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The periodogram of an i.i.d. sequence

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Abstract

Periodogram ordinates of a Gaussian white-noise computed at Fourier frequencies are well known to form an i.i.d. sequence. This is no longer true in the non-Gaussian case. In this paper, we develop a full theory for weighted sums of non-linear functionals of the periodogram of an i.i.d. sequence. We prove that these sums are asymptotically Gaussian under conditions very close to those which are sufficient in the Gaussian case, and that the asymptotic variance differs from the Gaussian case by a term proportional to the fourth cumulant of the white noise. An important consequence is a functional central limit theorem for the spectral empirical measure. The technique used to obtain these results is based on the theory of Edgeworth expansions for triangular arrays. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $(Z_t)_{t\in\mathbb{Z}}$ be a white noise with unit variance, i.e., an i.i.d. sequence such that $\mathbb{E}[Z_0]=0$ and $\mathbb{E}[Z_0^2]=1$. Define the discrete Fourier transform and the periodogram as

$$d_n(x) = (2\pi n)^{-1/2}\sum_{t=1}^{n} Z_t \mathrm{e}^{\mathrm{i}tx} \quad \text{and} \quad I_n(x) = |d_n(x)|^2.$$

The Fourier frequencies are usually defined as $x_k = 2\pi k/n$, $1\leqslant k\leqslant \tilde{n}$ where $\tilde{n}=[(n-1)/2]$ (the dependency with respect to n will be omitted). It is a well-known fact that if the variables Z_t are moreover jointly Gaussian, then the periodogram ordinates computed at Fourier frequencies are independent and $2\pi I_n(x_k)$ has a $\Gamma(1,1)$ distribution. The $\Gamma(a,\lambda)$ distribution is the distribution with density function $\Gamma(a)^{-1}\lambda^a x^{a-1}\mathrm{e}^{-\lambda x}$ ($x\in\mathbb{R}_+$, $a>0$ and $\lambda>0$) with respect to Lebesgue measure on $\mathbb{R}^+$, and $\Gamma(a)=\int_0^\infty x^{a-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is

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the Gamma function. Gaussianity and the specific choice of the Fourier frequencies are the fundamental reasons for this independence. For $0 \leqslant k < j \leqslant \tilde{n}$, it holds that

$$\mathbb{E}[d_n(x_k)d_n(\pm x_j)] = \frac{1}{2\pi n}\sum_{t=1}^{n} \mathrm{e}^{\mathrm{i}t(x_k \pm x_j)} = 0.$$

The last sum vanishes because of the specific choice of the Fourier frequencies. This implies uncorrelatedness of the variables $d_n(x_k)$, hence independence in the Gaussian case. This latter property no longer holds in the non-Gaussian case. For instance, let κ_4 denote the fourth cumulant of Z_0. An easy computation yields, for $0 \leqslant k < j \leqslant \tilde{n}$,

$$\mathrm{cov}(I_n(x_k), I_n(x_j)) = \frac{\kappa_4}{4\pi^2 n}.$$

The fourth cumulant of a standard Gaussian variable is 0, but it is not necessarily so for an arbitrary distribution. Nevertheless, the central limit theorem implies that for any fixed u, and pairwise distinct integers $k_1, \ldots, k_u$, $d_n(x_{k_1}), \ldots, d_n(x_{k_u})$ are asymptotically independent, in the sense that the asymptotic distribution of the $2u$ r.v.'s

$$\sqrt{2\pi}\,\mathrm{Re}\{d_n(x_{k_1})\}, \quad \sqrt{2\pi}\,\mathrm{Im}\{d_n(x_{k_1})\}, \ldots, \sqrt{2\pi}\,\mathrm{Re}\{d_n(x_{k_u})\}, \quad \sqrt{2\pi}\,\mathrm{Im}\{d_n(x_{k_u})\}$$

is that of $2u$ i.i.d. $\mathcal{N}(0, \frac{1}{2})$ random variables. This implies that $2\pi I_n(x_{k_1}), \ldots, 2\pi I_n(x_{k_u})$ are asymptotically independent exponentials. Anyhow, statistics of interest seldom involve a fixed finite number of periodogram ordinates. Among important problems, we can mention the following.

1.1. *Asymptotic distribution of the maximum*

In the Gaussian case, $M_n = 2\pi \max_{1 \leqslant k \leqslant \tilde{n}} I_n(x_k)$ has a $\Gamma(\tilde{n}, 1)$ distribution. Thus, $\lim_{n\to\infty} \mathbb{P}(M_n - \log(\tilde{n}) \leqslant x) = \mathrm{e}^{-\mathrm{e}^{-x}}$ (the standard Gumbel distribution). Davis and Mikosch (1999) have shown that this asymptotic property still holds true in the non-Gaussian case.

1.2. *Weighted sums of functionals of the periodogram*

Consider real numbers $\beta_{n,k}$ such that $\sum_{k=1}^{\tilde{n}} \beta_{n,k}^2 = 1$ and a function ϕ, and define

$$S_n(\phi) = \sum_{k=1}^{\tilde{n}} \beta_{n,k}\phi(2\pi I_n(x_k)).$$

In the Gaussian case, as already mentioned, the periodogram ordinates $I_n(x_1), \ldots, I_n(x_{\tilde{n}})$ are i.i.d. random variables, thus $S_n(\phi)$ is asymptotically Gaussian if $\mathbb{E}[\phi(I_n(x_1))] = 0$, $\mathbb{E}[\phi^2(I_n(x_1))] < \infty$ and under the Lindeberg condition

$$\lim_{n\to\infty} \max_{1 \leqslant k \leqslant \tilde{n}} |\beta_{n,k}| = 0.$$

The considerations above make one expect that this result still holds in the non-Gaussian case. However, no general result of this kind is known. We recall now some previous results. In the case of a linear functional, i.e., $\phi(x) = x$, if the weights $\beta_{n,k}$ are proportional to the value of a smooth function g at Fourier frequencies, it is easily proved, using for instance the martingale Central Limit Theorem as in Robinson

(1995) or the method of cumulants (see Brillinger, 1981), that $\sum_{k=1}^{\tilde{n}} \beta_{n,k} I_n(x_k)$ is asymptotically Gaussian, with an asymptotic variance that depends on κ_4. For instance, if $\beta_{n,k} = \tilde{n}^{-1/2}$, then $\tilde{n}^{-1/2} \sum_{k=1}^{\tilde{n}} (2\pi I_n(x_k) - 1)$ is asymptotically Gaussian with variance $1 + \kappa_4/2$. In the case of non-linear functionals, very little is known. The first attempt to derive an asymptotic theory for such sums in the non-linear case is due to Chen and Hannan (1980) in the case $\phi(x) = \log x$. They used the technique of Edgeworth expansions for triangular arrays of independent random variables to compute the variance of $\tilde{n}^{-1/2} \sum_{k=1}^{\tilde{n}} \{\log(2\pi I_{n,k}) - \log 2 + \gamma\}$, where γ is Euler's constant. If q_n denotes the joint density of $(\mathrm{Re}\{d_n(x_k)\}, \mathrm{Im}\{d_n(x_k)\}, \mathrm{Re}\{d_n(x_j)\}, \mathrm{Im}\{d_n(x_j)\})$, for $0 < k < j \leqslant \tilde{n}$, a second-order Edgeworth expansion of q_n yields (with the weights $\beta_{n,k}$ set equal to $\tilde{n}^{-1/2}$)

$$\mathrm{var}(S_n(\log)) = \frac{\pi^2}{6} + \frac{\kappa_4}{2} + \mathrm{O}(n^{-1/2}).$$

Note that $\pi^2/6$ is exactly the variance in the Gaussian case. The main restriction of this method is that the existence of the joint density q_n and the validity of its Edgeworth expansion require a regularity assumption on the distribution of Z_0, which nearly amounts to the existence of a density with respect to Lebesgue measure. The necessity of this assumption is not obvious, but in the case of non-regular functionals, some regularity assumption on the distribution of Z_0 is needed. If, for instance, the distribution of Z_0 has a positive mass at zero, then the log-periodogram cannot be computed. Recently, Velasco (2000) using the same method, proved a central limit theorem in the case of the function log and in the particular case where the number of non-vanishing coefficients $\beta_{n,k}$ is negligible with respect to n. The asymptotic variance is then $\pi^2/6$, the same as in the Gaussian case. The central limit theorem is proved using the method of moments, and Velasco assumes that $\mathbb{E}[|Z_0|^s]$ is finite for all s. This is obviously a strong assumption that one would like to get rid of.

1.3. Empirical spectral distribution function

Another important and unsolved problem was to prove a functional central limit theorem for the empirical spectral measure, defined as

$$\hat{F}_n(x) = \tilde{n}^{-1} \sum_{k=1}^{\tilde{n}} \mathbf{1}_{[0,x]}(2\pi I_n(x_k)), \quad x \geqslant 0.$$

Freedman and Lane (1980) and Kokoszka and Mikosch (2000) proved that under the only assumption that $\mathbb{E}(Z_t^2) < \infty$, $\sup_{x \geqslant 0} |\hat{F}_n(x) - F_1(x)|$ converges in probability to zero, where $F_1(x) = 1 - \mathrm{e}^{-x}$ is the standard exponential cumulative distribution function. Kokoszka and Mikosch (2000) strengthened this result and proved convergence of the first three moments of $\tilde{n}^{-1/2}(\hat{F}_n(x) - F_1(x))$ under the natural assumptions of finiteness of the six first moments of Z_0 (but under the unnecessary assumption that they all coincide with those of a $\mathcal{N}(0,1)$ distribution) and under the regularity assumption on the distribution of Z_0 mentioned above.

In this paper, using the ideas of Chen and Hannan (1980) and generalizing (and making more formal) the deep ideas of Velasco (2000), we present a full theory for weighted sums of (possibly) non-linear functionals of the periodogram of an i.i.d. sequence, and we solve the above-mentioned problems. We also bring a new tool to the study of this problem. While the cited authors used Edgeworth expansion of the joint density of a finite number of discrete Fourier transforms, which necessitates the regularity assumption, we use the results of Götze and Hipp (1978) on Edgeworth expansions for moments of smooth functions. This allows, in the case of smooth functionals, to get rid of the regularity assumption on the distribution of Z_0. This, in its turn, allows to use truncation arguments to also get rid of the assumption of finite moments of all order to obtain a central limit theorem by means of the method of moments.

Before concluding this section, let us mention that in statistical applications, the observations are not realizations of a white noise, but rather of a process X which admits a linear representation with the i.i.d. sequence Z:

$$X_t = \sum_{j\in\mathbb{Z}} a_j Z_{t-j}, \quad t\in\mathbb{Z},$$

where $(a_t)_{t\in\mathbb{Z}}$ is a sequence of real numbers such that $\sum_{t\in\mathbb{Z}} a_t^2 < \infty$. The spectral density of the process X is then

$$f_X = (2\pi)^{-1}\left|\sum_{j\in\mathbb{Z}} a_j e_j\right|^2,$$

where $e_j(x) = \mathrm{e}^{\mathrm{i}jx}$. If the coefficients a_j are absolutely summable, then f_X is continuous and the process X is said weakly dependent. If the coefficients a_j are not absolutely summable, then f_X may not be continuous and even have singularities, in which case the process X is usually said strongly dependent. The quantity of interest in this framework is thus not $S_n(\phi)$ but $S_n^X(\phi)$ defined by

$$S_n^X(\phi) = \sum_{k=1}^{\tilde{n}} \beta_{n,k}\phi(I_n^X(x_k)/f(x_k)).$$

The study of $S_n^X(\phi)$ is then based on the so-called Bartlett's decomposition (cf. Bartlett, 1995), which consists in relating the periodogram of X to that of Z:

$$I_n^X(x) = 2\pi f_X(x) I_n^Z(x) + R_n(x),$$

where the superscript indicates the process with respect to which the periodogram is computed. Then one can write

$$S_n^X(\phi) = S_n(\phi) + T_n,$$

$$T_n = \sum_{k=1}^{\tilde{n}} \beta_{n,k}\{\phi(I_n^X(x)/f(x_k)) - \phi(2\pi I_n^Z(x))\}.$$

Under reasonable regularity assumptions on ϕ, one can prove that T_n tends to zero in probability, and the remaining task is to obtain a central limit theorem for $S_n(\phi)$. The problem with this decomposition is that the remainder term R_n is rather large, even if the coefficients a_j decay very rapidly or are only finitely many. We will not give any statistical applications in this paper, but for the problem mentioned above, we can already say that our results yield a central limit theorem for the estimator of the innovation variance considered in Chen and Hannan (1980), and that we improve on Velasco (2000) since we prove that his central limit theorem holds if Z_0 has only a finite number of finite moments (the exact number depends on several parameters not specified here). Other applications for weak dependent linear processes are presented in Fay et al. (1999) and an application to the estimation of the dependence coefficient of a fractional process is presented in Hurvich et al. (2000).

The rest of the paper is organized as follows. Since the technique of Edgeworth expansion is applied to the distribution of the discrete Fourier transforms, we first state a very general theorem for functionals of the Fourier transforms. Another motivation is that it can be applied to modifications of the periodogram such as tapered periodogram, not considered here for the sake of brevity, but that are very important in statistical applications, especially for long-range dependent processes. In Section 3, we apply this result to general linear functionals of the periodogram and in Section 4, we state a functional central limit theorem for the empirical spectral distribution function. The proof of the main theorem, being very involved is split in several sections. The main technical tool, a moment expansion (Lemma 3) is stated in Section 6 and proved in Section 8. Even though it is just a technical lemma, we consider it as the actual main result of this paper, since all the other results easily derive from it, and because it offers the deepest insight into the dependence structure of periodogram ordinates at Fourier frequencies of a non-Gaussian i.i.d. sequence.

