delta(\sigma)$ is the type of the double coset of σ. We see that the double cosets play a role here analogous to that played by the conjugacy classes in the Frobenius map. Moreover, we will show that the zonal polynomials form an orthogonal basis with respect to the scalar product defined on the power sums by setting:

$$\langle \Psi_\lambda, \Psi_\mu \rangle_2 = \delta_{\lambda\mu} 2^{1(\lambda)} \tilde{\lambda}$$

where $1(\lambda)$ is the number of nonzero parts of λ and for $\lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$,

$$\tilde{\lambda} = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \, \alpha_2! \cdots \alpha_n!$$

To derive this last assertion, let us set some notations. Let us denote by d_ν the cardinality of the double coset of type ν. Let us denote by Θ_ν^λ the value of the function Θ^λ on the same double coset. Thus we can write

$$\langle Z_\lambda, Z_\mu \rangle_2 = \sum_\nu \Theta_\nu^\lambda \Theta_\nu^\mu d_\nu^2 2^{1(\nu)} \tilde{\nu}.$$

On the other hand, we have from equation (2.5) that

$$\langle \Theta^\lambda, \Theta^\mu \rangle_{S_{2n}} = \frac{1}{(2n)!} \sum_\nu \Theta_\nu^\lambda \Theta_\nu^\mu d_\nu = \delta_{\lambda\mu} \frac{1}{(2n)!} \frac{1}{h_{2\lambda}}.$$

Now a simple permutation counting argument gives

$$d_\nu 2^{1(\nu)} \tilde{\nu} = |B_n|^2,$$

thus

$$\langle Z_\lambda, Z_\mu \rangle_2 = \sum_\nu \Theta_\nu^\lambda \Theta_\nu^\mu d_\nu \, |B_n|^2 = \delta_{\lambda\mu} \frac{|B_n|^2}{h_{2\lambda}}$$

which yields the desired orthogonality.

Since $\downarrow \Theta_n = \Theta_n$, then for any f, g, we have

$$\langle \Theta_n f \Theta_n, g \rangle_{S_{2n}} = \langle f, \Theta_n g \Theta_n \rangle_{S_{2n}}. \tag{4.1}$$

This is an analogue of the Frobenius reciprocity for spherical functions when

$$g = \Theta_n g \Theta_n \in \mathcal{H}(B_n, S_{2n}).$$

Let us cut the interval $\{1, \ldots, 2n\}$ into two even intervals: $A = \{1, \ldots, 2k\}$ and $B = \{2k+1, \ldots, 2n\}$. The tensor product of two spherical functions, on S_A and S_B respectively, is the pointwise multiplication of the two functions. In S_{2n}, that gives

$$\Theta^\nu \Theta^\mu(\sigma) = \begin{cases} \Theta^\nu(\sigma_A)\Theta^\mu(\sigma_B) & \text{if } \sigma = \sigma_A\sigma_B \in S_A \times S_B, \\ 0 & \text{elsewhere.} \end{cases}$$

to induce this function to a spherical function on S_{2n} we multiply by Θ_n on both sides and we obtain

$$\Theta_n \Theta^\nu \Theta^\mu \Theta_n = \sum_{\lambda \in \Lambda[n]} b^\lambda_{\nu\mu} \Theta^\lambda. \tag{4.2}$$

Now let us return to the zonal polynomials. The multiplication of two zonal polynomials Z_ν, Z_μ, $\nu \in \Lambda[k]$ and $\mu \in \Lambda[n-k]$ gives

$$Z_\nu Z_\mu = \Big(\sum_{\sigma_A \in S_A} \Theta^\nu(\sigma_A) \Psi_{\delta(\sigma_A)} \Big) \Big(\sum_{\sigma_B \in S_B} \Theta^\mu(\sigma_B) \Psi_{\delta(\sigma_B)} \Big). \tag{4.3}$$

