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Circles minimize most knot energies

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Abstract

We define a new class of knot energies (known as *renormalization energies*) and prove that a broad class of these energies are uniquely minimized by the round circle. Most of O'Hara's knot energies belong to this class. This proves two conjectures of O'Hara and of Freedman, He, and Wang. We also find energies not minimized by a round circle. The proof is based on a theorem of Lükő on average chord lengths of closed curves. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

For the past decade, there has been a great deal of interest in defining new knot invariants by minimizing various functionals on the space of curves of a given knot type. For example, imagine a loop of string bearing a uniformly distributed electric charge, floating in space. The loop will repel itself, and settle into some least energy configuration. If the loop is knotted, the potential energy of this configuration will provide a measure of the complexity of the knot.

In 1991 Jun O'Hara began to formalize this picture [12,14] by proposing a family of energy functionals e_j^p (for $j,\ p > 0$) which are based on the physicists' concept of renormalization, and are

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defined by $e_j^p[c] := (1/j)(E_j^p[c])^{1/p}$, where

$$E_j^p[c] := \iint \left(\frac{1}{|c(s)-c(t)|^j} - \frac{1}{d(s,t)^j} \right)^p \mathrm{d}s\,\mathrm{d}t, \tag{1}$$

$c : S^1 \to \mathbf{R}^3$ is a unit-speed curve, $|c(s)-c(t)|$ is the distance between $c(s)$ and $c(t)$ in space, and $d(s,t)$ is the shortest distance between $c(s)$ and $c(t)$ along the curve. O'Hara showed [15] that these integrals converge if c is smooth, embedded, and $j < 2 + 1/p$. He also showed that a minimizing curve exists in each isotopy class when $jp > 2$.

It was then natural to try to find examples of these energy-minimizing curves in various knot types. O'Hara conjectured [13] in 1992 that the energy-minimizing unknot would be the round circle for all e_j^p energies with $p \geqslant 2/j \geqslant 1$, and wondered whether this minimum would be unique. Later that year, he provided some evidence to support this conjecture by proving [14] that the limit of e_j^p as $p \to \infty$ and $j \to 0$ was the logarithm of Gromov's *distortion*, which was known to be minimized by the round circle (see [10] for a simple proof).

Two years later, Freedman, He, and Wang investigated a family of energies almost identical to the e_j^p energies, proving that the e_2^1 energy was Möbius-invariant [4], and as a corollary that the overall minimizer for e_2^1 was the round circle. For the remaining e_j^1 energies, they were able to show that the minimizing curves must be convex and planar for $0 < j < 3$ (Theorem 8.4 in [4]). They conjectured that these minimizers were actually circles.

We generalize the energies of O'Hara and Freedman–He–Wang as follows:

Definition 1. Given a curve $c : S^1 \to \mathbf{R}^n$ parametrized by arclength, let $|c(s)-c(t)|$ be the distance between $c(s)$ and $c(t)$ in space, and $d(s,t)$ denote the shortest distance between s and t along the curve. Given a function $F : \mathbf{R}^2 \to \mathbf{R}$, the energy functional in the form

$$f[c] := \iint F(|c(s)-c(t)|, d(s,t))\,\mathrm{d}s\,\mathrm{d}t, \tag{2}$$

is called the *renormalization energy* based on F if it converges for all embedded $C^{1,1}$ curves.

The main result of this paper is that a broad class of these energies are uniquely minimized by the round circle.

Theorem 2. *Suppose $F(x,y)$ is a function from $\mathbf{R}^2$ to $\mathbf{R}$. If F is convex and decreasing in x^2 for $x^2 \in (0, y^2)$ and $y \in (0, \pi)$ then the renormalization energy based on F is uniquely minimized among closed unit-speed curves of length 2π by the round unit circle.*

It is easy to check that the hypotheses of Theorem 2 are slightly weaker than requiring that F be convex and decreasing in x. The theorem encompasses both O'Hara's and Freedman, He, and Wang's conjectures:

Corollary 3. *Suppose $0 < j < 2 + 1/p$, while $p \geqslant 1$. Then for every closed unit-speed curve c in $\mathbf{R}^n$ with length 2π,*

$$E_j^p[c] \geqslant 2^{3-jp}\pi \int_0^{\pi/2} \left(\left(\frac{1}{\sin s} \right)^j - \left(\frac{1}{s} \right)^j \right)^p \mathrm{d}s \tag{3}$$

with equality if and only if c is the circle.

We must include the condition $j < 2 + 1/p$ in our theorem, for otherwise the integral defining E_j^p does not converge. We do not know whether the condition $p \geqslant 1$ is sharp, since the energies are well-defined for $0 < p < 1$, but it is required for our proof.