2. Main result

Throughout the paper, m will denote a fixed positive integer and for all $n \geqslant 2m$, we denote $K := K(m,n) = [(n-m)/2m)]$. For $1 \leqslant k \leqslant K$, define the $2m$-dimensional vector

$$W_{n,k} = (2/n)^{1/2} \sum_{t=1}^{n} Z_t(\cos(tx_{m(k-1)+1}), \sin(tx_{m(k-1)+1}), \ldots, \cos(tx_{mk}), \sin(tx_{mk}))^{\mathrm{T}} \tag{1}$$

so that $2\pi \bar{I}_{n,k} = \|W_{n,k}\|^2/2$. In this section, we give conditions on triangular arrays of functions $(\psi_{n,k})_{1 \leqslant k \leqslant K}$ and of reals $(\beta_{n,k})_{1 \leqslant k \leqslant K}$ to obtain a central limit theorem for sums

$$S_n := \sum_{k=1}^{K} \beta_{n,k} \psi_{n,k}(W_{n,k}). \tag{2}$$

In the case of non-smooth functions, as mentioned in the introduction, a regularity assumption on the distribution of the white noise Z_0 is necessary.

(A1) There exists a real $p \geqslant 1$, such that $\int_{-\infty}^{+\infty} |\mathbb{E}(\mathrm{e}^{\mathrm{i}tZ_0})|^p \, \mathrm{d}t < \infty$.

Assumption (A1) ensures that $n^{-1/2} \sum_{t=1}^{n} Z_t$ has a density q_n for $n \geqslant p$ and that this density converges uniformly to the standardized Gaussian distribution (see, for example, Bhattacharya and Rao, 1976, Theorem 19.1, p. 189). It is a strengthening of the usual Cramér's condition which excludes "strongly lattice" variables. This condition is rather weak in the sense that it can hold even if the distribution of Z_0 does not have a density with respect to Lebesgue measure on the line. If for instance the distribution of Z_0 has a square integrable density, then (A1) holds with $p=2$.

The admissible functions will be either smooth functions or non-smooth functions that satisfy some integrability condition. The following definitions will be used throughout the paper. For integers v and r, let $\mathscr{S}_v^r$ be the space of r times differentiable function on $\mathbb{R}^{2m}$ such that for all $2m$-tuples of non-negative integers $\beta=(\beta_1,\dots,\beta_{2m})$ that satisfy $\beta_1+\cdots+\beta_{2m}\leqslant r$,

$$M_v(D^\beta\psi)<\infty,$$

where D^β denotes the partial derivative of ψ of order β_i with respect to the ith component, and for any function ϕ on $\mathbb{R}^{2m}$,

$$M_v(\phi)=\sup_{x\in\mathbb{R}^{2m}}\frac{|\phi(x)|}{1+|x|^v}.$$

The notation M_v comes from Götze and Hipp (1978). We will also use the following notation. For $\psi\in\mathscr{S}_v^r$, denote

$$M_{v,r}(\psi)=\sum_{\beta_1+\cdots+\beta_{2m}\leqslant r}M_v(D^\beta\psi). \tag{3}$$

To deal with the case of non-smooth functions, we introduce the following family of semi-norms. For any measurable function ψ on $\mathbb{R}^d$, define

$$N_{d,\alpha}(\psi)=\int_{\mathbb{R}^d}|\psi(x)|(1+|x|)^{-\alpha}\,\mathrm{d}x.$$

It is easily seen that any function ψ such that $N_{d,\alpha}(\psi)<\infty$ can be approximated in the sense of the norm $N_{d,\alpha}$ by a sequence of indefinitely differentiable (C^∞) functions with compact support.

Let $\xi=(\xi_1,\dots,\xi_{2m})^{\mathrm{T}}$ denote a $2m$-dimensional standard Gaussian vector, and define, when possible:

$$\|\psi\|^2=\mathbb{E}[\psi^2(\xi)], \tag{4}$$

$$C_2(\psi,j)=\mathbb{E}[(\xi_j^2-1)\psi(\xi)],\quad 1\leqslant j\leqslant 2m. \tag{5}$$

Recall now that the Hermite rank of a real-valued function ψ defined on $\mathbb{R}^{2m}$ such that $\|\psi\|<\infty$ is the smallest integer τ such that there exists a polynomial P of degree τ with $\mathbb{E}[P(\xi)\psi(\xi)]\neq 0$. In this section, for the sake of simplicity, only functions of Hermite rank at least 2 will be considered. A sufficient condition for a function ψ to have Hermite rank at least 2 is $\mathbb{E}[\psi(\xi)]=0$ and ψ is componentwise even. This condition usually holds in applications. The assumptions needed to prove the asymptotic normality of S_n (defined in Eq. (2)) are now stated.

(A2) $(\beta_{n,k})_{1\leqslant k\leqslant K}$ is a triangular array of real numbers such that $\sum_{k=1}^K\beta_{n,k}^2=1$ and

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant K}|\beta_{n,k}|=0.$$

(A3) There exists a real $\sigma>0$ such that

$$\lim_{n\to\infty}\sum_{k=1}^K\beta_{n,k}^2\|\psi_{n,k}\|^2=\sigma^2.$$

(A4) There exists a real τ such that

$$\lim_{n\to\infty} n^{-1}\kappa_4 \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l} \sum_{i,j=1,\dots,2m} C_2(\psi_{n,k},i)C_2(\psi_{n,l},j) = \tau\kappa_4$$

and $\sigma^2 + \kappa_4\tau/4 \neq 0$.

(A5)

$$\forall \varepsilon > 0, \quad \max_{1\leqslant k\leqslant K} |\beta_{n,k}| = \mathrm{O}(\mu_n^{-1/2+\varepsilon}),$$

where $\mu_n := \#\{k\colon 1\leqslant k\leqslant K, \beta_{n,k}\neq 0\}$.

Assumption (A2) implies the Lindeberg–Levy smallness condition and together with (A3) imply that S_n is asymptotically Gaussian when Z is Gaussian white noise. Assumption (A4) is necessary in the non-Gaussian case since it appears in the expansion of $\mathrm{var}(S_n)$. Note that it automatically holds if $\kappa_4 = 0$. Assumption (A5) means that $\mu_n(\max_{1\leqslant k\leqslant K}|\beta_{n,k}|)^2$ is bounded by a slowly varying function of μ_n. It holds when $\beta_{n,k}$ is defined as $g(y_k)/(\sum_{k=1}^K g^2(y_k))^{1/2}$ for most "reasonable" functions g (such as continuous functions on $[-\pi,\pi]$ or $g(x) = \log(x)$) and evenly spaced frequencies y_k, $1\leqslant k\leqslant K$. This assumption does not seem necessary, but we cannot prove our result without it. See the comment after Theorem 1. The next assumption is necessary to replace possibly non-smooth functions $\psi_{n,k}$ by smooth ones.

(A6) For all real $\varepsilon > 0$, there exist a sequence of C^∞ functions $\psi^\varepsilon_{n,k}$ with same compact support $\mathscr{K}_\varepsilon$ and with Hermite rank at least 2, reals $\sigma^2(\varepsilon) > 0$ and $\tau(\varepsilon)$ such that

$$\max_n \max_{1\leqslant k\leqslant K} \|\psi_{n,k} - \psi^\varepsilon_{n,k}\| \leqslant \varepsilon,$$

$$\forall \varepsilon > 0, \quad \forall r\in\mathbb{N}, \quad \exists C_{r,\varepsilon}, \quad \forall n, \geqslant 2m, \quad \forall k = 1,\dots,K, \quad M_{0,r}(\psi^\varepsilon_{n,k}) \leqslant C_{r,\varepsilon},$$

$$\lim_{n\to\infty} \sum_{k=1}^K \beta^2_{n,k}\|\psi^\varepsilon_{n,k}\|^2 = \sigma^2(\varepsilon),$$

$$\lim_{n\to\infty} n^{-1} \sum_{1\leqslant k<l\leqslant K} \beta_{n,k}\beta_{n,l} \sum_{i,j=1,\dots,2m} C_2(\psi^\varepsilon_{n,k},i)C_2(\psi^\varepsilon_{n,l},j) = \tau(\varepsilon).$$

Note that if (A3), (A4) and (A6) hold, then $\lim_{\varepsilon\to 0}\sigma^2(\varepsilon) = \sigma^2$ and $\lim_{\varepsilon\to 0}\kappa_4\tau(\varepsilon) = \kappa_4\tau$.

Theorem 1. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a unit variance white noise with finite moment of order μ. Let $(\beta_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array of reals and $(\psi_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array of functions such that Assumptions* (A2)–(A6) *hold. Assume either*

- (A1) *holds, there exists an integer $\alpha\geqslant 3$, an integer $\beta\geqslant 4$ and a constant C such that for all n and $1\leqslant k\neq l\leqslant K$,*

$$\int_{\mathbb{R}^{2m}} \psi^2_{n,k}(x)(1+|x|)^{-\alpha}\,\mathrm{d}x \leqslant C,$$

$$\int_{\mathbb{R}^{2m}\times\mathbb{R}^{2m}} |\psi_{n,k}(x)\psi_{n,l}(y)|(1+|x|+|y|)^{-\beta}\,\mathrm{d}x\,\mathrm{d}y \leqslant C, \tag{6}$$

and $\mathbb{E}(|Z_t|^{\alpha\vee\beta}) < +\infty$.

- *There exists a non-negative integer* v *such that for all* $1 \leqslant k \leqslant K$, $\psi_{n,k} \in \mathscr{S}_v^2$ *and* $\mu \geqslant 4v \vee 4$.

Then the distribution of S_n *is asymptotically centered Gaussian with variance* $\sigma^2 + \tau\kappa_4/4$. *Moreover, asymptotic normality still holds without Assumptions* (A4) *or* (A5) *in the following cases.*

- *If* $\mu_n = \mathrm{o}(n^{2/3})$ *then Assumption* (A5) *is not needed and Assumption* (A4) *holds with* $\tau = 0$.
- *If for all* $k \leqslant K$ *and all* $j = 1, \ldots, 2m$, $C_2(\psi_{n,k}, j) = 0$, *then Assumptions* (A4) *and* (A5) *are not needed and thus the central limit theorem holds under the same assumption on the weights* $\beta_{n,k}$ *and with the same limit as in the Gaussian case.*

Comments. This result gives a better understanding of the differences between the Gaussian and the non-Gaussian case. Recall that in the Gaussian case Assumption (A2) and (A3) yield the central limit theorem for S_n. Here, we need a stronger assumption on the functions considered, and also a restriction on the admissible weights. Note that Assumption (A1) holds in the Gaussian case. The strengthened assumptions on the functions considered are somehow necessary, since some conditions are needed to insure integrability of $\psi_{n,k}(W_{n,k})$. The conditions we impose are nearly minimal, and in the case of smooth functions, they are optimal in terms of the requirement on the moments of Z_0. Assumption (A5) is probably not necessary. As mentioned in the theorem, it is indeed not needed in some cases.

3. Non-linear functionals of the periodogram

Since Theorem 1 is stated for arbitrary m, we can derive a central limit theorem for non-linear functionals of the aggregated (or averaged, or pooled) periodogram. Let m be a fixed integer and recall that we defined $K = [(n-m)/2m]$. Define

$$\bar{I}_{n,k} = \sum_{s=(k-1)m+1}^{km} I_n(x_s), \quad 1 \leqslant k \leqslant K.$$

Let ϕ be a measurable function on $\mathbb{R}$ such that $\mathbb{E}[\phi^2(Y)] < \infty$ where Y is a $\Gamma(m,1)$ random variable, or, equivalently, Y is distributed as $|\xi|^2/2$, where ξ denote a $2m$-dimensional standard Gaussian vector, and $|\cdot|$ denotes the Euclidean norm. The following quantities are then well defined:

$$\gamma_m(\phi) = \mathbb{E}[\phi(Y)], \tag{7}$$

$$\sigma_m^2(\phi) = \mathrm{var}(\phi(Y)), \tag{8}$$

$$\rho_m(\phi) = \mathbb{E}[(Y-m)\phi(Y)]. \tag{9}$$

Let $(\beta_{n,k})_{1 \leqslant k \leqslant K}$ be a triangular array of real numbers such that (A2) holds. In the context of this section, Assumption (A3) will hold automatically, while (A4) will be a consequence of the following assumption.

(A7) There exists a real γ such that $\lim_{n\to\infty} n^{-1}\kappa_4 \sum_{k\neq l} \beta_{n,k}\beta_{n,l} = \gamma\kappa_4$.
Define finally

$$J_{m,n}(\phi) = \sum_{k=1}^{K} \beta_{n,k}\{\phi(2\pi \bar{I}_{n,k}) - \gamma_m(\phi)\}. \tag{10}$$

Theorem 2. *Let $(Z_t)_{t\in\mathbb{Z}}$ be a unit variance white noise with finite moment of order μ. Assume either*

- Smooth case. *ϕ is twice differentiable, there exists an integer v such that*

$$\max_{x\in\mathbb{R}} \frac{|\phi(x)| + |\phi'(x)| + |\phi''(x)|}{1+|x|^v} < \infty$$

and $\mu \geqslant 4v \vee 4$.

