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Journal of Combinatorial Theory, Series A 120 (2013) 433–452

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

RSK correspondence and classically irreducible Kirillov–Reshetikhin crystals ☆

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A R T I C L E I N F O

Article history:
Received 28 March 2012
Available online 2 October 2012

Keywords:
Quantum affine algebra
Kirillov–Reshetikhin crystal
RSK algorithm

A B S T R A C T

We give a new combinatorial model of the Kirillov–Reshetikhin crystals of type $A_n^{(1)}$ in terms of non-negative integral matrices based on the classical RSK algorithm, which has a simple description of the affine crystal structure without using the promotion operator. We have a similar description of the Kirillov–Reshetikhin crystals associated to exceptional nodes in the Dynkin diagrams of classical affine or non-exceptional affine type, which are called classically irreducible together with those of type $A_n^{(1)}$.

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1. Introduction

The Robinson–Schensted–Knuth (simply RSK) correspondence is a weight preserving bijection from the set $\mathcal{M}_{m\times n}$ of $m\times n$ non-negative integral matrices to the set $\mathcal{T}_{m\times n}$ of pairs of semistandard Young tableaux of the same shape with entries from m and n letters, respectively [1].

The RSK map κ has nice representation theoretic interpretations from a viewpoint of the Kashiwara's crystal base theory [2]. In [3], Lascoux shows that $\mathcal{M}_{m\times n}$ has a $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$-crystal structure and κ is an isomorphism of crystals, where one can define a $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$-crystal structure on $\mathcal{T}_{m\times n}$ in an obvious way following [4]. As an application, a non-symmetric Cauchy kernel expansion into a sum of product of Demazure characters is obtained. In [5], the author shows that κ can be extended to an isomorphism of $\mathfrak{gl}_{m+n}$-crystals. Here $\mathcal{M}_{m\times n}$ or $\mathcal{T}_{m\times n}$ can be regarded as a crystal associated to a generalized Verma module over $\mathfrak{gl}_{m+n}$. As an application, a weight generating function of plane partitions in a bounded region is given as a Demazure character of $\mathfrak{gl}_{m+n}$. (See also [6] for another application of RSK to the crystal base of a modified quantized enveloping algebra of type $A_{+\infty}$ and A_∞.)

☆ This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012-0002607).

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http://dx.doi.org/10.1016/j.jcta.2012.09.003

The purpose of this paper is to study the RSK correspondence further in this direction and discuss its connection with affine crystals. It is motivated by the observation that $\mathcal{M}_{r\times(n-r)}$ has a natural affine crystal structure of type $A_{n-1}^{(1)}$ for $n \geqslant 2$ and $1 \leqslant r \leqslant n-1$ by [5] and the symmetry of the Dynkin diagram of $A_{n-1}^{(1)}$. For $s \geqslant 1$, we let $\mathcal{M}_{r\times(n-r)}^{s}$ be the set of matrices in $\mathcal{M}_{r\times(n-r)}$ such that the length of a maximal decreasing subsequence of its row or column word is no more than s. Then as the main result in this paper, we show (Theorem 3.8) that as an affine crystal of type $A_{n-1}^{(1)}$,

$$\mathcal{M}_{r\times(n-r)}^{s} \otimes T_{s\omega_r} \cong \mathbf{B}^{r,s}, \tag{1.1}$$

where $\mathbf{B}^{r,s}$ is a perfect crystal [7] with highest weight $s\omega_r$ or the rectangular partition (s^r) as a classical $\mathfrak{gl}_n$-crystal, and $T_{s\omega_r} = \{t_{s\omega_r}\}$ is a crystal with $\mathrm{wt}(t_{s\omega_r}) = s\omega_r$, $\varepsilon_i(t_{s\omega_r}) = \varphi_i(t_{s\omega_r}) = -\infty$ for all i.

To prove (1.1), two RSK maps $\kappa^{\nwarrow}$ and $\kappa^{\searrow}$ are considered, which map a matrix in $\mathcal{M}_{r\times(n-r)}^{s}$ to a pair of semistandard Young tableaux of normal and anti-normal shape, respectively. They turn out to be the projections of $\mathcal{M}_{r\times(n-r)}^{s}$ to a classical crystal of type A_{n-1} corresponding to maximal parabolic subalgebras obtained from $A_{n-1}^{(1)}$ by removing the simple roots α_0 and α_r respectively. These two RSK maps play an important role in proving the regularity of $\mathcal{M}_{r\times(n-r)}^{s} \otimes T_{s\omega_r}$ and constructing the isomorphism in (1.1). Note that $\mathcal{M}_{r\times(n-r)}$ can be regarded as a limit of the crystals $\mathbf{B}^{r,s} \otimes T_{-s\omega_r}$ as s goes to infinity.

Let $\mathfrak{g}$ be an affine Kac–Moody algebra and let $U_q'(\mathfrak{g})$ be the quantized enveloping algebra associated to the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The finite dimensional irreducible $U_q'(\mathfrak{g})$-modules do not have crystal bases in general. But it was conjectured by Hatayama et al. [8,9] that a certain family of finite dimensional irreducible $U_q'(\mathfrak{g})$-modules $W^{r,s}$ called *Kirillov–Reshetikhin modules* (simply *KR modules*) [10] have crystal bases, where r denotes a simple root index of $\mathfrak{g}$ except 0 and s is an arbitrary positive integer. The conjectured crystals $\mathbf{B}^{r,s}$ are now called *KR crystals*.

For type $A_{n-1}^{(1)}$, the KR crystals $\mathbf{B}^{r,s}$ are the perfect crystals in (1.1). In this case, a combinatorial description of $\mathbf{B}^{r,s}$ was given by Shimozono [11] using semistandard Young tableaux of a rectangular shape and the Schützenberger's promotion operator [12]. But, the main advantage of our model using $r \times (n-r)$ integral matrices is that the description of its crystal structure is remarkably simple, where the crystal operators or Kashiwara operators corresponding to α_0 and α_r are given as adding ± 1 at the entries at southeast and northwest corners of a matrix, respectively (see Fig. 1).

Recently, the existence of KR crystals $\mathbf{B}^{r,s}$ for the other classical affine or non-exceptional affine type was proved by Okado and Schilling [13], and its combinatorial construction was given in [13,14], where the Kashiwara–Nakashima tableaux [4] were used to describe the classical crystal structure on $\mathbf{B}^{r,s}$.

We use (1.1) to obtain a new description of the KR crystals associated to so-called *exceptional nodes* in the Dynkin diagrams of classical affine type (see [14, Table 1]). These crystals together with $\mathbf{B}^{r,s}$ of type $A_{n-1}^{(1)}$ are called *classically irreducible* [15] since they are connected as a classical crystal, and they are also perfect crystals [7].

We use the Kashiwara's method of folding crystals [16] to construct $\mathbf{B}^{n,s}$ of type $D_{n+1}^{(2)}$ and $C_n^{(1)}$ in terms of symmetric non-negative integral matrices (Theorem 4.4), and we describe $\mathbf{B}^{n-1,s}$ and $\mathbf{B}^{n,s}$ of type $D_n^{(1)}$ in terms of semistandard Young tableaux of type A_{n-1} (Theorem 5.4). (See Figs. 2 and 3.) In both cases, the affine crystal structures are given explicitly as in $A_{n-1}^{(1)}$.

It would be nice to have a similar description of arbitrary KR crystals of classical affine type, but we do not know how to generalize the method here in a natural way.

2. Preliminary

2.1. Quantum groups and crystals

Let us give a brief review on crystals (cf. [17,18]). Let $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix with an index set I. Consider a quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ called a Cartan datum, where $P^\vee$ is a

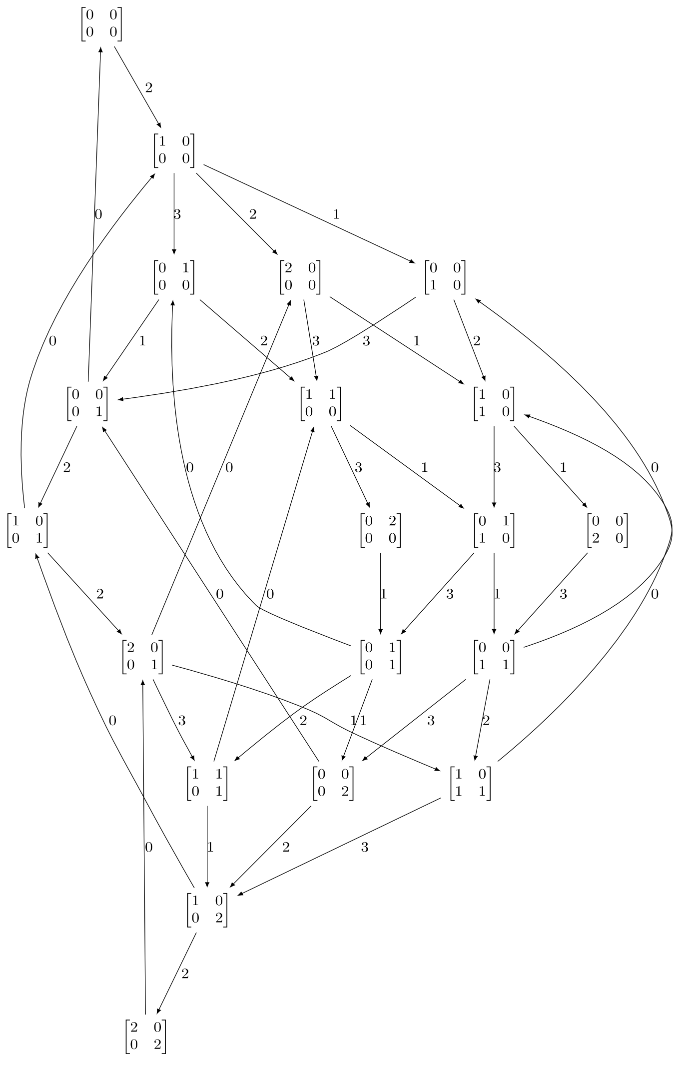


Fig. 1. The KR crystal $\mathbf{B}^{2,2}$ of type $A_3^{(1)}$ where the vertices are given in terms of non-negative integral 2×2 matrices with the length of column or row words no more than 2. This graph was implemented by SAGE.

free $\mathbb{Z}$-module of finite rank, $P=\mathrm{Hom}_{\mathbb{Z}}(P^\vee,\mathbb{Z})$, $\Pi^\vee=\{h_i \mid i\in I\}\subset P^\vee$, and $\Pi=\{\alpha_i \mid i\in I\}\subset P$ such that $\langle \alpha_j, h_i\rangle = a_{ij}$ for $i, j\in I$.

A crystal associated to $(A, P^\vee, P, \Pi^\vee, \Pi)$ is a set B together with the maps $\mathrm{wt}: B\to P$, $\varepsilon_i, \varphi_i : B\to \mathbb{Z}\cup\{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B\to B\cup\{\mathbf{0}\}$ $(i\in I)$ such that for $b\in B$ and $i\in I$

(1) $\varphi_i(b)=\langle \mathrm{wt}(b), h_i\rangle+\varepsilon_i(b)$,
(2) $\varepsilon_i(\widetilde{e}_i b)=\varepsilon_i(b)-1$, $\varphi_i(\widetilde{e}_i b)=\varphi_i(b)+1$, $\mathrm{wt}(\widetilde{e}_i b)=\mathrm{wt}(b)+\alpha_i$ if $\widetilde{e}_i b\neq \mathbf{0}$,
(3) $\varepsilon_i(\widetilde{f}_i b)=\varepsilon_i(b)+1$, $\varphi_i(\widetilde{f}_i b)=\varphi_i(b)-1$, $\mathrm{wt}(\widetilde{f}_i b)=\mathrm{wt}(b)-\alpha_i$ if $\widetilde{f}_i b\neq \mathbf{0}$,
(4) $\widetilde{f}_i b=b'$ if and only if $b=\widetilde{e}_i b'$ for $b, b'\in B$,
(5) $\widetilde{e}_i b=\widetilde{f}_i b=\mathbf{0}$ if $\varphi_i(b)=-\infty$,

where $\mathbf{0}$ is a formal symbol. Here we assume that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. Note that B is equipped with an I-colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $b' = \widetilde{f}_i b$ for $b, b' \in B$ and $i \in I$. We call B connected if it is connected as a graph, and normal if $\varepsilon_i(b) = \max\{k \mid \widetilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k \mid \widetilde{f}_i^k b \neq \mathbf{0}\}$ for $b \in B$ and $i \in I$. The dual crystal $B^\vee$ of B is defined to be the set $\{b^\vee \mid b \in B\}$ with $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$, $\varepsilon_i(b^\vee) = \varphi_i(b)$, $\varphi_i(b^\vee) = \varepsilon_i(b)$, $\widetilde{e}_i(b^\vee) = (\widetilde{f}_i b)^\vee$ and $\widetilde{f}_i(b^\vee) = (\widetilde{e}_i b)^\vee$ for $b \in B$ and $i \in I$. We assume that $\mathbf{0}^\vee = \mathbf{0}$.