We use several ideas from a prophetic paper of Lükő Gábor [11], written almost 30 years before the conjectures of O'Hara and Freedman, He, and Wang were made. Lükő [4] showed that among closed, unit-speed planar curves of length 2π, circles are the only maximizers of any functional in the form $\int\int f(|c(s) - c(t)|^2)\,\mathrm{d}s\,\mathrm{d}t$, where f is increasing and concave.

Our arguments are modeled in part on Hurwitz's proof of the planar isoperimetric inequality [8,9,3, p. 111]. In Section 2, we derive a Wirtinger-type inequality (Theorem 5), which we use in Section 3 to generalize Lükő's theorem (Theorem 8). We then apply this result to obtain sharp integral inequalities for average chord lengths and distortions. In the process, we find another proof that the curve of minimum distortion is a circle. In Section 4, we give the proof of the main theorem.

Our main result, Theorem 2, gives sufficient conditions for a renormalization energy to be minimized uniquely by round circles. Since these conditions seem rather weak, it is natural to ask whether our conditions are necessary as well. Section 5 examines this question by focusing on the renormalization energies based on the functions $F(x, y) = -x^p$. For $p \leqslant 2$, these energies obey the conditions of Theorem 2, while for $p > 2$ they do not. Numerical experiments suggest that for $p \leqslant p^*$, where $3.46 < p^* < 3.5721$, the minimizing curves continue to be round circles. Thus it seems that our conditions are *not* necessary for the theorem to hold.

2. A Wirtinger-type inequality

Definition 4. Let $\lambda : \mathbf{R} \to \mathbf{R}$ be given by

$$\lambda(s) := 2 \sin \frac{s}{2}. \tag{4}$$

For $0 \leqslant s \leqslant 2\pi$, $\lambda(s)$ is the length of the chord connecting the end points of an arc of length s in the unit circle.

Our main aim in this section is to prove the following inequality, modeled after a well known lemma of Wirtinger [3, p. 111]. For simplicity, we restrict our attention to closed curves of length 2π in $\mathbf{R}^n$.

[4] There are references in the literature to papers authored both by Lükő Gábor and by Gábor Lükő. We are informed that these people are identical and that Lükő is the family name; the confusion likely results from the Hungarian convention of placing the family name first.

Theorem 5. *Let* $c : S^1 := \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{R}^n$ *be an absolutely continuous function. If* $c'(t)$ *is square integrable, then for any* $s \in \mathbf{R}$

$$\int |c(t+s) - c(t)|^2 \, \mathrm{d}t \leqslant \lambda^2(s) \int |c'(t)|^2 \, \mathrm{d}t \tag{5}$$

with equality if and only if s *is an integral multiple of* 2π *or*

$$c(t) = a_0 + (\cos t)\, a + (\sin t) b \tag{6}$$

for some vectors a_0, $a, b \in \mathbf{R}^n$.

We give two proofs of this result, one based on the elementary theory of Fourier series, and one based on the maximum principle for ordinary differential equations.

Fourier series proof. We assume that $c : S^1 \to \mathbf{R}^n \subset \mathbf{C}^n$, as the complex form of the Fourier series is more convenient. $\mathbf{C}^n$ is equipped with its standard positive definite Hermitian inner product $\langle v, w\rangle = \sum_{k=1}^n v_k \bar{w}_k$, where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. This agrees with the usual inner product on $\mathbf{R}^n \subset \mathbf{C}^n$. The norm of $v \in \mathbf{C}^n$ is given by $|v| := \sqrt{\langle v, v\rangle}$, and $i := \sqrt{-1}$.

The facts about Fourier series required for the proof are as follows. If $\phi : S^1 \to \mathbf{C}^n$ is locally square integrable then it has a Fourier expansion

$$\phi(t) = \sum_{k=-\infty}^{\infty} \phi_k \mathrm{e}^{kti}$$

(the convergence is in L^2 and the series may not converge pointwise). The L^2 norm of ϕ is given by

$$\int |\phi(t)|^2 \, \mathrm{d}t = 2\pi \sum_{k=-\infty}^{\infty} |\phi_k|^2. \tag{7}$$

If ϕ is absolutely continuous and ϕ' is locally square integrable then ϕ' has the Fourier expansion $\phi'(t) = i \sum_{k=-\infty}^{\infty} k \phi_k \mathrm{e}^{kti}$ and therefore

$$\int |\phi'(t)|^2 \, \mathrm{d}t = 2\pi \sum_{k=-\infty}^{\infty} k^2 |\phi_k|^2 = 2\pi \sum_{k=1}^{\infty} k^2 (|\phi_{-k}|^2 + |\phi_k|^2) \tag{8}$$

as the contribution to the middle sum from the term $k = 0$ is zero.