- Non-smooth case. *Assumption* (A1) *holds, there exists an integer $\alpha \geqslant 3$ and an integer $\beta \geqslant 4$ such that*

$$\int_{\mathbb{R}^{2m}} \phi^2(|x|^2)(1+|x|)^{-\alpha}\, \mathrm{d}x < \infty,$$

$$\int_{\mathbb{R}^{4m}} |\phi(|x|^2)\phi(|y|^2)|(1+|x|+|y|)^{-\beta}\, \mathrm{d}x\, \mathrm{d}y < \infty$$

and $\mu \geqslant \alpha \vee \beta$.

Let $(\beta_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array satisfying Assumptions (A2), (A5) *and* (A7), *and assume that $\sigma_m^2(\phi) + \gamma\kappa_4\rho_m^2(\phi) \neq 0$. Then $J_{m,n}(\phi)$ converges in distribution to the centered Gaussian distribution with variance $\sigma_m^2(\phi)+\gamma\kappa_4\rho_m^2(\phi)$. Moreover, asymptotic normality still holds without Assumptions* (A5) *or* (A7) *in the following cases.*

- *If $\mu_n = \mathrm{o}(n^{2/3})$ then Assumption* (A5) *is not needed and Assumption* (A7) *holds with $\gamma = 0$.*
- *If $\rho_m(\phi) = 0$, then Assumptions* (A5) *and* (A7) *are not needed and thus the central limit theorem holds under the same assumption on the weights $\beta_{n,k}$ as in the Gaussian case and with the same limit $\sigma_m^2(\phi)$.*

Remark. Note that by definition, $\rho_m^2(\phi) \leqslant m\sigma_m^2(\phi)$ and equality holds only for $\phi(y) = c(y-m)$ for some constant c. If $\kappa_4 \neq 0$, then $|\gamma| \leqslant 1/2m$. Thus, $\sigma_m^2(\phi) + \gamma\kappa_4\rho_m^2(\phi) \geqslant \sigma_m^2(\phi)(1+\kappa_4/2) \geqslant 0$, and $\sigma_m^2(\phi) + \gamma\kappa_4\rho_m^2(\phi) = 0$ implies that $\phi(y) = c(y-m)$ and $\kappa_4 = -2$, which is equivalent to $\mathrm{var}(Z_0^2) = 0$, i.e., $Z_0 = \pm 1$ almost surely. This case is of a limited interest and can be studied directly.

Examples. In statistical applications, the most important case of a non-linear functional is the logarithm. It is well known that (cf. Johnson and Kotz, 1970) $\gamma_m(\log) = \Psi(m)$ and $\sigma_m^2(\log) = \Psi'(m)$, where Ψ denotes the digamma function: $\Psi(z) = \Gamma'(z)/\Gamma(z)$. An elementary computation yields

$$\rho_m(\log) = \mathbb{E}[(Y-m)\log Y] = ((m-1)!)^{-1} \int_0^\infty (y-m)\log(y) y^{m-1} \mathrm{e}^{-y}\, \mathrm{d}y = 1.$$

Thus, if Assumption (A1) holds, and if $\mathbb{E}[|Z_0|^{4m+1}] < \infty$, then

$$K^{-1/2} \sum_{k=1}^{K} \{\log(2\pi \bar{I}_{n,k}) - \Psi(m)\} \xrightarrow{(\mathrm{d})} \mathcal{N}\left(0, \psi'(m) + \frac{\kappa_4}{2m}\right).$$

This implies that the estimator of the innovation variance of Chen and Hannan (1980) is asymptotically Gaussian with variance $2m\Psi'(m) + \kappa_4$ (for a full treatment of this problem, see Fay et al. 1999). If $(\beta_{n,k})_{1 \leqslant k \leqslant K}$ is a triangular array of reals such that Assumptions (A2), (A5) and (A7) hold with $\gamma = 0$, then

$$\sum_{k=1}^{K} \beta_{n,k} \{\log(2\pi I_n(x_k)) - \Psi(m)\} \xrightarrow{(\mathrm{d})} \mathcal{N}(0, \psi'(m)).$$

Velasco proved this result in the specific context of the narrow-band log-periodogram estimator (the so-called GPH estimator) of the fractional differencing coefficient of a long-memory linear process, under the additional assumption that $\mathbb{E}[|Z_0|^s] < \infty$ for all $s > 0$.

Proof of Theorem 2. If ϕ satisfy the assumptions of Theorem 2, define, for $x \in \mathbb{R}^{2m}$, $\psi(x) = \phi(|x|^2/2) - \gamma_m(\phi)$ and $\psi_{n,k} = \psi$ for all n and $1 \leqslant k \leqslant K$. As mentioned above, ψ has Hermite rank 2 since $\mathbb{E}[\psi(\xi)] = 0$ and ψ is componentwise even. If the array $\beta_{n,k}$ satisfies Assumptions (A2) and (A7) then Assumptions (A3) and (A4) hold with $\sigma^2 = \sigma_m^2(\phi)$ and $\tau = 4\gamma\rho_m^2(\phi)$. Under the assumptions of Theorem 2, ϕ can be approximated by a sequence of C^∞ function ϕ^ε with same compact support, i.e.

$$\forall \varepsilon > 0, \quad \mathbb{E}[(\phi(|\xi|^2/2) - \phi^\varepsilon(|\xi|^2/2))^2] \leqslant \varepsilon.$$

Define then $\psi^\varepsilon(x) = \phi^\varepsilon(|x|^2/2)$ and $\psi_{n,k}^\varepsilon = \psi^\varepsilon$ for all n and k. It can be assumed, without loss of generality, that $\mathbb{E}[\phi(\xi)] = \mathbb{E}[\phi^\varepsilon(|\xi|^2/2)] = 0$. Thus, the functions $\psi_{n,k}$ and $\psi_{n,k}^\varepsilon$ all have Hermite rank at least 2, and Assumption (A6) holds. Thus, Theorem 2 follows from Theorem 1. Since the proof of Theorem 1 is based on the so-called method of moments, it is an immediate by-product that, under a relevant moment assumption, convergence of moments holds.

Proposition 1. *Let q be an integer. Under the assumptions of Theorem 2, if moreover $\mathbb{E}[Z_0^{4q \vee 4}] < \infty$ in the smooth case, or $\mathbb{E}[Z_0^{q\alpha}] < \infty$ and $\int_{\mathbb{R}^{2m}} (|\phi(x)| \vee |\phi(|x|^2)|^q (1 + |x|)^{-\alpha}\, \mathrm{d}x < \infty$ in the non-smooth case, then, if q is even,*

$$\lim_{n\to\infty} \mathbb{E}[S_n^q(\phi)] = \frac{(q)!}{2^{q/2}(q/2)!} (\sigma_m^2(\phi) + \gamma\kappa_4\rho_m^2(\phi))^{q/2},$$

and $\lim_{n\to\infty} \mathbb{E}[S_n^q(\phi)] = 0$ if q is odd.

4. Functional central limit theorem for the empirical spectral measure

The empirical spectral distribution function is defined as

$$\hat{F}_{m,n}(x) = K^{-1} \sum_{k=1}^{K} \mathbf{1}_{[0,x]}(2\pi \bar{I}_{n,k}), \quad x \geqslant 0,$$

where, as before $K=[(n-m)/2m]$ and $m\geqslant 1$ is an integer. In the case $m=1$, it has been shown by Freedman and Lane (1980) and Kokoszka and Mikosch (2000) that under the only assumption that $\mathbb{E}(Z_t^2)<\infty$, $\sup_{x\geqslant 0}|\hat{F}_{1,n}(x)-F_1(x)|$ converges in probability to zero, where $F_1(x)=1-\mathrm{e}^{-x}$ is the standard exponential cumulative distribution function. Kokoszka and Mikosch (2000) also proved that if the distribution of Z_0 satisfies the Cramer condition (A1), if $\mathbb{E}(|Z_t|^6)<\infty$ and the first six moments of Z_0 coincide with those of a standard normal variable, then $\lim_{n\to\infty} n^{s/2}\mathbb{E}[(\hat{F}_{1,n}(x)-F_1(x))^s]=0$ for $s=1,3$ and $\lim_{n\to\infty} n\mathbb{E}[(\hat{F}_{1,n}(x)-F_1(x))^2]=2F_1(x)(1-F_1(x))$. But these authors were unable to derive convergence in distribution of $\sqrt{n}(\hat{F}_{1,n}(x)-F_1(x))$ and asked if a functional central limit theorem can be proved. Applying Theorem 2, we prove here that under (A1) and a suitable moment condition, the functional central limit theorem holds, and that $n^q\mathbb{E}[(\hat{F}_{1,n}(x)-F_1(x))^{2q}]$ converges to $\{2F_1(x)(1-F_1(x))\}^q$ under the only additional assumption that $\kappa_4=0$. Define $F_m(x)=((m-1)!)^{-1}\int_0^x t^{m-1}\mathrm{e}^{-t}\,\mathrm{d}t$, the distribution function of the $\Gamma(m,1)$ distribution.

Theorem 3. *If Assumption* (A1) *holds and if* $\mathbb{E}(|Z_0|^8)<\infty$, *then* $\sqrt{n}(\hat{F}_{m,n}(x)-F_m(x))$ *converges in the space* $\mathscr{D}([0,\infty[)$ *of left-limited right-continuous* (*càdlàg*) *functions on* $[0,\infty)$ *to the Gaussian process* $G_m(x)$ *with covariance function*

$$\mathbb{E}[G_m(x)G_m(y)]=2mF_m(x\wedge y)(1-F_m(x\vee y))+\frac{\kappa_4\mathrm{e}^{-x-y}x^m y^m}{((m-1)!)^2}.$$

If $q\geqslant 4$ *is an integer such that* $\mathbb{E}(|Z_0|^q)<\infty$, *then*

$$\lim_{n\to\infty} n^{q/2}\mathbb{E}[(\hat{F}_{m,n}(x)-F_m(x))^q]$$

$$=\begin{cases}0 & \text{if } q \text{ is odd},\\ \left(2mF_m(x)(1-F_m(x))+\dfrac{\kappa_4\mathrm{e}^{-2x}x^{2m}}{((m-1)!)^2}\right)^{q/2} & \text{if } q \text{ is even}.\end{cases}$$

Remark.

- If $\kappa_4=0$ then the limit process is the same as if $(Z_t)_{t\in\mathbb{Z}}$ were a Gaussian white noise, or, equivalently, if the periodogram ordinates $2\pi\bar{I}_{n,k}$ were i.i.d. random variables with $\Gamma(m,1)$ distribution (i.i.d. exponentials in the case $m=1$). If enough moments of Z_0 are finite, the limiting moments are also the same as in the Gaussian case. Thus, the difference with the behaviour of an i.i.d. sequence appears only through the fourth cumulant.
- The proof of Theorem 3 is split into two parts. The convergence of finite distribution is an immediate consequence of Theorem 2 and holds under finiteness of the fourth moment of Z_0 only. Tightness is proved using the criterion for empirical processes of Shao and Yu (1996) and needs finiteness of the eighth moment of Z_0.

5. Proof of Theorem 1

Theorem 1 is proved by means of the method of moments and Edgeworth expansions. Thus, the first step is to prove a central limit theorem in the case of smooth functions and when all the moments of Z_0 are finite. Recall that we defined $K=[(n-m)/2m]$ and $S_n=\sum_{k=1}^{K}\beta_{n,k}\psi_{n,k}(W_{n,k})$, and $W_{n,k}$ is defined in (1).

Proposition 2. *Assume that $(Z_t)_{t\in\mathbb{Z}}$ is a centered unit variance white noise such that for all integers s, $\mathbb{E}(|Z_0|^s)<\infty$. Assume that the functions $\psi_{n,k}$ are C^∞ with same compact support and*

$$\forall r\in\mathbb{N},\quad \exists C_r,\quad \forall n,\quad \forall k\leqslant K,\quad M_{0,r}(\psi_{n,k})\leqslant C_r. \tag{11}$$

Let $(\beta_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array of reals such that Assumptions (A2)–(A5) *hold. Then S_n is asymptotically centered Gaussian with variance $\sigma^2+\tau\kappa_4/4$. Moreover, asymptotic normality still holds without Assumptions* (A4) *or* (A5) *in the following cases.*

- *If $\mu_n=\mathrm{o}(n^{2/3})$ then Assumption* (A5) *is not needed and Assumption* (A4) *holds with $\tau=0$.*
- *If for all $k\leqslant K$ and all $j=1,\dots,2m$, $C_2(\psi_{n,k},j)=0$, then Assumptions* (A4) *and* (A5) *are not needed.*

We must now relax the assumption that Z_0 has finite moments of all orders. Define $Z_t^{(M)}=\sigma_M^{-1}Z_t\mathbf{1}_{\{|Z_t|\leqslant M\}}$ and with $\sigma_M^2=\mathbb{E}((Z_t\mathbf{1}_{\{|Z_t|\leqslant M\}})^2)$. Define $W_{n,k}^{(M)}$ in the same way as $W_{n,k}$, replacing Z by $Z^{(M)}$. Without loss of generality, we can assume that for all M, $\mathbb{E}(Z_t^{(M)})=0$, since discrete Fourier transforms are computed at Fourier frequencies. Indeed, since for any Fourier frequency $x_k=2k\pi/n$, $(1\leqslant k<n/2)$, it holds that $\sum_{t=1}^{n}\mathrm{e}^{\mathrm{i}tx_k}=0$, we can replace $Z_t^{(M)}$ by $Z_t^{(M)}-\mathbb{E}[Z_t^{(M)}]$ in the definition of $W_{n,k}^{(M)}$.