Let B_1 and B_2 be crystals. A morphism $\psi : B_1 \to B_2$ is a map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$ such that

(1) $\psi(\mathbf{0}) = \mathbf{0}$,
(2) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq \mathbf{0}$,
(3) $\psi(\widetilde{e}_i b) = \widetilde{e}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\widetilde{e}_i b) \neq \mathbf{0}$,
(4) $\psi(\widetilde{f}_i b) = \widetilde{f}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\widetilde{f}_i b) \neq \mathbf{0}$,

for $b \in B_1$ and $i \in I$. We call ψ an embedding and B_1 a subcrystal of B_2 when ψ is injective, and call ψ strict if $\psi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for all $i \in I$, where we assume that $\widetilde{e}_i \mathbf{0} = \widetilde{f}_i \mathbf{0} = \mathbf{0}$. When ψ is a bijection, it is called an isomorphism. For $b_i \in B_i$ $(i = 1, 2)$, we say that b_1 is equivalent to b_2 if there exists an isomorphism of crystals $C(b_1) \to C(b_2)$ sending b_1 to b_2, where $C(b_i)$ is the connected component in B_i including b_i as an I-colored oriented graph.

A tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\begin{aligned}
&\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
&\varepsilon_i(b_1 \otimes b_2) = \max\bigl\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i\rangle\bigr\},\\
&\varphi_i(b_1 \otimes b_2) = \max\bigl\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i\rangle, \varphi_i(b_2)\bigr\},\\
&\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2),\\ b_1 \otimes \widetilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),\end{cases}\\
&\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2),\end{cases}
\end{aligned}$$

for $i \in I$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is a crystal.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra associated to A. Let $P^\vee$ be the dual weight lattice, $P = \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{h_i \mid i \in I\}$ the set of simple coroots, and $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots of $\mathfrak{g}$.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(q)$ generated by e_i, f_i and q^h for $i \in I$ and $h \in P^\vee$. For a dominant integral weight Λ, let $\mathbf{B}(\pm\Lambda)$ be the crystal of an irreducible highest (respectively lowest) weight $U_q(\mathfrak{g})$-module with highest (respectively lowest) weight $\pm\Lambda$. Then $\mathbf{B}(\pm\Lambda)$ is a crystal associated to $(A, P^\vee, P, \Pi^\vee, \Pi)$. We say that a crystal B is regular if it is isomorphic to the crystal of an integrable $U_q(\mathfrak{g}_J)$-module for any $J \subset I$ with $|J| \leqslant 2$, where $\mathfrak{g}_J$ is the Kac–Moody algebra associated to $A_J = (a_{ij})_{i,j \in J}$. Note that a regular crystal is normal.

For $\Lambda \in P$, we denote by $T_\Lambda = \{t_\Lambda\}$ a crystal with $\mathrm{wt}(t_\Lambda) = \Lambda$ and $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$ for $i \in I$.

2.2. Quantum affine algebras

Assume that A is a generalized Cartan matrix of affine type with an index set $I = \{0, 1, \ldots, n\}$ following [1, §4.8], and $\mathfrak{g}$ is the associated affine Kac–Moody algebra with the Cartan subalgebra $\mathfrak{h}$. Let $P^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i \oplus \mathbb{Z} d \subset \mathfrak{h}$ be the dual weight lattice of $\mathfrak{g}$, where d is given by $\langle \alpha_j, d\rangle = \delta_{0j}$ for $j \in I$. Let $\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^*$ be the positive imaginary null root of $\mathfrak{g}$ and let $\Lambda_i \in \mathfrak{h}^*$ $(i \in I)$ be the i-th fundamental weight such that $\langle \Lambda_i, h_j\rangle = \delta_{ij}$ for $j \in I$ and $\langle \Lambda_i, d\rangle = 0$. Then the weight lattice of $\mathfrak{g}$ is $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{1}{a_0}\delta$.

Let $P_{\mathrm{cl}} = P/(\mathbb{Q}\delta \cap P) = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ and $(P_{\mathrm{cl}})^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$, where we still denote the image of Λ_i in P_{cl} by Λ_i. Then we define $U_q'(\mathfrak{g})$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i and q^h for $i \in I$ and

$h \in (P_{\mathrm{cl}})^\vee$. We regard P_{cl} as the weight lattice of $U_q'(\mathfrak{g})$. For a proper subset $J \subset I$, let $\Pi_J^\vee = \{h_i \mid i \in J\}$ and $\Pi_J = \{\alpha_i \mid i \in J\}$, and let $U_q(\mathfrak{g}_J)$ be the subalgebra of $U_q'(\mathfrak{g})$ generated by e_i, f_i and q^h for $i \in J$ and $h \in (P_{\mathrm{cl}})^\vee$.

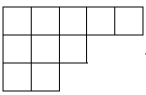
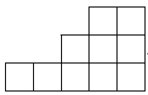
From now on, we mean by a $U_q'(\mathfrak{g})$-crystal (respectively $U_q(\mathfrak{g}_J)$-crystal) a crystal associated to $(A, (P_{\mathrm{cl}})^\vee, P_{\mathrm{cl}}, \Pi^\vee, \Pi)$ (respectively $(A_J, (P_{\mathrm{cl}})^\vee, P_{\mathrm{cl}}, \Pi_J^\vee, \Pi_J)$). For simplicity, we will often write the type of the generalized Cartan matrix A (or A_J) instead of $\mathfrak{g}$ (or $\mathfrak{g}_J$).

The following lemma plays an important role in this paper to have a combinatorial realization of KR crystals.

Lemma 2.1. *(See Lemma 2.6 in [15].) Let $\mathfrak{g}$ be of classical affine or non-exceptional affine type. Fix $r \in I \setminus \{0\}$ and $s \geqslant 1$. Then any regular $U_q'(\mathfrak{g})$-crystal that is isomorphic to the KR crystal $\mathbf{B}^{r,s}$ as a $U_q(\mathfrak{g}_{I\setminus\{0\}})$-crystal is also isomorphic to $\mathbf{B}^{r,s}$ as a $U_q'(\mathfrak{g})$-crystal.*

2.3. RSK algorithm

Let us recall some necessary background on semistandard tableaux following [19,20]. Let $\mathscr{P}$ be the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i\geqslant 1}$ with a Young diagram. We denote the length of λ by $\ell(\lambda)$ and the conjugate of λ by $\lambda' = (\lambda_i')_{i\geqslant 1}$. We let λ^π be the skew Young diagram obtained by 180°-rotation of λ. For example,

$(5,3,2) =$  , $(5,3,2)^\pi =$  .

Let $\mathbb{A}$ be a linearly ordered set. For a skew Young diagram λ/μ, let $SST_{\mathbb{A}}(\lambda/\mu)$ be the set of all semistandard tableaux of shape λ/μ with entries in $\mathbb{A}$. Let $\mathcal{W}_{\mathbb{A}}$ be the set of finite words in $\mathbb{A}$. For $T \in SST_{\mathbb{A}}(\lambda/\mu)$, let $w(T)$ be a word in $\mathcal{W}_{\mathbb{A}}$ obtained by reading the entries of T row by row from top to bottom, and from right to left in each row.

Let $\mathrm{sh}(T)$ denote the shape of a tableau T. If $\mathrm{sh}(T) = \nu$ (respectively ν^π) for some $\nu \in \mathscr{P}$, then we say that T is of normal (respectively anti-normal) shape. For $T \in SST_{\mathbb{A}}(\lambda/\mu)$, let $T^{\nwarrow}$ (respectively $T^{\searrow}$) be the unique semistandard tableau of normal (respectively anti normal) shape such that $w(T^{\nwarrow})$ (respectively $w(T^{\searrow})$) is Knuth equivalent to $w(T)$. Note that if $\mathrm{sh}(T^{\nwarrow}) = \nu$, then $\mathrm{sh}(T^{\searrow}) = \nu^\pi$.

For $T \in SST_{\mathbb{A}}(\lambda)$ and $a \in \mathbb{A}$, let $a \to T$ be the tableau obtained by applying the Schensted's column insertion of a into T. For $w = w_1 \cdots w_r \in \mathcal{W}_{\mathbb{A}}$, we define $\mathbf{P}(w) = (w_r \to (\cdots (w_2 \to w_1)\cdots))$.

Let $\mathbb{B}$ be another linearly ordered set. Let

$$\mathcal{M}_{\mathbb{A},\mathbb{B}} = \left\{ M = (m_{ab})_{a\in\mathbb{A}, b\in\mathbb{B}} \;\middle|\; m_{ab} \in \mathbb{Z}_{\geqslant 0},\ \sum_{a,b} m_{ab} < \infty \right\}. \tag{2.1}$$

Let $\Omega_{\mathbb{A},\mathbb{B}}$ be the set of biwords $(\mathbf{a},\mathbf{b}) \in \mathcal{W}_{\mathbb{A}} \times \mathcal{W}_{\mathbb{B}}$ such that (1) $\mathbf{a} = a_1 \cdots a_r$ and $\mathbf{b} = b_1 \cdots b_r$ for some $r \geqslant 0$, (2) $(a_1, b_1) \leqslant \cdots \leqslant (a_r, b_r)$, where for (a,b) and $(c,d) \in \mathbb{A} \times \mathbb{B}$, $(a,b) < (c,d)$ if and only if $(b < d)$ or $(b = d$ and $a > c)$. Then we have a bijection from $\Omega_{\mathbb{A},\mathbb{B}}$ to $\mathcal{M}_{\mathbb{A},\mathbb{B}}$, where $(\mathbf{a},\mathbf{b})$ is mapped to $M(\mathbf{a},\mathbf{b}) = (m_{ab})$ with $m_{ab} = |\{k \mid (a_k, b_k) = (a,b)\}|$. Note that the pair of empty words $(\emptyset,\emptyset)$ corresponds to zero matrix. Let $M \in \mathcal{M}_{\mathbb{A},\mathbb{B}}$ be given. Suppose that $M = M(\mathbf{a},\mathbf{b})$ and it transpose $M^t = M(\mathbf{c},\mathbf{d})$ with $(\mathbf{c},\mathbf{d}) \in \Omega_{\mathbb{B},\mathbb{A}}$. Let $\mathbf{P}(M) = \mathbf{P}(\mathbf{a})$ and $\mathbf{Q}(M) = \mathbf{P}(\mathbf{c})$. Then we have a bijection called the RSK correspondence:

$$\kappa : \mathcal{M}_{\mathbb{A},\mathbb{B}} \to \bigsqcup_{\lambda} SST_{\mathbb{A}}(\lambda) \times SST_{\mathbb{B}}(\lambda),$$

where M is mapped to $(\mathbf{P}(M), \mathbf{Q}(M))$, and the union is over all λ with $SST_{\mathbb{A}}(\lambda) \neq \emptyset$ and $SST_{\mathbb{B}}(\lambda) \neq \emptyset$.

3. KR crystals of type $A_{n-1}^{(1)}$

3.1. Affine algebra of type $A_{n-1}^{(1)}$

Assume that $\mathfrak{g} = A_{n-1}^{(1)}$ ($n \geqslant 2$) with $I = \{0, 1, \ldots, n-1\}$. We put $I_r = I \setminus \{r\}$ for $r \in I$, and $I_{0,r} = I_0 \cap I_r$ for $r \in I_0$. Note that $\mathfrak{g}_{I_0} \cong \mathfrak{g}_{I_r} = A_{n-1}$ and $\mathfrak{g}_{I_{0,r}} = A_{r-1} \oplus A_{n-r-1}$.

Let $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 1, \ldots, n-1$ and $\epsilon_n = \Lambda_0 - \Lambda_{n-1}$. Then $\epsilon_1 + \cdots + \epsilon_n = 0$ and $\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ forms a weight lattice of $\mathfrak{g}_{I_0}$. Note that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I_0$ and $\alpha_0 = \epsilon_n - \epsilon_1$ in P_{cl}. The fundamental weights for $\mathfrak{g}_{I_0}$ are $\omega_i = \Lambda_i - \Lambda_0 = \sum_{k=1}^i \epsilon_k$ for $i \in I_0$.

We regard $[n] = \{1 < \cdots < n\}$ as a $U_q(\mathfrak{g}_{I_0})$-crystal $\mathbf{B}(\omega_1)$ with $\mathrm{wt}(k) = \epsilon_k$, and $[\bar{n}] = \{\bar{n} < \cdots < \bar{1}\}$ as its dual crystal with $\mathrm{wt}(\bar{k}) = -\epsilon_k$. Then $\mathcal{W}_{[n]}$ and $\mathcal{W}_{[\bar{n}]}$ are regular $U_q(\mathfrak{g}_{I_0})$-crystals, where we identify $w = w_1 \cdots w_r$ with $w_1 \otimes \cdots \otimes w_r$.

The fundamental weights for $\mathfrak{g}_{I_r}$ are $\omega'_i = \Lambda_i - \Lambda_r$ for $i \in I_r$. Note that $\omega_r = -\omega'_0$. In this case, we may identify a $U_q(\mathfrak{g}_{I_r})$-crystal $\mathbf{B}(\omega'_{r+1})$, the crystal of the natural representation of $U_q(\mathfrak{g}_{I_r})$, with $[n]_{+r} = \{r+1 \prec \cdots \prec n \prec 1 \prec \cdots \prec r\}$.