Let $\sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{kti}$ be the Fourier expansion of $c(t)$, where $a_k \in \mathbf{C}^n$. Then

$$\begin{aligned} c(t+s/2) - c(t-s/2) &= \sum_{k=-\infty}^{\infty} (\mathrm{e}^{ksi/2} - \mathrm{e}^{-ksi/2}) a_k \mathrm{e}^{kti} \\ &= 2i \sum_{k=-\infty}^{\infty} \left(\sin \frac{ks}{2}\right) a_k \mathrm{e}^{kti}. \end{aligned}$$

Therefore, using (7), we have

$$\begin{aligned}\int |c(t+s)-c(t)|^2\,\mathrm{d}t &= \int \left|c\left(t+\frac{s}{2}\right)-c\left(t-\frac{s}{2}\right)\right|^2\,\mathrm{d}t\\ &= 2\pi|2i|^2\sum_{k=-\infty}^{\infty}\left(\sin^2\frac{ks}{2}\right)|a_k|^2\\ &= 8\pi\sum_{k=1}^{\infty}\left(\sin^2\frac{ks}{2}\right)(|a_{-k}|^2+|a_k|^2). \qquad (9)\end{aligned}$$

Also, by (8) and (4),

$$\begin{aligned}\lambda^2(s)\int |c'(t)|^2\,\mathrm{d}t &= \left(4\sin^2\frac{s}{2}\right)\left(2\pi\sum_{k=1}^{\infty}k^2(|a_k|^2+|a_{-k}|^2)\right)\\ &= 8\pi\sum_{k=1}^{\infty}\left(k^2\sin^2\frac{s}{2}\right)(|a_k|^2+|a_{-k}|^2). \qquad (10)\end{aligned}$$

Subtracting (9) from (10), we set

$$\begin{aligned}\rho_c(s) &:= \lambda^2(s)\int |c'(t)|^2\,\mathrm{d}t-\int |c(t+s)-c(t)|^2\,\mathrm{d}t\\ &= 8\pi\sum_{k=2}^{\infty}\left(k^2\sin^2\frac{s}{2}-\sin^2\frac{ks}{2}\right)(|a_{-k}|^2+|a_k|^2).\end{aligned}$$

Lemma 6 (below) implies that $\rho_c(s)\geqslant 0$ with equality if and only if s is a multiple of 2π, or $a_k=a_{-k}=0$ for all $k\geqslant 2$. The latter occurs if and only if

$$c(t)=a_{-1}\mathrm{e}^{-it}+a_0+a_1\mathrm{e}^{it}=a_0+(\cos t)a+(\sin t)b, \qquad (11)$$

where $a:=a_1+a_{-1}$ and $b:=i(a_1-a_{-1})$. $\square$

Lemma 6. *Let $k\geqslant 2$ be an integer. Then*

$$\sin^2(k\theta)\leqslant k^2\sin^2(\theta) \qquad (12)$$

with equality if and only if $\theta=m\pi$ for some integer m.

Proof. If $\theta=m\pi$, for some integer m, then equality holds in (12). If θ is not an integer multiple of π, we set $q_k(\theta):=|\sin(k\theta)/\sin(\theta)|$. Then $|\cos(\theta)|<1$, and the addition formula for sine yields

$$q_{k+1}(\theta)=|\cos(\theta)q_k(\theta)+\cos(k\theta)|<q_k(\theta)+1 \qquad (13)$$

Since $q_1(\theta)\equiv 1$, we then have $q_k(\theta)<k$ by induction, which completes the proof. $\square$

Maximum principle proof. This method is an adaptation of Lükő's approach [11]. In that paper, he solves a discrete version of the problem, showing that the average squared distance between the vertices of an n-gon of constant side length is maximized by the regular n-gon. He then obtains the main result by approximation. We go directly to the continuum case, which turns out to be simpler.

To simplify notation, let $L = \int |c'(t)|^2 \, \mathrm{d}t$. Let

$$f(s) := \int |c(t+s) - c(t)|^2 \, \mathrm{d}t,$$
$$\Lambda(s) := \lambda^2(s) \int |c'(t)|^2 = L\lambda^2(s).$$

We claim that f is C^2 with

$$f'(s) = 2 \int \langle c(t) - c(t-s), c'(t) \rangle \, \mathrm{d}t,$$
$$f''(s) = 2 \int \langle c'(t-s), c'(t) \rangle \, \mathrm{d}t,$$

and initial conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2 \int |c'(t)|^2 \, \mathrm{d}t = 2L. \tag{14}$$

These formulas are clear when c is C^2 and hold in the general case by approximating by C^2 functions. The explicit formula for f'' makes it clear that f is C^2.