Lemma 1. *Let $(Z_t)_{t\in\mathbb{Z}}$ be an i.i.d. sequence of zero-mean random variables with finite moment of order 4. Let $(\beta_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array of real numbers such that $\sum_{k=1}^{K}\beta_{n,k}^2=1$. Assume that $(\psi_{n,k})_{1\leqslant k\leqslant K}$ is a triangular array of twice continuously differentiable (C^2) functions with same compact support $\mathscr{K}$ and that there exists a constant C such that*

$$\forall n,\quad \forall k\leqslant K,\quad M_{0,2}(\psi_{n,k})\leqslant C. \tag{12}$$

Then

$$\lim_{M\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\sum_{k=1}^{K}\beta_{n,k}\{\psi_{n,k}(W_{n,k})-\psi_{n,k}(W_{n,k}^{(M)})\}\right)^2=0.$$

Proposition 3. *Assume that $(Z_t)_{t\in\mathbb{Z}}$ is a unit variance white noise such $\mathbb{E}(|Z_0|^4)<\infty$. Assume that for all $1\leqslant k\leqslant K\psi_{n,k}$ is compactly supported C^∞ and* (11) *holds. Let $(\beta_{n,k})_{1\leqslant k\leqslant K}$ be a triangular array of reals such that Assumptions* (A2)–(A5) *hold. Then S_n is asymptotically centered Gaussian with variance $\sigma^2+\tau\kappa_4/4$.*

Proof of Proposition 3. Define $S_n^{(M)}=\sum_{k=1}^{K}\beta_{n,k}\psi_{n,k}(W_{n,k}^{(M)})$. Applying Proposition 2 and Lemma 1, we get

$$\forall M\in\mathbb{N},\quad S_n^{(M)}\xrightarrow{(\mathrm{d})}\mathscr{N}(0,\sigma_M^2+\tau\kappa_4^{(M)}/4) \tag{13}$$

$$\lim_{M\to\infty} \limsup_{n} \mathbb{E}(S_n^{(M)} - S_n)^2 = 0, \tag{14}$$

where $\stackrel{(d)}{\to}$ denotes convergence in distribution and $\kappa_4^{(M)}$ is the fourth cumulant of $Z_0^{(M)}$. Since $\lim_{M\to\infty}\kappa_4^{(M)} = \kappa_4$, we can apply Theorem 4.2 in Billingsley (1968) to conclude the proof of Proposition 3.

To conclude the proof of Theorem 1, there only remains to replace the sequence $\psi_{n,k}$ by a sequence of smooth functions. □

Lemma 2. *Assume either*

- (A1) *holds, there exists an integer* $\alpha \geqslant 3$, *an integer* $\beta \geqslant 4$ *and a constant* C *such that for all* n *and* $1 \leqslant k \neq l \leqslant K$ (6) *holds and* $\mathbb{E}(|Z_t|^{\alpha\vee\beta}) < +\infty$;
- *there exists an integer* v *and a constant* C *such that for all* n *and* $1 \leqslant k \leqslant K$, $\psi_{n,k} \in \mathscr{S}_v^2$ *and* $M_{v,2}(\psi_{n,k}) \leqslant C$, *and* $\mathbb{E}(|Z_t|^{2v\vee 4}) < +\infty$.

Then $\max_n \max_{1\leqslant k\leqslant K} \|\psi_{n,k}\| < \infty$, *and for all triangular array of integers* $\beta_{n,k}$ *such that* $\sum_{k=1}^K \beta_{n,k}^2 = 1$, *for large enough* n,

$$\limsup_{n} \mathbb{E}\left[\left(\sum_{k=1}^{K} \beta_{n,k}\psi_{n,k}(W_{n,k})\right)^2\right] \leqslant \max_{n} \max_{1\leqslant k\leqslant K} \|\psi_{n,k}\|^2. \tag{15}$$

We can now conclude the proof of Theorem 1. Using the notations of Assumption (A6), denote $S_n(\varepsilon) = \sum_{k=1}^K \beta_{n,k}\psi_{n,k}^{\varepsilon}(W_{n,k})$. Applying Proposition 3, we have

$$\forall \varepsilon > 0, \quad S_n(\varepsilon) \stackrel{(d)}{\to} \mathscr{N}(0, \sigma^2(\varepsilon) + \tau(\varepsilon)\kappa_4/4).$$

Applying Lemma 2 (15) and Assumption (A6), we get

$$\limsup_{n} \mathbb{E}(S_n(\varepsilon) - S_n)^2 \leqslant \max_{n} \max_{1\leqslant k\leqslant K} \|\psi_{n,k} - \psi_{n,k}^{\varepsilon}\|^2 \leqslant \varepsilon^2.$$

Thus, $\lim_{\varepsilon\to 0} \limsup_n \mathbb{E}(S_n(\varepsilon) - S_n)^2 = 0$. Moreover, as noted above, under (A3) and (A4), $\lim_{\varepsilon\to 0} \sigma^2(\varepsilon) = \sigma^2$ and $\lim_{\varepsilon\to 0} \tau(\varepsilon) = \tau$. Thus, we can again conclude by applying Theorem 4.2 in Billingsley (1968).

6. Proof of Proposition 2 and of Lemmas 1 and 2

The proofs of Proposition 2 and of Lemmas 1 and 2 are based on a moment expansion for functions of the periodogram.

Lemma 3. *Let* $0 \leqslant s \leqslant d$ *be two integers. Let* $k = (k_1, \ldots, k_d)$ *be a* d*-tuple of pairwise distinct integers. Let* $\phi_1, \ldots, \phi_d$ *be* d *functions defined on* $\mathbb{R}^{2m}$ *and define* $\psi(\mathbf{x}) = \prod_{i=1}^d \phi_i(x_i)$, *with* $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^{2md}$. *Assume that one of the following assumption holds.*

(BR) (A1) *holds, and there exists an integer* $\alpha \geqslant s+2$ *such that* $N_{2md,\alpha}(\psi) < \infty$ *and* $\mathbb{E}[|Z_0|^{\alpha}] < \infty$.

(GH) *Denote* $r=(s-2md)^{+}+2$ (*with* $x^{+}=x\vee 0$). *Let* $v_1,\dots,v_d$ *be non-negative integers and denote* $v=(v_1+\cdots+v_d)\vee(s+2)$. *For all* $i=1,\dots,d$, $\phi_i\in\mathscr{S}^{r}_{v_i}$ *and* $\mathbb{E}(|Z_0|^{v})<\infty$.

Let τ_i *be the Hermite rank of* ϕ_i, $1\leqslant i\leqslant s$ *and* $\tau=\inf_{1\leqslant i\leqslant s}\tau_i$.

- *If* $\tau=2$ *or* 3 *then*

$$\mathbb{E}\left[\prod_{i=1}^{d}\phi_i(W_{n,k_i})\right]=n^{-s/2}\frac{s!\kappa_4^{s/2}}{2^{3s/2}(s/2)!}\sum_{j_1,\dots,j_s=1,\dots,2m}\prod_{j=1}^{s}C_2(\phi_i,j_i)\prod_{i=s+1}^{d}\mathbb{E}[\phi_i(\xi)]\mathbf{1}_{\{s\in 2\mathbb{N}\}}$$
$$+\sum_{r=[(2s+2)/3]}^{s}n^{-r/2}\mathbb{F}_{r,k}(\phi_1,\dots,\phi_d)+n^{-s/2}r_n(\phi_1,\dots,\phi_d,k)\varepsilon_n, \tag{16}$$

$$|\mathbb{F}_{r,k}(\phi_1,\dots,\phi_d)|\leqslant C\prod_{i=1}^{d}\|\phi_i\|\Delta_r(k), \tag{17}$$

where Δ_r *is uniformly bounded by one and vanishes outside a finite union of subspaces of* $\mathbb{R}^d$, *the greatest dimension of which is strictly less than* $d+(r-s)/2$.

- *If* $\tau\geqslant 4$, *then*

$$\mathbb{E}\left[\prod_{i=1}^{d}\phi_i(W_{n,k_i})\right]=\prod_{i=1}^{d}\mathbb{E}[\phi_i(\xi)]+n^{-s/2}r_n(\phi_1,\dots,\phi_d,k)\varepsilon_n. \tag{18}$$

ε_n *is a sequence which depends only on* d, s, $\alpha_1,\dots,\alpha_d$ *or* $v_1,\dots,v_d$ *and the distribution of* Z_0 *and such that* $\lim_n\varepsilon_n=0$. *The following bounds hold for* r_n:

- *if assumption* (BR) *holds*:

$$|r_n(\phi_1,\dots,\phi_d,k)|\leqslant N_{2md,\alpha}(\psi), \tag{19}$$

- *if assumption* (GH) *holds*:

$$|r_n(\phi_1,\dots,\phi_d,k)|=\prod_{i=1}^{d}M_{v_i,r}(\phi_i). \tag{20}$$

Remark.

- The constants involved in the above bounds are uniform w.r.t. n and $k_1,\dots,k_d$ but depend on d. This is why the central limit theorem must be proved by the method of moments. To use another classical method such as the Lindeberg method, or martingale techniques, bounds uniform with respect to d are necessary.
- In the context of Theorems 2 or 3, Lemma 3 is used with $\phi_1=\cdots=\phi_s=\psi$ for some function ψ such that $\|\psi\|<\infty$ and $C_2(\psi,1)=\cdots=C_2(\psi,2m):=C_2(\psi)$. Then the first term in the expansion of $\mathbb{E}[\prod_{i=1}^{d}\phi_i(W_{n,k_i})]$ becomes, if s is even and $\tau\geqslant 2$,

$$n^{-s/2}\frac{s!(m^2C_2^2(\psi)\kappa_4)^{s/2}}{2^{s/2}(s/2)!}\prod_{j=s+1}^{d}\mathbb{E}[\phi_j(\xi)].$$

- In (18), the product vanishes if $s>0$.
- The case $\tau=0$ is included in the case $s=0$.
- In view of Lemma 2, it is important that the bound (19) is explicit in terms of the norm $N_{2md,\alpha}(\psi)$.

6.1. Proof of Proposition 2

The proof is based on the method of moments. Denote $Y_{n,k}=\psi_{n,k}(W_{n,k})$ and $\sigma^2_{n,k}=\mathbb{E}[\psi^2_{n,k}(\xi)]$. With this notation, $S_n=\sum_{k=1}^K \beta_{n,k}Y_{n,k}$. Recall that $\sum_{k=1}^K \beta^2_{n,k}=1$ and denote $b_n=\max_{1\leqslant k\leqslant K}|\beta_{n,k}|$. Let $q\in\mathbb{N},\ q\geqslant 1$.

$$\mathbb{E}(S_n^q)=\sum_{v=1}^{q}\sum_{v,q}{}^{'}\frac{q!}{q_1!\cdots q_v!}\,\frac{1}{v!}\,A_n(q_1,\ldots,q_v),$$

$$A_n(q_1,\ldots,q_v)=\sum_{v,n}{}^{''}\prod_{i=1}^{v}\beta^{q_i}_{n,k_i}\,\mathbb{E}\left[\prod_{i=1}^{v}Y^{q_i}_{n,k_i}\right],$$

$\sum'_{v,q}$ extends on all v-tuples of positive integers $(q_1,\ldots,q_v)$ such that $q_1+\cdots+q_v=q$ and $\sum''_{v,n}$ extends on all v-uplets $(k_1,\ldots,k_v)$ of pairwise distinct integers in the range $\{1,\ldots,K\}$. For any v-tuple $(q_1,\ldots,q_v)$ such that $q_1+\cdots q_v=q$, let s be the number of indices i such that $q_i=1$. Under the assumptions of Proposition 2, the functions $\psi_{n,k}$ are uniformly smooth and bounded, thus Lemma 3 yields the following bound:

$$|A_n(q_1,\ldots,q_v)|\leqslant Cb_n^q\mu_n^{v-s/2}, \tag{21}$$

where the constant C depends on m, s, d, and the uniform bound for the functions $\psi_{n,k}$. If $s=0$, (21) is a consequence of the definition of μ_n and b_n. If $s>0$, we can apply Lemma 3

$$|A_n(q_1,\ldots,q_v)|\leqslant Cb_n\mu_n^v n^{-s/2}+C\sum_{r=[(2s+2)/3]}^{s} n^{-r/2}b_n^q\mu_n^{v+(r-s)/2}.$$

Since by definition $\mu_n\leqslant n$, this yields (21).

Let now u be the number of indices i such that $q_i=2$. Denote $w=v-s-u$. By definition, $q\geqslant s+2u+3w$. Thus, $v-s/2=s/2+u+w\leqslant q/2-w/2$. If $w>0$, then $v-s/2\leqslant q/2-\frac{1}{2}$, thus

$$|A_n(q_1,\ldots,q_v)|\leqslant C(b_n^2\mu_n)^{q/2}\mu_n^{-1/2}, \tag{22}$$

and this last term is o(1) under Assumption (A5).

Consider now $(q_1,\ldots,q_v)$ a v-tuple such that $w=0$, i.e., $s+u=v$ and $s+2u=q$.