3.2. Affine crystal $\mathcal{M}_{r\times(n-r)}$

For $1 \leqslant r \leqslant n-1$, let

$$\mathcal{M}_{r\times(n-r)} = \mathcal{M}_{[\bar{r}],[n]\setminus[r]} \tag{3.1}$$

(see (2.1)). First note that $\mathcal{M}_{r\times(n-r)}$ is a $U_q(A_{r-1})$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ ($1 \leqslant i \leqslant r-1$), where $\widetilde{x}_i M = M(\widetilde{x}_i \mathbf{a}, \mathbf{b})$ for $x = e, f$ and $M \in \mathcal{M}_{r\times(n-r)}$ with $M = M(\mathbf{a}, \mathbf{b})$. Here, we assume that $\widetilde{x}_i M = \mathbf{0}$ if $\widetilde{x}_i \mathbf{a} = \mathbf{0}$. In a similar way, we may view $\mathcal{M}_{r\times(n-r)}$ as a $U_q(A_{n-r-1})$-crystal with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ ($r+1 \leqslant i \leqslant n-1$) by considering the transpose of $M \in \mathcal{M}_{r\times(n-r)}$ as an element in $\mathcal{M}_{[n]\setminus[r],[\bar{r}]}$. Since $\mathfrak{g}_{I_{0,r}} = A_{r-1} \oplus A_{n-r-1}$, $\mathcal{M}_{r\times(n-r)}$ is a regular $U_q(\mathfrak{g}_{I_{0,r}})$-crystal with $\mathrm{wt}(M) = \sum_{i,j} m_{\bar{i}j}(\epsilon_j - \epsilon_i)$ for $M = (m_{\bar{i}j}) \in \mathcal{M}_{r\times(n-r)}$.

Now, let us define two more operators $\widetilde{x}_0$ and $\widetilde{x}_r$ ($x = e, f$) to make $\mathcal{M}_{r\times(n-r)}$ a $U'_q(A_{n-1}^{(1)})$-crystal. For $M = (m_{\bar{i}j}) \in \mathcal{M}_{r\times(n-r)}$, we define

$$\begin{aligned} &\widetilde{e}_r M = \begin{cases} M - E_{\bar{r}r+1}, & \text{if } m_{\bar{r}r+1} \geqslant 1, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad \widetilde{f}_r M = M + E_{\bar{r}r+1}, \\ &\widetilde{f}_0 M = \begin{cases} M - E_{\bar{1}n}, & \text{if } m_{\bar{1}n} \geqslant 1, \\ \mathbf{0}, & \text{otherwise}, \end{cases} \qquad \widetilde{e}_0 M = M + E_{\bar{1}n}, \end{aligned} \tag{3.2}$$

where $E_{\bar{i}j} \in \mathcal{M}_{r\times(n-r)}$ denotes the elementary matrix with 1 at the position $(\bar{i}, j)$ and 0 elsewhere. Put

$$\begin{aligned} &\varepsilon_r(M) = \max\{k \mid \widetilde{e}_r^k M \neq \mathbf{0}\}, \qquad \varphi_r(M) = \varepsilon_r(M) + \langle \mathrm{wt}(M), h_r \rangle, \\ &\varphi_0(M) = \max\{k \mid \widetilde{f}_0^k M \neq \mathbf{0}\}, \qquad \varepsilon_0(M) = \varphi_0(M) - \langle \mathrm{wt}(M), h_0 \rangle. \end{aligned}$$

Then we have

Proposition 3.1. *$\mathcal{M}_{r\times(n-r)}$ is a $U'_q(A_{n-1}^{(1)})$-crystal with respect to* wt, ε_i, φ_i *and* $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in I$).

3.3. Young tableau description of $\mathcal{M}_{r\times(n-r)}$ as a $U_q(A_{n-1})$-crystal

Let us give another description of $\mathcal{M}_{r\times(n-r)}$ in terms of semistandard tableaux. Consider

$$\mathcal{T}^{\searrow}_{r\times(n-r)} = \bigsqcup_{\ell(\lambda)\leqslant r, n-r} SST_{[\bar{r}]}(\lambda^\pi) \times SST_{[n]\setminus[r]}(\lambda^\pi). \tag{3.3}$$

By [4], $SST_{[\bar{r}]}(\lambda^\pi) \times SST_{[n]\setminus[r]}(\lambda^\pi)$ is a regular $U_q(\mathfrak{g}_{I_{0,r}})$-crystal and so is $\mathcal{T}^{\searrow}_{r\times(n-r)}$.

We will define $\widetilde{e}_r, \widetilde{f}_r$ on $\mathcal{T}^{\searrow}_{r\times(n-r)}$ to make $\mathcal{T}^{\searrow}_{r\times(n-r)}$ a $U_q(\mathfrak{g}_{I_0})$-crystal. Let us first recall a combinatorial algorithm often called a signature rule, which will be used throughout the paper. Suppose that $\sigma = (\dots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \dots)$ is a sequence (not necessarily finite) with $\sigma_k \in \{+, -, \cdot\}$ such that $\sigma_k = +$ or $\cdot$ for $k \gg 0$ and $\sigma_k = -$ or $\cdot$ for $k \ll 0$. In σ, we replace a pair $(\sigma_s, \sigma_{s'}) = (+, -)$, where $s < s'$ and $\sigma_t = \cdot$ for $s < t < s'$, with $(\cdot, \cdot)$, and repeat this process as far as possible until we get a sequence with no $-$ placed to the right of $+$. Such a reduced sequence will be denoted by $\widetilde{\sigma}$. When we have an infinite sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ (respectively $\sigma = (\dots, \sigma_2, \sigma_1)$), we also understand $\widetilde{\sigma}$ as a reduced sequence obtained by applying the signature rule to a doubly infinite sequence $(\dots, \cdot, \cdot, \cdot, \sigma_1, \sigma_2, \dots)$ (respectively $(\dots, \sigma_2, \sigma_1, \cdot, \cdot, \cdot, \dots)$).

Now, let $(S, T) \in \mathcal{T}^{\searrow}_{r\times(n-r)}$ be given. For $k \geqslant 1$, let s_k and t_k be the entries in the top of the k-th columns of S and T (enumerated from the right), respectively. We put

$$\sigma_k = \begin{cases} +, & \text{if the } k\text{-th column is empty},\\ +, & \text{if } s_k > \bar{r} \text{ and } t_k > r+1,\\ -, & \text{if } s_k = \bar{r} \text{ and } t_k = r+1,\\ \cdot, & \text{otherwise.} \end{cases}$$

Let $\widetilde{\sigma}$ be the reduced sequence obtained from $\sigma = (\sigma_1, \sigma_2, \dots)$ by the signature rule. Then we define $\widetilde{e}_r(S, T)$ to be the bitableaux obtained from (S, T) by removing $\boxed{\bar{r}}$ and $\boxed{r+1}$ in the columns of S and T corresponding to the right-most $-$ in $\widetilde{\sigma}$. If there is no such $-$ sign, then we define $\widetilde{e}_r(S, T) = \mathbf{0}$. We define $\widetilde{f}_r(S, T)$ to be the bitableaux obtained from (S, T) by adding $\boxed{\bar{r}}$ and $\boxed{r+1}$ on top of the columns of S and T corresponding to the left-most $+$ in $\widetilde{\sigma}$. Note that $\widetilde{f}_r^k(S, T) \neq \mathbf{0}$ for all $k \geqslant 1$.

We put $\varepsilon_r(S, T) = \max\{k \mid \widetilde{e}_r^k(S, T) \neq \mathbf{0}\}$ and $\varphi_r(S, T) = \varepsilon_r(S, T) + \langle \mathrm{wt}(S, T), h_r \rangle$, where $\mathrm{wt}(S, T) = \mathrm{wt}(S) + \mathrm{wt}(T)$. Then $\mathcal{T}^{\searrow}_{r\times(n-r)}$ is a $U_q(\mathfrak{g}_{I_0})$-crystal with respect to wt, ε_i, φ_i and $\widetilde{e}_i, \widetilde{f}_i$ $(i \in I_0)$.

Example 3.2. Suppose that $n = 6$ and $r = 3$. Consider

$$(S, T) = \left(\begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 4 & 4 & 4 \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \right).$$

Then

$$\widetilde{e}_3(S, T) = \left(\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 4 & 4 \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \right),$$

and

$$\widetilde{f}_3(S, T) = \left(\begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & \bar{3} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline \end{array}, \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & 4 & 4 & 4 \\ \hline 4 & 5 & 5 & 5 & 6 \\ \hline \end{array} \right).$$

Define

$$\kappa^{\searrow} : \mathcal{M}_{r\times(n-r)} \to \mathcal{T}^{\searrow}_{r\times(n-r)} \tag{3.4}$$

by $\kappa^{\searrow}(M) = (\mathbf{P}(M)^{\searrow}, \mathbf{Q}(M)^{\searrow})$. By [5, Theorem 3.6], we have the following.

Proposition 3.3. *$\kappa^{\searrow}$ is an isomorphism of $U_q(\mathfrak{g}_{I_0})$-crystals.*

Example 3.4. Let (S, T) be as in Example 3.2. Then $(S, T) = \kappa^{\searrow}(M)$, where

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

We have

$$\widetilde{e}_3 M = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

and $\kappa^{\searrow}(\widetilde{e}_3 M) = \widetilde{e}_3(S,T)$.

Next, let us consider

$$\mathcal{T}^{\nwarrow}_{r\times(n-r)} = \bigsqcup_{\ell(\lambda)\leqslant r,n-r} SST_{[\bar{r}]}(\lambda) \times SST_{[n]\setminus[r]}(\lambda). \tag{3.5}$$

As in $\mathcal{T}^{\searrow}_{r\times(n-r)}$, $\mathcal{T}^{\nwarrow}_{r\times(n-r)}$ is a regular $U_q(\mathfrak{g}_{I_{0,r}})$-crystal. Let us define $\widetilde{e}_0$, $\widetilde{f}_0$ on $\mathcal{T}^{\nwarrow}_{r\times(n-r)}$ to make $\mathcal{T}^{\nwarrow}_{r\times(n-r)}$ a $U_q(\mathfrak{g}_{I_r})$-crystal. Let $(S,T) \in \mathcal{T}^{\nwarrow}_{r\times(n-r)}$ be given. For $k \geqslant 1$, let s_k and t_k be the entries in the bottom of the k-th columns of S and T (enumerated from the left), respectively. We put

$$\sigma_k = \begin{cases} -, & \text{if the } k\text{-th column is empty,} \\ -, & \text{if } s_k < \bar{1} \text{ and } t_k < n, \\ +, & \text{if } s_k = \bar{1} \text{ and } t_k = n, \\ \cdot, & \text{otherwise.} \end{cases}$$

Let $\widetilde{\sigma}$ be the reduced sequence obtained from $\sigma = (\ldots, \sigma_2, \sigma_1)$ by the signature rule. We define $\widetilde{e}_0(S,T)$ to be the bitableaux obtained from (S,T) by adding $\boxed{\bar{1}}$ and $\boxed{n}$ to the bottom of the columns of S and T corresponding to the right-most $-$ in $\widetilde{\sigma}$. We define $\widetilde{f}_0(S,T)$ to be the bitableaux obtained from (S,T) by removing $\boxed{\bar{1}}$ and $\boxed{n}$ in the columns of S and T corresponding to the left-most $+$ in $\widetilde{\sigma}$. If there is no such $+$ sign, then we define $\widetilde{f}_0(S,T) = \mathbf{0}$. Note that $\widetilde{e}_0^k(S,T) \neq \mathbf{0}$ for all $k \geqslant 1$.

We put $\varphi_0(S,T) = \max\{k \mid \widetilde{f}_0^k(S,T) \neq \mathbf{0}\}$ and $\varepsilon_0(S,T) = \varphi_0(S,T) - \langle \mathrm{wt}(S,T), h_0\rangle$. Then $\mathcal{T}^{\nwarrow}_{r\times(n-r)}$ is a $U_q(\mathfrak{g}_{I_r})$-crystal with respect to wt, ε_i, φ_i and $\widetilde{e}_i, \widetilde{f}_i$ ($i \in I_r$).

Define

$$\kappa^{\nwarrow} : \mathcal{M}_{r\times(n-r)} \to \mathcal{T}^{\nwarrow}_{r\times(n-r)} \tag{3.6}$$

by $\kappa^{\nwarrow}(M) = (\mathbf{P}(M)^{\nwarrow}, \mathbf{Q}(M)^{\nwarrow}) = (\mathbf{P}(M), \mathbf{Q}(M))$. By the same argument as in [5, Theorem 3.6], we have the following.