Here, it is not difficult to see that the cycle structure of the graph associated to the double coset of $\sigma_A\sigma_B$ is the union of the cycle structure of the graphs corresponding to σ_A and σ_B. This gives that

$$\Psi_{\delta(\sigma_A\sigma_B)} = \Psi_{\delta(\sigma_A)} \Psi_{\delta(\sigma_B)}$$

and we may rewrite (4.3) in the form

$$Z_\nu Z_\mu = \sum_{\sigma = \sigma_A\sigma_B \in S_A \times S_B} \Theta^\nu(\sigma_A)\Theta^\mu(\sigma_B)\Psi_{\delta(\sigma)} = (2n)!\, \langle \Theta^\nu \Theta^\mu, \Psi_{\delta(-)} \rangle_{S_{2n}}.$$

Since $\Psi_{\delta(-)}$ is constant on the double cosets, we have

$$\Psi_{\delta(-)} = \Theta_n \Psi_{\delta(-)} \Theta_n.$$

Thus

$$\begin{aligned} Z_\nu Z_\mu &= (2n)!\, \langle \Theta^\nu \Theta^\mu, \Theta_n \Psi_{\delta(-)} \Theta_n \rangle_{S_{2n}} \\ &= (2n)!\, \langle \Theta_n \Theta^\nu \Theta^\mu \Theta_n, \Psi_{\delta(-)} \rangle_{S_{2n}} \\ &= \sum_{\lambda \in \Lambda[n]} b^\lambda_{\nu\mu} Z_\lambda \end{aligned}$$

where the $b^\lambda_{\nu\mu}$ are defined in (4.2). This is an analogue of the multiplication rule for two Schur functions,

$$S_\nu S_\mu = \sum_{\lambda \in \Lambda[n]} c^\lambda_{\nu\mu} S_\lambda$$

where

$$\mathrm{ind}(\chi^{\nu} \otimes \chi^{\mu})_{S_A \times S_B}^{S_n} = \sum_{\lambda \in \Lambda[n]} c_{\nu\mu}^{\lambda} \chi^{\lambda}.$$

The $b_{\nu\mu}^{\lambda}$ may be obtained as follows:

$$b_{\nu\mu}^{\lambda} = (2n)!\, h_{2\lambda} \langle \Theta_n \Theta^{\nu} \Theta^{\mu} \Theta_n,\ \Theta^{\lambda} \rangle_{S_{2n}} \tag{4.4}$$

or alternatively

$$b_{\nu\mu}^{\lambda} \Theta^{\lambda} = \Theta^{\lambda} \Theta_n \Theta^{\nu} \Theta^{\mu} \Theta_n \Theta^{\lambda} = \Theta^{\lambda} \Theta^{\nu} \Theta^{\mu} \Theta^{\lambda}.$$

We have hence proved the following theorem.

Theorem 4.1.

$$Z_{\nu} Z_{\mu} = \sum_{\lambda \in \Lambda[n]} b_{\nu\mu}^{\lambda} Z_{\lambda} \quad \textit{where } b_{\nu\mu}^{\lambda} = (2n)!\, h_{2\lambda} \langle \Theta_n \Theta^{\nu} \Theta^{\mu} \Theta_n,\ \Theta^{\lambda} \rangle_{S_{2n}}$$

To this date there is no combinatorial rule, like the Littlewood–Richardson rule, for computing the coefficients $b_{\nu\mu}^{\lambda}$. Theorem 3.4 suggests that the D^{λ}-tableaux should be involved in such a rule. We shall terminate this paper by showing a relation between the coefficients for the multiplication of zonal polynomials and the coefficients for the multiplication of Schur functions. For this, we need the following lemma.

Lemma 4.2. *For $f, g \in \mathcal{A}(G)$, (using the same setting as in Section 2)*

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{\lambda \in \Lambda} \frac{1}{h_{\lambda}} \mathrm{tr}\, U^{\lambda}(fg^*).$$

Proof. We first notice that

$$\langle f, g \rangle_G = \langle fg^*, \varepsilon \rangle_G = \frac{1}{|G|} fg^* |_{\varepsilon}. \tag{4.5}$$

On the other hand, for any h in $\mathcal{A}(G)$ and from (2.1)

$$h|_{\varepsilon} = \sum_{i,j,\lambda} u_{ij}^{\lambda}(f) e_{ij}^{\lambda}|_{\varepsilon}. \tag{4.6}$$