- If s is even, Lemma 3 yields

$$\begin{aligned}A_n(q_1,\ldots,q_v)&=A_n(1,\ldots,1,2,\ldots,2)\\&=n^{-s/2}\frac{s!\,\kappa_4^{s/2}}{2^{3s/2}(s/2)!}\sum_{v,n}{}^{''}\prod_{i=1}^{s}\left\{\beta_{n,k_i}\sum_{j_i=1}^{2m}C_2(\psi_{n,k_i},j_i)\right\}\prod_{i=s+1}^{v}\beta^2_{n,k_i}\sigma^2_{n,k_i}\\&\quad+\mathrm{O}((b_n^2\mu_n)^{q/2}\mu_n^{-1}).\end{aligned}$$

• If s is odd (when $w=0$ and q is odd), Lemma 3 yields

$$|A_n(q_1,\dots,q_v)| \leqslant C(b_n^2\mu_n)^{q/2}\mu_n^{-1}. \tag{23}$$

Now we have proved that if q is odd, then $\lim_{n\to\infty}\mathbb{E}[S_n^q]=0$. Indeed, if q is odd, then either $w>0$, or $w=0$ and s is odd. In both cases, (22) and (23) imply that $A_n(q_1,\dots,q_v)=\mathrm{o}(1)$. Consider now an even q and a v-tuple $(q_1,\dots,q_v)$ such that $w=0$. The leading term in the expansion of $\mathbb{E}(S_n^q)$ is thus, (note that $v=(q+s)/2$ and denote $t=s/2$),

$$\tilde{s}_{n,q} = \frac{q!}{(q/2)!2^{q/2}}\sum_{t=0}^{q/2}\binom{q/2}{t}\binom{q/2}{t}n^{-t}$$

$$\times \sum_{t+q/2,n}'' \prod_{i=1}^{2t}\beta_{n,k_i}\sum_{j_{ii}=1}^{2m}C_2(\psi_{n,k_i},j_{ii})\prod_{i=2t+1}^{t+q/2}\beta_{n,k_i}^2\sigma_{n,k_i}^2.$$

Denote

$$s_n^2=\sum_{k=1}^K\beta_{n,k}^2\|\psi_{n,k}\|^2+\frac{\kappa_4}{4n}\sum_{1\leqslant k\neq l\leqslant K}\beta_{n,k}\beta_{n,l}\sum_{i,j=1,\dots,2m}C_2(\psi_{n,k},i)C_2(\psi_{n,l},j).$$

s_n^2 is the leading term of $\mathbb{E}[S_n^2]$ and Assumptions (A3) and (A4) imply that $\lim_{n\to\infty}s_n^2=\sigma^2+\tau\kappa_4/4$. Since $b_n=\mathrm{o}(1)$ (Assumption (A2)), it also holds that

$$s_n^q=\sum_{t=0}^{q/2}\binom{\frac{q}{2}}{t}\left(\sum_{k=1}^K\beta_{n,k}^2\sigma_{n,k}^2\right)^{q/2-t}$$

$$\times\left(\frac{\kappa_4}{4n}\sum_{1\leqslant k_1\neq k_2\leqslant \mu_n}\beta_{n,k_1}\beta_{n,k_2}\sum_{j_1,j_2=1,\dots,2m}C_2(\psi_{n,k_1},j_1)C_2(\psi_{n,k_2},j_2)\right)^t$$

$$=\left(\frac{q!}{2^{q/2}(q/2)!}\right)^{-1}\tilde{s}_{n,q}(1+\mathrm{O}(b_n)),$$

and finally

$$\lim_{n\to\infty}\mathbb{E}(S_n^q)=\frac{q!}{2^{q/2}(q/2)!)}(\sigma+\tau\kappa_4/4)^q,$$

which concludes the proof of Proposition 2 in the general case.

Proof of Proposition 2 in the case $\mu_n=\mathrm{o}(n^{2/3})$. Let $q_1,\dots,q_u$ be such that $\#\{i,q_i=1\}=s$. Then $\sum_{\{i,q_i\geqslant 2\}}(q_i-2)=q-2u+s$. Since $\sum_{k=1}^K\beta_{n,k}^2=1$, we have, by definition of μ_n, $\sum_{k=1}^K|\beta_{n,k}|=\mathrm{O}(\mu_n^{1/2})$. Thus, keeping the same notations, we get

$$|A_n(q_1,\dots,q_v)|\leqslant C\mu_n^{s/2}b_n^{q-2u+s}n^{-s/3},$$

where the term $n^{-s/3}$ comes from the expansion in Lemma 3 since the terms in that expansion are of order $n^{-r/2}$ with $r\geqslant 2s/3$. Thus, if $s>0$,

$$|A_n(q_1,\dots,q_v)|=\mathrm{O}((\mu_n/n^{2/3})^{s/2})=\mathrm{o}(1).$$

If $s=0$ then either $u<q/2$ or $u=q/2$ and $q_1=\cdots=q_u=2$. In both cases, the condition $b_n=\mathrm{o}(1)$ yields the required limit.

Proof of Proposition 2 in the case $\tau\geqslant 4$. Assume that for all n and $1\leqslant k\leqslant K$, the Hermite rank of $\psi_{n,k}$ is at least 4. This yields

$$A_n(q_1,\dots,q_v)=\sum_{v,n}\prod_{i=1}^{v}\beta_{n,k_i}^{q_i}\left\{\prod_{i=1}^{v}\mathbb{E}[\psi_{n,k_i}^{q_i}(\xi)]+\mathrm{o}(n^{-s/2})\right\}.$$

The expectation term above vanishes when the number (s) of indices i such that $q_i=1$ is not zero. Thus, applying (18) and the definition of b_n and s, we get, for such v-tuples,

$$|A_n(q_1,\dots,q_v)|\leqslant C\varepsilon_n n^{-s/2}\left(\sum_{k=1}^{K}|\beta_{n,k}|\right)^{s}b_n^{v-s},$$

where $\lim_{n\to\infty}\varepsilon_n=0$. Since $\sum_{k=1}^{K}\beta_{n,k}^2=1$, applying Hölder inequality, we have $\sum_{k=1}^{K}|\beta_{n,k}|=\mathrm{O}(\sqrt{n})$, thus if $s\neq 0$, $A_n(q_1,\dots,q_v)=\mathrm{o}(1)$. If $s=0$ and $v<q/2$, then as before $A_n(q_1,\dots,q_v)=\mathrm{o}(1)$. If $s=0$ and $v=q/2$, then

$$A_n(2,\dots,2)=\sum_{q/2,n}\prod_{i=1}^{q/2}\sigma_{n,k}^2+\mathrm{o}(1).$$

The proof of Proposition 2 in the case $\tau\geqslant 4$ is concluded as in the general case by noting that under the Lindeberg condition $b_n=\mathrm{o}(1)$, $s_n^q=\sum_{q/2,n}\prod_{i=1}^{q/2}\sigma_{n,k}^2+\mathrm{o}(1)$.

6.2. Proof of Lemma 1

Define $\tilde{\sigma}_M^2=\mathbb{E}(Z_t^2\mathbf{1}_{\{|Z_t|>M\}})$ and $\tilde{Z}_t^{(M)}=\tilde{\sigma}_M^{-1}Z_t\mathbf{1}_{\{|Z_t|>M\}}$. Define $\tilde{W}_{n,k}^{(M)}$ with respect to the sequence $(\tilde{Z}_t^{(M)})$ as $W_{n,k}$ is defined with respect to (Z_t) in (1). With these notations, we have $W_{n,k}=\sigma_M W_{n,k}^{(M)}+\tilde{\sigma}_M\tilde{W}_{n,k}^{(M)}$ and

$$\mathbb{E}\left[\left(\sum_{k=1}^{K}\beta_{n,k}\{\psi_{n,k}(W_{n,k})-\psi_{n,k}(W_{n,k}^{(M)})\}\right)^2\right]$$

$$=\sum_{k=1}^{K}\beta_{n,k}^2\left\{\mathbb{E}[\psi_{n,k}^2(W_{n,k})]+\mathbb{E}[\psi_{n,k}^2(W_{n,k}^{(M)})]-2\mathbb{E}(\psi_{n,k}(W_{n,k})\psi_{n,k}(W_{n,k}^{(M)}))\right\}$$

$$+\sum_{1\leqslant k\neq l\leqslant K}\beta_{n,k}\beta_{n,l}\mathbb{E}\left[\{\psi_{n,k}(W_{n,k})-\psi_{n,k}(W_{n,k}^{(M)})\}\{\psi_{n,l}(W_{n,l})-\psi_{n,l}(W_{n,l}^{(M)})\}\right]$$

$$=:A_{n,M}+B_{n,M}.$$

By assumption, the functions $\psi_{n,k}$ are differentiable and their first derivatives are uniformly bounded with respect to n and k. Thus, applying the mean-value theorem, we get

$$0\leqslant A_{n,m}\leqslant C\sum_{k=1}^{K}\beta_{n,k}^2\mathbb{E}[\|W_{n,k}-W_{n,k}^{(M)}\|^2].$$

Since the r.v.'s $(Z_t, Z_t^{(M)})$ are i.i.d., we get, applying the definition of $W_{n,k}$ and $W_{n,k}^{(M)}$,

$$\mathbb{E}[\|W_{n,k} - W_{n,k}^{(M)}\|^2] \leqslant C\mathbb{E}[|Z_0 - Z_0^{(M)}|^2].$$

Since $\sum -k = 1^K \beta_{n,k}^2 = 1$, we get

$$\limsup_n A_{n,m} \leqslant C\mathbb{E}[|Z_0 - Z_0^{(M)}|^2],$$

Since $\mathbb{E}[Z_t^2] < \infty$, we can apply the bounded convergence theorem to obtain $\lim_{M\to\infty} \limsup_n A_{n,M} = 0$.

To deal with the second term $B_{n,M}$ we need an Edgeworth expansion up to the order n^{-1} of the expectations in $B_{n,m}$. These expansions will be shown to be valid in Section 8.1, and yield

$$\sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,l}(W_{n,l})]$$

$$= \frac{\kappa_4}{4n} \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l} \sum_{j_1,j_2=1,\dots,2m} C_2(\psi_{n,k}, j_1)C_2(\psi_{n,l}, j_2) + \mathrm{o}(1),$$

$$\sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\psi_{n,k}(W_{n,k}^M)\psi_{n,l}(W_{n,l}^M)]$$

$$= \frac{\kappa_4(M)}{4n} \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l} \sum_{j_1,j_2=1,\dots,2m} C_2(\psi_{n,k}, j_1)C_2(\psi_{n,l}, j_2) + \mathrm{o}(1),$$

$$\sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l}\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,l}(W_{n,l}^{(M)})]$$

$$= \frac{\mathrm{cum}(Z_0, Z_0, Z_0^{(M)}, Z_0^{(M)})}{4n} \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l}$$

$$\times \sum_{j_1,j_2=1,\dots,2m} C_2^{(M)}(\psi_{n,k}, j_1)C_2(\psi_{n,l}, j_2) + \mathrm{o}(1),$$

where $\kappa_4(M)$ is the fourth-order cumulant of $Z_0^{(M)}$ and $C_2^{(M)}(\psi, j) = \mathbb{E}[H_2(\xi_j^{(1)})\psi(\sigma_M\xi^{(1)} + \tilde{\sigma}_M\xi^{(2)})]$. Thus,

$$B_{n,m} = \frac{\kappa_4 + \kappa_4(M)}{4n} \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l} \sum_{j_1,j_2=1,\dots,2m} C_2(\psi_{n,k}, j_1)C_2(\psi_{n,l}, j_2)$$

$$-2\,\frac{\mathrm{cum}(Z_0, Z_0, Z_0^{(M)}, Z_0^{(M)})}{4n} \sum_{1\leqslant k\neq l\leqslant K} \beta_{n,k}\beta_{n,l}$$

$$\times \sum_{j_1,j_2=1,\dots,2m} C_2^{(M)}(\psi_{n,k}, j_1)C_2(\psi_{n,l}, j_2) + \mathrm{o}(1).$$

Under (12), the coefficients $C_2^{(M)}(\psi_{n,k}, j)$ converge to $C_2(\psi_{n,k}, j)$ as M tends to infinity, uniformly with respect to n and k. By the bounded convergence theorem, under the assumption that $\mathbb{E}[Z_0^4] < \infty$, the following limits also hold:

$$\lim_{M\to\infty} \kappa_4(M) = \lim_{M\to\infty} \mathrm{cum}(Z_0, Z_0, Z_0^{(M)}, Z_0^{(M)}) = \kappa_4.$$

Consequently, $\lim_{M\to\infty} \limsup_n |B_{n,M}| = 0$.

6.3. Proof of Lemma 2

Under the assumptions of Lemma 2, using Lemma 3, it is easily seen that for all $1 \leqslant k \neq j \leqslant K$, the following expansions are valid.

$$\mathbb{E}[\psi_{n,k}^2(W_{n,k})] = \|\psi_{n,k}\|^2 + \mathrm{O}(n^{-1/2}),$$

$$\mathbb{E}[\psi_{n,k}(W_{n,k})\psi_{n,j}(W_{n,j})] = \frac{\kappa_4}{4n} \sum_{1\leqslant i_1, i_2, \leqslant 2m} C_2(\psi_{n,k}, i_1) C_2(\psi_{n,j}, i_2)$$

$$+ n^{-1/2}\mathbb{F}(\psi_{n,k}, \psi_{n,j}) + \mathrm{o}(n^{-1}),$$

$$|\mathbb{F}(\psi_{n,k}, \psi_{n,j})| \leqslant C\|\psi_{n,k}\| \|\psi_{n,j}\| \Delta(k, j),$$

where Δ vanishes outside a finite union of subspace of $\mathbb{R}^2$ of dimension at most 1, and the terms $\mathrm{O}(n^{-1/2})$ and $\mathrm{o}(n^{-1})$ are uniform because of the assumptions of Lemma 2. Summing these expressions yields Lemma 2.