Proposition 3.5. *$\kappa^{\nwarrow}$ is an isomorphism of $U_q(\mathfrak{g}_{I_r})$-crystals.*

3.4. Main theorem

For $M \in \mathcal{M}_{r\times(n-r)}$ with $M = M(\mathbf{a},\mathbf{b})$, let $\ell(M)$ be the maximal length of weakly decreasing subwords of $\mathbf{a}$. For $s \geqslant 1$, let

$$\mathcal{M}^s_{r\times(n-r)} = \bigl\{ M \in \mathcal{M}_{r\times(n-r)} \bigm| \ell(M) \leqslant s \bigr\}. \tag{3.7}$$

Note that $\ell(M)$ is the number of columns in $\mathbf{P}(M)$ or $\mathbf{Q}(M)$ (cf. [19, §3.1]). We regard $\mathcal{M}^s_{r\times(n-r)}$ as a subcrystal of $\mathcal{M}_{r\times(n-r)}$ and define a $U'_q(A^{(1)}_{n-1})$-crystal

$$\mathcal{B}^{r,s} = \mathcal{M}^s_{r\times(n-r)} \otimes T_{s\omega_r}. \tag{3.8}$$

Lemma 3.6. *$\mathcal{B}^{r,s}$ is a regular $U'_q(A^{(1)}_{n-1})$-crystal that is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal.*

Proof. When restricted to $\mathcal{M}^s_{r\times(n-r)}$, we have the following bijections

$$\kappa^{\searrow} : \mathcal{M}^s_{r\times(n-r)} \to \mathcal{T}^{\searrow,s}_{r\times(n-r)}, \qquad \kappa^{\nwarrow} : \mathcal{M}^s_{r\times(n-r)} \to \mathcal{T}^{\nwarrow,s}_{r\times(n-r)}, \tag{3.9}$$

where

$$\mathcal{T}^{\searrow,s}_{r\times(n-r)} = \bigsqcup_{\substack{\ell(\lambda)\leqslant r,n-r\\ \lambda_1\leqslant s}} SST_{[\bar{r}]}\big(\lambda^\pi\big) \times SST_{[n]\setminus[r]}\big(\lambda^\pi\big),$$

$$\mathcal{T}^{\nwarrow,s}_{r\times(n-r)} = \bigsqcup_{\substack{\ell(\lambda)\leqslant r,n-r\\ \lambda_1\leqslant s}} SST_{[\bar{r}]}(\lambda) \times SST_{[n]\setminus[r]}(\lambda).$$

Since $\mathcal{T}^{\searrow,s}_{r\times(n-r)}$ (respectively $\mathcal{T}^{\nwarrow,s}_{r\times(n-r)}$) can be viewed as a subcrystal of $\mathcal{T}^{\searrow}_{r\times(n-r)}$ (respectively $\mathcal{T}^{\nwarrow}_{r\times(n-r)}$), $\kappa^{\searrow}$ (respectively $\kappa^{\nwarrow}$) is an isomorphism of $U_q(\mathfrak{g}_{I_0})$ (respectively $U_q(\mathfrak{g}_{I_r})$)-crystals.

First we claim that $\mathcal{T}^{\searrow,s}_{r\times(n-r)} \otimes T_{s\omega_r}$ is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal. Recall that $\mathbf{B}(s\omega_r)$ can be identified with $SST_{[n]}((s^r))$ [4].

Let $(S,T) \in \mathcal{T}^{\searrow,s}_{r\times(n-r)}$ be given where $\mathrm{sh}(S) = \mathrm{sh}(T) = \lambda^\pi$ for some $\lambda \in \mathscr{P}$ with $\lambda_1 \leqslant s$. Consider an isomorphism of $U_q(\mathfrak{g}_{\{1,\dots,r-1\}})$-crystals,

$$\varsigma : SST_{[\bar{r}]}\big(\lambda^\pi\big) \otimes T_{s\omega_r} \to SST_{[r]}\big(\lambda^c\big),$$

where $\lambda^c = (s^r) \setminus \lambda^\pi = (s-\lambda_r, \dots, s-\lambda_1)$ is a rectangular complement of λ^π in (s^r) (see [21, Lemma 5.8] for an explicit description of ς, which is given as σ^s). Let $S^c = \varsigma(S \otimes t_{s\omega_r})$ and let U be the semistandard tableau in $SST_{[n]}((s^r))$ obtained by gluing S^c and T. Therefore, the map sending $(S,T) \otimes t_{s\omega_r}$ to U defines a weight preserving bijection (with the same notation)

$$\varsigma : \mathcal{T}^{\searrow,s}_{r\times(n-r)} \otimes T_{s\omega_r} \to SST_{[n]}\big(\big(s^r\big)\big). \tag{3.10}$$

By definition, it is straightforward to check that ς commutes with $\widetilde{e}_r$ and $\widetilde{f}_r$, which therefore implies that it is an isomorphism of $U_q(\mathfrak{g}_{I_0})$-crystals.

Next consider $\mathcal{T}^{\nwarrow,s}_{r\times(n-r)} \otimes T_{s\omega_r} = \mathcal{T}^{\nwarrow,s}_{r\times(n-r)} \otimes T_{-s\omega'_0}$. We claim that $\mathcal{T}^{\nwarrow,s}_{r\times(n-r)} \otimes T_{s\omega_r}$ is isomorphic to $\mathbf{B}(-s\omega'_0)$ as a $U_q(\mathfrak{g}_{I_r})$-crystal. Since $\mathbf{B}(-s\omega'_0) = \mathbf{B}(s\omega'_t)$ where $t \equiv 2r \pmod n$, $\mathbf{B}(-s\omega'_0)$ can be identified with $SST_{[n]+r}((s^r))$.

Let $(S,T) \in \mathcal{T}^{\nwarrow,s}_{r\times(n-r)}$ be given where $\mathrm{sh}(S) = \mathrm{sh}(T) = \lambda$ for some $\lambda \in \mathscr{P}$ with $\lambda_1 \leqslant s$. By modifying the bijection in [21, Lemma 5.8] (exchanging $k^\vee$ and k), we have an isomorphism of $U_q(\mathfrak{g}_{\{1,\dots,r-1\}})$-crystals,

$$\overline{\varsigma} : SST_{[\bar{r}]}(\lambda) \otimes T_{s\omega_r} \to SST_{[r]}\big(\big(s^r\big)/\lambda\big).$$

Let $\overline{S}^c = \overline{\varsigma}(S \otimes t_{s\omega_r})$ and let U be the semistandard tableau in $SST_{[n]+r}((s^r))$ obtained by gluing $\overline{S}^c$ and T. Then the map sending $(S,T) \otimes t_{s\omega_r}$ to U defines a weight preserving bijection (with the same notation)

$$\overline{\varsigma} : \mathcal{T}^{\nwarrow,s}_{r\times(n-r)} \otimes T_{s\omega_r} \to SST_{[n]+r}\big(\big(s^r\big)\big). \tag{3.11}$$

As in (3.10), $\overline{\varsigma}$ commutes with $\widetilde{e}_0$ and $\widetilde{f}_0$ and it is an isomorphism of $U_q(\mathfrak{g}_{I_r})$-crystals.

Now, for a proper subset $J \subset I$ with $|J| \leqslant 2$, we have $J \subset I_0$ or $J \subset I_r$ or $J \subset \{0, r\}$. By (3.10) and (3.11), $\mathcal{B}^{r,s}$ is a crystal of an integrable $U_q(\mathfrak{g}_J)$-module. Hence it is a regular $U'_q(A^{(1)}_{n-1})$-crystal. □

Example 3.7. Assume that $n = 6$ and $r = 3$. Consider

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \in \mathcal{M}^4_{3\times 3}.$$

Then we have

$$\mathbf{P}(M)^{\searrow} = \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & \overline{3} & \overline{2} & \overline{2} \\ \hline \overline{3} & \overline{2} & \overline{1} & \overline{1} \\ \hline \end{array}, \qquad \mathbf{Q}(M)^{\searrow} = \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 4 & 4 & 4 \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array}.$$

Note that as an element in a $U_q(A_2)$-crystal, $\mathbf{P}(M)^{\searrow}$ is equivalent to

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

By gluing it with $\mathbf{Q}(M)^{\searrow}$, we have

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 4 & 4 & 4 \\ \hline 5 & 5 & 5 & 6 \\ \hline \end{array} \in \mathbf{B}(4\omega_3),$$

which is equivalent to $M \otimes t_{4\omega_3} \in \mathcal{B}^{3,4}$ as an element in a $U_q(\mathfrak{g}_{I_0})$ $(= U_q(A_5))$-crystal. If we view $M \in \mathcal{M}^5_{4\times 3}$, then $M \otimes t_{5\omega_3} \in \mathcal{B}^{3,5}$ corresponds to

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 & 4 \\ \hline 3 & 5 & 5 & 5 & 6 \\ \hline \end{array} \in \mathbf{B}(5\omega_3).$$

On the other hand, we have

$$\mathbf{P}(M)^{\nwarrow} = \begin{array}{|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{2} & \overline{2} \\ \hline \overline{2} & \overline{1} & \overline{1} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}, \qquad \mathbf{Q}(M)^{\nwarrow} = \begin{array}{|c|c|c|c|}\hline 4 & 4 & 4 & 6 \\ \hline 5 & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

Note that as an element in a $U_q(A_2)$-crystal, $\mathbf{P}(M)^{\nwarrow}$ is equivalent to

$$\begin{array}{|c|c|c|c|}\cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array}.$$

By gluing it with $\mathbf{Q}(M)^{\nwarrow}$, we have

$$\begin{array}{|c|c|c|c|}\hline 4 & 4 & 4 & 6 \\ \hline 5 & 5 & 5 & 1 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array} \in \mathbf{B}\big(-4\omega_0'\big) \cong \mathbf{B}\big(4\omega_0'\big),$$

which is equivalent to $M \otimes t_{4\omega_3} \in \mathcal{B}^{3,4}$ as an element in a $U_q(\mathfrak{g}_{I_3})$ $(= U_q(A_5))$-crystal.

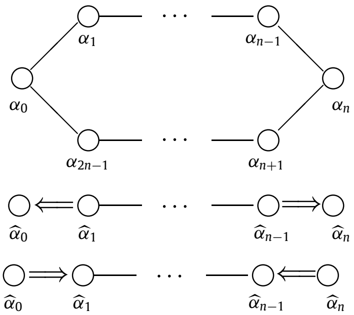
Theorem 3.8. *Let $\mathbf{B}^{r,s}$ be the KR crystal of type $A^{(1)}_{n-1}$ for $1 \leqslant r \leqslant n-1$ and $s \geqslant 1$. Then as a $U_q'(A^{(1)}_{n-1})$-crystal, we have $\mathcal{B}^{r,s} \cong \mathbf{B}^{r,s}$.*

Proof. Note that $\mathbf{B}^{r,s}$ is isomorphic to $\mathbf{B}(s\omega_r)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal [7]. Then it follows from Lemmas 2.1 and 3.6 that $\mathcal{B}^{r,s} \cong \mathbf{B}^{r,s}$. $\square$

4. Classically irreducible KR crystals of type $D^{(2)}_{n+1}$ and $C^{(1)}_n$

4.1. Affine algebras of type $D^{(2)}_{n+1}$ and $C^{(1)}_n$

Assume that $\mathfrak{g} = A^{(1)}_{2n-1}$ $(n \geqslant 2)$ with $I = \{0, 1, \ldots, 2n-1\}$ and the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, and $\widehat{\mathfrak{g}} = D^{(2)}_{n+1}$ or $C^{(1)}_n$ with $\widehat{I} = \{0, \ldots, n\}$ and the Cartan datum $(\widehat{A}, \widehat{P}^\vee, \widehat{P}, \widehat{\Pi}^\vee, \widehat{\Pi})$.



$A^{(1)}_{2n-1}$:

$D^{(2)}_{n+1}$:

$C^{(1)}_n$:

Throughout this section, we assume that $\epsilon \in \{1,2\}$ and $\widehat{\mathfrak{g}} = D^{(2)}_{n+1}$ (respectively $\widehat{\mathfrak{g}} = C^{(1)}_n$) when $\epsilon = 1$ (respectively $\epsilon = 2$). Put $\widehat{I}_r = \widehat{I} \setminus \{r\}$ $(r = 0, n)$ and $\widehat{I}_{0,n} = \widehat{I}_0 \cap \widehat{I}_n$. Note that $\widehat{\mathfrak{g}}_{\widehat{I}_0} \cong \widehat{\mathfrak{g}}_{\widehat{I}_n} = B_n$ (respectively C_n) when $\epsilon = 1$ (respectively $\epsilon = 2$) and $\widehat{\mathfrak{g}}_{\widehat{I}_{0,n}} = A_{n-1}$. We may assume that

$$\begin{aligned}
&\widehat{P}^\vee = \mathbb{Z}h_0 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d \subset P^\vee,\\
&\widehat{P} = \left\{\lambda \;\middle|\; \frac{1}{\epsilon}\langle\lambda, h_i\rangle \in \mathbb{Z}\ (i = 0, n),\ \langle\lambda, h_i\rangle = \langle\lambda, h_{2n-i}\rangle\ (i \in \widehat{I}_{0,n})\right\} \subset P,\\
&\widehat{\Pi}^\vee = \{\widehat{h}_i = h_i\ (i \in \widehat{I})\} \subset \Pi^\vee,\\
&\widehat{\Pi} = \{\widehat{\alpha}_i = \epsilon\alpha_i\ (i = 0, n),\ \widehat{\alpha}_i = \alpha_i + \alpha_{2n-i}\ (i \in \widehat{I}_{0,n})\} \subset \Pi.
\end{aligned}$$

The classical weight lattice of $\widehat{\mathfrak{g}}$ is $\widehat{P}_{\mathrm{cl}} = \bigoplus_{i\in\widehat{I}} \mathbb{Z}\widehat{\Lambda}_i$ and its dual classical weight lattice is $(\widehat{P}_{\mathrm{cl}})^\vee = \bigoplus_{i\in\widehat{I}} \mathbb{Z}h_i$, where $\widehat{\Lambda}_i = \epsilon\Lambda_i$ for $i = 0, n$ and $\widehat{\Lambda}_i = \Lambda_i + \Lambda_{2n-i}$ for $i \in \widehat{I}_{0,n}$. Note that $\widehat{\alpha}_i = \widehat{\epsilon}_i - \widehat{\epsilon}_{i+1}$ $(i \in I_{0,n})$, where $\widehat{\epsilon}_i = \epsilon_i - \epsilon_{2n-i+1}$ for $i = 1, \ldots, n$, $\widehat{\alpha}_0 = -\epsilon\widehat{\epsilon}_1$ and $\widehat{\alpha}_n = \epsilon\widehat{\epsilon}_n$ in $\widehat{P}_{\mathrm{cl}}$. We denote the fundamental weights for $\widehat{\mathfrak{g}}_{I_0}$ by $\widehat{\omega}_i = \omega_i + \omega_{2n-i}$ for $i \in \widehat{I}_{0,n}$ and $\widehat{\omega}_n = \epsilon\omega_n$, and those for $\widehat{\mathfrak{g}}_{I_n}$ by $\widehat{\omega}'_i = \omega'_i + \omega'_{2n-i}$ for $i \in \widehat{I}_{0,n}$ and $\widehat{\omega}'_0 = \epsilon\omega'_0 = -\widehat{\omega}_n$.