From representation theory [9–10], one can show that

$$e_{ij}^{\lambda}|_{\varepsilon} = \frac{\delta_{ij}}{h_{\lambda}}. \tag{4.7}$$

Combining (4.6) and (4.7) we get

$$h|_{\varepsilon} = \sum_{\lambda} \frac{1}{h_{\lambda}} \sum_{i} u_{ii}^{\lambda}(h) = \sum_{\lambda} \frac{1}{h_{\lambda}} \mathrm{tr}\, U^{\lambda}(h). \tag{4.8}$$

The result follows from (4.5) and (4.8). □

Theorem 4.3. $c^{2\lambda}_{2\nu,2\mu}=0 \;\Rightarrow\; b^{\lambda}_{\nu\mu}=0.$

Proof. From Theorem 4.1 and equation (3.2) we have

$$b^{\lambda}_{\nu\mu}=(2n)!\,h_{2\lambda}\{\Theta_n\Theta^{\nu}\Theta^{\mu}\Theta_n,\ \Theta^{\lambda}\rangle_{S_{2n}}$$

$$=(2n)!\left\langle \Theta_n\Theta_{|\nu|}\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}}\Theta_{|\mu|}\Theta_n,\ \chi^{2\lambda}\Theta_n\right\rangle_{S_{2n}}$$

The hyperocatahedral groups B_A and B_B acting on respectively the sets A and B subgroups of B_n, we thus have $\Theta_n\Theta_{|\nu|}=\Theta_n$ and $\Theta_{|\mu|}\Theta_n=\Theta_n$. Using (4.1) we obtain

$$b^{\lambda}_{\nu\mu}=(2n)!\left\langle\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}},\ \chi^{2\lambda}\Theta_n\right\rangle_{S_{2n}}.$$

Since $\Theta_n^*=\Theta_n$ and $(\chi^{2\lambda})^*=\chi^{2\lambda}$, Lemma 4.2 gives

$$b^{\lambda}_{\nu\mu}=\sum_{\kappa\in\Lambda[2n]}\frac{h_{2\lambda}}{h_\kappa}\operatorname{tr} U^{\kappa}\left(\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}}\Theta_n\frac{\chi^{2\lambda}}{h_{2\lambda}}\right)$$

$$=\sum_{\kappa\in\Lambda[2n]}\frac{h_{2\lambda}}{h_\kappa}\operatorname{tr}\left(U^{\kappa}\left(\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}}\Theta_n\right)U^{\kappa}\left(\frac{\chi^{2\lambda}}{h_{2\lambda}}\right)\right).$$

This implies that the only nonzero term in this last sum is when $\kappa=2\lambda$ and we get

$$b^{\lambda}_{\nu\mu}=\operatorname{tr}\left(U^{2\lambda}\left(\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}}\right)U^{2\lambda}(\Theta_n)\right). \tag{4.9}$$

Using Frobenius reciprocity we can write for $\sigma=\sigma_A\sigma_B\in S_A\times S_B$

$$U^{2\lambda}(\sigma)\cong\bigoplus_{\substack{\phi\in\Lambda[2k]\\ \eta\in\Lambda[2n-2k]}}(U^{\phi}(\sigma_A)\otimes U^{\eta}(\sigma_B))^{\otimes c^{2\lambda}_{\phi\eta}}.$$

From this we may state that

$$c^{2\lambda}_{2\nu,2\mu}=0\;\Rightarrow\; U^{2\lambda}\left(\frac{\chi^{2\nu}}{h_{2\nu}}\frac{\chi^{2\mu}}{h_{2\mu}}\right)=[0].$$

This last result combined with (4.9) completes the proof. □

Remark 4.4. We should mention that, working with the Littlewood–Richardson tableaux, one can show that we have

$$c^{2\lambda}_{2\nu,2\mu}=0\;\Leftrightarrow\; c^{\lambda}_{\nu\mu}=0.$$

Thus Theorem 4.3 may be restated as

$$c^{\lambda}_{\nu\mu}=0\;\Rightarrow\; b^{\lambda}_{\nu\mu}=0.$$

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