7. Proof of Theorem 3

We need only prove the tightness of the sequence $\xi_n(x) := \sqrt{K}\{\hat{F}_n(x) - F_m(x)\}$ in the space $\mathscr{D}([0, M])$ of right-continuous, left-limited functions on $[0, M]$. For that we must compute the moments of $\xi_n(x) - \xi_n(y)$ for some $0 \leqslant x < y \leqslant M$. Denote $\psi_{x,y}(t) = \mathbf{1}_{\{x<t\leqslant y\}} - (F_m(y) - F_m(x))$. Let q be a positive integer and let $m_{n,q}(x, y) = \mathbb{E}[(\xi_n(x) - \xi_n(y))^{2q}]$. Using the same notations as in the proof of Proposition 2, we have the expansion

$$m_{n,q}(x, y) = \sum_{v=1}^{q} \sum_{v,q}' \frac{q!}{q_1! \ldots q_v!} \frac{1}{v!} A_n(q_1, \ldots, q_v),$$

$$A_n(q_1, \ldots, q_v) = n^{-q/2} \sum_{v,n}'' \mathbb{E}\left[\prod_{i=1}^{v} \psi_{x,y}^{q_i}(2\pi \bar{I}_{n,k_i})\right].$$

We now use Lemma 3 to obtain an expansion of the expectation above under Assumption (A1). Denote $K(x, y) = \mathbb{E}[(Y - m)\mathbf{1}_{\{x\leqslant Y\leqslant y\}}$ where Y is a $\Gamma(m, 1)$ r.v. For a given v-tuple $(q_1, \ldots, q_v)$, as in the proof of Proposition 2, denote s *the* number of indices j such that $q_j = 1$. Assuming Z_0 has enough finite moments, we get

$$\mathbb{E}\left[\prod_{i=1}^{v} \psi_{x,y}^{q_i}(2\pi \bar{I}_{n,k_i})\right] = n^{-s/2} \frac{s! K^s(x, y) \kappa_4^{s/2}}{2^{s/2}(s/2)!} \prod_{\{j, q_j \geqslant 2\}} \mathbb{E}[\psi_{x,y}^{q_j}(|\xi|^2/2z)] \mathbf{1}_{\{s\in 2\mathbb{N}\}}$$

$$+ \sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \mathbb{F}_{r,k}(\psi_{x,y}^{q_1}, \ldots, \psi_{x,y}^{q_v})$$

$$+ n^{-s/2} r_n(\psi_{x,y}^{q_1}, \ldots, \psi_{x,y}^{q_v}, k).$$

We must now bound all these terms by powers of $y - x$. It is easily seen that there exists a constant C such that $|K(x, y)| \leqslant C(y - x)$ and $|\mathbb{E}[\psi_{x,y}^{q_j}(\xi)]| \leqslant C(y - x)$. Thus, the first term is bounded by $n^{-s/2}(y - x)^v$. Since it also holds that $\|\psi_{x,y}^q\|^2 \leqslant C(y - x)$ and $N_\alpha(\psi_{x,y}^q) \leqslant C(y - x)$ for any positive integer q, we get

$$\sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \mathbb{F}_{r,k}(\psi_{x,y}^{q_1}, \ldots, \psi_{x,y}^{q_v})$$

$$\leqslant C \sum_{r=[(2s+2)/3]}^{s} n^{-r/2} \Delta_r(k)(y - x)^v \leqslant C n^{v-s/2}(y - x)^v,$$

$$n^{-s/2} r_n(\psi_{x,y}^{q_1}, \ldots, \psi_{x,y}^{q_v}, k) \leqslant C n^{-(s+1)/2}(y - x)^v.$$

Altogether, we get

$$A_n(q_1, \ldots, q_v) \leqslant C n^{-q/2} n^{v-s/2}(y - x)^v = C n^{-q/2+v-s/2}(y - x)^v.$$

Since for a given q, v is at least equal to one and at most equal to q, we get for $|y - x| \leqslant 1/n$,

$$m_{n,q}(x, y) \leqslant C(n^{-q/2+1}|x - y| + |y - x|^{q/2}).$$

If $|y - x| \geqslant 1/n$, then since $v \leqslant (q + s)/2$, it always holds that

$$m_{n,q}(x, y) \leqslant C n^{-(q+s)/2}(n|y - x|)^v \leqslant C n^{-(q+s)/2}(n|y - x|)^{(q+s)/2} \leqslant C|y - x|^{q/2}.$$

Finally, we get, for $q = 4$, provided that $\mathbb{E}[|Z_0|^8] < \infty$,

$$\mathbb{E}[(\xi_n(x) - \xi_n(y))^4] \leqslant C(n^{-1}|x - y| + |y - x|^2). \tag{24}$$

Applying Theorem 2.1 (Remark 2.1) in Shao and Yu (1996), (24) ensures the tightness of the empirical spectral process.

8. Proof of Lemma 3

Let $k = (k_1, \ldots, k_d)$ be a d-tuple of pairwise distinct integers. Let $\xi^{(1)}, \ldots, \xi^{(d)}$ be d independent $2m$-dimensional standard Gaussian vectors and denote $\boldsymbol{\xi} = (\xi^{(1)}, \ldots, \xi^{(d)})^{\mathrm{T}}$. Denote $\psi(\boldsymbol{\xi}) = \prod_{j=1}^{d} \phi_j(\xi^{(j)})$. In Section 8.1, a general definition of the so-called "formal Edgeworth expansion" up to order μ^* is given, and it will be proved that assumptions (BR) and (GH) imply, respectively, that the assumptions of Theorem 19.4 in Bhattacharya and Rao (1976) and Theorem 3.17 in Götze and Hipp (1978) hold, so that these expansions are valid. We can then write

$$\mathbb{E}\left[\prod_{i=1}^{d} \phi_i(W_{n,k_i})\right] = \sum_{r=0}^{\mu^*} n^{-r/2} \mathbb{E}_{r,k}(\phi_1, \ldots, \phi_d) + n^{-s/2} \eta_n R_n(\phi_1, \ldots, \phi_d), \tag{25}$$

where the sequence η_n depends only on the distribution of Z_0 and μ^* and verifies $\lim_{n\to\infty}\eta_n = 0$. Moreover,

- under the assumption (BR), $\mu^* = \alpha - 2$ and

$$|R_n(\phi_1,\dots,\phi_d)| \leqslant N_\alpha(\psi),$$

- under the assumption (GH), $\mu^* = \nu - 2$ and

$$|R_n(\phi_1,\dots,\phi_d)| \leqslant \prod_{i=1}^{d} M_{v_i,r}(\phi_i).$$

We now give explicit expressions for the quantities $\mathbb{E}_{r,k}$. They derive from the general theory of Edgeworth expansions recalled in Section 8.1. In the context of discrete Fourier transforms computed at Fourier frequencies, we obtain

$$\mathbb{E}_{0,k}(\phi_1,\dots,\phi_d) = \prod_{i=1}^{d} \mathbb{E}(\phi_i(\xi_i)),$$

$$\mathbb{E}_{r,k}(\phi_1,\dots,\phi_d) = \sum_{t=1}^{r} \frac{1}{t!} \sum_{r,t}^{*} \frac{\chi_{v_1}(k)\dots\chi_{v_t}(k)}{v_1!\dots v_t!}\, \mathbb{E}[H_{v_1+\dots+v_t}(\xi)\psi(\xi)] \quad \text{for } r>0, \tag{26}$$

where $\sum_{r,t}^{*}$ extends over all t-tuples $\underline{v}$ of multi-indices $v_l := (v_l(1),\dots,v_l(2md)) \in \mathbb{N}^{2md}$, $l=1,\dots,t$ such that

$$|v_l| := v_l(1)+\dots+v_l(2md) \geqslant 3, \quad l=1,\dots,t \quad \text{and} \quad \sum_{l=1}^{t} |v_l| = r+2t. \tag{27}$$

and for $k \in \{1,\dots,K\}^{md}$ and $v \in \mathbb{N}^{2md}$, $\chi_v(k) = 2^{|v|/2}\kappa_{|v|}A_v(k)$ with

$$A_v(k) = n^{-1}\sum_{t=1}^{n}\prod_{j=1}^{d}\prod_{i=1}^{m} \cos(tx_{m(k_j-1)+i})^{v_{2m(j-1)+2i-1}} \sin(tx_{m(k_j-1)+i})^{v_{2m(j-1)+2i}}, \tag{28}$$

and $\kappa_{|v|}$ is the cumulant of order $|v|$ of Z_0 (see (50)).

If $\mu^* > s$, we first prove that the terms $(\mathbb{E}_{r,k})_{s+1\leqslant r\leqslant \mu^*}$ can be conveniently bounded. Clearly, $|A_v| \leqslant 1$, thus, for $0 \leqslant r \leqslant \mu^*$, there exists a constant C_d, uniform w.r.t. n and $k=(k_1,\dots,k_d)$ such that

$$|\mathbb{E}_{r,k}(\phi_1,\dots,\phi_d)| \leqslant C_d K(\phi_1,\dots,\phi_d),$$

with $K(\phi_1,\dots,\phi_d) = N_{2m,d,\alpha}(\psi)$ under Assumption (BR) and $K(\phi_1,\dots,\phi_d) = \prod_{i=1}^{d} M_{v,r}(\phi_i)$ under Assumption (GH). Thus, if $\mu^* > s$, we have

$$\left| \sum_{r=s+1}^{\mu^*} n^{-r/2}\mathbb{E}_{r,k}(\phi_1,\dots,\phi_d) \right| \leqslant Cn^{-(s+1)/2}K(\phi_1,\dots,\phi_d),$$

for some constant which depends only on the distribution of Z_0 and μ^*. Then

$$\left| \mathbb{E}\left[\prod_{i=1}^{d}\phi_i(W_{n,k_i})\right] - \sum_{r=0}^{s} n^{-r/2}\mathbb{E}_{r,k}(\phi_1,\dots,\phi_d) \right| \leqslant n^{-s/2}\varepsilon_n r_n(\phi_1,\dots,\phi_d), \tag{29}$$

where r_n satisfies either (19) or (20), and $\lim_{n\to\infty}\varepsilon_n = 0$.

For convenience, we now introduce the following definition. We will say that a function defined on $\mathbb{R}^d$ has the property $V(d,r,s)$ if it is uniformly bounded and if it vanishes outside a finite union of subspaces of $\mathbb{R}^d$ of dimension strictly less than $d-(r-s)/2$. To prove Lemma 3, we must thus prove that as functions of k, the quantities $\mathbb{E}_{r,.}(\phi_1,\dots,\phi_d)$ have the property $V(d,r,s)$, for $r<s$, and that, for $r=s$,

$$\mathbb{E}_{s,k}(\phi_1,\dots,\phi_d)=\frac{s!\kappa_4^{s/2}}{2^{3s/2}(s/2)!}\sum_{j_1,\dots,j_s=1,\dots,2m}\prod_{j=1}^{s}C_2(\phi_i,j_i)\prod_{i=s+1}^{d}\mathbb{E}[\phi_i(\xi)]\mathbf{1}_{\{s\in 2\mathbb{N}\}}$$

$$+\tilde{\mathbb{E}}_{s,k}(\phi_1,\dots,\phi_d),$$

where $\tilde{\mathbb{E}}_{s,.}(\phi_1,\dots,\phi_d)$ has property $V(d,s,s)$. To prove these properties, we must analyse separately the moments $\mathbb{E}[H_{v_1+\cdots+v_t}(\xi)\psi(\xi)]$ and the products of cumulants $\prod_{i=1}^{t}\chi_{v_i}$ that appear in the expression of the terms $\mathbb{E}_{r,k}$, in order to find conditions upon which these terms are non-vanishing.

Conditions for moments not to vanish. Since the ξ_i's are i.i.d. standard Gaussian, we have

$$\mathbb{E}[H_{v_1+\cdots+v_t}(\xi)\psi(\xi)]=\prod_{i=1}^{d}\mathbb{E}\left[\prod_{j=1}^{2m}H_{v_1(2m(i-1)+j)+\cdots+v_t(2m(i-1)+j)}(\xi_{2m(i-1)+j})\phi_i(\xi^{(i)})\right]. \tag{30}$$

Thus, the t-tuples of multi-indices $\underline{v}=(v_1,\dots,v_t)$ satisfying (27) and such that $\mathbb{E}[H_{v_1+\cdots+v_t}(\xi)\psi(\xi)]\neq 0$ must satisfy

$$|v_1+\cdots+v_t|\geqslant s\tau, \tag{31}$$

since by definition τ_i is the Hermite rank of ϕ_i $(1\leqslant i\leqslant s)$ and $\tau=\inf\{\tau_i, i=1,\dots,s\}$. Moreover, since $t\leqslant r\leqslant s$, the definition of $\sum_{r,t}^{*}$ implies that $|v_1+\cdots+v_t|=r+2t\leqslant 3r\leqslant 3s$. Thus,

- If $\tau\geqslant 4$, (31) is never fulfilled so that $E_{r,k}$ vanishes for all $r=1,\dots,s$ and all k, and (18) follows.
- If $\tau=3$, the coefficients $C(\phi_i,j)$ are identicaly vanishing and (31) implies that $r=s=t$ and $|v_l|=3$ for $l=1,\dots,s$. Thus, the leading term in the Edgeworth expansion (25) is

$$\mathbb{E}_{s,k}(\phi_1,\dots,\phi_d)=\frac{1}{s!}\sum_{s,s}^{*}\frac{\prod_{l=1}^{s}\chi_{v_l}(k)}{\prod_{l=1}^{s}v_l!}\mathbb{E}[H_{v_1+\dots+v_s}(\xi)\psi(\xi)].$$

- In the case $\tau=2$, all terms $\mathbb{E}_{r,k}$, $2s/3\leqslant r\leqslant s$, can be non vanishing. Note first that for $\tau=2$, (31) yields

$$\sum_{l=1}^{t}|v_l|=r+2t\geqslant 2s. \tag{32}$$

Moreover, it can be seen from (30) that the following condition must also hold:

$$\sum_{j=1}^{2m}\sum_{l=1}^{t}v_l(2m(i-1)+j)\geqslant 2,\quad 1\leqslant i\leqslant s; \tag{33}$$

Conditions for products of cumulants not to vanish. We first state a lemma which easily follows from the orthogonality properties of the sine and cosine functions computed at the Fourier frequencies.