4.2. Crystals of symmetric matrices

Put

$$\widehat{\mathcal{M}}_n = \{M = (m_{\bar{i}\bar{j}}) \in \mathcal{M}_{n\times n} \mid m_{\bar{i}\bar{j}} = m_{\bar{j}\bar{i}} \text{ and } \epsilon | m_{\bar{i}\bar{i}} \text{ for } i, j \in [n]\}. \tag{4.1}$$

Define

$$\widehat{e}_i = \begin{cases} (\widetilde{e}_i)^\epsilon, & \text{for } i = 0, n,\\ \widetilde{e}_i\widetilde{e}_{2n-i}, & \text{for } i \in \widehat{I}_{0,n}, \end{cases} \qquad \widehat{f}_i = \begin{cases} (\widetilde{f}_i)^\epsilon, & \text{for } i = 0, n,\\ \widetilde{f}_i\widetilde{f}_{2n-i}, & \text{for } i \in \widehat{I}_{0,n}. \end{cases}$$

Note that $\mathcal{M}_{n\times n}$ is a $U'_q(A^{(1)}_{2n-1})$-crystal with respect to wt, ε_i, φ_i and $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I)$ by Proposition 3.1. Then it is not difficult to see that $\widehat{\mathcal{M}}_n \cup \{\mathbf{0}\}$ is invariant under $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in \widehat{I}$ (cf. [5, Proposition 5.14]). For $M \in \widehat{\mathcal{M}}_n$, define $\widehat{\mathrm{wt}}(M) = \mathrm{wt}(M)$,

$$\widehat{\varepsilon}_i(M) = \begin{cases} \frac{1}{\epsilon}\varepsilon_i(M), & \text{if } i = 0, n,\\ \varepsilon_i(M), & \text{if } i \in \widehat{I}_{0,n}, \end{cases} \qquad \widehat{\varphi}_i(M) = \begin{cases} \frac{1}{\epsilon}\varphi_i(M), & \text{if } i = 0, n,\\ \varphi_i(M), & \text{if } i \in \widehat{I}_{0,n}. \end{cases}$$

Hence $\widehat{\mathcal{M}}_n$ is a $U'_q(\widehat{\mathfrak{g}})$-crystal with respect to $\widehat{\mathrm{wt}}$, $\widehat{\varepsilon}_i$, $\widehat{\varphi}_i$, $\widehat{e}_i$, $\widehat{f}_i$ $(i \in \widehat{I})$.

Consider

$$\widehat{\mathcal{T}}^{\searrow}_n = \bigsqcup_{\ell(\lambda)\leqslant n} SST_{[\bar{n}]}(\epsilon\lambda^\pi), \qquad \widehat{\mathcal{T}}^{\nwarrow}_n = \bigsqcup_{\ell(\lambda)\leqslant n} SST_{[\bar{n}]}(\epsilon\lambda), \tag{4.2}$$

where $2\lambda = (2\lambda_i)_{i\geqslant 1}$ for $\lambda = (\lambda_i)_{i\geqslant 1} \in \mathscr{P}$. They are regular $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_{0,n}})$-crystals with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in \widehat{I}_{0,n}$). Here $\mathrm{wt}(T) = -\sum_{i\in[n]} m_{\bar{i}}\widehat{\epsilon}_i$, for $T \in \widehat{\mathcal{T}}_n^{\searrow}$ or $\widehat{\mathcal{T}}_n^{\nwarrow}$, where $m_{\bar{i}}$ is the number of $\bar{i}$'s appearing in T.

Let us define $\widetilde{e}_n$, $\widetilde{f}_n$ on $\widehat{\mathcal{T}}_n^{\searrow}$ corresponding to $\widehat{\alpha}_n$ as follows: Let $T \in \widehat{\mathcal{T}}_n^{\searrow}$ be given. Suppose that $\epsilon = 1$. For $k \geqslant 1$, let t_k be the entry in the top of the k-th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \ldots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_k > \bar{n} \text{ or the } k\text{-th column is empty,} \\ -, & \text{if } t_k = \bar{n}. \end{cases}$$

Then we define $\widetilde{e}_n T$ to be the tableau obtained from T by removing $\boxed{\bar{n}}$ in the column corresponding to the right-most $-$ in $\widetilde{\sigma}$. If there is no such $-$ sign, then we define $\widetilde{e}_n T = \mathbf{0}$. We define $\widetilde{f}_n T$ to be the tableau obtained from T by adding $\boxed{\bar{n}}$ on top of the column corresponding to the left-most $+$ in $\widetilde{\sigma}$. Suppose that $\epsilon = 2$. For each $k \geqslant 1$, let (t_{2k}, t_{2k-1}) the pair of entries in the top of the $2k$-th and $(2k-1)$-st columns of T (from the right), respectively. Note that t_{2k} and t_{2k-1} are placed in the same row and $t_{2k} \leqslant t_{2k-1}$. Consider $\sigma = (\sigma_1, \sigma_2, \ldots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_{2k}, t_{2k-1} > \bar{n} \text{ or the } (2k-1)\text{-st column is empty,} \\ -, & \text{if } t_{2k} = t_{2k-1} = \bar{n}, \\ \cdot, & \text{otherwise.} \end{cases}$$

Then we define $\widetilde{e}_n T$ and $\widetilde{f}_n T$ in the same way as in $\epsilon = 1$ with $\boxed{\bar{n}}$ replaced by $\boxed{\bar{n}}\boxed{\bar{n}}$.

Hence $\widehat{\mathcal{T}}_n^{\searrow}$ is a $U_q'(\widehat{\mathfrak{g}}_{\widehat{I}_0})$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in \widehat{I}_0$), where $\varepsilon_n(T) = \max\{k \mid \widetilde{e}_n^k T \neq \mathbf{0}\}$ and $\varphi_n(T) = \varepsilon_n(T) + \langle \mathrm{wt}(T), \widehat{h}_n\rangle$.

Proposition 4.1. *The map $\widehat{\kappa}^{\searrow} : \widehat{\mathcal{M}}_n \to \widehat{\mathcal{T}}_n^{\searrow}$ given by $\widehat{\kappa}^{\searrow}(M) = \mathbf{P}(M)^{\searrow}$ is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$-crystals.*

Proof. It follows from [22, Propositions 3.5 and 6.5]. □

Next, let us define $\widetilde{e}_0$, $\widetilde{f}_0$ on $\widehat{\mathcal{T}}_n^{\nwarrow}$ corresponding to $\widehat{\alpha}_0$ as follows: Let $T \in \widehat{\mathcal{T}}_n^{\nwarrow}$ be given. Suppose that $\epsilon = 1$. For $k \geqslant 1$, let t_k be the entry in the bottom of the k-th column of T (enumerated from the left). Consider $\sigma = (\ldots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_k < \bar{1} \text{ or the } k\text{-th column is empty,} \\ +, & \text{if } t_k = \bar{1}. \end{cases}$$

Then we define $\widetilde{e}_0 T$ to be the tableau obtained from T by adding $\boxed{\bar{1}}$ to the bottom of the column corresponding to the right-most $-$ in $\widetilde{\sigma}$. We define $\widetilde{f}_0 T$ to be the tableau obtained from T by removing $\boxed{\bar{1}}$ in the column corresponding to the left-most $+$ in $\widetilde{\sigma}$. If there is no such $+$ sign, then we define $\widetilde{f}_0 T = \mathbf{0}$. Suppose that $\epsilon = 2$. For $k \geqslant 1$, let (t_{2k-1}, t_{2k}) be the pair of entries in the bottom boxes of the $(2k-1)$-st and $2k$-th columns of T (from the left), respectively. Note that t_{2k-1} and t_{2k} are placed in the same row and $t_{2k-1} \geqslant t_{2k}$. Consider $\sigma = (\ldots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_{2k-1}, t_{2k} < \bar{1} \text{ or the } (2k-1)\text{-st column is empty,} \\ +, & \text{if } t_{2k-1} = t_{2k} = \bar{1}, \\ \cdot, & \text{otherwise.} \end{cases}$$

Then we define $\widetilde{e}_n T$ and $\widetilde{f}_n T$ in the same way as in $\epsilon = 1$ with $\boxed{\bar{1}}$ replaced by $\boxed{\bar{1}}\boxed{\bar{1}}$.

Hence $\widehat{\mathcal{T}}_n^{\nwarrow}$ is a $U_q'(\widehat{\mathfrak{g}}_{\widehat{I}_n})$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ ($i \in \widehat{I}_n$), where $\varphi_0(T) = \max\{k \mid \widetilde{f}_0^k T \neq \mathbf{0}\}$ and $\varepsilon_0(T) = \varphi_0(T) - \langle \mathrm{wt}(T), \widehat{h}_0\rangle$. Then we have

Proposition 4.2. *The map $\widehat{\kappa}^{\nwarrow} : \widehat{\mathcal{M}}_n \to \widehat{\mathcal{T}}_n^{\nwarrow}$ given by $\widehat{\kappa}^{\nwarrow}(M) = \mathbf{P}(M)^{\nwarrow}$ is an isomorphism of $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_n})$-crystals.*

4.3. KR crystals $\mathbf{B}^{n,s}$

For $s \geqslant 1$, let $\widehat{\mathcal{M}}_n^s = \widehat{\mathcal{M}}_n \cap \mathcal{M}_{n\times n}^{\epsilon s}$. We regard $\widehat{\mathcal{M}}_n^s$ as a subcrystal of $\widehat{\mathcal{M}}_n$ and consider a $U_q'(\widehat{\mathfrak{g}})$-crystal

$$\mathcal{B}^{n,s} = \widehat{\mathcal{M}}_n^s \otimes T_{s\widehat{\omega}_n}. \tag{4.3}$$

Lemma 4.3. *$\mathcal{B}^{n,s}$ is a regular $U_q'(\widehat{\mathfrak{g}})$-crystal that is isomorphic to $\mathbf{B}(s\widehat{\omega}_n)$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$-crystal.*

Proof. By (3.9), we have bijections

$$\widehat{\kappa}^{\searrow} : \widehat{\mathcal{M}}_n^s \to \widehat{\mathcal{T}}_n^{\searrow,s}, \qquad \widehat{\kappa}^{\nwarrow} : \widehat{\mathcal{M}}_n^s \to \widehat{\mathcal{T}}_n^{\nwarrow,s}, \tag{4.4}$$

where $\widehat{\mathcal{T}}_n^{\searrow,s}$ (respectively $\widehat{\mathcal{T}}_n^{\nwarrow,s}$) is the set of tableaux $T \in \widehat{\mathcal{T}}_n^{\searrow}$ of $\mathrm{sh}(T) = \epsilon\lambda^{\pi}$ (respectively $\epsilon\lambda$) with $\lambda \subset (\epsilon s^n)$. We may regard $\widehat{\mathcal{T}}_n^{\searrow,s}$ and $\widehat{\mathcal{T}}_n^{\nwarrow,s}$ as subcrystals of $\widehat{\mathcal{T}}_n^{\searrow}$ and $\widehat{\mathcal{T}}_n^{\nwarrow}$, respectively. Then by Propositions 4.1 and 4.2, the bijections in (4.4) are isomorphisms of $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$ and $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_n})$-crystals, respectively. On the other hand, by [5, Remark 5.16] (or as a special case of [22, Theorem 6.4] when λ is the empty partition), we have $\mathcal{B}^{n,s} \cong \widehat{\mathcal{T}}_n^{\searrow,s} \otimes T_{s\widehat{\omega}_n} \cong \mathbf{B}(s\widehat{\omega}_n)$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$-crystal, and $\mathcal{B}^{n,s} \cong \widehat{\mathcal{T}}_n^{\nwarrow,s} \otimes T_{s\widehat{\omega}_n} \cong \mathbf{B}(-s\widehat{\omega}_0') \cong \mathbf{B}(s\widehat{\omega}_0')$ as a $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_n})$-crystal. This implies that $\mathcal{B}^{n,s}$ is regular. $\square$

Theorem 4.4. *Let $\mathbf{B}^{n,s}$ be the KR crystal of type $\widehat{\mathfrak{g}}$ for $s \geqslant 1$. Then as a $U_q'(\widehat{\mathfrak{g}})$-crystal, we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$.*

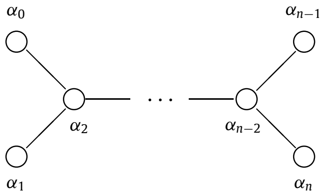
Proof. Since $\mathbf{B}^{n,s} \cong \mathbf{B}(s\widehat{\omega}_n)$ as an $U_q(\widehat{\mathfrak{g}}_{\widehat{I}_0})$-crystal (cf. [14]), we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$ by Lemmas 2.1 and 4.3. $\square$

5. Classically irreducible KR crystals of type $D_n^{(1)}$

5.1. Affine algebra of type $D_n^{(1)}$

Assume that $\mathfrak{g} = D_n^{(1)}$ ($n \geqslant 4$) with $I = \{0, 1, \ldots, n\}$. Put $I_r = I \setminus \{r\}$ ($r = 0, n$), and $I_{0,n} = I_0 \cap I_n$. Note that $\mathfrak{g}_{I_0} \cong \mathfrak{g}_{I_n} = D_n$ and $\mathfrak{g}_{I_{0,n}} = A_{n-1}$.