Next we derive a differential inequality for f, using an elementary geometric fact (which appears in a slightly different form in Lükő's paper as Lemma 7).

Lemma 7. *For any tetrahedron A, B, C, D in $\mathbf{R}^n$,*

$$|AC|^2 + |BD|^2 \leqslant |BC|^2 + |AD|^2 + 2|AB|\,|CD| \tag{15}$$

with equality if and only if AB and DC are parallel as vectors.

Proof. Denote the vectors AB, BC, CD, DA by v_1, v_2, v_3, v_4. Then $\sum v_i = 0$, and

$$\begin{aligned}
|AC|^2 + |BD|^2 &= \frac{1}{2}(|v_1 + v_2|^2 + |v_2 + v_3|^2 + |v_3 + v_4|^2 + |v_4 + v_1|^2) \\
&= \sum_{i=1}^{4} |v_i|^2 + \langle v_1, v_2 \rangle + \langle v_2, v_3 \rangle + \langle v_3, v_4 \rangle + \langle v_4, v_1 \rangle \\
&= \sum_{i=1}^{4} |v_i|^2 + \langle v_1 + v_3, v_2 + v_4 \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{4} |v_i|^2 - |v_1 + v_3|^2 \\
&\leqslant \sum_{i=1}^{4} |v_i|^2 - (|v_1| - |v_3|)^2 \\
&= |v_2|^2 + |v_4|^2 + 2|v_1||v_3| = |BC|^2 + |AD|^2 + 2|AB|\,|CD|.
\end{aligned}
$$

Equality holds if and only if $v_3 = -\rho v_1$ for some $\rho > 0$, which is equivalent to AB and DC being parallel as vectors. □

For any t, s and h, we can apply Lemma 7 to the tetrahedron $c(t)$, $c(t+s+h)$, $c(t+s)$, $c(t+h)$ to derive the equation

$$
\begin{aligned}
&|c(t+s) - c(t)|^2 + |c(t+s+h) - c(t+h)|^2 \leqslant |c(t+s+h) - c(t+s)|^2 \\
&\quad + |c(t+h) - c(t)|^2 + 2|c(t+s+h) - c(t)|\,|c(t+s) - c(t+h)|.
\end{aligned}
$$

Holding s, h fixed and integrating with respect to t,

$$
\begin{aligned}
2f(s) &\leqslant 2f(h) + 2\int |c(t+s+h) - c(t)|\,|c(t+s) - c(t+h)|\,\mathrm{d}t \\
&\leqslant 2f(h) + 2\sqrt{f(s+h)f(s-h)}
\end{aligned}
$$

by the Cauchy–Schwartz inequality. Therefore, $f(s) \leqslant f(h) + \sqrt{f(s+h)f(s-h)}$. For any fixed s, this can be rewritten

$$
g(h) := \tfrac{1}{2}(\log f(s+h) + \log f(s-h)) - \log(f(s) - f(h)) \geqslant 0.
$$

When s is not a multiple of 2π, $f(s) > 0$ and g is well-defined for small h. Further, g has a local minimum at $h = 0$, and so the second derivative of g is non-negative at zero. Using (14), this tells us that

$$
\frac{\mathrm{d}^2}{\mathrm{d}s^2} \log f(s) \geqslant \frac{-2L}{f(s)}. \tag{16}
$$

Meanwhile, $\Lambda(s)$ satisfies the differential equation

$$
\frac{\mathrm{d}^2}{\mathrm{d}s^2} \log \Lambda(s) = \frac{-2L}{\Lambda(s)}. \tag{17}
$$

We are trying to show that $f(s) \leqslant \Lambda(s)$ and that if equality holds for any $s \in (0, 2\pi)$, then $f(s) \equiv \Lambda(s)$. Let

$$
u(s) = \log \frac{f(s)}{\Lambda(s)} = \log f(s) - \log \Lambda(s).
$$

In these terms, we want to show that $u(s) \leqslant 0$ and that if $u(s) = 0$ for some $s \in (0, 2\pi)$ then $u \equiv 0$. Using (16) and (17),

$$u''(s) \geqslant \frac{-2L}{f(s)} + \frac{2L}{\Lambda(s)} = \frac{2L}{f(s)}\left(\frac{f(s)}{\Lambda(s)} - 1\right) = \frac{2L}{f(s)}(\mathrm{e}^{u(s)} - 1) \geqslant \frac{2L}{f(s)} u(s).$$