Lemma 4. *Let $(k_1,\dots,k_d)$ be a d-tuple of pairwise distinct integers in $\{1,\dots,K\}$ and $v\in\mathbb{N}^{2md}$. Then there exist a constant $\gamma_v\in\mathbb{R}$ and a function δ_v such that*

$$A_v(k)=\gamma_v+\delta_v(k), \tag{34}$$

where $|\gamma_v|\leqslant 1$, $\gamma_v=0$ if v has at least one odd component (i.e. $v\in\mathbb{N}^{2md}\setminus(2\mathbb{N})^{2md}$), and

$\Delta 1$ *δ_v depends only on v;*
$\Delta 2$ *δ_v identically vanishes outside a finite union of strict hyperplanes of $\mathbb{R}^d$;*
$\Delta 3$ *$\forall k\in\mathbb{N}^d$, $|\delta_v(k)|\leqslant 1$.*

The exact value of the constant γ_v is irrelevant, except in the case $|v|=4$, and the components of v are all equal to zero, except two which are equal to 2. In that case, $\gamma_v=1/4$. Thus, for each v, $A_v(\cdot)$ is constant outside a finite union of strict hyperplanes of $\{1,\dots,K\}^d$. To illustrate these properties, we give two examples in the case $m=1$, $d=2$. Assume n is even and let $v=(2,0,1,0)$. Then $|v|=3$ and

$$\begin{aligned}A_v(k)&=n^{-1}\sum_{t=1}^{n}\cos^2(tx_{k_1})\cos(tx_{k_2})\\&=\frac{1}{4n}\sum_{t=1}^{n}\{2\cos(tx_{k_2})+\cos(2tx_{k_1}+tx_{k_2})+\cos(2tx_{k_1}-tx_{k_2})\}.\end{aligned}$$

Thus, $A_v(k)=1/4+\delta_v(k)$, where δ_v vanishes outside the lines $k_2=0$ and $2k_1\pm k_2=0$, where the equalities must hold modulo n. Consider now $v=(2,0,2,0)$. Then $|v|=4$ and

$$\begin{aligned}A_v(k)&=n^{-1}\sum_{t=1}^{n}\cos^2(tx_{k_1})\cos^2(tx_{k_2})\\&=\frac{1}{4}+\frac{1}{8n}\sum_{t=1}^{n}\{2\cos(2tx_{k_1})+2\cos(2tx_{k_2})\\&\quad+\cos(2t(x_{k_1}+x_{k_2}))+\cos(2t(x_{k_1}-x_{k_2}))\}.\end{aligned}$$

Thus, $A_v(k)=1/4+\delta_v(k)$, where δ_v vanishes outside the sets $2k_1=0\,(\mathrm{mod}\,n)$, $k_2=0\,(\mathrm{mod}\,n)$, $2k_2=0\,(\mathrm{mod}\,n)$, $k_1\pm k_2=0\,(\mathrm{mod}\,n)$.

In the case $\tau=3$, Property $\Delta 2$ in Lemma 4 implies that $\mathbb{E}_{s,.}(\phi_1,\dots,\phi_d)$ has property $V(d,s,s)$, and this concludes the proof of Lemma 3 in the case $\tau=3$.

From now on, it is assumed that $\tau=2$. For a given $\underline{v}=(v_1,\dots,v_t)$ satisfying (27) and (33), that will be refered to as *admissible* hereafter, we want to find conditions on the multi-index of integers k that insure that

$$\prod_{l=1}^{t}\chi_{v_l}(k)=\prod_{l=1}^{t}\kappa_{|v_l|}2^{|v_l|/2}A_{v_l}(k)=\prod_{l=1}^{t}2^{|v_l|/2}\kappa_{|v_l|}\times\prod_{l=1}^{t}(\gamma_{v_l}+\delta_{v_l}(k))\neq 0. \tag{35}$$

More precisely, Lemma 4 imply that the multi-indice k such that (35) holds belong to a finite union of subspaces of $\mathbb{R}^d$. We must find an upper bound for the dimension of these subspaces. We will denote $d(\underline{v})$ the greatest dimension of these subspaces. Lemma 3 will derive from a sharp estimate of $d(\underline{v})$. First of all, remark for any admissible $\underline{v}$ that

R1 If one of the multi-indices $v_1, \dots, v_t$ has at least one odd component, then property $\Delta 2$ of the functions δ_v yields $d(\underline{v}) < d$.
R2 If $|v_l| \geqslant 4$, $l = 1, \dots, t$, then $r + 2t = |v_1 + \cdots + v_t| \geqslant 4t$ and $r + 2t \leqslant 2r \leqslant 2s$. By (32), it follows that $r = s = 2t$, thus s is necessarily even and $|v_l| = 4$, $l = 1, \dots, s/2$.
R3 $|\prod_{l=1}^{t} \chi_{v_l}(k)| \leqslant 2^{t+r/2} \prod_{l=1}^{t} |\kappa_{|v_l|}|$

It will also be convenient to consider $\underline{v}$ as an array with $2md$ columns and t lines

$$\underline{v} = \begin{pmatrix} v_1(1) & \cdots & v_1(2md) \\ \vdots & & \vdots \\ v_t(1) & \cdots & v_t(2md) \end{pmatrix}.$$

Using array terminology, condition (32) means that the sum over all entries $\underline{v}$ is no less than $2s$, (27) implies that the sum of the entries of each line is at least 3 and (33) implies that the sum of the entries of the s first sets of $2m$ consecutive columns : 1 to $2m, 2m+1$ to $4m, \dots, 2(s-1)m+1$ to $2sm$, is at least 2. An array with only even integer entries will be said *even*.

Consider successively the cases (a) $r = s$ and s is odd, (b) $r = s$ and s is even, and (c) $r < s$, which is the most involved situation.

(a) If $r = s$ and s is odd and if $\underline{v}$ satisfies (27), then necessarily one v_l at least has an odd component, for any $t \in \{1, \dots, s\}$. In that case, Remark R1 shows that

$$|\mathbb{E}_{s,k}(\phi_1, \dots, \phi_d)| \leqslant C \|\phi_1\| \cdots \|\phi_d\| \Delta_s(k),$$

where Δ_s has the property $V(d,s,s)$, which is the claimed result in this case.

(b) If $r = s$ and s is even, then for any $t \in \{1, \dots, s\}$ and any non-even (with at least one odd entry) admissible $\underline{v}$, the product of cumulants $\prod_{l=1}^{t} \chi_{v_l}(k)$ has the property $V(d,s,s)$ by virtue of Lemma 4 (because at least one of the v_l's has an odd component) and is bounded uniformly in k, n. It follows that the contribution of non-even $\underline{v}$'s to $\mathbb{E}_{s,k}$, say $\mathbb{F}_{s,k}^{(1)}$, satisfies

$$|\mathbb{F}_{s,k}^{(1)}| \leqslant C \|\phi_1\| \cdots \|\phi_d\| \Delta_s^{(1)}(k) \tag{36}$$

for some constant C and a function $\Delta_s^{(1)}$ having the property $V(d,s,s)$. Consider now even and admissible $\underline{v}$'s (without any odd entry). Necessarily $|v_l| \geqslant 4$ for $l = 1, \dots, t$. By Remark R2, this implies that s is even, $t = s/2$, and for all $l = 1, \dots, s/2$, $|v_l| = 4$. Notice now that no entry of $\underline{v}$ can be equal to 4, otherwise (33) would not hold. Thus, for all $l = 1, \dots, s/2$, the entries of v_l are all vanishing except exactly two of them which are equal to 2, which implies that $\gamma_{v_l} = \frac{1}{4}$. In this case, (33) is equivalent to the fulfillment of

$$\forall i \in \{1, \dots, s\}, \quad \exists! j_i(\underline{v}) \in \{1, \dots, 2m\}, \quad v_1 + \cdots + v_{s/2}(2m(i-1) + j_i(\underline{v})) = 2, \tag{37}$$

$$\forall i \in \{1,\dots,s\}, \quad \forall j \in \{1,\dots,2m\}\backslash\{j_i(\underline{v})\}, \quad v_1 + \cdots + v_{s/2}(2m(i-1)+j) = 0, \tag{38}$$

$$\forall i \in \{s+1,\dots,d\}, \quad \forall j \in \{1,\dots,2m\}, \quad v_1 + \cdots + v_{s/2}(2m(i-1)+j) = 0. \tag{39}$$

For such a $\underline{v}$, we get

$$\mathbb{E}[H_{v_1+\cdots+v_t}(\xi)\psi(\xi)] = \prod_{i=1}^{s} C_2(\phi_i, j_i(\underline{v})) \prod_{i=s+1}^{d} \mathbb{E}[\phi_i(\xi^{(i)})],$$

$$\frac{\prod_{l=1}^{s/2} \chi_{v_l}(k)}{\prod_{j=1}^{s} v_l!} = \frac{\kappa_4^{s/2}}{4^{s/2}} \prod_{l=1}^{s/2}(1+4\delta_v(k)) = \frac{\kappa_4^{s/2}}{4^{s/2}} + \bar{\delta}_{\underline{v}}(k)$$

for some function $\bar{\delta}_{\underline{v}}$ bounded uniformly in k, n and which has the property $V(d,s,s)$. Conversely, to each s-tuple $(j_1,\dots,j_s) \in \{1,\dots,2m\}^s$, there correspond exactly $2^{-s/2}s!$ even $\underline{v}$'s that satisfy (27) and such that (37)–(39) hold with $j_i(\underline{v}) = j_i$, $i = 1,\dots,s$. Then, the overall contribution of even $\underline{v}$'s to $\mathbb{E}_{s,k}$ is

$$\frac{s!\kappa_4^{s/2}}{(s/2)!2^{3s/2}} \left\{ \sum_{c_1,\dots,c_s=1,\dots,2m} \prod_{i=1}^{s} C_2(\phi_i, c_i) \right\} \prod_{j=s+1}^{d} \mathbb{E}[\phi_j(\xi^{(j)})] + \mathbb{F}_{s,k}^{(2)} \tag{40}$$

for some function $\mathbb{F}_{s,k}^{(2)}$ such that

$$\mathbb{F}_{s,k}^{(2)} \leqslant C\|\phi_1\| \cdots \|\phi_d\| \Delta_s^{(2)}(k) \tag{41}$$

for some constant C and a function $\Delta_s^{(2)}$ having the property $V(d,s,s)$. Eqs. (36), (40) and (41) yield

$$\mathbb{E}_{s,k}(\phi_1,\dots,\phi_d) = \frac{s!\kappa_4^{s/2}}{(s/2)!2^{3s/2}} \left\{ \sum_{c_1,\dots,c_s=1,\dots,2m} \prod_{i=1}^{s} C_2(\phi_i, c_i) \right\}$$
$$\times \prod_{j=s+1}^{d} \mathbb{E}[\phi_j(\xi^{(j)})] + \mathbb{F}_{s,k}(\phi_1,\dots,\phi_d)$$

with

$$|\mathbb{F}_{s,k}(\phi_1,\dots,\phi_d)| \leqslant C\|\phi_1\| \cdots \|\phi_d\| \Delta_s(k),$$

where Δ_s vanishes outside a finite union of subspaces of $\mathbb{R}^d$ of dimension strictly less than d, i.e., has property $V(d,s,s)$.

(c) There only remains to consider the case $r < s$. Since a finite sum of functions which have the property $V(d,r,s)$ still has the property $V(d,r,s)$, by definition of $\mathbb{E}_{r,\cdot}$ and by definition of $d(\underline{v})$, we will have shown that $\mathbb{E}_{r,\cdot}$ has the property $V(d,r,s)$, thereby concluding the proof of Lemma 3 if we prove that

$$\text{for all admissible } \underline{v} \text{ and for all } r < s,\ d(\underline{v}) < d + (r-s)/2. \tag{42}$$

The rest of this section is devoted to the proof of (42). Let $l(\underline{v})$ be the number of indices i such that v_l has at least one odd component. Property R2 implies that if $r < s$ and if $\underline{v}$ satisfies (27), then necessarily there exists at least one v_l such that

$|v_l| = 3$, hence with an odd component. Thus, $l(\underline{v}) > 0$. By definition of $l(\underline{v})$, we have $r + 2t \geqslant 3l(\underline{v}) + 4(t - l(\underline{v}))$, whence $r + l(\underline{v}) \geqslant 2t$. Define $q = r + 2t - 2s$. By (32), q is nonnegative. Since $r < s$, we get for any $d \geqslant s$,

$$r + l(\underline{v}) \geqslant 2t = 2s + q - r > s + q,$$

$$l(\underline{v}) - q > s - r,$$

$$d - (l(\underline{v}) - q)/2 < d + (r - s)/2. \tag{43}$$

If we can prove that the following bound holds:

$$d(\underline{v}) \leqslant d - (l(\underline{v}) - q)/2 \tag{44}$$

then (43) implies (42). Thus, proving (44) will conclude the proof of Lemma 3.