$D_n^{(1)}$:



Let $\epsilon_1 = \Lambda_1 - \Lambda_0$, $\epsilon_2 = \Lambda_2 - \Lambda_1 - \Lambda_0$, $\epsilon_k = \Lambda_k - \Lambda_{k-1}$ for $k = 3, \ldots, n-2$, $\epsilon_{n-1} = \Lambda_{n-1} + \Lambda_n - \Lambda_{n-2}$ and $\epsilon_n = \Lambda_n - \Lambda_{n-1}$. Then $\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ forms a weight lattice of $\mathfrak{g}_{I_0}$. Note that $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in I_{0,n}$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$, and $\alpha_0 = -\epsilon_1 - \epsilon_2$ in P_{cl}. The fundamental weights for $\mathfrak{g}_{I_0}$ are $\omega_i = \sum_{k=1}^i \epsilon_k$ for $i = 1, \ldots, n-2$, $\omega_{n-1} = (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2$ and $\omega_n = (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2$. We denote the fundamental weights for $\mathfrak{g}_{I_n}$ by ω_i' for $i \in I_n$, where $\omega_i' = \omega_i$ for $i \in I_{0,n}$ and $\omega_0' = -\omega_n$.

5.2. Young tableau descriptions of $\mathbf{B}(s\omega_n)$ *and* $\mathbf{B}(-s\omega_0')$

Consider

$$\mathcal{T}_n^{\searrow} = \bigsqcup_{\substack{\lambda_i' :\ \text{even} \\ \ell(\lambda) \leqslant n}} SST_{[\bar{n}]}\left(\lambda^{\pi}\right). \tag{5.1}$$

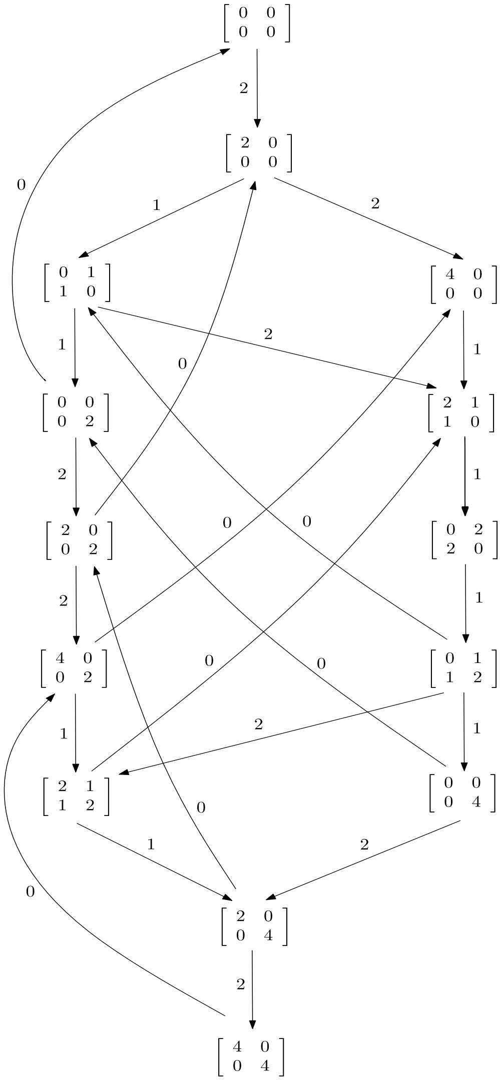


Fig. 2. The KR crystal graph $\mathbf{B}^{2,2}$ of type $C_2^{(1)}$.

It is a regular $U_q(\mathfrak{g}_{I_{0,n}})$-crystal with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I_{0,n}$, where $\mathrm{wt}(T) = -\sum_{i\in[n]} m_{\bar{i}}\epsilon_i$ ($m_{\bar{i}}$ is the number of $\bar{i}$'s in T) for $T \in \mathcal{T}_n^{\searrow}$.

Let $T \in \mathcal{T}_n^{\searrow}$ be given. For $k \geqslant 1$, let t_k be the entry in the top of the k-th column of T (enumerated from the right). Consider $\sigma = (\sigma_1, \sigma_2, \ldots)$, where

$$\sigma_k = \begin{cases} +, & \text{if } t_k > \overline{n-1} \text{ or the } k\text{-th column is empty},\\ -, & \text{if the } k\text{-th column has both } \overline{n-1} \text{ and } \bar{n} \text{ as its entries},\\ \cdot, & \text{otherwise}. \end{cases}$$

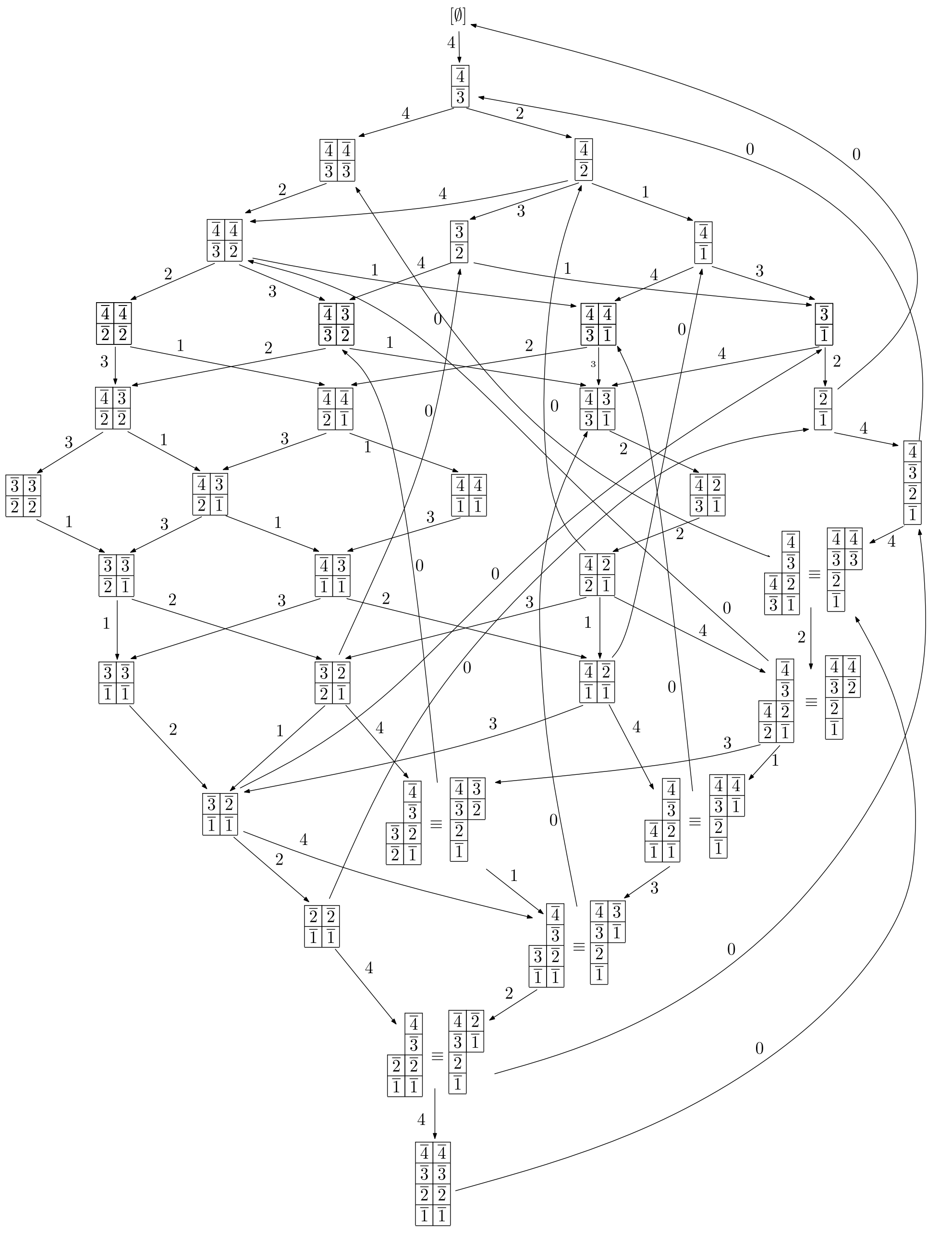


Fig. 3. The KR crystal graph $\mathbf{B}^{4,2}$ of type $D_4^{(1)}$. Here $\equiv$ denotes the Knuth equivalence or $U_q(A_3)$-crystal equivalence.

Define $\widetilde{e}_n T$ and $\widetilde{f}_n T$ as in the case of $\widehat{\mathcal{T}}_n^{\searrow}$ (see Section 4) with $\boxed{\bar{n}}$ replaced by $\begin{array}{|c|}\hline \bar{n} \\ \hline \overline{n-1} \\ \hline \end{array}$. Then $\mathcal{T}_n^{\searrow}$ is a $U_q(\mathfrak{g}_{I_0})$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I_0)$, where $\varepsilon_n(T) = \max\{k \mid \widetilde{e}_n^k T \neq \mathbf{0}\}$ and $\varphi_n(T) = \varepsilon_n(T) + \langle \mathrm{wt}(T), h_n \rangle$.

For $s \geqslant 1$, let $\mathcal{T}_n^{\searrow,s}$ be the set of tableaux $T \in \mathcal{T}_n^{\searrow}$ of shape λ^π with $\lambda \subset (s^n)$, and consider $\mathcal{T}_n^{\searrow,s}$ as a subcrystal of $\mathcal{T}_n^{\searrow}$.

Lemma 5.1. *$\mathcal{T}_n^{\searrow,s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal.*

Proof. First we prove the case when $s = 1$. Recall that $\mathbf{B}(\omega_n)$ is the crystal of the spin representation of $U_q(\mathfrak{g}_{I_0})$, and by [4] it can be identified with $\{v = (i_1, \ldots, i_n) \mid i_k = \pm 1, i_1 \cdots i_n = 1\}$, where $\mathrm{wt}(v) = \frac{1}{2}\sum_{k=1}^n i_k \epsilon_k$ and

$$\tilde{e}_k v = \begin{cases} (\ldots, -i_k, -i_{k+1}, \ldots), & \text{if } k \in I_{0,n} \text{ and } (i_k, i_{k+1}) = (-1, 1), \\ (\ldots, -i_{n-1}, -i_n), & \text{if } k = n \text{ and } (i_{n-1}, i_n) = (-1, -1), \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$
$$\tilde{f}_k v = \begin{cases} (\ldots, -i_k, -i_{k+1}, \ldots), & \text{if } k \in I_{0,n} \text{ and } (i_k, i_{k+1}) = (1, -1), \\ (\ldots, -i_{n-1}, -i_n), & \text{if } k = n \text{ and } (i_{n-1}, i_n) = (1, 1), \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

Note that $\mathcal{T}_n^{\searrow,1}$ is the set of semistandard tableaux with a single column of even length no more than n. Define $\rho : \mathcal{T}_n^{\searrow,1} \otimes T_{\omega_n} \to \mathbf{B}(\omega_n)$ by $\rho(T \otimes t_{\omega_n}) = (i_1, \ldots, i_n)$, where $i_k = -1$ if and only if $\bar{k}$ appears in T. Note that the empty tableau is mapped to $(1, \ldots, 1)$ of weight ω_n. Then ρ is an isomorphism of $U_q(\mathfrak{g}_{I_0})$-crystals.