By two applications of L'Hospital's rule, we compute $\lim_{s\to 0} u(s) = 0$. Thus, the limit $\lim_{s\to 2\pi} u(s)$ is zero, as well. So if u is ever positive, it will have a positive local maximum at some point $s_0 \in (0, 2\pi)$. At that point,

$$0 \geqslant u''(s_0) \geqslant \frac{2L}{f(s_0)} u(s_0) > 0,$$

which is a contradiction. So u is non-positive on $(0, 2\pi)$. Further, if u is zero at any point in $(0, 2\pi)$, the strong maximum principle [22, Theorem 17, p. 183] implies that u vanishes on the entire interval. Thus $f(s) \leqslant \Lambda(s)$ with equality at any point of $(0, 2\pi)$ if and only if $f(s) \equiv \Lambda(s)$.

Last, we show that if $f(s) = \int |c(t+s) - c(t)|^2 \, \mathrm{d}t \equiv \lambda^2(s) \int |c'(t)|^2 \, \mathrm{d}t = \Lambda(s)$, then c is an ellipse. By our work above, if $f = \Lambda$, then for each fixed s, c maximizes $\int |c(t+s) - c(t)|^2 \, \mathrm{d}t$ subject to the constraint that $\int |c'(t)|^2 \, \mathrm{d}t$ is held constant. The Lagrange multiplier equation for this variational problem is

$$c''(t) = M(c(t+s) - 2c(t) + c(t-s)),$$

where M is a constant depending on s. When $s = \pi$ we can use the fact that c has period 2π and this becomes

$$c''(t) = 2M(c(t+\pi) - c(t)). \tag{18}$$

Differentiating twice with respect to t, and using both the periodicity and (18),

$$\begin{aligned} c''''(t) &= 2M(c''(t+\pi) - c''(t)) \\ &= 4M^2(c(t) - c(t-\pi) - c(t+\pi) + c(t)) \\ &= -8M^2(c(t+\pi) - c(t)) \\ &= -4Mc''(t). \end{aligned}$$

So c'' satisfies the equation $g'' = -4Mg$ and has period 2π. This implies that $4M = k^2$ for some $k \in \mathbf{Z}$, and $c''(t) = (\cos kt)V + (\sin kt)W$ with V and W in $\mathbf{R}^n$. But $k = \pm 1$, for otherwise $f(2\pi/k) = 0 \neq \Lambda(2\pi/k)$, a contradiction. Taking two antiderivatives,

$$c(t) = a_0 + tb_0 + (\cos t)a + (\sin t)b,$$

with a_0, b_0, a, b in $\mathbf{R}^n$. Periodicity implies that $b_0 = 0$, completing the proof. □

We remark that by (11), extremals for the inequality of Theorem 5 are either ellipses or double coverings of line segments, depending on whether a and b are linearly independent. Thus, the set of extremal curves is invariant under affine maps of $\mathbf{R}^n$. When the extremal is an ellipse, the parameterization is a constant multiple of the *special affine arclength* (cf. [2, p. 7], [21, p. 56]). It would be interesting to find an affine invariant interpretation of inequality (5) or of the deficit $\rho_c(s)$ used in the first proof—especially when c is a convex planar curve.

3. Inequalities for concave functionals

We now apply Theorem 5 to obtain an inequality for chord lengths. Recall Definition 4, that $\lambda(s)$ is the length of a chord of arclength s on the unit circle.

Theorem 8. *Let c be a closed, unit-speed curve of length 2π in $\mathbf{R}^n$. For $0 < s < 2\pi$, if $f:\mathbf{R}\to\mathbf{R}$ is increasing and concave on $(0, d(0,s)^2]$, where $d(s,t)$ is the shortest distance along the curve between $c(s)$ and $c(t)$, then*

$$\frac{1}{2\pi}\int f(|c(t+s)-c(t)|^2)\,\mathrm{d}t \leqslant f(\lambda^2(s)) \tag{19}$$

and equality holds if and only if c is the unit circle.