Proof of (44). Denote $m(\underline{v}) = d - d(\underline{v})$, the minimum codimension of any subspace of $\mathbb{R}^d$ on which the product $\prod_{l=1}^{t} A_{v_l}$ does not identically vanish (this quantity is referred to as the NRES – number of restrictions – in Velasco, 2000). With this notation, (44) becomes

$$m(\underline{v}) \geqslant (l(\underline{v}) - q)/2. \tag{45}$$

Note that if the array $\underline{v}'$ is obtained from $\underline{v}$ by removing some lines, then $m(\underline{v}') \leqslant m(\underline{v})$.

(i) Our first argument is that if there are at most two odd component in any single column of the array $\underline{v}$, then $m(\underline{v})$ is at least equal to $l(\underline{v})/2$, since each line of the array (i.e. each v_l) with at least one odd component yields one restriction, and different lines will yield different restrictions, except if their odd components are in the same columns. Thus, (44) holds in this case.

(ii) If there exists at least one column with at least three odd components, let $z(\underline{v})$ denote the number of such columns and let $y(\underline{v})$ denote the total number of odd components in these columns. We now prove by induction on $y(\underline{v})$ that the following inequality holds:

$$m(\underline{v}) \geqslant \{(l(\underline{v}) - (y(\underline{v}) - 2z(\underline{v})))/2\} \vee 1. \tag{46}$$

We have proved this property for $y(\underline{v}) = 0$, but we cannot start the induction at 0 since if $y(\underline{v}) \neq 0$, then $y(\underline{v}) \geqslant 3$. Thus, we prove the property for $y(\underline{v}) = 3$, which implies $z(\underline{v}) = 1$. In that case, we can cancel one line of the array in such a way as to obtain a new array $\underline{v}'$ with $l(\underline{v}') = l(\underline{v}) - 1$ and $y(\underline{v}) = z(\underline{v}) = 0$. For that array, we have $m(\underline{v}') \geqslant l(\underline{v}')/2$, then

$$m(\underline{v}) \geqslant m(\underline{v}') \geqslant l(\underline{v}')/2 = (l(\underline{v}) - 1)/2 = (l(\underline{v}) - (y(\underline{v}) - 2z(\underline{v})))/2.$$

Induction. Let $y \geqslant 4$ and assume that the induction assumption is true for any $Y \in \{0\} \cup \{3, \ldots, y - 1\}$. Let $\underline{v}$ be an array such that $y(\underline{v}) = y$. As above, we cancel one of the line of the array and we obtain a new array $\underline{v}'$ with $l(\underline{v}') = l(\underline{v}) - 1$, $y(\underline{v}') < y(\underline{v})$ and $z(\underline{v}') \leqslant z(\underline{v})$. If $y(\underline{v}') = 0$, then $m(\underline{v}) \geqslant m(\underline{v}') \geqslant l(\underline{v}')/2 = (l(\underline{v}) - 1)/2 \geqslant (l(\underline{v}) - (y(\underline{v}) - 2z(\underline{v})))/2$ since by definition $y(\underline{v}) \geqslant 3z(\underline{v})$ and thus $y(\underline{v}) - 2z(\underline{v}) \geqslant 1$ as soon as $z(\underline{v}) \geqslant 1$.

If $y(\underline{v}') \neq 0$, then $3 \leqslant y(\underline{v}') \leqslant y(\underline{v}) - 1$ and we can apply the induction assumption. Thus we get

$$m(\underline{v}) \geqslant m(\underline{v}') \geqslant (l(\underline{v}') - (y(\underline{v}') - 2z(\underline{v}')))/2 = (l(\underline{v}) - (y(\underline{v}') - 2z(\underline{v}') + 1))/2.$$

Thus, we must prove that $y(\underline{v}') - 2z(\underline{v}') + 1 \leqslant y(\underline{v}) - 2z(\underline{v})$, i.e., $2(z(\underline{v}) - z(\underline{v}') + 1 \leqslant y(\underline{v}) - y(\underline{v}')$. If $z(\underline{v}) = z(\underline{v}')$, this is obvious since $y(\underline{v}') < y(\underline{v})$. If $z(\underline{v}') < z(\underline{v})$, then $y(\underline{v}) - y(\underline{v}') \geqslant 3(z(\underline{v}) - z(\underline{v}')) \geqslant 2(z(\underline{v}) - z(\underline{v}')) + 1$. This proves that the induction assumption holds for y.

Thus, (46) holds and to prove that (45) holds, we must now check that for an admissible array $\underline{v}$, we have $y(\underline{v}) - 2z(\underline{v}) \leqslant q$. Denote $w(\underline{v})$ the number of indices $j \in \{1, \dots, d\}$ such that the sum of all the entries of the columns $2m(j-1)+1, \dots, 2mj$ is exactly 1. Since the Hermite rank of $\phi_1, \dots, \phi_s$ is at least 2, then it is necessary that $w(\underline{v}) \leqslant d - s$, i.e., $d - w(\underline{v}) \geqslant s$. Thus, we have

$$2s + q = y(\underline{v}) + w(\underline{v}) + 2(d - z(\underline{v}) - w(\underline{v})) = 2d - w(\underline{v}) + y(\underline{v})$$

$$-2z(\underline{v}) \geqslant 2s + y(\underline{v}) - 2z(\underline{v}),$$

and thus $y(\underline{v}) - 2z(\underline{v}) \leqslant q$.

This concludes the proof of (46), and thus of Lemma 3. □

8.1. Validity of Edgeworth expansions

In this section, we prove that the Edgeworth expansions used in the previous sections are valid. Chen and Hannan (1980, Lemma 2) have adapted Theorem 19.3 of Bhattacharya and Rao (1976) to prove that under Assumption (A1), the Edgeworth expansion of the joint density of an arbitrary number of discrete Fourier transform is valid up to the order 2. That was all they needed since they considered the function log and were only proving consistency of their estimator. To consider more general functions, we should check the validity of the expansion up to an arbitrary order. We will omit this proof since the arguments of Chen and Hannan (1980) are easily generalized. We will only check the validity of Edgeworth expansions of moments using the result of Götze and Hipp (1978). We first state a version of Theorem 3.17 in Götze and Hipp (1978) with stronger assumptions, but which are easy to check in our context. Let $(\zeta_{n,k})_{1 \leqslant k \leqslant n}$ be a triangular array of independent a-dimensional vectors. Define $S_n = n^{-1} \sum_{k=1}^{n} \zeta_{n,k}$ and let $Q_s(\psi)$ be the formal Edgeworth expansion of $\mathbb{E}[\psi(S_n)]$ up to the order s, defined as (cf. Bhattacharya and Rao, 1976)

$$Q_s(\psi) = \sum_{r=0}^{s} n^{-r/2} \mathbb{E}_r(\psi), \tag{47}$$

$$\mathbb{E}_0(\psi) = \mathbb{E}(\psi(\xi)),$$

$$\mathbb{E}_r(\psi) = \sum_{t=1}^{r} \frac{1}{t!} \sum_{r,t}^{*} \frac{\chi_{v_1} \cdots \chi_{v_t}}{v_1! \dots v_t!} \mathbb{E}[H_{v_1+\cdots+v_t}(\xi)\psi(\xi)], \tag{48}$$

where ξ denotes a standard a-dimensional Gaussian vector, and with the following notations:

- $\sum_{r,t}^{*}$ extends over all t-tuples of multi-indices $v_1,\dots,v_t$ such that $v_i := (v_i(1),\dots, v_i(a)) \in \mathbb{N}^a$,

$$|v_i| := v_i(1) + \cdots + v_i(a) \geqslant 3, \quad \sum_{i=1}^{t} |v_i| = r + 2t, \tag{49}$$

- for $v \in \mathbb{N}^a$, $v! = \prod_{j=1}^{a} v(j)!$
- H_v is a multidimensional Hermite polynomial, i.e., $H_v(x_1,\dots,x_a) = H_{v(1)}(x_1)\dots H_{v(a)}(x_a)$,
- for $v = (v(1),\dots,v(a)) \in \mathbb{N}^a$, χ_v is the following cumulant:

$$\chi_v = \frac{1}{n}\mathrm{cum}\left(\underbrace{\sum_{t=1}^{n}\zeta_{n,k}^{(1)},\dots,\sum_{t=1}^{n}\zeta_{n,k}^{(1)}}_{v(1)\ \text{times}},\dots,\underbrace{\sum_{t=1}^{n}\zeta_{n,k}^{(a)},\dots,\sum_{t=1}^{n}\zeta_{n,k}^{(a)}}_{v(a)\ \text{times}}\right). \tag{50}$$

Denote $|x| = (x_1^2 + \cdots + x_a^2)^{1/2}$ the Euclidean norm of an a-dimensional vector and define, whenever possible,

$$\rho_{n,3} = n^{-1}\sum_{k=1}^{n}\mathbb{E}(|\zeta_{n,k}|^3), \tag{51}$$

$$\Delta_{n,s} = n^{-1}\sum_{k=1}^{n}\mathbb{E}[|\zeta_{n,k}|^s\{n^{-1/2}|\zeta_{n,k}|\mathbf{1}_{\{|\zeta_{n,k}|\leqslant n^{1/2}\}} + \mathbf{1}_{\{|\zeta_{n,k}|>n^{1/2}\}}\}]. \tag{52}$$

Theorem 4. *Let ψ be a C^{r+2} function on $\mathbb{R}^a$ and p be an integer such that for all $\beta \in \mathbb{N}^a$ with $\sum_{i=1}^{a}\beta_i = r+2$,*

$$\int_{-\infty}^{+\infty}\frac{|D^\beta\psi(x)|}{1+|x|^p}\,\mathrm{d}x \leqslant C(\psi),$$

for some finite constant $C(\psi)$. Assume that the variables $\zeta_{n,k}$ have finite moment of order $s+2$. If $\lim_{n\to\infty}\Delta_{n,s+2} = 0$, then there exists a constant C which depends only on a and the distribution of Z_0 such that, for large enough n,

$$|\mathbb{E}(\psi(S_n)) - Q_s(\psi)| \leqslant C(M_s(\psi) + C(\psi))\Delta_{n,s+2}n^{-s/2} + C\rho_{n,3}^{r+k+1}n^{-(r+a+1)/2}).$$

In particular, if $\rho_{n,3}$ is uniformly bounded, then $\mathbb{E}(\psi(S_n)) - Q_s(\psi) = \mathrm{o}(n^{-s/2})$ as soon as ψ is $C^{(s-a)^+ + 2}$ and the constants involved in the term $\mathrm{o}(n^{-s/2})$ depend only on the derivatives of ψ up to the order $(s-a)^+ + 2$. We now check that $\rho_{n,3}$ is bounded and $\lim_{n\to\infty}\Delta_{n,s+2} = 0$ in our context. For $k = (k_1,\dots,k_u)$, define

$$\zeta_{n,t} = \sqrt{2}Z_t(\cos(tx_{m(k_1-1)+1}),\sin(tx_{m(k_1-1)+1}),\dots,\cos(tx_{mk_u}),\sin(tx_{mk_u}))^{\mathrm{T}}.$$

Then $|\zeta_{n,t}|^2 = 2um|Z_t|^2$. In the context of Lemma 1, we must also consider

$$\begin{aligned}\zeta_{n,t}^{(M)} = \sqrt{2}(&Z_t^{(M)}\cos(tx_{m(k_1-1)+1}), Z_t^{(M)}\sin(tx_{m(k_1-1)+1}),\dots,Z_t^{(M)}\cos(tx_{mk_1}),\\ &Z_t^{(M)}\sin(tx_{mk_1}),\tilde{Z}_t^{(M)}\cos(tx_{m(k_2-1)+1}),\tilde{Z}_t^{(M)}\sin(tx_{m(k_2-1)+1}),\dots,\\ &\tilde{Z}_t^{(M)}\cos(tx_{mk_2}),\tilde{Z}_t^{(M)}\sin(tx_{mk_2}))^{\mathrm{T}}.\end{aligned}$$

In that case, we have

$$|\zeta_{n,t}^{(M)}|^2 = 2m(|Z_t^{(M)}|^2 + |\tilde{Z}_t^{(M)}|^2) = m|Z_t|^2(\sigma_M^{-2}\mathbf{1}_{\{|Z_t|\leqslant M\}} + \tilde{\sigma}_M^{-2}\mathbf{1}_{\{|Z_t|>M\}}).$$

Thus, in both cases, $\rho_{n,3}$ is bounded. To prove that $\lim_{n\to\infty} \Delta_{n,s+2} = 0$, note that for any sequence i.i.d. (Y_t) with finite moment of order s, the following limits hold:

$$\lim_{n\to\infty} n^{-1}\sum_{t=1}^{n} \mathbb{E}[|Y_t|^s \mathbf{1}_{\{|Y_t|>n^{1/2}\}}] = 0 \text{ and } \lim_{n\to\infty} n^{-3/2}\sum_{t=1}^{n} \mathbb{E}[|Y_t|^{s+1} \mathbf{1}_{\{|Y_t|\leqslant n^{1/2}\}}] = 0.$$

This is obvious since the variables Y_t are identically distributed, thus these sums are equal respectively, to $\mathbb{E}[|Y_1|^s \mathbf{1}_{\{|Y_1|>n^{1/2}\}}]$ and $n^{-1/2}\mathbb{E}[|Y_1|^{s+1} \mathbf{1}_{\{|Y_1|\leqslant n^{1/2}\}}]$. Since $|Y_1|^s \mathbf{1}_{\{|Y_1|>n^{1/2}\}}$ and $n^{-1/2}|Y_1|^{s+1} \mathbf{1}_{\{|Y_1|\leqslant n^{1/2}\}}$ converge almost surely to 0 and both sequences are bounded for all n by $|Y_1|^s$, their expectations tend to 0 as n tends to infinity by the bounded convergence theorem.

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