For $s \geqslant 1$, consider the map

$$\iota_s : \mathcal{T}_n^{\searrow,s} \otimes T_{s\omega_n} \to \left(\mathcal{T}_n^{\searrow,1}\right)^{\otimes s} \otimes T_{s\omega_n} \cong \left(\mathcal{T}_n^{\searrow,1} \otimes T_{\omega_n}\right)^{\otimes s} \cong \mathbf{B}(\omega_n)^{\otimes s},$$

where for $\iota_s(T \otimes t_{s\omega_n}) = T^1 \otimes \cdots \otimes T^s \otimes t_{s\omega_n}$ (T^i is the i-th column of T from the right). Then it is straightforward to check that ι_s is a strict embedding of $U_q(\mathfrak{g}_{I_0})$-crystals, and its image is isomorphic to the connected component of $\emptyset^{\otimes s} \otimes t_{s\omega_n}$, where $\emptyset$ is the empty tableau. Since $\emptyset^{\otimes s} \otimes t_{s\omega_n}$ is a highest weight element of weight $s\omega_n$ in $\mathbf{B}(\omega_n)^{\otimes s}$, $\mathcal{T}_n^{\searrow,s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(s\omega_n)$. $\square$

Next, consider

$$\mathcal{T}_n^{\nwarrow} = \bigsqcup_{\substack{\lambda_i' : \text{ even} \\ \ell(\lambda) \leqslant n}} SST_{[\bar{n}]}(\lambda). \tag{5.2}$$

As in $\mathcal{T}_n^{\searrow}$, it is a regular $U_q(\mathfrak{g}_{I_{0,n}})$-crystal. Let $T \in \mathcal{T}_n^{\nwarrow}$ be given. For $k \geqslant 1$, let t_k be the entry in the bottom of the k-th column of T (enumerated from the left). Consider $\sigma = (\ldots, \sigma_2, \sigma_1)$, where

$$\sigma_k = \begin{cases} -, & \text{if } t_k < \bar{2} \text{ or the } k\text{-th column is empty}, \\ +, & \text{if the } k\text{-th column has both } \bar{1} \text{ and } \bar{2} \text{ as its entries}, \\ \cdot, & \text{otherwise}. \end{cases}$$

Define $\tilde{e}_0 T$ and $\tilde{f}_0 T$ as in the case of $\widehat{\mathcal{T}}_n^{\nwarrow}$ (see Section 4) with $\boxed{\bar{1}}$ replaced by $\begin{array}{|c|}\hline \bar{2} \\ \hline \bar{1} \\ \hline \end{array}$. Then $\mathcal{T}_n^{\nwarrow}$ is a $U_q(\mathfrak{g}_{I_n})$-crystal with respect to wt, ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ ($i \in I_n$), where $\varphi_0(T) = \max\{k \mid \tilde{f}_0^k T \neq \mathbf{0}\}$ and $\varepsilon_0(T) = \varphi_0(T) - \langle \mathrm{wt}(T), h_0 \rangle$.

For $s \geqslant 1$, let $\mathcal{T}_n^{\nwarrow,s}$ be the set of tableaux $T \in \mathcal{T}_n^{\nwarrow}$ of shape λ with $\lambda \subset (s^n)$ consider $\mathcal{T}_n^{\nwarrow,s}$ as a subcrystal of $\mathcal{T}_n^{\nwarrow}$.

Lemma 5.2. *$\mathcal{T}_n^{\nwarrow,s} \otimes T_{s\omega_n}$ is isomorphic to $\mathbf{B}(-s\omega_0')$ as a $U_q(\mathfrak{g}_{I_n})$-crystal.*

Proof. The proof is similar to that of Lemma 5.1. $\square$

5.3. KR crystals $\mathbf{B}^{n,s}$

For a semistandard tableau T of skew shape, let $[T]$ denote the equivalence class of T with respect to Knuth equivalence. For $n \geqslant 4$, let

$$\mathcal{T}_n = \{[T] \mid T \in \mathcal{T}_n^{\searrow}\} = \{[T] \mid T \in \mathcal{T}_n^{\nwarrow}\}. \tag{5.3}$$

Recall that under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I_{0,n}$, any $T' \in [T]$ generates the same crystal as T. Hence, $\mathcal{T}_n$ has a well-defined $U_q(\mathfrak{g}_{I_{0,n}})$-crystal structure. Now, for $i = 0, n$ and $x = e, f$, we define

$$\widetilde{x}_i[T] = \begin{cases} [\widetilde{x}_0 T^{\nwarrow}], & \text{if } i = 0, \\ [\widetilde{x}_n T^{\searrow}], & \text{if } i = n, \end{cases} \tag{5.4}$$

where we assume that $[\mathbf{0}] = \mathbf{0}$. Put

$$\begin{aligned} &\mathrm{wt}([T]) = \mathrm{wt}(T), \qquad \varepsilon_i([T]) = \varepsilon_i(T), \qquad \varphi_i([T]) = \varphi_i(T) \quad (i \in I_{0,n}), \\ &\varepsilon_n([T]) = \varepsilon_n(T^{\searrow}), \qquad \varphi_n([T]) = \varphi_n(T^{\searrow}), \\ &\varepsilon_0([T]) = \varepsilon_n(T^{\nwarrow}), \qquad \varphi_0([T]) = \varphi_n(T^{\nwarrow}). \end{aligned} \tag{5.5}$$

Then, $\mathcal{T}_n$ is a $U_q'(\mathfrak{g})$-crystal with respect to wt, ε_i, φ_i, $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in I)$.

Now, for $s \geqslant 1$, we put $\mathcal{T}_n^s = \{[T] \mid T \in \mathcal{T}_n^{\searrow,s}\} = \{[T] \mid T \in \mathcal{T}_n^{\nwarrow,s}\}$, which is a subcrystal of $\mathcal{T}_n$, and then define

$$\mathcal{B}^{n,s} = \mathcal{T}_n^s \otimes T_{s\omega_n}. \tag{5.6}$$

Lemma 5.3. *$\mathcal{B}^{n,s}$ is a regular $U_q'(\mathfrak{g})$-crystal that is isomorphic to $\mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal.*

Proof. By definition of $\mathcal{B}^{n,s}$ and Lemmas 5.1 and 5.2, we have $\mathcal{B}^{n,s} \cong \mathcal{T}_n^{\searrow,s} \otimes T_{s\omega_n} \cong \mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal, and $\mathcal{B}^{n,s} \cong \mathcal{T}_n^{\nwarrow,s} \otimes T_{s\omega_n} \cong \mathbf{B}(-s\omega_0')$ as a $U_q(\mathfrak{g}_{I_n})$-crystal. This implies that $\mathcal{B}^{n,s}$ is regular. □

Theorem 5.4. *Let $\mathbf{B}^{n,s}$ be the KR crystal of type $\mathfrak{g} = D_n^{(1)}$ for $s \geqslant 1$. Then as a $U_q'(\mathfrak{g})$-crystal, we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$.*

Proof. Since $\mathbf{B}^{n,s} \cong \mathbf{B}(s\omega_n)$ as a $U_q(\mathfrak{g}_{I_0})$-crystal (cf. [14]), we have $\mathcal{B}^{n,s} \cong \mathbf{B}^{n,s}$ by Lemmas 2.1 and 5.3. □

Remark 5.5. One may expect a matrix realization of $\mathbf{B}^{n,s}$ as in the cases of $A_{n-1}^{(1)}$, $D_{n+1}^{(2)}$ and $C_n^{(1)}$. In fact, there is a variation of RSK map which is a bijection from $\mathcal{T}_n$ to a set of symmetric non-negative integral matrices with trace zero and also an isomorphism of $U_q(A_{n-1})$-crystals (see [21, Proposition 3.13] when $m = 0$). But there does not seem to be a natural extension to an isomorphism of $U_q(D_n)$-crystals (and hence $U_q(D_n^{(1)})$-crystals).

5.4. KR crystals $\mathbf{B}^{n-1,s}$

Let us give a combinatorial description of $\mathbf{B}^{n-1,s}$ to complete the list of KR crystals associated to exceptional nodes in the Dynkin diagram of classical affine type. In this case, we put

$$\mathcal{B}^{n-1,s} = \widetilde{\mathcal{T}}_n^s \otimes T_{s\omega_n}, \tag{5.7}$$

where $\widetilde{\mathcal{T}}_n^s$ is defined in the same way as $\mathcal{T}_n^s$ in Section 5.3 with λ_i' being odd for all i (see (5.2)). Then

$$\mathcal{B}^{n-1,s} \cong \mathbf{B}^{n-1,s}, \tag{5.8}$$

where $\mathbf{B}^{n-1,s}$ is the KR crystal isomorphic to $\mathbf{B}(s\omega_{n-1})$ as a $U_q(\mathfrak{g}_{I_0})$-crystal. The proof is almost identical to that of Theorem 5.4. So we leave the details to the reader.

6. Remarks on $\widetilde{e}_0$ and $\widetilde{f}_0$

6.1. Lusztig involution

Let η be the involutive automorphism of $U_q(A_{n-1})$ given by $\eta(e_i) = f_{n-i}$, $\eta(f_i) = e_{n-i}$, and $\eta(q^{h_i}) = q^{-h_{n-i}}$ $(i = 1, \ldots, n-1)$. Let w_0 be the longest element in the Weyl group of A_{n-1}. Recall that $w_0(\alpha_i) = -\alpha_{n-i}$ for $i = 1, \ldots, n-1$. Let B be a crystal of a finite dimensional $U_q(A_{n-1})$-module. Then by [23, Proposition 21.1.2], we have an induced map

$$\eta : B \to B \tag{6.1}$$

such that $\eta^2(b) = b$, $\mathrm{wt}(\eta(b)) = w_0(\mathrm{wt}(b))$, $\eta(\widetilde{e}_i(b)) = \widetilde{f}_{n-i}\eta(b)$ and $\eta(\widetilde{f}_i b) = \widetilde{e}_{n-i}\eta(b)$ for $b \in B$ and $i = 1, \ldots, n-1$. Similarly, one can define η on a crystal of a finite dimensional $U_q(A_{m-1} \oplus A_{n-1})$-module for $m, n \geqslant 2$.

In [24], it is shown that η coincides with the Schützenberger's involution (see e.g. [19]) when $B = SST_{[n]}(\lambda)$ for $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leqslant n$. Indeed, for $T \in SST_{[n]}(\lambda)$, let T' be the tableau obtained by 180°-rotation of T and replacing i with $n-i+1$. Then $\eta(T) = (T')^{\nwarrow}$.

Based on our combinatorial descriptions, we have the following characterization of $\widetilde{e}_0$ and $\widetilde{f}_0$ on classically irreducible KR crystals in terms of η on an underlying classical crystal of type A.

Proposition 6.1. *Let $\mathbf{B}^{r,s}$ be a classically irreducible KR crystal of type $\mathfrak{g}$ $(s \geqslant 1)$ (that is, for $r = 1, \ldots, n-1$ when $\mathfrak{g} = A^{(1)}_{n-1}$, $r = n$ when $\mathfrak{g} = D^{(2)}_{n+1}$, $C^{(1)}_n$, $r = n, n-1$ when $\mathfrak{g} = D^{(1)}_n$, and $s \geqslant 1$). Let η denote the involution (6.1) on $\mathbf{B}^{r,s}$ as a crystal of type $\mathfrak{g}_J$ with $J = I \setminus \{0, r\}$. Then we have on $\mathbf{B}^{r,s}$*

$$\widetilde{e}_0 = \eta \circ \widetilde{f}_r \circ \eta, \qquad \widetilde{f}_0 = \eta \circ \widetilde{e}_r \circ \eta.$$

Proof. We assume that $x = e$ (respectively f) when $y = f$ (respectively e) throughout the proof.

Case 1. $\mathbf{B}^{r,s}$ of type $A^{(1)}_{n-1}$ for $r = 1, \ldots, n-1$ and $s \geqslant 1$. Note that $\mathfrak{g}_J = A_{r-1} \oplus A_{n-r-1}$. Consider $\pi : \mathcal{M}_{r\times(n-r)} \to \mathcal{M}_{r\times(n-r)}$, where $\pi(M)$ is obtained by 180°-rotation of M. By definition of $\widetilde{e}_0$ and $\widetilde{f}_0$ on $\mathcal{M}_{r\times(n-r)}$, we have $\widetilde{x}_0 = \pi \circ \widetilde{y}_r \circ \pi$.

Let $M = M(\mathbf{a}, \mathbf{b})$ be given with $\mathbf{a} = \overline{i_1} \cdots \overline{i_k}$. Then $\pi(M) = M(\mathbf{a}^\pi, \mathbf{b}^\pi)$ with $\mathbf{a}^\pi = \overline{r - i_k + 1} \cdots \overline{r - i_1 + 1}$. Also, if $M^t = M(\mathbf{c}, \mathbf{d})$ with $\mathbf{c} = j_1 \cdots j_l$, then $\pi(M^t) = M(\mathbf{c}^\pi, \mathbf{d}^\pi)$ with $\mathbf{c}^\pi = (n - j_l + r + 1) \cdots (n - j_1 + r + 1)$. This implies that

$$\widetilde{x}_i M \neq \mathbf{0} \iff \widetilde{y}_{n-i+r}\pi(M) \neq \mathbf{0}, \tag{6.2}$$

for $i \in I_{0,r}$, where the indices are assumed to be in $\mathbb{Z}_n$. On the other hand, we have

$$\widetilde{x}_i M \neq \mathbf{0} \iff \widetilde{y}_{n-i+r}\eta(M) \neq \mathbf{0}, \tag{6.3}$$

for $i \in I_{0,r}$.

Let $M = (m_{\overline{i}j})$ be a $\mathfrak{g}_J$-highest weight element in $\mathcal{M}_{r\times(n-r)}$, where $m_{\overline{i}j} = 0$ unless $i = j$, and $m_{\overline{r}r+1} \geqslant m_{\overline{r-1}r+2} \geqslant m_{\overline{r-2}r+2} \geqslant \cdots$. It is easy to see that $\pi(M) = \eta(M)$. Then it follows from (6.2) and (6.3) that $\pi = \eta$ and hence $\widetilde{x}_0 = \eta \circ \widetilde{y}_r \circ \eta$ on $\mathcal{M}_{r\times(n-r)}$. Since $\mathbf{B}^{r,s}$ is a subcrystal of $\mathcal{M}_{r\times(n-r)} \otimes T_{s\omega_r}$, we have $\widetilde{x}_0 = \eta \circ \widetilde{y}_r \circ \eta$ on $\mathbf{B}^{r,s}$.