Proof. The shortest distance between $c(t)$ and $c(t+s)$ along the curve is $d(0,s)$. Thus, the squared chord length $|c(t+s)-c(t)|^2$ is in $(0, d(0,s)^2]$, except when $s=0$. Being undefined at this point does not affect the existence of the integrals. Using Jensen's inequality for concave functions [18, p. 115], Theorem 5, that f is increasing, and that $|c'(t)| = 1$ for almost all t, we have

$$\begin{aligned}
\frac{1}{2\pi}\int f(|c(t+s)-c(t)|^2)\,\mathrm{d}t &\leqslant f\left(\frac{1}{2\pi}\int |c(t+s)-c(t)|^2\,\mathrm{d}t\right)\\
&\leqslant f\left(\frac{\lambda^2(s)}{2\pi}\int |c'(t)|^2\,\mathrm{d}t\right)\\
&= f(\lambda^2(s)).
\end{aligned}$$

If equality holds in (19), then the above string of inequalities implies that equality holds between the two middle terms, i.e., equality holds in (5). Thus, since $0 < s < 2\pi$, we may apply Theorem 5 to conclude that $c(t)$ must be as in (6). Since c has unit speed, it follows that

$$c'(t) = -(\sin t)a + (\cos t)b$$

is a unit vector for all t, which forces the vectors a and b to be orthonormal, and so implies that c is the unit circle. Conversely, if c is the unit circle, then $|c(t+s)-c(t)| = \lambda(s)$ for all t and therefore equality holds in (19). □

Letting $f(x) = \sqrt{x}$ in Theorem 8, we obtain the following inequality:

Corollary 9. *Let c be a closed, unit-speed curve of length 2π in $\mathbf{R}^n$. Then for any s,*

$$\frac{1}{2\pi}\int |c(t+s)-c(t)|\,\mathrm{d}t \leqslant \lambda(s), \tag{20}$$

with equality if and only if c is the unit circle.

Next we apply Theorem 8 to obtain sharp inequalities for Gromov's *distortion* [6,10]. By definition, the distortion of a curve is the maximum value of the ratio of the distance along the curve to the distance in space for all pairs of points on the curve. As we mentioned above, distortion is a limit of O'Hara energies: $\exp(e_0^{\infty}(c)) = \mathrm{distort}(c)$ [15, p. 150].

Inequality (22) is due to Gromov [7, pp. 11–12] (but see also [10]). As always, while we state our results for curves of length 2π, the corresponding result holds for curves of arbitrary length.

Corollary 10. *For every closed, unit-speed curve c of length 2π in $\mathbf{R}^n$*

$$\mathrm{distort}_s(c) := \sup_{t\in\mathbf{R}} \frac{s}{|c(t+s)-c(t)|} \geqslant \frac{s}{\lambda(s)}, \tag{21}$$

$$\mathrm{distort}(c) := \sup_{s\in(0,\pi]} \mathrm{distort}_s(c) \geqslant \frac{\pi}{2} \tag{22}$$

with equalities if and only if c is the unit circle.

Proof. In both cases equality is clear for the unit circle. By the mean value property of integrals and inequality (20),

$$\frac{1}{\mathrm{distort}_s(c)} = \inf_{t\in\mathbf{R}} \frac{|c(t+s)-c(t)|}{s} \leqslant \frac{1}{2\pi s}\int |c(t+s)-c(t)|\,\mathrm{d}t \leqslant \frac{\lambda(s)}{s},$$

establishing (21). Further, equality in (21) implies equality in (20), which, by Theorem 8, happens if and only if c is the unit circle.

The proof of (22) follows easily from (21):

$$\mathrm{distort}(c) = \sup_{s\in(0,\pi]} \mathrm{distort}_s(c) \geqslant \mathrm{distort}_{\pi}(c) \geqslant \frac{\pi}{\lambda(\pi)} = \frac{\pi}{2},$$

and again equality implies in particular that $\mathrm{distort}_{\pi}(c) = \pi/\lambda(\pi)$, which, by (21), happens if and only if c is the unit circle. □

For general maps $f : M \to \mathbf{R}^n$ of a compact m dimensional Riemannian manifold to Euclidean space, Gromov [6, p. 115] has given the general lower bound

$$\mathrm{distort}(f)^2 \geqslant \frac{\lambda_1(M)\iint_{M\times M} d(x,y)^2\,\mathrm{d}x\,\mathrm{d}y}{2m\,\mathrm{Vol}(M)^2},$$

where $d(x,y)$ is the Riemannian distance between x and y and $\lambda_1(M)$ is the first eigenvalue of the Laplace operator on M. When $M = S^1$ this reduces to $\mathrm{distort}(f) \geqslant \pi/\sqrt{6}$.

4. Proof of the main theorem

We are now ready to prove the main theorem. We start by recalling its statement.