Case 2. $\mathbf{B}^{n,s}$ of type $D^{(2)}_{n+1}$, $C^{(1)}_n$ for $s \geqslant 1$. The proof is similar to Case 1.

Case 3. $\mathbf{B}^{r,s}$ of type $D^{(1)}_n$ for $r = n, n-1$ and $s \geqslant 1$. Let us prove the case $\mathbf{B}^{n,s}$. The proof for $\mathbf{B}^{n-1,s}$ is almost the same.

Let $[T] \in \mathcal{T}_n$ be given. Define a map $\pi : \mathcal{T}_n \to \mathcal{T}_n$, where $\pi([T]) = [T']$ and T' is obtained by 180°-rotation of T and replacing each entry $\overline{i}$ in T with $\overline{n-i+1}$. By definition, $\widetilde{x}_i T \neq \mathbf{0}$ if and only if $\widetilde{y}_{n-i} T' \neq \mathbf{0}$ $(i = 1, \ldots, n-1)$. This implies that $[T'] = [\eta(T)]$. Moreover, if T is of normal shape, then we have by definition of $\widetilde{x}_0$ and $\widetilde{y}_n$ (see Section 5.2) $\widetilde{x}_0([T]) = (\pi \circ \widetilde{y}_n \circ \pi)([T])$. Since the action of η is also well-defined on $\mathcal{T}_n$ (that is, $\eta([T]) = [\eta(T)]$), we conclude that $\widetilde{x}_0 = \eta \circ \widetilde{y}_n \circ \eta$. Since $\mathbf{B}^{n,s}$ is a subcrystal of $\mathcal{T}_n \otimes T_{s\omega_n}$, we have $\widetilde{x}_0 = \eta \circ \widetilde{y}_n \circ \eta$ on $\mathbf{B}^{n,s}$. $\square$

6.2. A connection with the Schützenberger's promotion operator

Let **pr** be the Schützenberger's promotion operator on $SST_{[n]}(\lambda)$ for $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leqslant n$ [12], which satisfies for $T \in SST_{[n]}(\lambda)$ with $\mathrm{wt}(T) = m_1\epsilon_1 + m_2\epsilon_2 + \cdots + m_n\epsilon_n$

(1) $\mathrm{wt}(\mathbf{pr}(T)) = m_n\epsilon_1 + m_1\epsilon_2 + \cdots + m_{n-1}\epsilon_n$,
(2) $\mathbf{pr}(\widetilde{e}_i T) = \widetilde{e}_{i+1}(\mathbf{pr}(T))$ and $\mathbf{pr}(\widetilde{f}_i T) = \widetilde{f}_{i+1}(\mathbf{pr}(T))$ for $i = 1, \ldots, n-2$.

Note that **pr** is the unique map on $SST_{[n]}(\lambda)$ satisfying (1) and (2), and **pr** is of order n if and only if λ is a rectangle (see [15, Proposition 3.2]). It is shown in [11] that on $\mathbf{B}^{r,s}$ of type $A^{(1)}_{n-1}$ ($r = 1, \ldots, n-1$, $s \geqslant 1$)

$$\widetilde{e}_0 = \mathbf{pr}^{-1} \circ \widetilde{e}_1 \circ \mathbf{pr}, \qquad \widetilde{f}_0 = \mathbf{pr}^{-1} \circ \widetilde{f}_1 \circ \mathbf{pr}.$$

Suppose that $\mathfrak{g} = A^{(1)}_{n-1}$. For $k \in I$, let η_k denote the involution (6.1) on crystals of type $\mathfrak{g}_{I_{0,k}}$. Here $\mathfrak{g}_{I_{0,0}} = \mathfrak{g}_{I_0}$. Let $\lambda \in \mathscr{P}$ be given with $\ell(\lambda) \leqslant n$. Put $\xi = \eta_1 \circ \eta_0$. By definition of ξ, it is straightforward to check that

(1) $\mathrm{wt}(\xi(T)) = m_n\epsilon_1 + m_1\epsilon_2 + \cdots + m_{n-1}\epsilon_n$,
(2) $\xi(\widetilde{e}_i T) = \widetilde{e}_{i+1}(\xi(T))$ and $\xi(\widetilde{f}_i T) = \widetilde{f}_{i+1}(\xi(T))$ for $i = 1, \ldots, n-2$.

By the uniqueness of **pr**, we have $\mathbf{pr} = \eta_1 \circ \eta_0$ on $SST_{[n]}(\lambda)$.

Lemma 6.2. *We have $\eta_0 \circ \widetilde{e}_0 = \widetilde{f}_0 \circ \eta_0$ on $\mathbf{B}^{r,s}$.*

Proof. First, we claim that

$$\widetilde{e}_0 = \eta_1 \circ \widetilde{f}_1 \circ \eta_1, \qquad \widetilde{f}_0 = \eta_1 \circ \widetilde{e}_1 \circ \eta_1. \tag{6.4}$$

Note that $\mathbf{pr}^n = \mathrm{id}_{\mathbf{B}^{r,s}}$. We have $\mathbf{pr} \circ \widetilde{e}_{n-1} = \mathbf{pr}^{n-1} \circ \widetilde{e}_1 \circ \mathbf{pr}^{-n+2} = \mathbf{pr}^{-1} \circ \widetilde{e}_1 \circ \mathbf{pr}^2 = \widetilde{e}_0 \circ \mathbf{pr}$. Since $\mathbf{pr} = \eta_1 \circ \eta_0$, we have $\widetilde{e}_0 = \eta_1 \circ \eta_0 \circ \widetilde{e}_{n-1} \circ \eta_0 \circ \eta_1 = \eta_1 \circ \eta_0 \circ \eta_0 \circ \widetilde{f}_1 \circ \eta_1 = \eta_1 \circ \widetilde{f}_1 \circ \eta_1$. Similarly, we have $\widetilde{f}_0 = \eta_1 \circ \widetilde{e}_1 \circ \eta_1$. Now, by (6.4), we have

$$\eta_0 \circ \widetilde{e}_0 = \eta_0 \circ \mathbf{pr}^{-1} \circ \widetilde{e}_1 \circ \mathbf{pr} = \eta_0 \circ \eta_0 \circ \eta_1 \circ \widetilde{e}_1 \circ \eta_1 \circ \eta_0 = \widetilde{f}_0 \circ \eta_0. \qquad \square$$

Proposition 6.3. *Let $\mathbf{B}^{r,s}$ be a KR crystal of type $A^{(1)}_{n-1}$ for $1 \leqslant r \leqslant n-1$ and $s \geqslant 1$. Then we have $\mathbf{pr}^k = \eta_k \circ \eta_0$, on $\mathbf{B}^{r,s}$ for $1 \leqslant k \leqslant n-1$.*

Proof. It is not difficult to see that the highest (respectively lowest) weight elements in $\mathbf{B}^{r,s}$ as a $U_q(\mathfrak{g}_{I_{0,k}})$-crystal are parametrized by the partitions $\lambda \subset (s^r)$, say $b^{\mathrm{h.w.}}_\lambda$ (respectively $b^{\mathrm{l.w.}}_\lambda$). Note that $\eta_k \circ \widetilde{x}_i = \widetilde{y}_{n+k-i} \circ \eta_k$ for $i \in I_{0,k}$ and $\eta_k(b^{\mathrm{h.w.}}_\lambda) = b^{\mathrm{l.w.}}_\lambda$ for $\lambda \subset (s^r)$. Here $x = e$ (respectively f) when $y = f$ (respectively e), and the indices are assumed to be in $\mathbb{Z}_n$.

Let $\xi_k = \mathbf{pr}^k \circ \eta_0$. It is straightforward to check that $\xi_k \circ \widetilde{x}_i = \widetilde{y}_{n+k-i} \circ \xi_k$ for $i \in I_{0,k}$. This implies that $\xi_k(b^{\mathrm{h.w.}}_\lambda)$ is a lowest weight element as a $U_q(\mathfrak{g}_{I_{0,k}})$-crystal and $\mathrm{wt}(\xi_k(b^{\mathrm{h.w.}}_\lambda)) = \mathrm{wt}(b^{\mathrm{l.w.}}_\lambda)$. Hence, we have $\xi_k(b^{\mathrm{h.w.}}_\lambda) = b^{\mathrm{l.w.}}_\lambda$, and $\xi_k(b) = \eta_k(b)$ for $b \in \mathbf{B}^{r,s}$. $\square$

Corollary 6.4. *Under the above hypothesis, we have $\widetilde{e}_0 = \eta_k \circ \widetilde{f}_k \circ \eta_k$ and $\widetilde{f}_0 = \eta_k \circ \widetilde{e}_k \circ \eta_k$ on $\mathbf{B}^{r,s}$ for $1 \leqslant k \leqslant n-1$.*

Proof. Since $\mathbf{pr}^{-k} \circ \widetilde{e}_k \circ \mathbf{pr}^k = \widetilde{e}_0$, we have $\eta_0 \circ \eta_k \circ \widetilde{e}_k \circ \eta_k \circ \eta_0 = \widetilde{e}_0$ by Proposition 6.3. Hence, we have $\eta_k \circ \widetilde{e}_k \circ \eta_k = \eta_0 \circ \widetilde{e}_0 \circ \eta_0 = \widetilde{f}_0$ by Lemma 6.2. Similarly, we have $\widetilde{f}_0 = \eta_k \circ \widetilde{e}_k \circ \eta_k$. $\square$

Remark 6.5. By Proposition 6.3, η_0 and η_1 on $\mathbf{B}^{r,s}$ generate the action of the dihedral group of order $2n$. When $k = r$, Corollary 6.4 also implies Proposition 6.1 for type $A^{(1)}_{n-1}$.

Acknowledgments

Part of this work was done during the author was visiting University of California, Berkeley in 2010–2011. The author would like to thank A. Schilling for valuable discussion and kind explanation of KR crystals together with her recent preprint [15].

References

[1] V. Kac, Infinite-Dimensional Lie Algebras, third edition, Cambridge University Press, Cambridge, 1990.
[2] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991) 465–516.
[3] A. Lascoux, Double crystal graphs, in: Studies in Memory of Issai Schur, in: Progr. Math., vol. 210, Birkhäuser, 2003, pp. 95–114.
[4] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (1994) 295–345.
[5] J.-H. Kwon, Demazure crystals of generalized Verma modules and a flagged RSK correspondence, J. Algebra 322 (2009) 2150–2179.
[6] J.-H. Kwon, Crystal bases of modified quantized enveloping algebras and a double RSK correspondence, J. Combin. Theory Ser. A 118 (2011) 2131–2156.
[7] S.-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (1992) 499–607.
[8] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Y. Yamada, Remarks on fermionic formula, Contemp. Math. 248 (1999) 243–291.
[9] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, Paths, crystals and fermionic formulae, in: MathPhys Odyssey 2001, in: Prog. Math. Phys., vol. 23, Birkhäuser Boston, Boston, MA, 2002, pp. 205–272.
[10] A.N. Kirillov, N.Yu. Reshetikhin, Representations of Yangians and multiplicities of the inclusions of the irreducible components of the tensor product of representations of simple Lie algebras, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 160 (1987), Anal. Teor. Chisel i Teor. Funktsii. 8, 211–221, 301, translation in J. Soviet Math. 52 (3) (1990) 3156–3164.
[11] M. Shimozono, Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties, J. Algebraic Combin. 15 (2002) 151–187.
[12] M.-P. Schützenberger, Promotion des morphisms d'ensembles ordonnes, Discrete Math. 2 (1972) 73–94.
[13] M. Okado, A. Schilling, Existence of Kirillov–Reshetikhin crystals for nonexceptional types, Represent. Theory 12 (2008) 186–207.
[14] G. Fourier, M. Okado, A. Schilling, Kirillov–Reshetikhin crystals for non-exceptional types, Adv. Math. 222 (2009) 1080–1116.
[15] A. Schilling, P. Tingley, Demazure crystals, Kirillov–Reshetikhin crystals, and the energy functions, Electron. J. Combin. 19 (2012) P2.
[16] M. Kashiwara, Similarity of crystal bases, Contemp. Math. 194 (1996) 177–186.
[17] J. Hong, S.-J. Kang, Introduction to Quantum Groups and Crystal Bases, Grad. Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.
[18] M. Kashiwara, On crystal bases, in: Representations of Groups, in: CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.
[19] W. Fulton, Young Tableaux, London Math. Soc. Stud. Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
[20] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second edition, Oxford University Press, 1995.
[21] J.-H. Kwon, Crystal graphs for Lie superalgebras and Cauchy decomposition, J. Algebraic Combin. 25 (2007) 57–100.
[22] J.-H. Kwon, Littlewood identity and crystal bases, Adv. Math. 230 (2012) 699–745.
[23] G. Lusztig, Introduction to Quantum Groups, Birkhäuser, Boston, 1993.
[24] A. Berenstein, A. Zelevinsky, Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics, Duke Math. J. 82 (1996) 473–502.