Theorem 11. *Suppose $F(x,y)$ is a function from $\mathbf{R}^2$ to $\mathbf{R}$. If F is convex and decreasing in x^2 for $x^2 \in (0,y^2]$ and $y \in (0,\pi)$ then the renormalization energy based on F,*

$$f[c] := \iint F(|c(s)-c(t)|, d(t,s))\,\mathrm{d}t\,\mathrm{d}s,$$

is uniquely minimized among closed unit-speed curves of length 2π by the round unit circle.

Proof. Making the substitution $s \mapsto s-t$, $t \mapsto t$, changing the order of integration, and using the fact that $d(s,t) = d(s+a,t+a)$ for any a, we have

$$\iint F(|c(s)-c(t)|, d(s,t))\,\mathrm{d}s\,\mathrm{d}t = \iint F(|c(t+s)-c(t)|, d(0,s))\,\mathrm{d}t\,\mathrm{d}s.$$

For each $s \in (0,2\pi)$, if we let $f(x) = -F(x, d(0,s))$, then

$$\int F(|c(t+s)-c(t)|, d(0,s))\,\mathrm{d}t = -\int f(|c(t+s)-c(t)|^2)\,\mathrm{d}t$$

and f is increasing and concave on $(0, d(0,s)^2]$. By Theorem 8,

$$-\int f(|c(t+s)-c(s)|^2)\,\mathrm{d}t \geqslant -2\pi f(\lambda^2(s)) \tag{23}$$

with equality if and only if c is the unit circle. Integrating this from $s=0$ to 2π tells us that $f[c]$ is greater than or equal to the corresponding value for the unit circle, with equality if and only if (23) holds for almost all $s \in [0,2\pi]$. But if equality holds for some value of $s \in (0,2\pi)$, then c is the unit circle. □

We now prove the corollary.

Corollary 12. *Suppose $0 < j < 2 + 1/p$, while $p \geqslant 1$. Then for every closed unit-speed curve c in $\mathbf{R}^n$ with length 2π,*

$$E_j^p[c] \geqslant 2^{3-jp}\pi \int_0^{\pi/2} \left(\left(\frac{1}{\sin s}\right)^j - \left(\frac{1}{s}\right)^j \right)^p \mathrm{d}s. \tag{24}$$

with equality if and only if c is the circle.

Proof. If we let

$$F(x,y) := \left(\frac{1}{x^j} - \frac{1}{y^j}\right)^p,$$

then using (1), we see that $E_j^p[c]$ is the renormalization energy based on F. We must show that $F(\sqrt{x}, y)$ is convex and decreasing in x for $x \in (0,y^2]$ for all $y \in (0,\pi)$. If we let $u := x^2$, it suffices to check the signs of the first and second partial derivatives of $F(\sqrt{u}, y)$ with respect to u for $u \in (0,y^2)$.

When $p \geqslant 1$, $y \neq 0$, and $u \in (0, y^2)$,

$$\frac{\partial F(\sqrt{u}, y)}{\partial u} = -\frac{jp}{2u^{(j+2)/2}}\left(\frac{1}{u^{j/2}} - \frac{1}{y^j}\right)^{p-1} < 0$$

and

$$\frac{\partial^2 F(\sqrt{u}, y)}{\partial u^2} = \frac{j(j+2)p}{4u^{(j+4)/2}}\left(\frac{1}{u^{j/2}} - \frac{1}{y^j}\right)^{p-1} + \frac{j^2 p(p-1)}{4u^{(j+2)}}\left(\frac{1}{u^{j/2}} - \frac{1}{y^j}\right)^{p-2} > 0.$$

Since $u^{j/2} = x^j$ can be arbitrarily close to y^j if the curve is nearly straight, examining this equation shows that the condition $p \geqslant 1$ is required to enforce the convexity of F in u.

So for every $y \neq 0$, F is decreasing and convex in x^2 on $(0, y^2]$. Further, a direct calculation shows that $\int F(\lambda^2(s), s)\,\mathrm{d}s < \infty$ when $j < 2 + 1/p$.

Thus F satisfies the hypotheses of Theorem 11. Computing the energy of the round circle by changing the variable $s \mapsto 2s$ and noting that the resulting integrand is symmetric about $s = \pi/2$, we have

$$E_j^p[c] \geqslant 2\pi \int F(\lambda^2(s), d(0, s))\,\mathrm{d}s = 2^{3-jp}\pi \int_0^{\pi/2} \left[\left(\frac{1}{\sin s}\right)^j - \left(\frac{1}{s}\right)^j\right]^p \mathrm{d}s,$$

with equality if and only if c is the unit circle. □

5. Analysis of our results

We have now finished the proof of our main theorem, and shown that a family of e_j^p energies satisfy the hypotheses of our theorem. This process has established sufficient conditions for a renormalization energy based on a function $F(x, y)$ to be uniquely minimized by round circles: $F(x, y)$ must be decreasing and convex in x^2 for each value of y. Are these conditions necessary?

To examine this question, we focus on the natural renormalization energies based on the functions $F_p(x, y) = -x^p$ for various values of $p > 0$. Each function $F_p(x, y)$ is decreasing in x^2. If $p < 2$, it is also convex in x^2, while for $p > 2$ it is not. This energy corresponds to the average chord length of the curve C, measured in the L^p norm. One can show that a minimizer exists for each of these renormalization energies. These minimizers must all be convex plane curves by Reshetnyak's theorem on inextensible mappings [16,17]. (A weaker version of this theorem, which is also sufficient for our purposes, is known as Sallee's stretching theorem [19]. See also [5].) If each minimizer is unique, the family of minimizers is continuous in p by a compactness argument.

We believe that there is a critical exponent p^* so that the F_p energies are minimized by round circles exactly when $p \leqslant p^*$.

Proposition 13. *The critical exponent* p^* (*if it exists*) *obeys* $2 \leqslant p^* < 3.5721$.

Proof. The first half of the inequality is our main theorem. The second half comes from a direct computation: for $p > 3.5721$, the double-covered line segment has less energy than the round circle. □

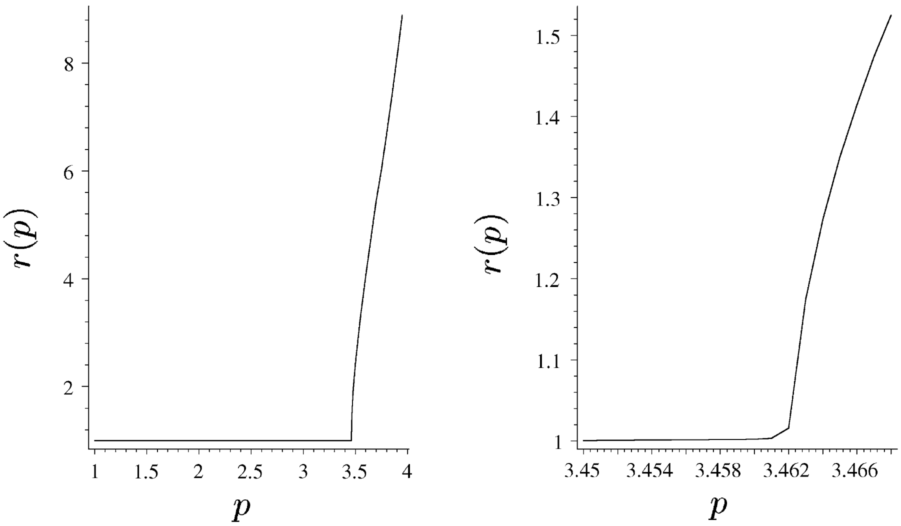

Fig. 1. This figure shows two plots of the ratio $r(p)$ of the widest and narrowest projections of the computed maximizers of average chord length to the pth power for values of p between 1 and 4.

Using Brakke's *Evolver* [1], we found numerical approximations to the minimizing curves for values of p between 2 and 4. To measure the shape of these curves, we computed the ratio $r(p)$ of the largest and smallest diameter of any projection of each curve to any line in the plane. Since all of these curves are convex, a value of $r(p)$ close to one indicates a curve close to a circle. By this metric, the computed minimizers are numerically very near to circles for $2 \leqslant p \leqslant 3.45$. (A graph of the results appears in Fig. 1.) This leads us to the following conjecture, which is somewhat surprising!

Conjecture 1. *All the renormalization energies based on $F_p(x, y) = -x^p$ are minimized by round circles for $p < 3.46$. (In particular, the convexity condition of Theorem 2 is not necessary.)*

We remark that Proposition 13 shows that some conditions *are* necessary for our main theorem to hold; that is, not every renormalization energy is minimized by a round circle!

We also observe that our conditions come from the use of the squared chord length in Theorem 4 and Jensen's inequality in Theorem 8. Neither of these techniques seems amenable to further improvement, so a proof of Conjecture 1 will have to come from altogether new ideas.

We conclude with a list of open problems:

Open Problem 1. *Find necessary conditions for a renormalization energy to be uniquely minimized by the round circle.*

Open Problem 2. *Find the critical exponent p^*, if it exists.*

Open Problem 3. *Describe the shape of the minimizers for the renormalization energies based on $F_p = -x^p$ for $p > p^*$.* Numerical evidence argues that these are stretched oval shapes, but they do not seem to be ellipses, as one might have conjectured from reading Theorem 5.

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